Speculation in First-Price Auctions with Resale

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Abstract

We analyze first-price auctions with two asymmetric bidders, where the winner can offer the good for resale to the loser. One bidder has a private value for the good, the other bidder—the speculator—has zero value. We show that, independently of the resale market rules, the speculator's expected profit equals zero. Nevertheless, the opportunity for resale can create a role for an active speculator, destroy the efficiency of the auction, and increase the initial seller's expected revenue.

1 Introduction

A seller can often not prevent her goods from being resold in the future. It is therefore important to study properties of standard sales mechanisms when resale is possible. Furthermore, models of markets with resale should allow for entry of agents with no (consumption) value for the goods traded in the market. Such agents might be called *speculators* because their only reason for buying is reselling. One would like to understand how the presence of speculators affects the market, and whether or not speculators can make profits.

We analyze the role of speculators in first-price auctions with resale. To keep the analysis as simple as possible, we consider an environment with just two bidders, one bidder with a private value and one speculator; our qualitative results continue to hold in a model with multiple speculators. Both bidders can participate in the auction. Subsequently, the winner can offer the good to the loser. No information becomes public after the auction

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beyond what is revealed via the winning bid. I.e., we consider a first-price auction as implemented via a Dutch auction.

The resale market can take any form: the resale seller may be able to offer the good in an optimal mechanism (given her posterior beliefs), she may be restricted to use a simple mechanism like posting a fixed price, or bargaining might enfold, to give just a few examples. Each resale market affects the bidding incentives in the auction differently. Our solution concept is perfect Bayesian equilibrium in the multi-stage game that begins with the auction and ends after the resale market has closed.

In the first part of the paper we show that, independently of the resale market rules, the speculator's expected profit equals zero in any equilibrium. To understand the significance of this result, suppose that the private-value bidder ignores possible entry of a speculator. She would then bid 0 and expect to win the auction. Hence, by making a small bid the speculator could win for sure and possibly make a large profit by reselling. We show that the speculator's equilibrium expected profit nevertheless equals zero. This contrasts the result that speculators can make large profits in second-price and English auctions, for a wide range of resale mechanisms (Garratt and Tröger, 2003).

In the second part, we provide a detailed analysis of the first-price auction with a specific resale market rule: the resale seller offers the good by posting a fixed price; i.e., the resale mechanism is an optimal take-it-or-leave-it offer. This mechanism is particularly simple and is also—given the posterior beliefs in an equilibrium—optimal for the resale seller across all conceivable mechanisms if the distribution for the bidder's value has a monotone hazard rate.

We construct an equilibrium for the game where the resale mechanism is an optimal take-it-or-leave-it offer and show that the equilibrium is essentially unique. In equilibrium, the speculator wins the auction with positive probability, the post-resale allocation is inefficient with positive probability, and the expected revenue of the initial seller is positive. In the absence of a resale opportunity, the private-value bidder would win the auction for sure, the allocation would be efficient, and the initial seller would collect no revenue. Therefore, the opportunity for resale changes the properties of the first-price auction in three important ways: it creates a role for an active speculator, it destroys the efficiency of the auction, and it increases the initial seller's expected revenue.

The result that a resale opportunity can be detrimental to efficiency contradicts the widespread belief that a seller who is interested in efficiency (e.g., a government agency) should embrace resale. Second-price and English auctions also have equilibria such that the opportunity for resale is detrimental for efficiency (Garratt and Tröger, 2003). A crucial difference is that in second-price and English auctions with a resale opportunity an efficient equilibrium continues to exist.

Attracting additional bidders to an auction is traditionally considered desirable (Bulow and Klemperer, 1996). Our results confirm this sentiment in the revenue dimension, but not in the efficiency dimension. Sellers who are interested in efficiency should be wary *which* additional bidders they are attracting.

Haile (1999, 2003), Gupta and Lebrun (1999), and Krishna (2002, Ch. 4.4) analyze first-price auctions with resale. None of these papers considers possible entry of speculators. Gupta and Lebrun (1999) consider asymmetric bidders and assume common knowledge of values in the resale market. In this model, the post-resale allocation is efficient while it would be inefficient in the absence of a resale opportunity; i.e., resale is beneficial for efficiency. Haile (1999, 2003) analyzes resale in symmetric environments where some buyers cannot participate in the initial auction and/or at the time of the initial auction each bidder is uncertain about her own value.[1] Assuming Haile's symmetric separating equilibria, the opportunity for resale is clearly beneficial for efficiency. Krishna (2002, Ch. 4.4) considers asymmetric bidders and a resale market where, as in our model, the auction winner makes a take-it-or-leave-it offer to the loser. He shows that the game has no efficient equilibrium, but stops short of actually constructing an equilibrium. For more literature on auctions with resale, see Haile (2003) and Garratt and Tröger (2003).

In Section 2 we describe the market and the show that the private-value bidder's bid function is weakly increasing in the winning range. In Section 3 we show that speculation is not profitable. In Section 4 we analyze resale via an optimal take-it-or-leave-it offer. In Section 5 we make concluding remarks.

2 Model

We consider two risk-neutral agents, *buyer* 1 and *speculator* s, who are interested in a single indivisible *good*.[2] The good is initially owned by a seller

[1] Note that in Haile's terminlogy, the term "use value" stands for our term "value." His term "valuation" refers to the opportunity cost of not winning the initial auction.

[2] Including multiple speculators into the model would leave the qualitative results unchanged, but complicate the notation. Equilibria with multiple active speculators can

who offers it via a first-price auction without reserve price.[3] The buyer has the random value $\tilde{\theta}_1 \in [0,1]$ for the good. The speculator has value 0 for the good. Let $F(\cdot)$ denote the distribution function for $\tilde{\theta}_1$. We assume that $F(\cdot)$ is continuous, $F(0) = 0$, $F(1) = 1$, and $F(\cdot)$ has a positive and bounded density on $[0,1]$. Before period 1, the buyer privately learns the realization of her value $\theta_1 = \tilde{\theta}_1$. In period 1, a first-price auction without reserve price takes place. The buyer and the speculator simultaneously submit bids $b_1, b_s \geq 0$. The highest bidder becomes the new owner of the good. To avoid some technicalities, we assume that if there is a tie then the buyer becomes the winner. We also assume that the bid of the loser remains private after the auction; i.e., we consider a first-price auction as implemented via a Dutch auction (see Section 5 on changing this assumption).

There is a resale period 2 where the period-1 winner can offer the good to the loser. Period-2 payoffs are discounted according to some factor $\delta \leq 1$. If the buyer is the period-1 winner then it is optimal for her to consume the good herself immediately; we will assume this in the following. If the speculator wins (i.e., $b_1 < b_s$) she can offer the good for resale. We do not make any assumptions on the resale market rules: either the speculator or the buyer may be a able to commit to a mechanism, or the agents may bargain.

Let us consider period 2 assuming that the speculator has won in period 1. Then, b_s is common knowledge and the speculator's posterior belief about the buyer's value can depend on b_s. On the other hand, b_1 is not payoff relevant in period 2. Therefore, we can assume that the actions taken by the buyer and the speculator in period 2 determine an expected payment function $P(\cdot \mid b_s)$ and a probability-of-sale function $Q(\cdot \mid b_s)$ such that the buyer's expected payment in period 2 is $P(\theta_1 \mid b_s)$, and her probability of obtaining the good in period 2 is $Q(\theta_1 \mid b_s)$. If period 2 includes multiple time-discounted stages at which the good may be sold and payments may be made, $Q(\theta_1 \mid b_s)$ and $P(\theta_1 \mid b_s)$ are defined as the present value (at the beginning of period 2) of the probability of sale, and payment, respectively.

By the revelation principle, the buyer's incentive compatibility constraints

$$\begin{aligned}&\forall b_s \geq 0,\ \theta_1, \theta_1' \in [0,1] :\\ &\qquad Q(\theta_1 \mid b_s)\theta_1 - P(\theta_1 \mid b_s) \geq Q(\theta_1' \mid b_s)\theta_1 - P(\theta_1' \mid b_s) \qquad (1)\end{aligned}$$

exist, with each speculator's expected profit being equal to zero.

