

# Imperfect Competition *à la* Negishi, also with Fixed Costs\*

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## Abstract

The paper studies equilibria for economies with imperfect competition and non-convex technologies. Following Negishi, firms maximise profits under downward-sloping perceived demand functions. Negishi's assumptions, in particular the assumption of a single monopolistic competitor in each market, are relaxed. Existence of equilibria is obtained, under otherwise standard assumptions, for productions sets defined in each firm by the union of a convex technology and a technology subject to fixed costs. In the light of a counterexample, it is assumed that fixed factors are distinct from variable factors. Technically, the proof rests on pricing rules.

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*Keywords:* Imperfect competition, fixed costs, general equilibrium, perceived demand, pricing rules.

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\*This is a revised version of the paper presented at the European Conference on General Equilibrium, Athens, May 2002. The main revision consists in using pricing rules that allow for some negative prices, a device evoked in a conversation with Jean-Marc Bonnisseau in Athens. That device permitted major weakenings of the assumptions. The perceptive remarks of an anonymous referee are acknowledged most gratefully. The paper presented at Athens is available from the authors.

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# 1 Motivation and Guidelines

This paper is a modest contribution to the challenging topic of general equilibrium under imperfect competition and non-convex technologies. The topic is important because fixed costs and increasing returns to scale – hence, barriers to entry and concentrated production – are a major source of market power. Currently available results are meager. Without attempting to review these, we note that some models - e.g. Novshek and Sonnenschein (1978) or Hart (1979) – deal with many firms that are or become small relative to the aggregate economy. In contrast, others – e.g. Heller (1993) – concentrate on a single monopolist embedded in an otherwise competitive economy. Others still - e.g. Blanchard and Kiyotaki (1987, section IV) – rely on specific models endowed with strong symmetry properties. In contrast, we introduce a methodological approach that holds promises of substantial generality, comparable to that of competitive analysis – even if we do not exploit these promises fully here.

Our approach builds on two main contributions: (i) the perceived-demand approach to monopolistic competition, introduced in the seminal paper by Negishi (1961); (ii) the pricing-rules approach to firm behaviour, introduced in the seminal paper by Dierker, Guesnerie and Neufeind (1985) and applied to competitive-like equilibrium under non-convex technologies by Bonnisseau and Cornet (1988), Dehez and Drèze (1988a,b).

The motivation for adopting the *perceived-demand* approach is twofold. First, it simplifies the general equilibrium analysis, a welcome feature given the analytical difficulties introduced by non-convexities. Second, it is in our opinion more realistic, in many situations, than the alternative, objective-demand approach. Endowing firms with the ability to compute general equilibrium reactions to their own market behaviour typically stretches the concept of rationality far beyond realistic limits.<sup>1</sup> Of course, Negishi's formulation has its

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<sup>1</sup>Negishi (1972, p. 107) goes as far as claiming: “It is widely agreed that the only

own shortcomings, in particular bounded rationality (more on this below) and failure to model the uncertainties surrounding the estimation of demand elasticities.<sup>2</sup> But the gain in realism remains substantial. And the two approaches are not logically inconsistent, as evidenced by the work of Silvestre (1977) and Gary-Bobo (1989) who impose that perceived demand curves reflect correctly both the level *and the slope* of demand at equilibrium. We retain the former requirement; we have not attempted to incorporate the latter.

*Pricing rules* provide a powerful tool in modeling firm behaviour. A pricing rule associates, with every point on the frontier of a firm's production set, the set of prices at which the firm is willing to produce at that point. In the case of *smooth convex* technologies, the pricing rule of a *competitive* firm gives the unique relative prices at which a production point yields maximum profit.<sup>3</sup> This suggests an alternative interpretation of the competitive equilibrium:<sup>4</sup> firms meet demand and adjust prices towards competitive levels – thus bypassing the contradiction between price setting and competition stressed by Arrow (1959).

Pricing rules have proved valuable in the general-equilibrium analysis of economies with non-convex technologies.<sup>5</sup> Their use rests on a suitably defined normal cone to an arbitrary closed set. One example is the work of Dehez and Drèze (1988a,b) on “competitive equilibria with quantity-taking producers”: (i) producers announce prices for their outputs and satisfy the demand which materialises at these prices; (ii) these output prices are *competitive*, in the

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demand (supply) function that is relevant in the theory of monopolistic (monopsonic) competition is the perceived, imagined or subjective demand (supply) functions which express the expectation of the firm as to the relationship between the price it charges (offers) and the quantity of its output (input) the market will buy (sell)”. In a footnote, he quotes Bushaw-Clover, Davis-Whinston, Kaldor and Triffin in support of his assertion. Contemporary authors, including ourselves, would be less sanguine...

<sup>2</sup>For a very modest step in that direction, see Drèze (1979).

<sup>3</sup>These relative prices are given by the normal vector to the production set.

<sup>4</sup>That alternative interpretation is given in Dehez and Drèze (1988a).

<sup>5</sup>Cf. *Journal of Mathematical Economics* 17-1(1988) and in particular Bonnisseau and Cornet (1988), Vohra (1988).

sense that they are *the lowest prices at which the producers remain willing to satisfy demand*. The pricing rules formalise condition (ii). An equilibrium exists, under otherwise standard assumptions, whether production sets are convex or not. Two properties of the pricing rules are used in the existence proof: (P1) the rules are defined by correspondences which are upper hemicontinuous (u.h.c), convex-, compact- and non-empty valued (c.c.n.v); (P2) the rules imply non-negative profits.

Under monopolistic competition *à la* Negishi, condition (ii) above must be modified to become: (ii') *at these output prices, profits, evaluated on the basis of the perceived-demand functions, are maximal*. Such a condition raises a specific difficulty. A pricing rule must be defined at every point on the frontier of the firm's production set. For some of these points, there do not exist non-negative prices sustaining the point as a profit maximising production plan. For instance, if marginal revenue becomes negative when output  $y$  exceeds some level  $\bar{y}$ , then the set of non-negative prices defined by (ii') is empty for all  $y > \bar{y}$ . Yet, the pricing rule should be non-empty valued also there.

To resolve that dilemma, we define pricing rules under which the prices stipulated at points where (ii') is violated cannot prevail *at equilibrium*. The rules are *defined* everywhere, but market equilibrium is not possible everywhere. In this exploratory paper, we assume *strict monotonicity of preferences*, so that equilibrium prices are strictly positive. And we impose: (P3) the rules are such that the first-order conditions (FOC) for profit maximisation under subjectively-perceived demand functions are verified, *or else some price is non positive*. Accordingly, the FOC are verified at equilibrium.

The resulting equilibria thus have a "bounded rationality" connotation, because: (i) first-order conditions are by nature local, and a local optimum need not be global; (ii) the demand elasticities need not be perceived correctly. Item (i) seems inescapable, in the absence of convexity. From a technical view-

point, existence of several isolated local optima, only some of which are global, introduces discontinuities in the profit-maximising supply correspondence that preclude recourse to Kakutani's fixed-point theorem. From an economic viewpoint, global optimisation imposes two unrealistic demands on firm behaviour, namely ability to solve a global optimisation problem not always amenable to standard techniques; and ability to perceive demand characteristics at allocations arbitrarily remote from experience.

In section 2, we record the immediate result that an equilibrium exists, *under otherwise standard assumptions plus strict monotonicity of preferences* (A1), when firms hold pricing rules verifying (P1), (P2) and (P3).<sup>6</sup> The rest of the paper deals with primitive assumptions under which such pricing rules exist.

In order to pave the ground, we first develop a pricing-rule approach to Negishi's model, under convex technologies (section 3). Negishi (1961, 1972) assumes that: (i) there is at most one monopolistic competitor in each market; (ii) perceived demand functions are linear and consistent with observations (in levels, not necessarily slopes); (iii) the perceived-demand coefficients are continuous functions of market prices and market net demands to the firm. We generalise (i) by allowing every firm to hold non-competitive net-demand perceptions on all markets but one;<sup>7</sup> and we generalise (iii) by allowing the coefficients to depend – continuously – on the full allocation. These generalisations are of interest in their own right.

But we use three ancillary assumptions:

- (A1) every commodity is strictly desired by at least one consumer;
- (A2) perceived inverse demand functions are continuous;
- (A3) each firm holds competitive price perceptions for at least one commodity.

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<sup>6</sup>Reminder: pricing rules are an abstract tool of the economist exploring a technical issue; in particular, they may be devoid of behavioural connotations out of equilibrium.

