

COMMENT ON “THE IRREDUCIBLE ROLE OF DERIVED MARGINAL UTILITY IN DYNAMIC STOCHASTIC PROGRAMMING”

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Abstract. This comment shows why the Lagrange method can be simpler than dynamic programming in solving dynamic optimization problems.

We are indebted to Professor Paul Samuelson (1996) for clarifying the relationship between dynamic programming and the Lagrange method (Chow, 1992, 1993, 1997). This comment supplements his paper by pointing out why the Lagrange method can be simpler.

Consider the well-known one-sector optimal growth model

$$\max E \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{subject to} \quad k_{t+1} = z_t k_t^\alpha - c_t$$

where β is the discount factor, c_t is consumption, and k_t is capital stock at the beginning of period t . The output $z_t k_t^\alpha$ includes depreciated capital stock; z_t is an i.i.d. random shock with $E[\ln z_t] = 0$. By dynamic programming, one solves the following Bellman equation for the value function $V(k, z)$:

$$V(k_t, z_t) = \max_{c_t} [\ln c_t + \beta E_t V(k_{t+1}, z_{t+1})].$$

Guessing correctly the form of V ,

$$V(k, z) = a + b \ln k + d \ln z$$

one uses the equation for k_{t+1} to evaluate $E_t V(k_{t+1}, z_{t+1})$, substitutes the result into the Bellman equation, and equates coefficients on both sides of the Bellman equation. After some lengthy algebra, one obtains

$$a = [1 - \beta]^{-1} [\ln(1 - \alpha\beta) + \alpha\beta(1 - \alpha\beta)^{-1} \ln(\alpha\beta)]$$

$$b = \alpha(1 - \alpha\beta)^{-1}$$

$$d = (1 - \alpha\beta)^{-1}.$$

Given the value function, the optimal c_t is obtained by maximizing the right-hand side of the Bellman equation.

By the Lagrange method, one differentiates the Lagrange expression

$$\mathcal{L} = \sum_{t=0}^{\infty} E_t \{ \beta^t \ln z_t - \beta^{t+1} \lambda_{t+1} [k_{t+1} - z_t k_t^\alpha + c_t] \}$$

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to obtain the first-order conditions for c and λ :

$$\beta^{-t} \partial \mathcal{L} / \partial c_t = c_t^{-1} - \beta E_t \lambda(k_{t+1})$$

$$\beta^{-t} \partial \mathcal{L} / \partial k_t = -\lambda_t + \beta \alpha k_t^{\alpha-1} z_t E_t \lambda(k_{t+1}).$$

Guessing correctly the form $\lambda = gk^{-1}$ for the Lagrange function, one solves the two first-order conditions to obtain $g = (1 - \alpha\beta)^{-1} \alpha$ and

$$c = (1 - \alpha\beta) z k^\alpha$$

$$\lambda = (1 - \alpha\beta)^{-1} \alpha k^{-1}.$$

The Lagrange method requires only one partial derivative $\partial V / \partial k$ of the value function; the value function includes knowledge of $\partial V / \partial z$ and the integration constant a , both of which are unnecessary in solving the dynamic optimization problem.

REFERENCES

- Chow, G. C. (1992) "Dynamic Optimization Without Dynamic Programming," *Economic Modelling* 9, January, 3-9.
- (1993) "Optimal Control Without Solving the Bellman Equation," *Journal of Economic Dynamics and Control* 17, 621-30.
- (1997) *Dynamic Economics: Optimization by the Lagrange Method*, New York: Oxford University Press.
- Samuelson, P. A. (1996) "The Irreducible Role of Derived Marginal Utility in Dynamic Stochastic Programming," *Pacific Economic Review* 1(2), September.