[3] We expect our qualitative results to remain valid if the auction is augmented by a small reserve price. A revenue-maximizing reserve price would shut out the speculator, but would also fail to achieve an efficient allocation.

are satisfied. Moreover, voluntary participation in the resale market implies the participation constraints

$$\forall b_s \geq 0,\ \theta_1 \in [0,1]:\ Q(\theta_1 \mid b_s)\theta_1 - P(\theta_1 \mid b_s) \geq 0. \tag{2}$$

Optimality and participation requirements for the speculator's resale actions may be introduced as well, but they would play no role for our results. A standard argument using (1) implies that $Q(\theta_1 \mid b_s)$ and $P(\theta_1 \mid b_s)$ are weakly increasing in θ_1.

A strategy for the buyer is given by a bid function $b_1(\cdot)$. A strategy for the speculator is given by a random bid $\tilde{b}_s$ (we allow for randomization of the speculator because otherwise there may be no equilibrium). Let $H(\cdot)$ denote the distribution function for $\tilde{b}_s$. Given the speculator's strategy, the expected payoff of the buyer with value θ_1 and bid b_1 is

$$\begin{aligned} u_1(b_1,\theta_1) &= H(b_1)(\theta_1 - b_1) \\ &\quad + (1 - H(b_1))\ \delta\ E[Q(\theta_1 \mid \tilde{b}_s)\theta_1 - P(\theta_1 \mid \tilde{b}_s) \mid \tilde{b}_s > b_1]. \end{aligned} \tag{3}$$

Given the buyer's strategy, the expected payoff of the speculator with bid b_s is

$$u_s(b_s) = \Pr[b_s > b_1(\tilde{\theta}_1)] \left(\delta E[P(\tilde{\theta}_1 \mid b_s) \mid b_s > b_1(\tilde{\theta}_1)] - b_s \right). \tag{4}$$

A profile $(b_1(\cdot), \tilde{b}_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ is called a *perfect Bayesian equilibrium* if (1), (2),

$$\forall \theta_1 \in [0,1]:\ b_1(\theta_1) \in \arg\max_{b_1 \geq 0} u_1(b_1, \theta_1), \tag{5}$$

and

$$\Pr[\tilde{b}_s \in \arg\max_{b \geq 0} u_s(b)] = 1 \tag{6}$$

are satisfied.

In Lemma 1 we show that in equilibrium the buyer's bid function is weakly increasing in the winning range. The main idea of the proof is to use the incentive compatibility constraints (1) in order to show that the buyer's payoff function has increasing differences; i.e., that the payoff gain from a bid increase is increasing in the buyer's value.

Lemma 1 *Let $(b_1(\cdot), \tilde{b}_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium and let $H(\cdot)$ denote the distribution function for $\tilde{b}_s$. Then*

$$\forall \theta_1, \theta_1' \in [0,1]:\ \textit{if } H(b_1(\theta_1')) > 0 \textit{ and } \theta_1 > \theta_1' \textit{ then } b_1(\theta_1) \geq b_1(\theta_1').$$

Proof. We begin by showing that

$$\begin{aligned}&\forall b_1, b_1' \geq 0,\ \ \theta_1, \theta_1' \in [0,1]: \\ &\quad \text{if } b_1 > b_1',\ \ \theta_1 > \theta_1',\ \text{ and } \ \Pr[b_1 \geq \tilde{b}_s] > \Pr[b_1' \geq \tilde{b}_s] \\ &\quad \text{then } u_1(b_1, \theta_1) - u_1(b_1, \theta_1') > u_1(b_1', \theta_1) - u_1(b_1', \theta_1').\end{aligned} \tag{7}$$

We have

$$\begin{aligned}&u_1(b_1, \theta_1) - u_1(b_1, \theta_1') \\ &= \Pr[b_1 \geq \tilde{b}_s](\theta_1 - \theta_1') + \delta \int_{(b_1,\infty)} (Q(\theta_1 \mid b_s)\theta_1 - Q(\theta_1' \mid b_s)\theta_1' \\ &\qquad\qquad\qquad\qquad - (P(\theta_1 \mid b_s) - P(\theta_1' \mid b_s)))\mathrm{d}H(b_s).\end{aligned}$$

From this we get

$$\begin{aligned}&u_1(b_1, \theta_1) - u_1(b_1, \theta_1') - (u_1(b_1', \theta_1) - u_1(b_1', \theta_1')) \\ &= \Pr[b_1' < \tilde{b}_s \leq b_1](\theta_i - \theta_i') \\ &\quad -\delta \int_{(b_1', b_1]} (Q(\theta_1 \mid b_s)\theta_1 - Q(\theta_1' \mid b_s)\theta_1' \\ &\qquad\qquad\qquad\qquad - (P(\theta_1 \mid b_s) - P(\theta_1' \mid b_s)))\mathrm{d}H(b_s), \\ &\overset{(1)}{\geq} \Pr[b_1' < \tilde{b}_s \leq b_1](\theta_i - \theta_i') - \delta \int_{(b_1', b_1]} \underbrace{Q(\theta_1 \mid b_s)}_{\leq 1}(\theta_1 - \theta_1')\mathrm{d}H(b_s), \\ &\geq \Pr[b_1' < \tilde{b}_s \leq b_1](1-\delta)(\theta_1 - \theta_1') \ > 0,\end{aligned}$$

which implies (7). To prove the lemma, suppose that $b_1(\theta_1) < b_1(\theta_1')$. Define $b_1 = b_1(\theta_1')$ and $b_1' = b_1(\theta_1)$. Then we have $\Pr[b_1 \geq \tilde{b}_s] > \Pr[b_1' \geq \tilde{b}_s]$ because otherwise type θ_1' could improve her expected payoff by lowering her bid from b_1 to b_1'. From (5) we get

$$u_1(b_1, \theta_1) - u_1(b_1, \theta_1') \leq 0 \leq u_1(b_1', \theta_1) - u_1(b_1', \theta_1').$$

This contradicts (7) and thereby completes the proof of the lemma. *QED*

For any given equilibrium $(b_1(\cdot), \tilde{b}_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ and all $b_s \geq 0$, define[4]

$$\phi(b_s) = \sup\{\theta_1 \in [0,1] \mid b_1(\theta_1) < b_s\}.$$

[4] Let $\sup \emptyset = 0$.

By Lemma 1, the set of buyer types against which the speculator wins with bid b_s is $[0, \phi(b_s))$ or $[0, \phi(b_s)]$, for all b_s with $H(b_s) > 0$. Thus, in equilibrium we have

$$\forall b_s \geq 0,\ H(b_s) > 0 :\ u_s(b_s) = F(\phi(b_s)) \left(\delta E[P(\tilde{\theta}_1 \mid b_s) \mid \tilde{\theta}_1 \leq \phi(b_s)] - b_s\right).$$

Now consider any perfect Bayesian equilibrium and a speculator's bid b_s that has a positive probability of winning. After winning with b_s, the speculator's posterior distribution $\Pi(\cdot \mid b_s)$ for the buyer's value is given for all $\theta_1 \in [0,1]$ by

$$\Pi(\theta_1 \mid b_s) = \Pr[\tilde{\theta}_1 \leq \theta_1 \mid b_1(\tilde{\theta}_1) < b_s] = \frac{\min\{F(\theta_1), F(\phi(b_s))\}}{F(\phi(b_s))}. \tag{8}$$

Finally, note that if the resale mechanism is an optimal take-it-or-leave-it offer function $T(\cdot)$ then

$$\forall b_s \geq 0 :\ T(b_s) \in \arg\max_{p \geq 0} (1 - \Pi(p \mid b_s))p, \tag{9}$$

where for bids b_s that win with probability 0 the distribution $\Pi(\cdot \mid b_s)$ is arbitrary. The direct mechanisms that correspond to the optimal take-it-or-leave-it offers are given as follows.

$$\forall \theta_1 \geq T(b_s) :\ P(\theta_1 \mid b_s) = T(b_s),\ Q(\theta_1 \mid b_s) = 1, \tag{10}$$

$$\forall \theta_1 < T(b_s) :\ P(\theta_1 \mid b_s) = 0,\ Q(\theta_1 \mid b_s) = 0. \tag{11}$$

Note that (10) and (11) imply (1) and (2). A strategy-belief profile

$$(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$$

is called a *perfect Bayesian equilibrium with optimal take-it-or-leave-it offers* if conditions (5), (6), (8), (9), (10), and (11) are satisfied.