<sup>7</sup>This generalisation defines in what sense we extend Negishi's approach from *monopolistic* to *imperfect* competition.

(A1) has been motivated above. (A2) and (A3), already used by Negishi, are clearly innocuous. Under standard assumptions plus (A1)-(A3), a Negishi equilibrium exists, when production sets are convex (theorem 3.1).

Turning to non-convex technologies, the challenge is to ascertain the existence of pricing rules endowed with properties (P1), (P2), (P3). Under linear demand functions, whether or not properties (P1)-(P3) are mutually consistent depends upon the production set. We give in section 4 a simple example of a production set for which there does not exist a pricing rule with the desired properties. In the light of that example, we define a restricted class of technologies, for which (P1)-(P3) are mutually consistent.

Our restricted class consists of technologies with fixed costs where (A4) , for each firm: (i) the fixed inputs are distinct from the variable inputs or the outputs; and (ii) the production set is the union of two convex sets, one and one only of which contains the origin. The extension to an arbitrary finite union of convex sets, each allowing for fixed costs, is at hand. Unless one of these contains the origin, there always exist (local) equilibria where the firm is inactive: upper hemi-continuity implies that the pricing rule places no restriction on admissible prices there (see section 4 for illustration); and there will exist an equilibrium for the sub-economy from which a given firm, or set of firms, is deleted.<sup>8</sup>

In section 5, we construct pricing rules verifying (P1)-(P3) for the case where the fixed inputs consist of a *single* commodity with *competitive* price perceptions. These two restrictions are used for expositional convenience and are amenable to generalisation. Under standard assumptions and (A1)-(A4), a Negishi equilibrium exists (theorem 5.1).

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<sup>8</sup>An alternative, introduced in Madden (1984), consists in showing that an equilibrium with non-zero production (hence fixed investments) exists, provided the consumption sector of the economy is large enough. Madden verifies this by replicating the consumption sector  $n$  times and proving the existence of a finite integer  $\bar{n}$  such that,  $\forall n \geq \bar{n}$ , the economy replicated  $n$  times admits an equilibrium with positive production. The same route could be followed here.

Our excuse for introducing property (i) is simple: it holds trivially whenever fixed costs correspond to *investments* in facilities giving access to superior operating technologies. Because investments precede operations in time, the fixed inputs are automatically separated from variable inputs and outputs through time indices.<sup>9</sup> Thus, *our specification covers in particular all the cases where increasing returns result from fixed investments*; that is, it covers many (most?) interesting cases.

## 2 Existence of Equilibria with Pricing Rules

Following Bonnisseau and Cornet (1988)<sup>10</sup> or Dehez and Drèze (1988a), hereafter DD, to which we refer for details, consider an economy with  $\ell$  commodities,  $n$  producers and  $m$  consumers. Producer  $j$  is characterised by a production set  $Y^j$ , a closed subset of  $R^\ell$  such that  $Y^j + R_-^\ell \subset Y^j$  (free disposal)<sup>11</sup> and  $Y^j \cap R_+^\ell = \{0\}$  (absence of free production, possibility of inaction). Consumer  $i$  is characterised by:

- a consumption set  $X^i$ , a closed subset of  $R^\ell$ , convex and bounded below;
- a preference relation  $\succsim_i$  on  $X^i$ , complete, continuous and convex;<sup>12</sup>
- an initial endowment  $\omega^i$  in the interior of  $X^i$ ;
- shares in firm profits  $\theta^i = (\theta^{i1}, \dots, \theta^{in}) \geq 0$ ,  $\sum_i \theta^{ij} = 1, j = 1, \dots, n$ .

It is further assumed that,  $\forall k \in R^\ell$ , the set

$$\{(y^1, \dots, y^n) \in \Pi Y^j \mid \Sigma y^j \geq k\} \text{ is bounded } \subset R^{\ell n},$$

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<sup>9</sup>An interesting set of technologies close to our class consists of *ex ante* convex technologies with fixed costs *ex post*. Under incomplete markets, these technologies may lead to second-best efficient departures from marginal cost pricing in some states, as suggested in Drèze (2001, section 3.3).

<sup>10</sup>Bonnisseau and Cornet do not impose non-negative profits, only bounded losses; but the rules of income formation for households do not embody limited liability of shareholders.

<sup>11</sup>Alternatively stated,  $Y^j$  is “comprehensive”.

<sup>12</sup>DD assume non-satiation; we substitute (A1) for that requirement.

implying that the set of feasible allocations

$$Z = \{(x^1, \dots, x^m, y^1, \dots, y^n) \in \Pi X^i \times \Pi Y^j \mid \Sigma x^i \leq \Sigma \omega^i + \Sigma y^j\}$$

is a bounded set  $\subset R^{\ell(m+n)}$ .

We refer to that set of assumptions as (DD), and we add the following:<sup>13</sup>

**(A1)** For each commodity  $k \in \{1, \dots, \ell\}$ , there exists a consumer  $i \in \{1, \dots, m\}$ , whose preferences are strictly monotonic with respect to  $x_k$  :

$$x > \hat{x} \text{ with } x_k > \hat{x}_k \text{ implies } x \succ_i \hat{x} \forall x, \hat{x} \in X_i.$$

Let

$$\partial Y^j := \{y \in Y^j \mid \nexists \hat{y} \in Y^j, \hat{y} \gg y\},$$

$$l' := (1, \dots, 1) \in R_+^\ell,$$

$$S := \{v \in R_+^\ell \mid l'v = 1\},$$

$$\mathcal{S} := \{v \in R^\ell \mid l'v = 1\};$$

denote by  $\bar{p}$  a price vector in  $R_+^\ell$  and by  $\bar{z} = (\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$  an allocation in  $Z$ .

**Definition 2.1** A pricing rule for firm  $j$   $\phi^j : \partial Y^j \times Z \times R_+^\ell \rightarrow \mathcal{S}$  is a correspondence assigning to each production plan  $y^j \in \partial Y^j$  a set of prices  $\phi^j(y^j; \bar{z}, \bar{p}) \in \mathcal{S}$ , given the market data  $(\bar{z}, \bar{p})$ .<sup>14</sup>

We formalise the assumptions (P1)-(P2) as follows, for each  $j$ :

(P1) The correspondence  $\phi^j : \partial Y^j \times Z \times R_+^\ell \rightarrow \mathcal{S}$ , is u.h.c. and c.c.n.v.

(P2) For all  $p \in \phi^j(y^j; \bar{z}, \bar{p})$ ,  $p'y^j \geq 0$ .

As for assumption (P3), it is implicit in the following

**Definition 2.2** An equilibrium with pricing rules is defined by a price vector  $\bar{p} \in S$  and an allocation  $\bar{z} \in Z$  such that:

<sup>13</sup>Vector inequalities are  $\geq$ ,  $>$ ,  $\gg$ . Rows vectors are primed.

<sup>14</sup>See remark 2.1 below. In DD,  $\phi^j = \phi^j(y^j)$ , but the immediate generalisation to  $\phi^j = \phi^j(y^j; \bar{z}, \bar{p})$  is mentioned in footnote 18 there.

- (E1) for each  $i$ ,  $\bar{x}^i$  is  $\succ_i$ -maximal in  $\{x^i \in X^i \mid \bar{p}'x^i \leq \bar{p}'\omega^i + \sum_j \theta^{ij} \bar{p}'\bar{y}^j\}$ ;  
(E2)  $\bar{p} \gg 0$  and, for each  $j$ ,  $\bar{p} \in \phi^j(\bar{y}^j; \bar{z}, \bar{p})$ ;  
(E3)  $\sum_i \bar{x}^i \leq \sum_i \omega^i + \sum_j \bar{y}^j$ .

**Theorem 2.1** *Under assumptions (DD), (A1), (P1) and (P2), there exists an equilibrium.*

**Proof** Follows from existence theorem 2 in DD. All the assumptions there are retained, and (P1) validates the generalisation from  $\phi^j(y^j)$  to  $\phi^j(y^j; \bar{z}, \bar{p})$ . One only needs to verify that  $\bar{p} \gg 0$ . As noted above, the set of feasible allocations is bounded in  $R^{\ell(m+n)}$ . But (A1) implies unbounded demands  $x_j^i$  (some  $i$ ) at  $\bar{p}_j \leq 0$ , contradicting (E3). ■

**Remark 2.1** DD define a pricing rule as a correspondence with values in  $S$ , the unit simplex. Careful examination of the proof of their theorem 2 reveals that our alternative definition is still conducive to existence. If  $\phi^j : \partial Y^j \times Z \times R_+^\ell \rightarrow S$  satisfies (P1)-(P2), the result still holds. This mildly surprising remark, which allows for pricing rules specifying *some* negative prices, will prove essential to our analysis in section 3.

