Efficiency and Information Aggregation in Auctions with Costly Information *

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May 1998, Revision: July 2002
Forthcoming in the *Review of Economic Design*

Abstract

Consider an auction in which k identical objects are sold to $n > k$ bidders who each have a value for one object which can have both private and common components to it. Private information concerning the common component of the object is not exogenously given, but rather endogenous and bidders face a cost to becoming informed. If the cost of information is not prohibitively high, then the equilibrium price in a uniform price auction will not aggregate private information, in contrast to the costless information case. Moreover, for a wide class of auctions if the cost of information is not prohibitively high then the objects can only be allocated in a weakly efficient sense, and then only if the equilibrium proportion of endogenously informed agents is vanishing as the economy grows. In spite of these results, it is shown that there is a mechanism for which there exist equilibria and for which (weak) efficiency is achieved as the economy grows in the face of endogenous information acquisition.

Keywords: Auctions, Efficiency, Information Acquisition, Information Aggregation
JEL classification numbers: C72, D44, D82

*Financial support under NSF grant SBR-9507912 is gratefully acknowledged.

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1 Introduction

The property of Pareto efficiency is of fundamental importance in the design and evaluation of markets. As auctions are used to allocate goods and services in a significant number of markets, ranging from government securities to art and procurement, it is important to understand the efficiency properties of auctions.

In what follows two types of efficiency of markets are examined. The first type is informational efficiency or the ability of a market to form prices in a way that aggregates privately held information concerning the value of a good, so that in a large society the price approximates the true (realized) value of the good.[1] This type of efficiency is the centerpiece of "efficient markets hypotheses". The second type is allocative efficiency or the ability of a market to allocate goods to those agents who value them most highly. This second type of efficiency is essentially Pareto efficiency, although the versions discussed in this paper are asymptotic and approximate. While the reasons for caring about allocative efficiency are self-evident, the motivation behind informational efficiency is less transparent, as one might not care about the informational content of prices, given that goods are efficiently allocated. However, a case may be made for informational efficiency to the extent that information obtained from prices in one market can be important in guiding decisions concerning investment or portfolio holdings in markets of other goods (or possibly future purchases/sales of goods in the given market).[2]

Pesendorfer and Swinkels (1997, 2000) have recently shown that uniform price auctions are approximately informationally and allocatively efficient with large numbers of agents. [3] Pesendorfer and Swinkels (1997) show that if both the number of objects and the number of bidders minus the number of objects become large, then uniform price auctions are informational efficient in a common value setting.[4] The key insight is that it is under this double

[1] A weaker requirement is to ask for the price to reflect the expected value of the good conditional on the join of agents' information. However, in large societies these two requirements are essentially equivalent as a law of large numbers applies. This discussion largely presumes a common valuation to the good. In cases where valuations are heterogeneous, a reasonable requirement would be that the price reflect the valuation to the marginal consumer, where marginal consumer is defined under an efficient allocation of the goods.

[2] Some discussion of this appears in Jackson and Peck (1999).

[3] See also Swinkels (1999) for analysis of discriminatory auctions.

[4] Milgrom (1979, 1981) identifies a necessary and sufficient condition for informational efficiency in auctions where a fixed finite number of objects are for sale but with an increasing number of bidders. (See Wilson (1977) for earlier work under sufficient conditions for such a result.) Pesendorfer and Swinkels (1997) work with signal structures that fail to satisfy this strong condition and hence the necessity of their double largeness condition.

largeness condition that a bidder's knowledge of being pivotal from a large number of bidders (and objects) together with knowledge of his or her own signal allows him or her to correctly estimate the value of the object, even though the single signal may be very noisy. Pesendorfer and Swinkels (2000) build on some of the insight concerning informational efficiency to study allocative efficiency in large uniform price auctions, where bidders may have both private information about a common component of an object's value, as well as a separate private valuation for another component of the object's value.[5] Such a setting presents a tough hurdle for allocative efficiency as a bidder may have, for instance, a high private value and a low estimate for the common value, or vice versa. Thus, an auction must sort out private information concerning the common component of the value from the private values themselves, in order to allocate goods efficiently.[6] The remarkable result demonstrated by Pesendorfer and Swinkels (2000) in the context of a uniform price (Vickrey style) auction, is that as the number of bidders and objects grow in a proportional manner (so that the ratio of objects to bidders is bounded from 0 and 1) then the auction is allocatively efficient in the limit, despite the two independent sources of private information. The Pesendorfer and Swinkels result derives from the fact that bidders tend to sort themselves primarily according to private values as their own information concerning the common value is swamped by the information of being pivotal. In the limit the price depends only on bidders with intermediate private values. The price then aggregates agents' private information about the common value component of the object, and in fact comes to reflect the ex post valuation of the marginal bidder under the efficient allocation. Moreover, Pesendorfer and Swinkels show that this allocative efficiency result holds even with costs to acquiring information.

While the efficiency results of Pesendorfer and Swinkels (1997, 2000) seem to reassure us about both the informational and allocative efficiency of uniform price auctions, there are

[5]A non-exhaustive list of papers discussing efficiency in auctions (with endowed information) includes Vickrey (1961), Myerson (1981), Riley and Samuelson (1981), Holmstrom and Myerson (1983), and more recently Maskin (1992), Dasgupta and Maskin (1998), and Jehiel and Moldovanu (2001). These recent papers are discussed in the concluding remarks.