Optimal take-it-or-leave-it offers are a particularly important example because these mechanisms are resale-revenue maximizing for the resale seller among all conceivable mechanisms, provided $F(\cdot)$ has a monotone hazard rate. Indeed, if the hazard rate for $F(\cdot)$ is strictly increasing, then the same is true for the posteriors $\Pi(\cdot \mid b_s)$ defined by (8) and we can apply Myerson's (1981) result.

3 Speculation is not profitable

In this section we show that the expected payoff of the speculator is zero in any equilibrium. Lemma 2 shows that in the winning range the buyer will not bid above her value. Lemma 3 shows that with positive probability the speculator will make arbitrarily small bids. Thus all buyer types (except $\theta_1 = 0$) bid below their values (Lemma 4). Proposition 1 concludes that the speculator's expected payoff is 0 in every equilibrium.

Lemma 2 below shows that in the winning range the buyer will not bid above her value. This follows because, starting from a candidate above-value bid, a deviation to a bid equal to her value replaces a sure loss by a non-negative payoff in the event that the speculator bids in the interval between her value and the candidate bid. Note that in the event where the speculator overbids the buyer, lowering her bid does not change the buyer's payoff because the losing bid is not observable to the speculator.

Lemma 2 *Let $(b_1(\cdot), \tilde{b}_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium, and let $H(\cdot)$ denote the distribution function for $\tilde{b}_s$. Then $b_1(\theta_1) \leq \theta_1$ for all $\theta_1 \in (0,1]$ with $H(b_1(\theta_1)) > 0$. If $H(b_1(0)) > 0$ then $b_1(0) = 0$.*

Moreover, $\phi(b_s) \geq b_s$ for all b_s with $H(b_s) > 0$.

Proof. Suppose that $b_1(\theta_1) > \theta_1 > 0$. Then a deviation to the bid $b_1 = \theta_1$ is profitable because

$$\begin{aligned}
&u_1(\theta_1, \theta_1) - u_1(b_1(\theta_1), \theta_1) \\
&= H(\theta_1)(b_1(\theta_1) - \theta_1) + (H(b_1(\theta_1)) - H(\theta_1)) \cdot \\
&\qquad \Big(\delta\ E[Q(\theta_1 \mid \tilde{b}_s)\theta_1 - P(\theta_1 \mid \tilde{b}_s) \mid \theta_1 < \tilde{b}_s \leq b_1(\theta_1)] - (\theta_1 - b_1(\theta_1))\Big) \\
&\overset{(2)}{\geq} H(\theta_1)(b_1(\theta_1) - \theta_1) + (H(b_1(\theta_1)) - H(\theta_1))(b_1(\theta_1) - \theta_1) \\
&\quad \geq H(b_1(\theta_1))(b_1(\theta_1) - \theta_1) > 0.
\end{aligned}$$

In the same manner one shows that for type $\theta_1 = 0$ a deviation to $b_1 = 0$ is profitable.

To prove the moreover-part, let $\theta_1 < b_s$ for some b_s with $H(b_s) > 0$. Then $b_1(\theta_1) < b_s$ by what we have just shown. Therefore, $\phi(b_s) \geq b_s$. *QED*

The next lemma shows that the speculator will make arbitrarily small bids. Suppose this were not so. Then buyers with low values never win the original auction but always wait for resale. However, the resale price must be sufficiently high so that the speculator recovers the price paid in the original auction. Due to the participation constraint for the resale mechanism, no buyer's expected resale payment can be higher than her value. Therefore, among the buyer types who wait for resale, those with a relatively high value must make a resale payment above the speculator's lowest price paid in the original auction. For those buyers, it is profitable to deviate to a higher bid that makes them win the original auction with positive probability.

Lemma 3 *Let* $(b_1(\cdot), \tilde{b}_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ *be a perfect Bayesian equilibrium. Then* $\forall b > 0 : \Pr[\tilde{b}_s < b] > 0$.

Proof. Suppose that there exists $\underline{b}_s > 0$ such that $\Pr[\tilde{b}_s < \underline{b}_s] = 0$. Let $\underline{b}_s$ be maximal with this property. Let $u_s^* \geq 0$ denote the equilibrium expected payoff for the speculator; i.e., $\Pr[u_s(\tilde{b}_s) = u_s^*] = 1$.

First consider the case $H(\underline{b}_s) = 0$ (i.e., no atom at $\underline{b}_s$). By assumption, there exists a sequence $(b^m)_{m\in\mathbb{N}}$ such that $b^m \to \underline{b}_s$ as $m \to \infty$, and $u_s(b^m) = u_s^*$ and $b^m > \underline{b}_s$ for all m. Define $\theta^* = \inf_{b>\underline{b}_s} \phi(b)$. Lemma 2 implies $\theta^* \geq \underline{b}_s > 0$. Moreover, an indirect argument using Lemma 1 shows that

$$\forall \theta_1 < \theta^* : \ b_1(\theta_1) \leq \underline{b}_s. \tag{12}$$

For all $b > \underline{b}_s$ with $u_s(b) = u_s^*$ we have

$$0 \leq u_s(b) = \underbrace{F(\phi(b))}_{\geq F(\theta^*)>0} \left(\delta E[P(\tilde{\theta}_1 \mid b) \mid \tilde{\theta}_1 \leq \phi(b)] - b\right).$$

Therefore,

$$\forall b > \underline{b}_s, \ u_s(b) = u_s^* : \ E[P(\tilde{\theta}_1 \mid b) \mid \tilde{\theta}_1 \leq \phi(b)] \geq \frac{\underline{b}_s}{\delta}. \tag{13}$$

From (2) we get

$$E[P(\tilde{\theta}_1 \mid b) \mid \tilde{\theta}_1 < \frac{\underline{b}_s}{2}] \leq \frac{\underline{b}_s}{2}. \tag{14}$$

Note also that

$$\Pr[\tilde{\theta}_1 < \frac{\underline{b}_s}{2} \mid \tilde{\theta}_1 \leq \phi(b)] = \frac{F(\underline{b}_s/2)}{F(\phi(b))} \geq F(\underline{b}_s/2). \tag{15}$$

Taken together, the inequalities (13), (14), and (15) imply that

$$\forall b > \underline{b}_s, \ u_s(b) = u_s^* : \ E[P(\tilde{\theta}_1 \mid b) \mid \frac{\underline{b}_s}{2} \leq \tilde{\theta}_1 \leq \phi(b)] \geq \frac{\underline{b}_s}{\delta} + \xi, \tag{16}$$

where

$$\xi := F(\underline{b}_s/2) \left(\frac{\underline{b}_s}{\delta} - \frac{\underline{b}_s}{2} \right) > 0.$$

There exists $\hat{b} > \underline{b}_s$ such that

$$\forall b \in (\underline{b}_s, \hat{b}) : \ \Pr[\theta^* \leq \tilde{\theta}_1 \leq \phi(b) \mid \frac{\underline{b}_s}{2} \leq \tilde{\theta}_1 \leq \phi(b)] < \xi/3, \tag{17}$$

and $\hat\theta \in (\underline{b}_s/2, \theta^*)$ such that

$$\forall b > \underline{b}_s: \ \Pr[\hat\theta \le \tilde\theta_1 \le \theta^* \mid \frac{\underline{b}_s}{2} \le \tilde\theta_1 \le \phi(b)] < \xi/3. \tag{18}$$

Taken together, (17) and (18) imply that

$$\forall b \in (\underline{b}_s, \hat b): \ \Pr[\hat\theta \le \tilde\theta_1 \le \phi(b) \mid \frac{\underline{b}_s}{2} \le \tilde\theta_1 \le \phi(b)] < \frac{2\xi}{3}$$