**Remark 2.2** As noted above, DD assume that the set of feasible allocations,  $Z$ , is bounded. Accordingly, the proof of theorem 2 there relies on the standard technique of restricting attention to *allocations* belonging to a compact set. Consequently, we need only verify the properties of pricing rules over compact sets, in order to invoke theorem 2.1. Although  $S$  is not compact, our correspondences  $\phi^j(y; \bar{z}, \bar{p})$  will be compact-valued, and the image in  $S$  of a compact set of triplets  $(y; \bar{z}, \bar{p})$  is itself compact (Hildenbrand, 1974, proposition 3, p.24).

### 3 Negishi Equilibria under Convex Technology

#### 3.1

In order to apply theorem 2.1 to imperfect competition *à la Negishi*, it is necessary and sufficient to specify pricing rules that satisfy postulates (P1)-(P2) and that (i) embody profit maximisation under downward sloping perceived-demand functions; (ii) entail some non-positive prices at production plans incompatible with profit maximisation.

At a market allocation  $(\bar{z}, \bar{p})$ , Negishi (1961, 1972) considers indirect perceived-demand functions defined as

$$p^j(y^j; \bar{z}, \bar{p}) = H^j(\bar{z}, \bar{p})y^j + K^j(\bar{z}, \bar{p}), \quad (3.1)$$

where the matrix  $H^j(\cdot)$  is negative semi-definite.<sup>15</sup> The interpretation is that the r.h.s of (3.1) gives the prices  $p^j$  at which firm  $j$  expects to be able to trade the quantities  $y^j$ , when the market data are  $(\bar{z}, \bar{p}) \in Z \times S$ . The vector  $K^j(\bar{z}, \bar{p})$  is such that  $p^j(\bar{y}^j; \bar{z}, \bar{p}) \equiv \bar{p}^j$  (consistency with observations).

Because  $H^j(\cdot)$  is NSD, the profit function is concave, and the first-order conditions (FOC) are *sufficient* for global profit maximisation, when  $Y^j$  is convex. Accordingly, we may use that property in condition (i) above and rephrase it as (i') embody the FOC for profit maximisation under the perceived-demand functions (3.1).

**Remark 3.1**<sup>16</sup> We have defined (3.1) with  $\bar{p} \in S$ , thus under a specific price normalisation. It is well known that an oligopolistic equilibrium is not invariant to the choice of a price normalisation. This has been brought out by

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<sup>15</sup>Negishi (1972, p.111) assumes that the matrix  $H^j$  has zero entries for commodities of which firm  $j$  is not the single monopolistic supplier, and owns a negative definite principal minor for the other commodities. He invokes gross substitutability of direct demand functions as a justification. If the matrix of partial derivatives of the direct demand functions were negative definite, with non-positive off-diagonal elements, then its inverse  $H$  would be negative definite with *all entries* non positive – by application of a theorem of Stieltjes (1886-7).

<sup>16</sup>We thank an anonymous referee for prompting this clarification.

Bronsard (1971) for the case of monopolies, then by Gabszewicz-Vial (1972) for the case of Cournot-Nash oligopolistic equilibria. There are two dimensions to price normalisation – one trivial and one substantive. The *trivial* dimension concerns the overall price level. Instead of imposing  $\bar{p} \in S$ , one could impose  $k\bar{p} \in S, k \in R_+$ . This does not affect the real equilibria: relative prices remain unaffected. The *substantive* dimension is the choice of  $\bar{p} \in S$  as contrasted with, say  $\bar{p}_i = 1$  for some  $i$  in  $L = \{1, \dots, \ell\}$ , or  $(\sum_i p_i^2) = 1$ , for instance. As shown in the references above, oligopolistic or monopolistic equilibria are not invariant to these alternative specifications. As one transparent illustration of the unavoidable arbitrariness in the choice of normalisation, let  $\bar{p} \in S$  and recognise that the bearing of that specification depends upon the arbitrary quantity units in which commodities are measured. By inflating the quantity unit of commodity  $i$ , one can bring the condition  $\bar{p} \in S$  arbitrarily close to imposing  $\bar{p}_i = 1$ , with consequences for equilibrium brought out in the references above.<sup>17</sup> The normalisation issue is basically the same under perceived-demand functions as under objective demand-functions: just consider the case where perceived and objective demands coincide! Thus, the Negishi equilibria inherit the normalisation-dependence of Cournot-Nash equilibria. The present paper is no exception.

Returning to (3.1), the fact that  $\bar{p} \in S$  does *not* imply that  $p^j(y^j; \bar{z}, \bar{p}) \in S$ . (Think about a single output with inelastic demand and inputs supplied at constant prices; the output price varies with  $y^j$  at unchanged input prices, so that  $p^j \notin S$  for  $y^j \neq \bar{y}^j$ .) But irrelevance of the overall price level requires

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<sup>17</sup>The practical side of the normalisation issue is intriguing, since economic agents in a market economy are not aware of any price normalisation. But it is suggestive to think about oligopolistic or monopolistic firms as pursuing a real (as opposed to nominal) profit motive. In a monetary economy, they will accordingly deflate future (possibly state-contingent) profits by a price index. The weights of individual commodities in the price index play a role comparable to the quantity units of the previous paragraph. Thus, indices with different weights may lead to different oligopolistic equilibria.

that (3.1) be homogeneous of degree 1 in  $\bar{p}$ . That is,  $\forall k \in R_+$  :

$$\begin{aligned} p^j(y^j; \bar{z}, k\bar{p}) &= H^j(\bar{z}, k\bar{p})y^j + K^j(\bar{z}, k\bar{p}) \\ &= kH^j(\bar{z}, \bar{p})y^j + kK^j(\bar{z}, \bar{p}) = kp^j(y^j; \bar{z}, \bar{p}). \end{aligned} \quad (3.2)$$

The perceived-profit function is

$$\Pi^j(y^j; \bar{z}, k\bar{p}) = y^j p^j(y^j; \bar{z}, k\bar{p}) = ky^j p^j(y^j; \bar{z}, \bar{p}). \quad (3.3)$$

The FOC for a maximum of  $\Pi^j$  on  $Y^j$  are again independent of the price level  $k$ . They require existence of a vector  $q^j$  in the normal cone to  $Y^j$  at  $y^j$ ,  $N^j(y^j)$ , such that<sup>18</sup>

$$p^j(y^j; \bar{z}, k\bar{p}) + H^j(\bar{z}, k\bar{p})y^j = q^j. \quad (3.4)$$

Because  $Y^j$  is comprehensive convex,  $q^j > 0$  for every  $q^j \in N^j(y^j)$ , and  $N^j(y^j) \cap S \neq \emptyset$ .

In (3.4), the norm of  $q^j$  is implicitly related to  $k$ . Let

$$\bar{q}^j \in N^j(y^j) \cap S, \quad \iota' \bar{q}^j = 1,$$

and write (3.4) as

$$\begin{aligned} kp^j(y^j; \bar{z}, \bar{p}) + kH^j(\bar{z}, \bar{p})y^j &= \lambda_k \bar{q}^j, \\ p^j(y^j; \bar{z}, \bar{p}) + H^j(\bar{z}, \bar{p})y^j &= \frac{\lambda_k}{k} \bar{q}^j. \end{aligned} \quad (3.5)$$

In order to obtain  $p^j(y^j; \bar{z}, \bar{p}) \in \mathcal{S}$ , it is necessary that

$$\frac{\lambda_k}{k} = 1 + \iota' H^j(\bar{z}, \bar{p})y^j. \quad (3.6)$$

This yields the first-order conditions in normalised form:

$$\begin{aligned} p^j(y^j; \bar{z}, \bar{p}) &= \bar{q}^j(1 + \iota' H^j(\bar{z}, \bar{p})y^j) - H^j(\bar{z}, \bar{p})y^j \\ &:= \bar{q}^j a^j(y^j; \bar{z}, \bar{p}) - H^j(\bar{z}, \bar{p})y^j \end{aligned} \quad (3.7)$$

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<sup>18</sup>See e.g., Clarke (1983), proposition 2.3.1 and corollary to proposition 2.4.3.

for  $y^j$  such that  $a^j(y^j; \bar{z}, \bar{p}) > 0$  (see below).