This together with $P(\tilde\theta_1 \mid b) \le 1$ (from (2)) and (16) yields

$$\forall b \in (\underline{b}_s, \hat b), \ u_s(b) = u_s^*: \ E[P(\tilde\theta_1 \mid b) \mid \frac{\underline{b}_s}{2} \le \tilde\theta_1 \le \hat\theta] \ge \frac{\underline{b}_s}{\delta} + \xi/3.$$

This implies $P(\hat\theta \mid b)] \ge \underline{b}_s/\delta + \xi/3$ because $P(\cdot \mid b)$ is weakly increasing by (1). Now let $b_1 \in (\underline{b}_s, \min\{\hat b, \underline{b}_s + \delta\xi/3\})$. Then

$$\begin{aligned}&u_1(b_1, \hat\theta) - u_1(b_1(\hat\theta), \hat\theta)\\ &= \Pr[\tilde b_s \le b_1]\left(\hat\theta - b_1 - \delta E\left[Q(\hat\theta \mid \tilde b_s)\hat\theta - P(\hat\theta \mid \tilde b_s) \mid \tilde b_s \le b_1\right]\right)\\ &\ge \Pr[\tilde b_s \le b_1](\underline{b}_s + \delta\xi/3 - b_1) \ > 0,\end{aligned}$$

which contradicts (5).

In the case $H(\underline{b}_s) > 0$ the proof is similar. One defines $\theta^* = \phi(\underline{b}_s)$ and shows that the deviation $b_1 = \underline{b}_s$ is profitable for some type $\hat\theta < \theta^*$. *QED*

Lemma 4 *Let $(b_1(\cdot), \tilde b_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium. Then $b_1(\theta_1) < \theta_1$ for all $\theta_1 \in (0, 1]$, and $b_1(0) = 0$.*

Proof. Suppose that $b_1(\theta_1) \ge \theta_1 > 0$. Then $b_1(\theta_1) > 0$, implying $H(b_1(\theta_1)) > 0$ by Lemma 3. Now Lemma 2 shows that $b_1(\theta_1) = \theta_1$. Finally, a computation similar to that in the proof of Lemma 2 shows that a deviation to the bid $b_1 = \theta_1/2$ is profitable—contradiction. The proof for the case $\theta_1 = 0$ is similar. *QED*

The proposition below shows that the speculator's expected payoff equals zero; i.e., she cannot make profits.[5] Suppose she does make positive profits.

[5] The result that the speculator cannot make profits does not mean that she would abstain if there were a small positive participation cost. Assuming the participation cost is incurred simultaneously to the bid, we expect an equilibrium where the speculator randomizes between participating and abstaining, and, conditional on participation, obtains a positive expected profit.

Then by Lemma 3 even arbitrarily small bids of the speculator have a probability of winning that remains bounded away from 0. Therefore, buyers with small value must be bidding 0. Also, the resale payment of some 0-bidding buyer types must be bounded away from zero for arbitrarily small bids of the speculator because otherwise the speculator could not make profits. For these buyer types, a deviation to a small positive bid can be shown to be profitable.

Proposition 1 *Let $(b_1(\cdot), \tilde{b}_s, P(\cdot \mid \cdot), Q(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium. Then the function $u_s(\cdot)$ defined by (4) satisfies*

$$\max_{b \geq 0} u_s(b) = 0.$$

Proof. Let $u_s^* = \max_{b \geq 0} u_s(b)$ and suppose that $u_s^* > 0$. By Lemma 3 there exists a sequence $(b^m)_{m \in \mathbb{N}}$ such that $b^m \to 0$ as $m \to \infty$, and $u_s(b^m) = u_s^*$ for all m. Note that $b^m > 0$ for all m because otherwise $u_s^* = u_s(0) = 0$ (using our assumption that the buyer wins all ties). Define $\theta^* = \inf_{b>0} \phi(b)$. We have

$$\Pr[\tilde{\theta}_1 \leq \theta^*] = \lim_{m \to \infty} \Pr[\tilde{\theta}_1 \leq \phi(b^m)] \geq u_s^* > 0.$$

Therefore, $\theta^* > 0$. Note that Lemma 1 implies

$$\forall \theta_1 < \theta^* : \ b_1(\theta_1) = 0. \tag{19}$$

There exists $\hat{b} > 0$ such that

$$\forall b \in (0, \hat{b}) : \ \Pr[\theta^* \leq \tilde{\theta}_1 \leq \phi(b)] < u_s^*/3$$

and $\hat{\theta} \in (0, \theta^*)$ such that

$$\Pr[\hat{\theta} \leq \tilde{\theta}_1 \leq \theta^*] < u_s^*/3.$$

Therefore, for all $b \in (0, \hat{b})$ with $u_s(b) = u_s^*$ we have

$$\begin{aligned} u_s^* \ &\leq \ E[1_{\tilde{\theta}_1 \leq \phi(b)} P(\tilde{\theta}_1 \mid b)] \\ &= E[1_{\tilde{\theta}_1 < \hat{\theta}} P(\tilde{\theta}_1 \mid b)] + E[1_{\hat{\theta} \leq \tilde{\theta}_1 \leq \theta^*} P(\tilde{\theta}_1 \mid b)] + E[1_{\theta^* < \tilde{\theta}_1 \leq \phi(b)} P(\tilde{\theta}_1 \mid b)] \\ &< E[1_{\tilde{\theta}_1 < \hat{\theta}} P(\tilde{\theta}_1 \mid b)] + \frac{2}{3} u_s^*, \end{aligned}$$

or

$$E[1_{\tilde{\theta}_1 < \hat{\theta}} P(\tilde{\theta}_1 \mid b)] > u_s^*/3.$$

This implies $P(\hat{\theta} \mid b)] > u_s^*/3$ because $P(\cdot \mid b)$ is weakly increasing by (1). Now let $b_1 \in (0, \min\{\hat{b}, \delta u_s^*/3\})$. Then

$$\begin{aligned}
&u_1(b_1, \hat{\theta}) - u_1(0, \hat{\theta}) \\
&= \Pr[\tilde{b}_s \leq b_1] \left(\hat{\theta} - b_1 - \delta E\left[Q(\hat{\theta} \mid \tilde{b}_s)\hat{\theta} - P(\hat{\theta} \mid \tilde{b}_s) \mid \tilde{b}_s \leq b_1\right]\right) \\
&\geq \Pr[\tilde{b}_s \leq b_1](\delta u_s^*/3 - b_1) \ > 0,
\end{aligned}$$

which contradicts (5). *QED*

Having shown that speculation is not profitable, we now turn to the question how exactly the auction is distorted by the presence of a speculator.

4 Existence and Uniqueness of Equilibrium with optimal take-it-or-leave-it offers

From now on we focus on a specific resale mechanism: an optimal take-it-or-leave-it offer. We proceed by establishing in Lemma 5 to Lemma 12 various properties that equilibria must have and then show that an essentially unique equilibrium exists (Proposition 2). The lemmas' proofs make frequent use of the fact that arbitrarily small bids of both players have a positive probability of winning (this follows from Lemma 3 and 4).

Lemma 5 below shows that positive types make positive bids; i.e., the buyer's bid function is strictly increasing at 0. If not then the speculator would make positive profits from any sufficiently small positive bid.

Lemma 5 *Let $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. Then for all $\theta_1 > 0$ we have $b_1(\theta_1) > 0$.*

Proof. Suppose that $b_1(\theta^*) = 0$ for some $\theta^* > 0$. Because the speculator can make the take-it-or-leave-it offer $T = \theta^*/2$, we have

$$u_s(b_s) \geq (F(\phi(b_s)) - F(\theta^*/2))\delta\theta^*/2 - F(\phi(b_s))b_s$$

for all $b_s > 0$ and thus

$$\lim \inf_{b_s \to 0, b_s > 0} u_s(b_s) \geq (F(\theta^*) - F(\theta^*/2))\delta\theta^*/2 > 0,$$

in contradiction to Proposition 1. *QED*

The three main properties of equilibrium that were mentioned in the Introduction follow from Lemma 5. The initial seller's expected revenue is positive. With positive probability, the speculator submits a positive bid (because otherwise $b_1(1) = 0$). After winning, the speculator makes a positive resale offer because otherwise she could not recover the price paid in the auction. As a result, the speculator keeps the good with positive probability so that the post-resale allocation is inefficient.

The next lemma shows that the buyer's bid function is strictly increasing. The proof supposes that some positive bid b^* occurs with positive probability. By bidding slightly above b^* the speculator wins against a discretely "better" pool of buyer types than by bidding slightly below. It is thus not optimal for the speculator to submit any bid slightly below b^*, and thus the buyer can improve her payoff by lowering her bid below b^*.

Lemma 6 *Let $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. Then for all $\theta_1, \theta_1' \in [0, 1]$ and $\theta_1' > \theta_1$, we have $b_1(\theta_1') > b_1(\theta_1)$.*

Proof. In the case $\theta_1 = 0$ we have $b_1(\theta_1) = 0 < b_1(\theta_1')$ by Lemma 4 and Lemma 5.

Now consider the case $\theta_1 > 0$. Suppose there exists $\theta_1' > \theta_1$ with $b_1(\theta_1) = b_1(\theta_1') = b^*$. Note that $b^* > 0$ by Lemma 5. We have[6]

$$\overline{\phi} := \lim_{b \searrow b^*} \phi(b) \geq \theta_1' > \theta_1 \geq \lim_{b \nearrow b^*} \phi(b) =: \underline{\phi}.$$

Using the definition of $T(\cdot)$, one can show that there exists $\underline{b} < b^*$ and $\underline{T} > 0$ such that $T(b) \geq \underline{T}$ for all $b > \underline{b}$.

Also, $\Pr[\tilde{b}_s \in [b^* - \underline{\epsilon}, b^*]] > 0$ for all $\underline{\epsilon} > 0$ (because otherwise $u_1(b^* - \underline{\epsilon}, \theta_1) > u_1(b^*, \theta_1) = u_1(b_1(\theta_1), \theta_1)$ which contradicts (5)). Therefore, there exists a sequence $(b^m)_{m \in \mathbb{N}}$ such that $b^m \to b^*$ as $m \to \infty$, and $b^m \in (0, b^*)$ and $u_s(b^m) = 0$ for all m (using Proposition 1).

For all $b_s > 0$ we have

$$u_s(b_s) = F(\phi(b_s)) \left(\delta \frac{F(\phi(b_s)) - F(T(b_s))}{F(\phi(b_s))} T(b_s) - b_s \right).$$

Because after bidding $b^* + 1/m$ the speculator can make the offer $T(b^m)$ rather than her optimal offer $T(b^* + 1/m)$, we have for all $b^m > \underline{b}$:

$$\frac{F(\phi(b^* + \frac{1}{m})) - F(T(b^* + \frac{1}{m}))}{F(\phi(b^* + \frac{1}{m}))} T(b^* + \frac{1}{m})$$

[6] By $\searrow$ and $\nearrow$ we denote the limits from the right and left, respectively.

$$\geq \frac{F(\phi(b^* + \frac{1}{m})) - F(T(b^m))}{F(\phi(b^* + \frac{1}{m}))} T(b^m)$$

$$\geq \underbrace{\frac{F(\phi(b^m)) - F(T(b^m))}{F(\phi(b^m))} T(b^m)}_{=b^m/\delta} + \underbrace{F(\underline{T}) \left(\frac{1}{F(\underline{\phi})} - \frac{1}{F(\overline{\phi})} \right) \underline{T}}_{>0}$$

Therefore, $u_s(b^* + 1/m) > 0$ for all sufficiently large m, contradicting Proposition 1. *QED*

The next lemma shows that the speculator's bid distribution is continuous, except for a possible atom at 0. If there were an atom at a positive bid b^* then buyer types who are in equilibrium supposed to bid just below b^* would rather deviate and win against the bid b^*. The resale offer would be more expensive then buying in the original auction because otherwise the speculator makes losses when she bids b^*.

Lemma 7 *Let* $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ *be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. Then the distribution* $H(\cdot)$ *for* $\tilde{b}_s$ *is continuous on* $(0, \infty)$.

Proof. Suppose that there exists $b^* > 0$ where $H(\cdot)$ is not continuous; i.e., $\Pr[\tilde{b}_s = b^*] > 0$.

Define $\theta^m = \phi(b^*) - 1/m$ for all m large enough that $\theta^m > 0$. Let $\bar{b} = \lim_{m\to\infty} b_1(\theta^m)$. We have $\bar{b} = b^*$ because otherwise $\Pr[b_1(\tilde{\theta}_1) \in (\bar{b}, b^*)] = 0$ which would imply $u_s((b^* + \bar{b})/2) > u_s(b^*)$.

Note that $T(b^*) > b^*/\delta$ because otherwise $u_s(b^*) < 0$. Also, $T(b^*) < \theta^m$ for large m because otherwise $T(b^*) \geq \phi(b^*)$, implying $u_s(b^*) < 0$. For large m we have

$$\begin{aligned} &u_1(b^*, \theta^m) - u_1(b_1(\theta^m), \theta^m) \\ &\geq H(b_1(\theta^m))(b_1(\theta^m) - b^*) + \Pr[\tilde{b}_s \in (b_1(\theta^m), b^*)](-1) \\ &\quad + \Pr[\tilde{b}_s = b^*]((1-\delta)\theta^m + (\delta T(b^*) - b^*)). \end{aligned}$$

Therefore,

$$\liminf_{m\to\infty} u_1(b^*, \theta^m) - u_1(b_1(\theta^m), \theta^m) \geq \Pr[\tilde{b}_s = b^*](\delta T(b^*) - b^*) > 0.$$

I.e., for large m type θ^m has a profitable deviation. *QED*

The next Lemma shows that the buyer's bid function is continuous and $\phi(\cdot)$ is its strictly increasing inverse. If the bid function did jump the speculator would also not bid in the resulting gap of the buyer's bid distribution. This contradicts the assumption of optimal bidding for types just above the jumping type.

Lemma 8 *Let* $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ *be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. Then the bid function* $b_1(\cdot)$ *is continuous.*

Moreover, for all $\theta_1 \in (0,1]$ *we have* $\phi(b_1(\theta_1)) = \theta_1$. *Also, for all* $b, b' \in (0, b_1(1)]$ *with* $b' > b$ *we have* $\phi(b') > \phi(b)$.

Proof. From Lemma 4 it follows that $b_1(\cdot)$ is continuous at 0. Suppose $b_1(\cdot)$ is not continuous at some $\theta^* > 0$. One case is that

$$b_1(\theta^*) > \underbrace{\lim_{\theta_1 \nearrow \theta^*} b_1(\theta_1)}_{=:\underline{b}}.$$

This implies $\Pr[\tilde{b}_s \in (\underline{b}, b_1(\theta^*))] = 0$ because otherwise the speculator could improve her payoff by lowering her bid. But then type θ^* can improve her payoff by lowering her bid—contradiction.

The other case to consider is

$$b_1(\theta^*) < \underbrace{\lim_{\theta_1 \searrow \theta^*} b_1(\theta_1)}_{=:\bar{b}}.$$

This implies $\Pr[\tilde{b}_s \in (b_1(\theta^*), \bar{b}]] = 0$. But then any type $\theta_1 > \theta^*$ that is sufficiently close to θ^* can improve her payoff by lowering her bid—contradiction.

The moreover-part now follows easily using Lemma 6. *QED*

The next lemma describes the unique candidate for an equilibrium bid function. The bid function is fully determined by the 0-profit condition for the speculator.

Lemma 9 *Let* $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ *be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. Then*

$$\forall \theta_1 \in (0,1]: \;\; b_1(\theta_1) = \delta \max_{p_2 \in [0,1]} \frac{F(\theta_1) - F(p_2)}{F(\theta_1)} p_2. \tag{20}$$

Moreover, $u_s(b_s) = 0$ *for all* $b_s \in [0, b_1(1)]$.