These expressions admit standard interpretations. Looking at the  $i$ -th component of the vector equalities (3.4), for  $y_i^j > 0$  ( $i$  is an output), the l.h.s. measures the marginal revenue of  $y_i^j$  and the r.h.s. its marginal cost, in a multi-product framework.

The term

$$H_i^j(\bar{z}, \bar{p})y^j := \sum_k H_{ik}^j y_k^j = \sum_k \frac{\partial p_i^j}{\partial y_k^j} y_k^j$$

measures the difference between the selling price and the marginal revenue, a negative term under normal circumstances. The same difference is added to the marginal cost in the r.h.s. of (3.7); it corresponds there to the monopolistic markup of price above marginal cost.<sup>19</sup>

Less familiar is the term  $1 + \iota' H^j(\bar{z}, \bar{p})y^j := a^j(y^j; \bar{z}, \bar{p})$ , which multiplies  $\bar{q}^j$  in the r.h.s. of (3.7). Its presence is due to the fact that  $\bar{q}^j \in S$  is a measure of relative, not absolute marginal costs. The additional term scales marginal cost commensurably with  $p^j$ . But a new issue arises: is that term positive? We have argued above that  $H_i^j(\bar{z}, \bar{p})y^j \leq 0$  should be the rule for an output  $i$  “under normal circumstances”. That term is proportional to  $y^j$ , and could become large negative for large  $y^j$ , entailing  $1 + \iota' H^j(\bar{z}, \bar{p})y^j < 0$ . Because  $\bar{q}^j > 0$ ,  $a^j(y^j; \bar{z}, \bar{p}) > 0$  is necessary for  $\bar{q}^j a^j(y^j; \bar{z}, \bar{p}) \in N^j(y^j)$ , hence is necessary for (3.7) to represent correctly the FOC.

The interpretation of this issue is straightforward. If  $y_i^j > 0$  is so large that  $1 + \iota' H^j(\bar{z}, \bar{p})y^j < 0$ , the production plan  $y^j$  generates negative marginal revenues “overall”,<sup>20</sup> and  $y^j$  cannot be a profit-maximising production plan.<sup>21</sup>

<sup>19</sup>For  $y_i^j < 0$  ( $i$  is an input),  $H_i^j$  measures the difference between the buying price and the marginal cost of procurement, a positive term under normal circumstances for a monopsonist.

<sup>20</sup>“Overall”:  $1 + \iota' H^j(\cdot)y^j = \sum_i [p_i^j + H_i^j(\cdot)y^j]$  = sum of marginal revenues of outputs and marginal procurement costs of inputs. The sum over commodities of these terms is meaningful in the same sense that a sum of prices defines a meaningful price index. If  $H_i^j(\cdot)y_i^j$  is normally negative for outputs and positive for inputs,  $a^j(\cdot) < 0$  is normally brought about by negative marginal revenues for outputs.

<sup>21</sup>At  $y^j \in \partial Y^j$ , profits increase in a direction pointing inward the comprehensive set  $Y^j$

(We have already alluded to that situation in section 1.) The pricing rule will have to specify some non-positive prices at such a  $y^j$ .

**Remark 3.2**<sup>22</sup> The FOC (3.4), hence (3.7), do not embody the requirement (3.1) that  $p^j = H^j(\bar{z}, k\bar{p})y^j + K^j(\bar{z}; k\bar{p})$ . Indeed, that requirement is meaningful only when  $p^j = k\bar{p}$ , a property that holds only at equilibrium, where furthermore  $k = 1, p^j = \bar{p}$ . To repeat, the pricing rule is a technical device used by the economist to prove existence of equilibria. Consistency requirements need only be verified *at equilibrium*. Such is indeed the case here, because  $H^j(\bar{z}, \bar{p})$  and  $K^j(\bar{z}, \bar{p})$  are assumed to verify  $\bar{p} = H^j(\bar{z}, \bar{p})y^j + K^j(\bar{z}, \bar{p})$  identically in  $\bar{z}, \bar{p}$  and  $y^j$  consistent with  $\bar{z}$ : at equilibrium,  $y^j = \bar{y}^j$  enters the definition of  $\bar{z} = (\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ . Accordingly, at equilibrium, the FOC for profit maximisation will be satisfied, for the normalisation  $\iota'\bar{p} = 1$  (see remark 3.1 above).

### 3.2

We now state formally:

**(A2)** For each  $j$ , the mapping  $(\bar{z}, \bar{p}) \rightarrow H^j(\bar{z}, \bar{p})$  is continuous, and the matrix  $H^j(\bar{z}, \bar{p})$  is NSD finite identically in  $(\bar{z}, \bar{p})$ .

**(A3)** For each  $j$ , the matrix  $H^j(\bar{z}, \bar{p})$  owns at least one row with zero entries identically in  $(\bar{z}, \bar{p})$ .

Our pricing rule is

$$\phi^j(y^j; \bar{z}, \bar{p}) = \left\{ p \in R^\ell \mid \exists \bar{q} \in N^j(y) \cap S : \right. \\ \left. p = \bar{q}^j \max(0, 1 + \iota' H^j(\bar{z}, \bar{p}) y^j) - \frac{H^j(\bar{z}, \bar{p}) y^j}{\max(1, -\iota' H^j(\bar{z}, \bar{p}) y^j)} \right\}. \quad (3.8)$$

**Lemma 3.1** Under assumptions (A2) and (A3), the pricing rules (3.8) satisfies (P1), (P2) and (P3), with  $\iota'p \equiv 1$ , for  $y^j, \bar{z}$  and  $\bar{p}$  in compact sets.

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(a direction bringing about production inefficiency); thus,  $y^j$  is not profit maximising.

<sup>22</sup>We thank an anonymous referee for prompting this clarification.

**Proof** We omit the superscript  $j$  and, where unnecessary, the reference to  $(\bar{z}, \bar{p})$ .

(1) Ad  $\iota'p \equiv 1$ .

$$\iota'p = \iota'\bar{q} \max(0, 1 + \iota'Hy) - \frac{\iota'Hy}{\max(1, -\iota'Hy)} \text{ with } \iota'\bar{q} = 1;$$

If  $1 + \iota'Hy > 0, -\iota'Hy < 1$ , then  $\iota'p = 1 + \iota'Hy - \iota'Hy = 1$ ;

if  $1 + \iota'Hy < 0, -\iota'Hy > 1$ , then  $\iota'p = 0 - \frac{\iota'Hy}{-\iota'Hy} = 1$ ;

if  $1 + \iota'Hy = 0, -\iota'Hy = 1$ , then  $\iota'p = 0 - \iota'Hy = 1$ .

(2) Ad (P1).

For any given  $(\bar{z}, \bar{p}) \in Z \times S$ , and for all  $y$  in a closed cube in  $R^\ell$  with finite length, every entry  $H_i y$ ,  $i = 1, \dots, \ell$ , is uniformly bounded; also,  $N(y)$  is closed; so  $\phi$  is compact-valued. It is convex-valued because  $N(y)$  is convex-valued. It is non-empty by construction. It is u.h.c in  $y$  because

-  $N(y)$  is u.h.c in  $y$ ,  $\max(0, 1 + \iota'Hy)$  is continuous, and so their product is u.h.c (Hildenbrand, 1974, p.25);

-  $\frac{-H_i y}{\max(1, -\iota'Hy)}$  is continuous in  $y$ ;

- the sum of these two terms is u.h.c (Hildenbrand, 1974, p.25).

The correspondence  $\phi(y; \bar{z}, \bar{p})$  is u.h.c in  $(\bar{z}, \bar{p})$  for all  $y$ , because  $p$  is continuous in  $H(\bar{z}, \bar{p})$  and  $H$  is continuous by (A2).

(3) Ad (P2).

Profits,  $\Pi(y, p) = y'p = y'\bar{q} \max(0, 1 + \iota'Hy) - \frac{y'Hy}{\max(1, -\iota'Hy)}$ , are non-negative:

-  $y'\bar{q} \geq 0$  by definition of  $N(y)$ , and  $0 \in Y$ ;

-  $-y'Hy \geq 0$  since  $H$  is NSD (A2).

(4) Ad (P3).

When  $1 + \iota'Hy > 0$ , the pricing rule implements the FOC (3.7), which

is equivalent to (3.4) at equilibrium (see remark 3.2 above). When  $1 + \ell'Hy \leq 0$ ,  $p_i = \frac{-H_i y}{-\ell'Hy}$  and  $p_i \equiv 0$  for  $i$  such that  $H_i = 0$  (A3). ■

### 3.3

**Definition 3.1** A Negishi equilibrium is defined by a price vector  $\bar{p} \in S$  and an allocation  $\bar{z} \in Z$  such that:

(E1) For each  $i$ ,  $x^i$  is  $\succsim_i$  - maximal in

$$\{x^i \in X^i \mid \bar{p}'x^i \leq \bar{p}'\omega^i + \sum_j \theta^{ij} \bar{p}'\bar{y}^j\};$$

(E2) For each  $j$ ,  $\bar{y}^j$  maximises locally on  $Y^j$

$$\Pi^j(y^j; \bar{z}, \bar{p}) = p^{jj}(y^j; \bar{z}, \bar{p})y^j = [H^j(\bar{z}, \bar{p})y^j + K^j(\bar{z}, \bar{p})]' y^j;$$

(E3)  $\sum_i \bar{x}^i \leq \sum_i \omega^i + \sum_j \bar{y}^j$ .

**Theorem 3.1** Under assumptions (DD), (A1), (A2) and (A3), if each  $Y^j$  is a convex set, there exists a Negishi equilibrium, and the maximum in (E.2) is global.

**Proof** Follows from theorem 2.1, remarks 2.1-2.2 and lemma 3.1. ■

## 4 A Pricing Rule May not Exist with Fixed Costs

In this section, we provide an elementary example of a production set for which there does not exist a pricing rule verifying (P1)-(P3) of section 1.

There are two commodities, an output  $y$  and an input  $x$ , with respective prices  $p$  and  $r$ . The production set, also depicted in figure 1, is defined by:

$$Y = \{(y, x) \in R \times R_- \mid y \leq \max[0, -(x + c)]\}.$$

Thus,  $y \leq 0 \forall x \geq -c$ ,  $y \leq -(x+c) \forall x \leq -c$ . The perceived inverse demand function is here defined, for a fixed  $(\bar{z}, \bar{p})$ , by

$$p(y) = \bar{p} + b(y - \bar{y}), b < 0; \quad r(x) = \bar{r}.$$

Profits are

$$\Pi(x, y) = p(y)y + \bar{r}x = \begin{cases} \bar{r}x & \text{if } x \geq -c \\ p(y)y - \bar{r}(y+c) & \text{if } x \leq -c. \end{cases}$$

The first-order condition for profit maximisation, given  $x \leq -c$ , is

$$\frac{d\Pi}{dy} = by + p(y) - \bar{r} = 0, \quad p(y) = \bar{r} - by. \quad (4.1)$$

This corresponds to the normalised vector in  $\mathcal{S}$

$$p = \frac{1-by}{2}, \quad r = \frac{1+by}{2}. \quad (4.2)$$

The corresponding profits are

$$\Pi = \frac{1-by}{2} y - \frac{1+by}{2} (y+c) = -by^2 - \frac{1+by}{2} c.$$

Profits are non-negative, hence maximal at a solution of (4.1), provided

$$-b^{-1} \geq y \geq \{[bc - (b^2c^2 - 8bc)^{1/2}] / 4b\} := \hat{y}.$$

To illustrate, if  $b = -1$ , then  $1 \geq \hat{y} \geq 0$  for  $c \leq 1$ .

We wish to construct, on the boundary  $\partial Y$  of  $Y$ , a pricing rule which is u.h.c (P1), which yields non-negative profits (P2), and is such that either  $p = \frac{1-by}{2}$ ,  $r = \frac{1+by}{2} \geq 0$ , or else  $pr \leq 0$  (P3). In attempting to do so, a contradiction arises at the point  $(\hat{y}, -\hat{y} - c)$ , labeled  $a$  in figure 1. For  $\frac{-1}{b} \geq y \geq \hat{y}$ , the FOC imposes the unique prices given by (4.2). In particular,  $\hat{p} = p(\hat{y}) = \frac{1-b\hat{y}}{2} > 0$  and  $\hat{r} = r(\hat{y}) = \frac{1+b\hat{y}}{2} > 0$  for  $\hat{y} < \frac{-1}{b}$ . For  $\hat{y} > y \geq 0$ , the FOC yields negative profits, and (P2)-(P3) impose  $pr \leq 0$  with  $p \geq 0, r \leq 0$ .

Let  $y^\nu$ ,  $\nu = 1, 2, \dots$ , tend to  $\hat{y}$ , with  $y^\nu < \hat{y} \forall \nu$ ; and let  $(p^\nu, r^\nu) \in \phi(y^\nu)$ , so that  $p^\nu \geq 0, r^\nu \leq 0$ . If  $\phi(y)$  is u.h.c, there exists a limit  $(\bar{p}, \bar{r})$  (not necessarily unique), with  $\bar{p} \geq 0, \bar{r} \leq 0$ . In order for  $\phi(y)$  to be u.h.c and c.c.n.v at  $\hat{y}$ , it must be the case that  $(p_\alpha, r_\alpha) = \alpha(\hat{p}, \hat{r}) + (1 - \alpha)(\bar{p}, \bar{r})$  belongs to  $\phi(\hat{y}) \forall \alpha \in [0, 1]$ . But this allows for strictly positive  $(p_\alpha, r_\alpha)$  different from  $(\hat{p}, \hat{r})$ , which violates (P3). So, there does not exist a pricing rule satisfying (P1)-(P3), in this example.

This explains why we introduce the restriction that fixed inputs and variable inputs be disjoint sets of commodities.

## 5 Negishi Equilibria with Fixed Costs

We now prove existence of equilibria *à la* Negishi for a class of technologies with fixed costs, where for each firm fixed inputs are distinct from variable inputs or outputs, and the production set is the union of two convex sets of which one contains the origin. For simplicity of exposition, we consider a single fixed input, and a fixed investment threshold. We leave open the extension to more complex technologies - where for instance fixed investments could be chosen from some feasible set, each choice giving access to a convex production set for variable inputs and outputs;<sup>23</sup> or where  $Y$  is the union of several convex sets, each allowing for fixed investments.

More specifically, let  $f(j)$  be the index of the fixed input to firm  $j$ , and write  $y^j = (y_f^j, y_{-f}^j)$  where the subscript  $f$  stands for  $f(j)$ .

**(A4)** For each  $j$ ,  $Y^j = Y^{j1} \cup Y^{j2}$ , where

$$Y^{j1} = \left\{ y^j \in R^\ell \mid y_f^j \leq -c^j \leq 0, y_{-f}^j \in Y_{-f}^{j1} \subset R^{\ell-1} \right\},$$

---

<sup>23</sup>The extension to several fixed inputs with fixed investment thresholds is straightforward, but the expository cost exceeds the benefits.

$$Y^{j2} = \{y^j \in R^\ell \mid y_f^j \leq 0, y_{-f}^j \in Y_{-f}^{j2} \subset R^{\ell-1}\},$$

$$Y_{-f}^{ji} \text{ is a closed convex set such that } Y_{-f}^{ji} + R_{-}^{\ell-1} \subset Y_{-f}^{ji},$$

$$Y_{-f}^{ji} \cap R_{+}^{\ell-1} = \{0\}, \quad i = 1, 2, \text{ and } Y_{-f}^{j1} + Y_{-f}^{j2} \subset Y_{-f}^{j1}.$$

That is,  $Y^j$  is the union of two convex sets, one of which ( $Y^{j1}$ ) stipulates a “fixed cost”  $y_f^j \leq -c^j$ , while the other ( $Y^{j2}$ ) contains the origin.<sup>24</sup> Both sets satisfy free disposal and admit  $y_{-f} = 0$ . The inclusion  $Y_{-f}^{j1} + Y_{-f}^{j2} \subset Y_{-f}^{j1}$  reflects the elementary fact that a production plan which is feasible without fixed investments remains feasible with fixed investment. (This is not essential, but natural.)

**(A3’)** For each  $j$ , the matrix  $H^j(\bar{z}, \bar{p})$  owns at least two rows with zero entries identically in  $(\bar{z}, \bar{p})$ , one of these being row  $f$ .

This imposes competitive price perceptions for the fixed input - a convenient, though not essential specification.