Proof. For all $(p_2, \theta_1) \in [0,1] \times (0,1]$, define the function $g(p_2, \theta_1) = \delta(F(\theta_1) - F(p_2))p_2/F(\theta_1)$. For all $\theta_1 \in (0,1]$, define the function $g_1(\theta_1) = \max_{p_2 \in [0,1]} g(p_2, \theta_1)$. By a well-known lemma, $g_1(\cdot)$ is continuous because $g(\cdot)$ is continuous. By Lemma 8, $b_1(\cdot)$ is continuous as well. Therefore, to prove (20) it is sufficient to find a set $D \subseteq (0,1]$ such that D is dense in $(0,1]$ and $b_1(\theta_1) = g_1(\theta_1)$ holds for all $\theta_1 \in D$. Let

$$D = \{\theta_1 \in (0,1] \mid u_s(b_1(\theta_1)) = 0\}.$$

Suppose D is not dense in $(0,1]$. Then there exist $\theta_1', \theta_1'' \in (0,1]$ such that $\theta_1'' > \theta_1'$ and $D \cap [\theta_1', \theta_1''] = \emptyset$; i.e., $u_s(b_1(\theta_1)) < 0$ for all $\theta_1 \in [\theta_1', \theta_1'']$ by Proposition 1. This implies $\Pr[\tilde{b}_s \in [b_1(\theta_1'), b_1(\theta_1'')]] = 0$, which contradicts the optimality of the bid $b_1(\theta_1'')$ for type θ_1''.

By definition of D, we have

$$\forall \theta_1 \in D: \ \delta \max_{p_2 \in [0,1]} \frac{F(\phi(b_1(\theta_1))) - F(p_2)}{F(\phi(b_1(\theta_1)))} p_2 - b_1(\theta_1) = 0.$$

Now the moreover-part of Lemma 8 shows (20).

The moreover-part of the statement is immediate from (20). *QED*

The next lemma describes the candidate equilibrium take-it-or-leave-it offer functions and uses monotone comparative statics to show that any offer function is strictly increasing in the speculator's bid. The candidate offer function is uniquely determined up to a countable set of points; this will turn out to be the only aspect of the equilibrium that is not uniquely determined.

Lemma 10 *Let $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. Then*

$$\forall b_s \in (0, b_1(1]) : \ T(b_s) \in \underbrace{\arg \max_{p \in (0, \phi(b_s))} (F(\phi(b_s)) - F(p))p}_{=\mathcal{T}(b_s)}. \tag{21}$$

Moreover, $T(\cdot)$ is strictly increasing on $(0, b_1(1)]$. Also, everywhere except for a countable set of points, $\mathcal{T}(\cdot)$ is single-valued.

Proof. From (9) one directly obtains (21). Consider $b_s, b_s' \in (0, b_1(1)]$ with $b_s > b_s'$. Then $T > T'$ for all $T \in \mathcal{T}(b_s)$, $T' \in \mathcal{T}(b_s')$, by Edlin and Shannon (1998, Theorem 1). Therefore, $T(\cdot)$ is strictly increasing.

For any $m \in \mathbb{N}$ define the set

$$J_m = \{b_s \in (0, b_1(1)] \mid \lim_{\hat{b}_s \nearrow b_s} \left(\sup \mathcal{T}(\hat{b}_s)\right) < \lim_{\hat{b}_s \searrow b_s} \left(\inf \mathcal{T}(\hat{b}_s)\right) - \frac{1}{m}\}$$

Because $\mathcal{T}(b_s) \subseteq [0,1]$ for all $b_s \in (0, b_1(1)]$, the sets J_m are finite for all $m \in \mathbb{N}$. Thus, $J = \cup_{m \in \mathbb{N}} J_m$ is countable. For all $b_s \in (0, b_1(1)] \setminus J$, we have $\sup \mathcal{T}(\hat{b}_s) = \inf \mathcal{T}(\hat{b}_s)$, showing that the set $\mathcal{T}(b_s)$ is single-valued. *QED*

The next lemma establishes a differential equation for the distribution $H(\cdot)$ of speculator's bids. The distribution is determined by the condition that the bidder's bid is (locally) optimal for each bidder type. Using this condition, we also prove differentiability of $H(\cdot)$ up to a countable set of points and local Lipschitz continuity.

Lemma 11 *Let $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers. For all $\epsilon > 0$, the distribution function $H(\cdot)$ for $\tilde{b}_s$ is Lipschitz continuous on $[\epsilon, b_1(1)]$.*

Moreover, $H(\cdot)$ is differentiable for all $b_s \in (0, b_1(1))$ up to a countable set of points, and

$$H'(b_s) = \frac{H(b_s)}{\delta T(b_s) + \phi(b_s)(1-\delta) - b_s}. \tag{22}$$

Proof. We have

$$\begin{aligned} &\forall b_1 \in (0, b_1(1)],\ \theta_1 > T(b_1):\ u_1(b_1, \theta_1) = H(b_1)(\theta_1 - b_1) \\ &\quad + (H(T^{-1}(\theta_1)) - H(b_1))\delta\left(\theta_1 - E[T(\tilde{b}_s) \mid b_1 < \tilde{b}_s \le T^{-1}(\theta_1)]\right) \\ &= H(b_1)(\theta_1 - b_1) + (H(T^{-1}(\theta_1)) - H(b_1))\delta\theta_1 - \delta \int_{b_1}^{T^{-1}(\theta_1)} T(b)\mathrm{d}H(b), \end{aligned} \tag{23}$$

where

$$T^{-1}(\theta_1) := \sup\{b_s \in (0, b_1(1)] \mid T(b_s) \le \theta_1\} \tag{24}$$

is the maximum speculator's bid that makes the resale offer acceptable for type θ_1.

Consider any $b_1, b_1' \in (0, b_1(1)]$ with $b_1' < b_1$. Defining $\theta_1 = \phi(b_1)$, we have $\theta_1 > T(b_1) > T(b_1')$ and thus (23) implies

$$\begin{aligned} 0 \quad \le \quad & u_1(b_1, \theta_1) - u_1(b_1', \theta_1) \\ & = (H(b_1) - H(b_1'))\theta_1(1-\delta) - (H(b_1)b_1 - H(b_1')b_1') \\ & \quad + \delta \underbrace{\int_{(b_1', b_1]} T(b)\mathrm{d}H(b)}_{\le T(b_1)(H(b_1) - H(b_1'))} \end{aligned}$$

$$\leq \underbrace{(H(b_1) - H(b_1'))}_{>0} \left(-H(b_1') \frac{b_1 - b_1'}{H(b_1) - H(b_1')} - b_1 + \theta_1(1-\delta) + \delta T(b_1) \right).$$

Therefore,

$$\begin{aligned} &\forall b_1, b_1' \in (0, b_1(1)],\ b_1' < b_1 : \\ &\frac{H(b_1) - H(b_1')}{b_1 - b_1'} \geq \frac{H(b_1')}{\delta T(b_1) + \phi(b_1)(1-\delta) - b_1}. \end{aligned} \tag{25}$$

Analogously, one finds that[7]

$$\begin{aligned} &\forall b_1, b_1' \in (0, b_1(1)],\ b_1' < b_1,\ \phi(b_1') > T(b_1),\ \delta T(b_1') + \phi(b_1')(1-\delta) - b_1 > 0 : \\ &\frac{H(b_1) - H(b_1')}{b_1 - b_1'} \leq \frac{H(b_1)}{\delta T(b_1') + \phi(b_1')(1-\delta) - b_1}. \end{aligned} \tag{26}$$

Fix any $\epsilon > 0$. For all $b_1 \in [\epsilon, b_1(1)]$, using $u_s(b_1) = 0$ from Lemma 9, we have

$$\frac{F(\phi(b_1) - F(T(b_1))}{F(\phi(\epsilon))} T(1) \geq \frac{F(\phi(b_1) - F(T(b_1))}{F(\phi(b_1))} T(b_1) = \frac{b_1}{\delta} \geq \frac{\epsilon}{\delta}.$$

This together with the fact that the density for $F(\cdot)$ is bounded above implies

$$\exists \eta > 0 \forall b_1 \in [\epsilon, b_1(1)] :\ \phi(b_1) - T(b_1) > \eta. \tag{27}$$