The boundary  $\partial Y^j = \{y^j \in Y^j \mid \nexists \hat{y} \in Y^j, \hat{y} \gg y^j\}$  of  $Y^j$  can be described as follows in terms of  $Y_{-f}^{j1}$ ,  $Y_{-f}^{j2}$  and their boundaries  $\partial Y_{-f}^{ji} \subset R^{\ell-1}$ ,  $i = 1, 2$ :

$$\begin{aligned} \partial Y^j &= \{y^j \in R^\ell \mid y_f^j < -c^j, y_{-f}^j \in \partial Y_{-f}^{j1}\} \cup \\ &\quad \{y^j \in R^\ell \mid y_f^j = -c^j, y_{-f}^j \in \partial Y_{-f}^{j1} \setminus (Y_{-f}^{j2} \setminus \partial Y_{-f}^{j2})\} \cup \\ &\quad \{y^j \in R^\ell \mid 0 > y_f^j > -c^j, y_{-f}^j \in \partial Y_{-f}^{j2}\} \cup \\ &\quad \{y^j \in R^\ell \mid y_f^j = 0, y_{-f}^j \in Y_{-f}^{j2}\}. \end{aligned}$$

To define a pricing rule  $\phi^j : \partial Y^j \times Z \times S \rightarrow \mathcal{S}$ , we can rely on (3.8) for the first and third elements in the union of sets defining  $\partial Y^j$ ; but we need to extend that specification so as to cover the second and fourth elements, while preserving upper hemi-continuity at  $y_f^j = -c^j$  and at  $y_f^j = 0$ .

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<sup>24</sup>The total cost function implied by (A4) is not convex for  $c^j > 0$ . It contains the origin. Fixed costs, a property of  $Y^{j1}$ , are avoided when  $y^j \in Y^{j2}$ .

**Lemma 5.1** *Under (A2), (A3') and (A4), there exist pricing rules verifying (P1), (P2) and (P3), for  $y, \bar{z}$  and  $\bar{p}$  in compact sets.*

**Proof** The proof is constructive. A suitable pricing rules is defined successively for all  $y^j \in \partial Y^j$  such that:

1.  $0 > y_f^j > -c^j$ ;
2.  $y_f^j < -c^j$ ;
3.  $y_f^j = -c^j$ ;
4.  $y_f^j = 0$ .

The proof applies to an arbitrary firm, so we omit the superscript  $j$ . Similarly, we omit explicit reference to  $(\bar{z}, \bar{p})$ . We write  $N^i(y_{-f})$  for the normal cone to  $Y_{-f}^i$  at  $y_{-f}$  and  $\phi_{-f}^i(y_{-f})$  for the correspondence defined by (3.8) with  $\bar{q} \in N^i(y_{-f}) \cap S$ .

**1.** When  $0 > y_f > -c$ , then  $y_{-f} \in \partial Y_{-f}^2$ . We set  $p_f = 0$ ,  $p_{-f} \in \phi_{-f}^2(y_{-f})$ ; that is

$$\phi(y \mid 0 > y_f > -c, y_{-f} \in \partial Y_{-f}^2) = \{p \in \mathcal{S} \mid p_f = 0, p_{-f} \in \phi_{-f}^2(y_{-f})\}. \quad (5.1)$$

Because lemma 3.1 applies to  $\phi_{-f}^2(y_{-f})$ , it applies to (5.1).

**2.** When  $y_f < -c$ , then  $y_{-f} \in \partial Y_{-f}^1$ , and we define:

$$\phi(y \mid y_f < -c, y_{-f} \in \partial Y_{-f}^1) = \{p \in \mathcal{S} \mid p_f = 0, p_{-f} \in \phi_{-f}^1(y_{-f})\}. \quad (5.2)$$

Again, lemma 3.1 applies to (5.2).

**3.** When  $y_f = -c$ , then  $y_{-f} \in Y_{-f}^1 \setminus (Y_{-f}^2 \setminus \partial Y_{-f}^2)$ , and  $y$  is efficient in production if and *only if*  $y_{-f} \in \partial Y_{-f}^1$ . Otherwise, the FOC conditions cannot be satisfied at strictly positive prices, and  $y$  cannot be part of an equilibrium. This provides leeway in the (purely technical) definition of the pricing rule. We use that leeway, when  $y_{-f} \notin \partial Y_{-f}^1$ , by setting  $p_f = 0$  and extending the correspondence suitably. It is then convenient to distinguish three subcases.

**3.1** For  $y_{-f} \in \partial Y_{-f}^1$ , set  $p_f \in [0, \frac{p'_{-f} y_{-f}}{c}]$  and  $p_{-f} = (1 - p_f) \hat{p}_{-f}$  with  $\hat{p}_{-f} \in \phi_{-f}^1(y_{-f})$ ; alternatively stated, for some  $\hat{p}_{-f} \in \phi_{-f}^1(y_{-f})$ , set  $p_f \in$

$\left(0, \frac{\hat{p}'_{-f} y_{-f}}{c + \hat{p}'_{-f} y_{-f}}\right)$  and  $p_{-f} = (1 - p_f) \hat{p}_{-f}$ . This defines the composition of correspondences:

$$\begin{aligned} \phi(y \mid y_f = -c, y_{-f} \in \partial Y_{-f}^1) &= \{p \in \mathcal{S} \mid \exists \hat{p}_{-f} \in \phi_{-f}^1(y_{-f}) : \\ & p_f \in \left(0, \frac{\hat{p}'_{-f} y_{-f}}{c + \hat{p}'_{-f} y_{-f}}\right), p_{-f} = (1 - p_f) \hat{p}_{-f}\}, \end{aligned} \quad (5.3)$$

which is u.h.c (Hildenbrand, 1974, p.22) and n.c.c.v. To verify convex valuedness, let  $\hat{p}_{-f}^i \in \phi_{-f}^1(y_{-f})$ ,  $i = 1, 2$ ; and let  $p^i$  verify  $p_{-f}^i = (1 - p_f^i) \hat{p}_{-f}^i$  with  $p_f^i \in \left(0, \frac{\hat{p}'^i_{-f} y_{-f}}{c + \hat{p}'^i_{-f} y_{-f}}\right)$ . Then, for all  $\lambda \in [0, 1]$ ,  $p^\lambda := \lambda p^1 + (1 - \lambda) p^2$  verifies  $p_{-f}^\lambda = (1 - p_f^\lambda) \hat{p}_{-f}^\lambda$  for  $\hat{p}_{-f}^\lambda = \nu^\lambda \hat{p}_{-f}^1 + (1 - \nu^\lambda) \hat{p}_{-f}^2$  with  $\nu^\lambda = \frac{\lambda(1 - p_f^1)}{\lambda(1 - p_f^1) + (1 - \lambda)(1 - p_f^2)}$ , implying  $p_f^\lambda \in \left(0, \frac{\hat{p}'^\lambda_{-f} y_{-f}}{c + \hat{p}'^\lambda_{-f} y_{-f}}\right)$ . Also, the upper bound on  $p_f$  guarantees  $p' y \geq 0$ . Lemma 3.1 thus applies to the pricing rule (5.3).

**3.2** For  $y_{-f} \notin \partial Y_{-f}^1, y_{-f} \in \partial Y_{-f}^2$ , let

$$\phi(y \mid y_f = -c, y_{-f} \in \partial Y_{-f}^2) = \{p_f \in \mathcal{S} \mid p_{-f} = 0, p_{-f} \in \phi_{-f}^2(y_{-f})\}. \quad (5.4)$$

Again, lemma 3.1 applies.

**3.3** For  $y_{-f} \notin \partial Y_{-f}^1 \cup \partial Y_{-f}^2$ , the construction of the pricing rule is more intricate. For given  $y = (-c, y_{-f})$ , we define:

$$\hat{y}_{-f}^1 = y_{-f} + \iota_{-f} d^1 \in \partial Y_{-f}^1, d^1 > 0;$$

$$\hat{y}_{-f}^2 = y_{-f} - \iota_{-f} d^2 \in \partial Y_{-f}^2, d^2 > 0.$$

That is,  $\hat{y}_{-f}^1$  is the intersection of the half-ray  $y_{-f} + \iota_{-f} d^1$  with  $\partial Y_{-f}^1$ . The intersection is non empty because  $Y$  is closed and  $\{\hat{y} \in Y \mid \hat{y} \geq y\}$  is bounded; it is unique, by definition of  $\partial Y_{-f}^1$ ;  $\hat{y}_{-f}^1$  is a continuous function of  $y$ , for parallel half rays.

Similar properties hold for  $\hat{Y}_{-f}^2$ , since  $y_{-f} \notin \partial Y_{-f}^2$  and  $Y_{-f}^2 \supset R_{-f}^{\ell-1}$ . There exists  $\tilde{p}_{-f}^2 = y_{-f} - \iota_{-f} \tilde{d}^2 \in Y_{-f}^2$  (for instance,  $\tilde{y}_{-f}^2 \in R_{-f}^{\ell-1}$  for  $\tilde{d}^2$  large enough). If  $\tilde{y}_{-f}^2 \in \partial Y_{-f}^2$ , take  $\hat{y}_{-f}^2 = \tilde{y}_{-f}^2$ . Otherwise, there exists (by the reasoning of the previous paragraph)  $y_{-f}^2 = \tilde{y}_{-f}^2 + \iota_{-f} \hat{d}^2 \in \partial Y_{-f}^2, \hat{d}^2 > 0$ . Thus,  $\hat{y}_{-f}^2 = y_{-f} - \iota_{-f} d^2 - \hat{d}^2 = y_{-f} - \iota_{-f} d^2$ , where  $d^2 > 0$  because  $y_{-f} \notin Y_{-f}^2$ .

Let

$$\begin{aligned}\phi(y \mid y_f = -c, y_{-f} \in Y_{-f}^1 \setminus (\partial Y_{-f}^1 \cup \partial Y_{-f}^2)) &= \{p \in \mathcal{S} \mid p_{-f} = 0, \\ &\exists p_{-f}^1 \in \phi_{-f}^1(y_{-f} + \iota_{-f}d^1), d^1 > 0 \ni y_{-f} + \iota_{-f}d^1 \in \partial Y_{-f}^1, \\ &\exists p_{-f}^2 \in \phi_{-f}^2(y_{-f} - \iota_{-f}d^2), d^2 > 0 \ni y_{-f} - \iota_{-f}d^2 \in \partial Y_{-f}^2 :\end{aligned}$$

$$p_{-f} = \frac{d^2}{d^1 + d^2} p_{-f}^1 + \frac{d^1}{d^1 + d^2} p_{-f}^2 \}. \quad (5.5)$$

Thus,  $\phi(-c, y_{-f})$  is a set of vectors  $(0, p_{-f})$  where  $p_{-f}$  is a convex combination of elements from  $\phi_{-f}^1(\hat{y}_{-f}^1)$  and  $\phi_{-f}^2(\hat{y}_{-f}^2)$  for  $\hat{y}_{-f}^1, \hat{y}_{-f}^2$  as defined above. The weights,  $\frac{d^2}{d^1+d^2} = 1 - \frac{d^1}{d^1+d^2}$  and  $\frac{d^1}{d^1+d^2} = 1 - \frac{d^2}{d^1+d^2}$ , are declining functions of the relative distances of  $y_{-f}$  from  $\partial Y_{-f}^1$  and  $\partial Y_{-f}^2$  respectively. If  $y_{-f}^\nu, \nu = 1, 2, \dots$ , tends to  $\bar{y}_{-f} \in \partial Y_{-f}^i, i \in \{1, 2\}$ , then  $1 - \frac{d^{i\nu}}{d^{1\nu}+d^{2\nu}} \rightarrow 1, \hat{y}_{-f}^{i\nu} \rightarrow \bar{y}_{-f}$ , and  $\phi(-c, y_{-f}^\nu) \rightarrow \{p \in \mathcal{S} \mid p_f = 0, p_{-f} \in \phi_{-f}^i(\bar{y}_{-f})\}$ .

So, the correspondence  $\phi(y)$  defined by (5.5) is u.h.c – see proposition 5 in Hildenbrand (1974, p.25). It is clearly compact- and non-empty valued; it is convex-valued, because the set of convex combinations of elements from two convex sets is itself convex. So (P1) is verified by (5.5).

Ad (P2), note that  $p_{-f}^i \hat{y}_{-f}^i \geq 0$  for  $p_{-f}^i \in \mathcal{S}$ , by construction of  $\phi_{-f}^i(\hat{y}_{-f}^i), i = 1, 2$ , hence for all  $p \in \phi(y)$  as defined by (5.5), and

$$\begin{aligned}p'y = p'_{-f}y_{-f} &= \frac{d^2}{d^1 + d^2} p'_{-f}^1(y_{-f} - \iota_{-f}d^1) + \frac{d^1}{d^1 + d^2} p'_{-f}^2(y_{-f} + \iota_{-f}d^2) \\ &\geq \frac{-d^2d^1}{d^1 + d^2} + \frac{d^1d^2}{d^1 + d^2} = 0.\end{aligned}$$

So, (P2) is verified by (5.5). And (P3) is verified because  $p_f \equiv 0$ . Thus, lemma 3.1 holds for  $\phi(y)$  defined by (5.5).

It may also be noted that (5.4) is a special case of (5.5), which could as well have been defined for  $y_{-f} \in Y_{-f}^1 \setminus \partial Y_{-f}^1$ .

**4.** When  $y_f = 0, y_{-f} \in Y_{-f}^2$ , it is again convenient to distinguish two sub-cases.

**4.1** For  $y_{-f} \in \partial Y_{-f}^2$ , let

$$\begin{aligned} \phi(y \mid y_f = 0, y_{-f} \in \partial Y_{-f}^2) &= \{p \in \mathcal{S} \mid \exists \hat{p}_{-f} \in \phi_{-f}^2(y_{-f}) : \\ p_f &\in [0, 1], p_{-f} = (1 - p_f) \hat{p}_{-f}\}. \end{aligned} \quad (5.6)$$

By the argument spelled out under **3.1** above, lemma 3.1 applies to this correspondence.

**4.2** For  $y_{-f} \notin \partial Y_{-f}^2$ , let

$$\phi(y \mid y_f = 0, y_{-f} \in Y_{-f}^2 \setminus \partial Y_{-f}^2) = \{p \in \mathcal{S} \mid p_f = 1, p_{-f} = 0\}. \quad (5.7)$$

Lemma 3.1 now applies trivially.

**5.** The correspondence  $\phi(y)$  defined by (5.1)-(5.7) satisfies (P1), (P2) and (P3) on the seven regions – labelled 1, 2, 3.1, 3.2, 3.3, 4.1 and 4.2 – defining a partition of  $\partial Y$ . Accordingly, it satisfies (P2) and (P3) everywhere, and it is c.c.n.v everywhere. To verify upper hemi-continuity at the common boundaries of these seven regions, we note the following:

- (1) the relevant connections (the common boundaries) concern 1 and 3.2, 1 and 4.1, 2 and 3.1, 3.1 and 3.3, 3.2 and 3.3, 4.1 and 4.2;
- (2) 1 and 3.2 rely on the identical pricing rules (5.1) and (5.4);
- (3) 1 connects to 4.1 for  $p_f = 0$  in (5.6);
- (4) 2 connects to 3.1 for  $p_f = 0$  in (5.3);
- (5) 3.1 connects to 3.3, hence also to 3.2, for  $p_f = 0$  in (5.3) with  $d^1 = 0$  in (5.5);
- (6) 3.2 connects to 3.3 with  $d^2 = 0$  in (5.5);
- (7) 4.1 connects to 4.2 for  $p_f = 1$  in (5.6).

This completes the proof of lemma 5.1 ■

**Theorem 5.1** *Under assumptions (DD), (A1), (A2), (A3') and (A4), there exists a Negishi equilibrium.*

**Proof** Follows from theorem 2.1, remarks 2.1-2.2 and corollary 5.1. ■

The pricing rule is summarised in table 1 and illustrated in figures 2 and 3 for  $\ell = 3$  : one output 1, one variable input 2 and one fixed input 3. The technology without fixed costs is linear, for ease of interpretation of the figure. Similarly, it is assumed that the matrix  $H$  verifies  $H_{11} = b < 0$ ,  $H_{ij} = 0$  otherwise. Accordingly, the locus “ $1 + \iota'Hy = 0$ ” is simply  $1 + by_1 = 0$ ,  $y_1 = -b^{-1}$ . It is represented by the dashed lines along which the horizontal plane  $y_1 = -b^{-1}$  intersects the boundary of the production set. Figure 2 identifies the seven regions 1, 2, ..., 4.2 of table 1. Figure 3 records the non-positive prices that implement (P3) when the FOC are not satisfied. Only the conditions holding in the general case are used.