By uniform continuity of $\phi(\cdot)$ on the compact set $[\epsilon, b_1(1)]$, there exists $\xi > 0$ such that for all $b_1, b_1' \in [\epsilon, b_1(1)]$ with $b_1 - b_1' \in (0, \xi)$,

$$\phi(b_1) - \phi(b_1') < \eta/2. \tag{28}$$

Subtracting (28) from (27) yields

$$\exists \xi > 0\ \forall b_1, b_1' \in [\epsilon, b_1(1)],\ b_1 - b_1' \in (0, \xi) : \phi(b_1') - T(b_1) > \eta/2. \tag{29}$$

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[7]Define $\theta_1 = \phi(b_1')$. Then (23) implies

$$\begin{aligned} 0 \geq\ & u_1(b_1, \theta_1) - u_1(b_1', \theta_1) \\ &= (H(b_1) - H(b_1'))\theta_1(1-\delta) - (H(b_1)b_1 - H(b_1')b_1') \\ &\quad + \underbrace{\delta \int_{(b_1', b_1]} T(b) \mathrm{d}H(b)}_{\geq T(b_1')(H(b_1) - H(b_1'))} \\ &\geq (H(b_1) - H(b_1')) \left(-H(b_1) \frac{b_1 - b_1'}{H(b_1) - H(b_1')} - b_1 + \theta_1(1-\delta) + \delta T(b_1') \right). \end{aligned}$$

Now fix a $b^* \in [\epsilon, b_1(1)]$. W.l.o.g., $\xi < b^*$. Define $\tau = \delta F(T(\epsilon))T(\epsilon) > 0$. For all $b_1 \in [\epsilon, b_1(1)]$, using $u_s(b_1') = 0$, we find

$$\delta T(b_1') - b_1' = \delta \frac{F(T(b_1'))}{F(\phi(b_1'))} T(b_1') \geq \tau$$

Therefore,

$$\begin{aligned} &\forall b_1, b_1' \in [\epsilon, b_1(1)],\ b_1 - b_1' \in (0, \tau/2): \\ &\quad \delta T(b_1') + \underbrace{\phi(b_1')(1-\delta)}_{\geq 0} - b_1 \geq \tau - (b_1 - b_1') \geq \tau/2. \end{aligned} \tag{30}$$

Taken together, (26), (29), and (30) imply that

$$\forall b_1, b_1' \in [\epsilon, b_1(1)],\ b_1 - b_1' \in (0, \min\{\xi, \tau/2\}): \ \frac{H(b_1) - H(b_1')}{b_1 - b_1'} \leq \frac{2}{\tau}, \tag{31}$$

which shows Lipschitz continuity. From (26) and (25) it is now immediate that $H(\cdot)$ is differentiable at all points in $(\epsilon, b_1(1))$ where $T(\cdot)$ is continuous, with the derivative given by (22). Because $T(\cdot)$ is strictly increasing, it is continuous up to a countable set of points. *QED*

The next lemma determines the unique candidate distribution function for the speculator's bid. Using the previous Lemma and an additional Lipschitz condition, one shows that the differential equation given in the previous lemma has a unique solution.

Lemma 12 *Let $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ be a perfect Bayesian equilibrium with optimal take-it-or-leave-it offers and let $H(\cdot)$ denote the distribution function for $\tilde{b}_s$. Then*

$$\forall b_s \in (0, b_1(1)]: \ H(b_s) = e^{-\int_{b^s}^{b_1(1)} \frac{1}{(1-\delta)\phi(b) - b + \delta T(b)} db}, \tag{32}$$

and

$$H(0) = \lim_{b_s \to 0, b_s > 0} H(b_s) < 1. \tag{33}$$

Proof. We have $H(0) < 1$ because otherwise $b_1(1) = 0$, contradicting Lemma 5. The limit formula for $H(0)$ follows from right-continuity of distribution functions. One can check that (32) satisfies the differential equation (22) for all but a countable set of points on $(0, b_1(1))$, is Lipschitz continuous on $[\epsilon, b_1(1)]$ for all $\epsilon > 0$, and satisfies the boundary condition $H(b_1(1)) =$

1. It remains to verify that no other distribution function satisfies these requirements. Consider any small $\epsilon > 0$. Note that $\delta T(b_s) \geq b_s$ for all $b_s \in (0, b_1(1)]$. Thus, for all real numbers $H, \hat{H}$ with $H > \hat{H}$ and all $b_s \in [\epsilon, b_1(1)]$, the Lipschitz condition

$$\frac{H - \hat{H}}{\delta T(b_s) + \phi(b_s)(1-\delta) - b_s} \leq (H - \hat{H}) \frac{1}{\phi(\epsilon)(1-\delta)}$$

is satisfied. Moreover, Lemma 11 implies that $H(\cdot)$ is Lipschitz on $[\epsilon, b_1(1)]$. A standard argument in the theory of differential equations shows that these conditions imply the uniqueness of the solution (32) on $[\epsilon, b_1(1)]$. Because ϵ is arbitrary, uniqueness on $(0, b_1(1)]$ follows. *QED*

The lemmas so far have shown that if an equilibrium exists then it is unique except that multiple optimal resale offers $T(b_s)$ may exist for a countable number of bids $b_s > 0$. Proposition 2 summarizes these results and also states the existence of an equilibrium.

Proposition 2 *A perfect Bayesian equilibrium exists for the game with optimal take-it-or-leave-it offers.*

In any perfect Bayesian equilibrium $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$, the buyer's bid function is given by $b_1(0) = 0$ and (20). The resale offer $T(b_s)$ satisfies (21), and is uniquely determined up to a countable number of points $b_s > 0$. The distribution function $H(\cdot)$ for the speculator's bid $\tilde{b}_s$ has the support $[0, b_1(1)]$ and is given by (32) and (33). For all $b_s > 0$, the posterior $\Pi(\cdot \mid b_s)$ is given by (8), where $\phi(\cdot)$ denotes the inverse of $b_1(\cdot)$.

Proof. Define $b_1(\cdot)$ by $b_1(0) = 0$ and (20). Define $T(\tilde{b}_s)$ as an arbitrary solution to (21). Define $H(\cdot)$ by (32), and let $\tilde{b}_s$ be a random variable that is independent of $\tilde{\theta}_1$ and has distribution $H(\cdot)$. For all $b_s > 0$, define $\Pi(\cdot \mid b_s)$ by (8) (note that the speculator cannot win with $b_s = 0$).

We first have to verify that this equilibrium $(b_1(\cdot), \tilde{b}_s, T(\cdot), \Pi(\cdot \mid \cdot))$ is well-defined. For all $\theta_1 > 0$ we have $0 < b_1(\theta_1)$, implying that $b_1(\cdot)$ is strictly increasing at 0. Now let $\theta_1 > \theta_1' > 0$, let p be a maximand for θ_1 in (20), and let p' be a maximand for θ_1'. Then we have $0 < p' < \theta_1'$ and thus

$$\frac{F(\theta_1') - F(p')}{F(\theta_1')} p' < \frac{F(\theta_1) - F(p')}{F(\theta_1)} p' \leq \frac{F(\theta_1) - F(p)}{F(\theta_1)} p,$$

implying that $b_1(\cdot)$ is strictly increasing. Continuity of $b_1(\cdot)$ was shown in the proof of Lemma 9. Thus, the inverse $\phi(\cdot)$ of $b_1(\cdot)$ is well-defined, continuous and strictly increasing. Note that

$$b_1(\theta_1) = \delta \frac{F(\theta_1) - F(T(b_1(\theta_1)))}{F(\theta_1)} T(b_1(\theta_1)) < \delta T(b_1(\theta_1))$$

for all $\theta_1 > 0$, implying $b < \delta T(b)$ and thus $Q(b) > 0$ for all $b \in (0, b_1(1)]$, where we define

$$Q(b) = (1-\delta)\phi(b) - b + \delta T(b).$$

In particular, the integrand in the definition of $H(\cdot)$ is well-defined. Moreover, as in the proof of Lemma 10 one sees that $T(\cdot)$ is strictly increasing. Thus, $1/Q(\cdot)$ is continuous up to a countable number of points and hence integrable. Moreover, $H(\cdot)$ is strictly increasing because $1/Q(\cdot) > 0$.