## 6 Summary

We have investigated the existence of imperfect-competition equilibria *à la* Negishi under some non-convexities in production. We rely on “pricing rules”, which have proved useful in earlier work on equilibria with non-convex technologies. Under our rules, *either* profits are locally maximal given linear perceived-demand functions, *or* else some price is zero or negative. Assuming that all commodities are strictly desired, profits are locally maximal at equilibrium. (A global maximum is not at hand without convexity.)<sup>25</sup>

We focus on production sets consisting of the sum of two convex sets, one of which allows for fixed costs. Our methodology lends itself to generalisations,

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<sup>25</sup>An equilibrium with  $y^j \in \partial Y^{j1}$  could yield lower profits than some element of  $\partial Y^{j2}$  and conversely; such global comparisons are not introduced here.

like a finite sum of convex sets, each allowing (or not) for fixed costs. In the light of an example (section 4), we assume that fixed costs (investments) involve specific commodities. As an intermediate step, we study existence of Negishi equilibria under convex technologies. Our assumptions generalise those of Negishi. We do not retain his requirement of a single monopolistic competitor in each market. Under extremely general assumptions, Negishi equilibria exist, both in the convex case and in the presence of fixed costs. (Competitive equilibria are a special case of Negishi equilibria.)

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**TABLE 1**

	Case	$y_f$	$y_{-f} \in$	$p_f$	$p_{-f}$
(5.1)	1	$0 > y_f > -c$	$\partial Y_{-f}^2$	0	$\phi_{-f}^2$
(5.2)	2	$< -c$	$\partial Y_{-f}^1$	0	$\phi_{-f}^1$
(5.3)	3.1	$= -c$	$\partial Y_{-f}^1$	$[0, \frac{\hat{p}_{-f} y_{-f}}{c + \hat{p}_{-f} y_{-f}}]$	$(1 - p_f) \phi_{-f}^1$
(5.4)	3.2	$-c$	$\partial Y_{-f}^2$	0	$\phi_{-f}^2$
(5.5)	3.3	$-c$	$Y_{-f} \setminus (\partial Y_{-f}^1 \cup \partial Y_{-f}^2)$	0	see (5.5)
(5.6)	4.1	$= 0$	$\partial Y_{-f}^2$	$[0, 1]$	$(1 - p_f) \phi_{-f}^2$
(5.7)	4.2	0	$Y_{-f}^2 \setminus \partial Y_{-f}^2$	1	0

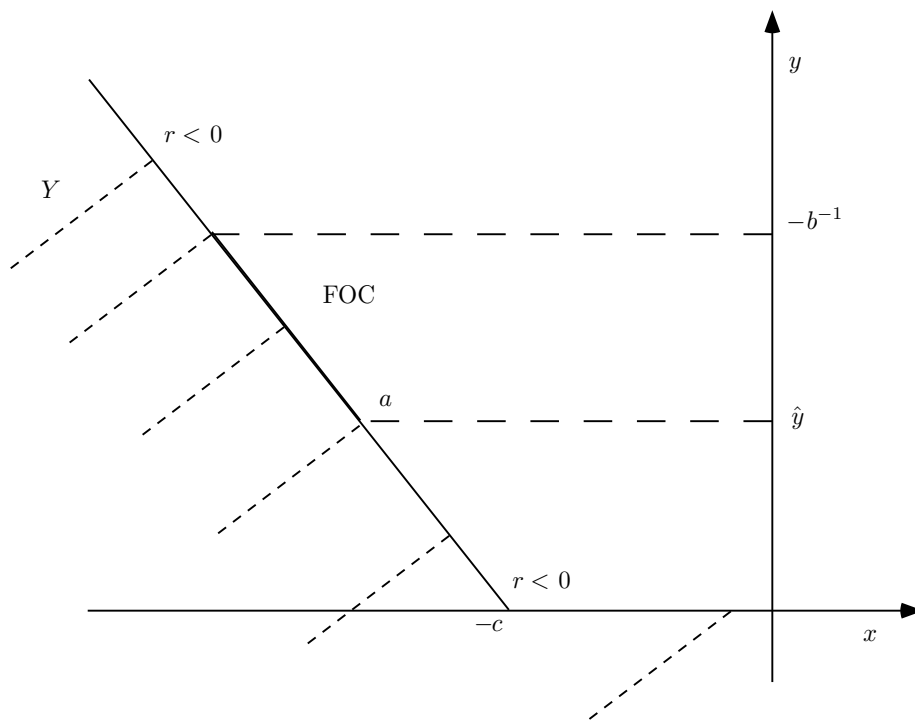


Figure 1

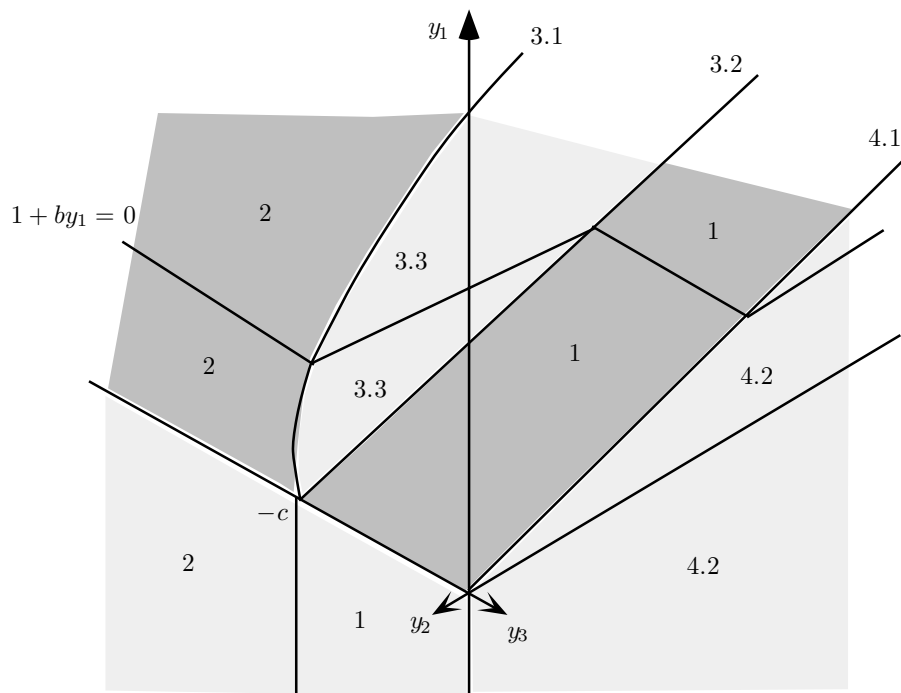


Figure 2

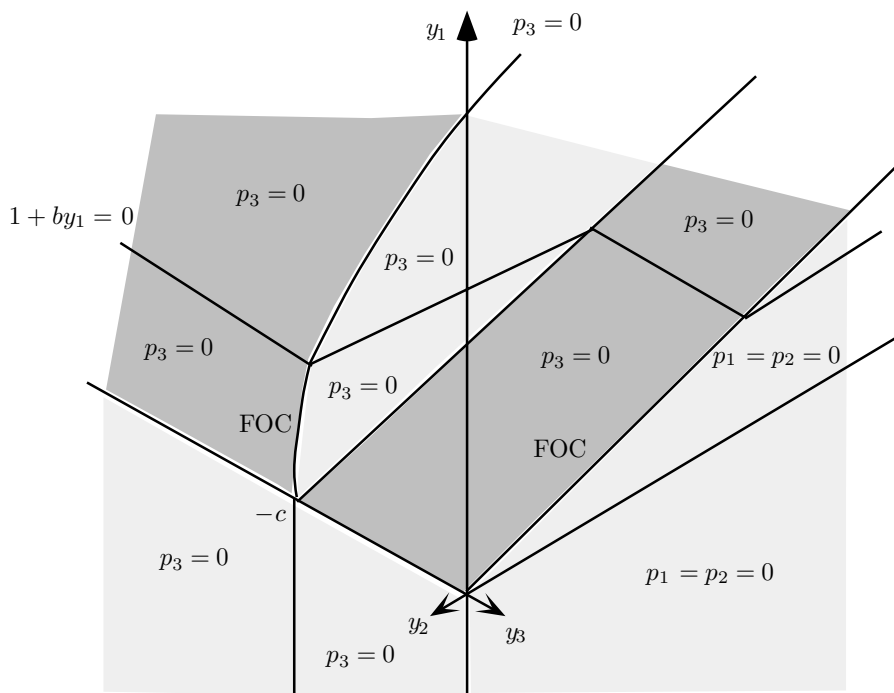


Figure 3