By definition, $H(\cdot)$ is continuous from the right at 0. Note that for any given $\epsilon > 0$ we have

$$\forall b \in [\epsilon, b_1(1)] : \ \frac{1}{Q(b)} \leq \frac{1}{(1-\delta)\phi(b)} \leq \frac{1}{(1-\delta)\phi(\epsilon)}$$

Thus, the function $Q(\cdot)$ is bounded on $[\epsilon, b_1(1)]$ and thus $H(\cdot)$ is continuous in the range $[\epsilon, b_1(1)]$. Because ϵ is arbitrary, we can conclude that $H(\cdot)$ is a continuous distribution function on $(0, \infty)$. Moreover,

$$h(b) = \frac{H(b)}{Q(b)} \quad (b \in (0, b_1(1)))$$

defines a density for $H(\cdot)$ because $H'(b) = h(b)$ for all $b > 0$ where $1/Q(b)$ is continuous.

The only equilibrium condition which does not follow directly from the construction of the equilibrium is (5). Clearly, $b_1(0) = 0$ is the optimal bid for type 0. Let us now check optimality of $b_1 = b_1(\theta_1)$ for types $\theta_1 > 0$.

In the following we use the definition (24). The expected payoff of type $\theta_1 > 0$ with any bid $b_1 > T^{-1}(\theta_1)$ is

$$u_1(b_1, \theta_1) = H(b_1)(\theta_1 - b_1).$$

This expression is strictly decreasing in the range $b_1 \geq b_1(1)$ because $H(b_1(1)) = 1$. It is also strictly decreasing in the range $b_1 \in [T^{-1}(\theta_1), b_1(1)]$ because it is continuous and for all b_1 up to a countable set we have

$$\begin{aligned} \frac{\partial u_1}{\partial b_1}(b_1, \theta_1) &= h(b_1)(\theta_1 - b_1) - H(b_1) \\ &= h(b_1)(\theta_1 - b_1 - Q(b_1)) \end{aligned}$$

$$
\begin{aligned}
&= \quad h(b_1)(\theta_1 - \phi(b_1) - \delta(T(b_1) - \phi(b_1))) \\
&\overset{T(b_1)>\theta_1}{<} \quad h(b_1)(1-\delta)(\theta_1 - \phi(b_1)) \\
&\overset{\phi(b_1)>T(b_1)>\theta_1}{<} \quad 0.
\end{aligned}
$$

In the range $b_1 \in [0, T^{-1}(\theta_1)]$, the utility function takes the form

$$
\begin{aligned}
u_1(b_1, \theta_1) &= H(b_1)(\theta_1 - b_1) \\
&\quad + (H(T^{-1}(\theta_1)) - H(b_1))\delta \left(\theta_1 - E[T(\tilde{b}_s) \mid b_1 < \tilde{b}_s \leq T^{-1}(\theta_1)]\right) \\
&= H(b_1)(\theta_1 - b_1) + (H(T^{-1}(\theta_1)) - H(b_1))\delta\theta_1 - \delta \int_{b_1}^{T^{-1}(\theta_1)} T(b)h(b)\mathrm{d}b
\end{aligned}
$$

Moreover, for all $\epsilon > 0$, the function $u_1(b_1, \theta_1)$ is Lipschitz continuous in b_1 on $[\epsilon, T^{-1}(\theta_1)]$, by Lemma 11. Therefore, for all $b_1^* \in (0, T^{-1}(\theta_1)]$,

$$
u_1(b_1^*, \theta_1) = u_1(T^{-1}(\theta_1), \theta_1) - \int_{b_1^*}^{T^{-1}(\theta_1)} \frac{\partial u_1}{\partial b_1}(b_1, \theta_1)\mathrm{d}b_1. \tag{34}
$$

Moreover, for all b_1 up to a countable set we have

$$
\frac{\partial u_1}{\partial b_1}(b_1, \theta_1) = h(b_1)\left((1-\delta)\theta_1 - b_1 + \delta T(b_1) - \frac{H(b_1)}{h(b_1)}\right) = h(b_1)(1-\delta)(\theta_1 - \phi(b_1)).
$$

One sees that $\partial u_1/\partial b_1 < 0$ if $\theta_1 < \phi(b_1)$, $=$ if $=$, and $>$ if $<$. Therefore, (34) together with $b_1(\theta_1) \leq T^{-1}(\theta_1)$ implies that $b_1^* = b_1(\theta_1)$ is the optimal bid among the bids $\neq 0$, for all $\theta_1 > 0$. A deviation to the bid 0 is not profitable because $u_1(b_1, \theta_1)$ is continuous in b_1 at $b_1 = 0$. This completes the existence proof.

The uniqueness-part follows from Lemma 9, Lemma 10, and Lemma 12. *QED*

The equilibrium that we have constructed survives multiple speculators. A second speculator would also make 0 expected profits from any bid in $[0, b_1(1)]$. Therefore, it is optimal to stay out. On the other hand, there are many equilibria with multiple speculators which have the same form as the one that we have constructed, except that the distribution function for the *maximum* among all speculators' bids must be defined by (32) and (33).

5 Conclusion

We have shown that the opportunity for resale can decrease the expected social surplus in a first-price auction with asymmetric bidders. At the same time, the resale opportunity can increase the seller's expected revenue. Having obtained these results for the extreme asymmetric case of one privately informed bidder and one 0-value bidder, similar results might be expected for any first-price auction with "sufficiently asymmetric" independent- private-value bidders.[8] Computing equilibria might, however, be difficult, given the complexity of first-price auctions with asymmetric bidder even when resale is not possible. Gupta and Lebrun (1999) compute equilibria when resale is possible, but assume common knowledge of values at the resale stage. Not making this simplifying assumption, the current paper contains the first construction of an equilibrium in a first-price auction with asymmetric bidders and resale.

Our other main result is that a 0-value trader cannot make profits in a first-price auction. We have obtained this result only for the special case of 2-bidder auction with a single privately informed bidder. On the other hand, no restricting assumptions on the resale market rules were needed.

Throughout the paper, we have assumed that the losing bids remain private. The analysis would be quite different if the losing bids were publicly announced. In particular, a strictly increasing bid function would fully reveal a buyer's value to the resale market. No equilibrium for the case of asymmetric bidders and publicly announced losing bids is currently known in the literature.[9]

Arguably, the most basic insight of our paper is that first-price auctions are sometimes *not speculation-proof*, in the sense that if possible entry of a speculator (i.e., a 0-value trader) is ignored (so that private-value buyers' strategies are best-responses only to each other), a speculator will be attracted to the auction and make profits via resale. A first-price auction with a *single* private-value buyer is not speculation-proof because the buyer would bid 0 in the absence of a speculator. Auctions with *multiple symmetric* independent-private-value buyers are sometimes not speculation-proof either. This can be seen in the case where losing bids are announced. Suppose, for example, there are two buyers with independent private values, each distributed uniformly on $[0, 1]$, and the resale seller makes an optimal

[8]The other extreme is the case of symmetric bidders. Haile (1999, Theorem 1) has shown for any number of bidders that the no-resale equilibrium remains valid, and thus no resale occurs, when resale becomes possible; this holds even if the resale seller can extract all gains from resale trade.

[9]Krishna (2002, Section 4.4) for two asymmetric bidders with independent private values distributed on the same support, shows that any equilibrium will be inefficient if the resale mechanism is an optimal take-it-or-leave-it offer. However, Krishna does not show the existence of an equilibrium.

take-it-or-leave-it offer. Suppose buyers ignore possible entry of a speculator. It is then optimal for each buyer to bid 1/2 her value and consume the good if she wins. Therefore, a speculator who steps in with the bid $b_s = 1/2$ wins for sure and from the observed losing bids she infers the maximum value among buyers, which has the expectation 2/3. The speculator's expected payoff equals $\delta(2/3) - b_s$, which is positive for δ close to 1. Thus, the auction is not speculation-proof.

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