[6]To see the complication in more detail, consider the intuition behind the efficiency of a single unit English auction, which is allocatively efficient in some cases when types are unidimensional. A bidder can see the prices at which other agents drop out and from that can infer the relevant content of other agents' information. With a multi-dimensional type space this sort an inversion is no longer possible. For instance in a procurement auction, firms bidding on a contract consider both the structure of their costs of production (which may reflect current capacity constraints or other idiosyncratic features) and their estimate concerning the materials, labor, etc. necessary to complete the job. A bidder may not be able to infer to what extent a competitor's bid is reflective of their current capacity versus their information about the common cost of the job.

two critical features that are worrisome.

The first worrisome feature is that the informational efficiency results are only demonstrated when information is costless. Matthews (1984) has pointed out that informational efficiency in some auction settings is critically dependent on such an assumption. Matthews provides an example of a first-price auction with many bidders where the informational efficiency results of Wilson (1977) and Milgrom (1979) are upset by making information acquisition costly to the bidders. In the equilibrium, insufficient information is acquired for the aggregate of agents' information to reflect the true value of the object, and in fact the equilibrium price does not even reflect the expected value conditional on the join of the agents' information. To some extent, the example has a similar intuition to that underlying the Grossman-Stiglitz (1981) paradox: if information were fully reflected in the price, then no agent should want to pay to acquire information. Thus, there is an important question as to whether informational efficiency is possible in the face of costs of information acquisition.

The second worrisome feature is that of existence of equilibrium. While Pesendorfer and Swinkels (2000) showed that a sequence of symmetric equilibria in a Vickrey style auction (if they exist) will allocate objects efficiently when there is a cost to information, their result does not provide for existence of equilibrium. This issue of existence of equilibrium is not simply a detail, as in fact symmetric equilibria may fail to exist. Jackson (1999) shows that there even in very simple examples symmetric equilibria do not exist in a Vickrey auction when a private and common component to valuations matter in preferences. The difficulty stems from the fact that multi-dimensional signals do not sort themselves nicely into bids. This can result in non-monotonicities and discontinuities in the information inferred from winning as a function of a bid. These problems can lead to non-existence, as demonstrated in the aforementioned examples. Thus, to establish that allocative efficiency is possible in the face of costly information, one needs to establish equilibrium existence in addition to other properties of equilibrium.

In this paper I examine the informational and allocative efficiency of auctions (and general mechanisms) with large numbers of bidders and objects in the presence of costly information acquisition.

To be more precise the paper begins by examining informational efficiency in large common value uniform price auctions. A very simple proof confirms Matthew's intuition and shows that informational efficiency must fail with any cost to information, regardless of the specifics of the setting.[7]

[7] A couple of people have remarked to me that one might expect bidders to be incidentally endowed with enough private information to result in informational efficiency. However, the fact that there is an industry

Next, the paper moves on to examine the issue of allocative efficiency in the presence of costly information. First, it is shown that if the cost of information is not prohibitively high (costs are low enough so that with positive probability at least one bidder becomes informed), then a strong type of allocative efficiency where all objects are approximately efficiently allocated cannot hold in the face of costly information. This holds for a general class of mechanisms, not just uniform price auctions. To understand this, suppose that every object were to be approximately efficiently allocated. Then the allocation must depend almost entirely on the private values of the bidders, and not on any information about the common value that they may have observed. Through incentive compatibility conditions implied by the equilibrium, this also implies that a bidder's expected payment is almost independent of any acquired information. Bidders thus have no incentive to acquire information given that it has some positive cost. So no information is acquired. This leads to a contradiction as some information will be acquired in equilibrium if the cost to information is not prohibitively high.

This is not the end of the story, however, as one can consider a weaker definition of allocative efficiency that does not require that every object be approximately efficiently allocated, but only a proportion approaching one of the objects be approximately efficiently allocated. Since only a limited number of agents are acquiring information, then there are still many objects to be allocated to uninformed agents. This however, would require the proportion of informed agents to go to zero in equilibrium. It is shown that this is a necessary condition for a weak form of allocative efficiency to hold in any mechanism.

Finally, the last theorem in the paper shows that a weak form of allocative efficiency is attained by an equilibrium of a specific type of auction mechanism. While the mechanism discussed here is less standard than a Vickrey auction studied by Pesendorfer and Swinkels (2000), it is still quite simple and more importantly it provides for existence of equilibria for all admissible preferences and so it establishes that weak allocative efficiency is generally possible. The efficiency comes from the fact that a dwindling proportion of bidders have an incentive to gather information and the auction almost becomes one of entirely uninformed agents. Objects may be misallocated to informed bidders, but that is negligible in the limit.

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of analysts who are paid to acquire information on various securities (including government securities sold through auctions) is inconsistent with informational efficiency holding, since if one expects the price to be informationally efficient in a uniform price auction, then one is better off not acquiring any costly information.

2 Definitions

A finite number, k, of indivisible objects are to be sold to n individuals. Each individual wishes to buy at most one object.

Preferences

Each agent i has a utility for the object which is described by $u(t_i, q)$, where $t_i \in [0, 1]$ is a private component and $q \in [0, 1]$ is a common component. It is assumed u is (jointly) continuous and nondecreasing in (t_i, q) and that it is strictly increasing in at least one of these two parameters. Utility is normalized so that $u(0, 0) = 0$ and $u(1, 1) = 1$. The agent's utility for obtaining the object and paying a price p is $u(t_i, q) - p$.

Effectively, the parameters t_i can be thought of as introducing heterogeneity in preferences.

Uncertainty

Individuals' private parameters are random. Agent i's private parameter is described by the random variable T_i. The T_i's are independently and identically distributed with distribution function $F(\cdot)$. Assume that F has a continuous density function f that is positive on all of $[0, 1]$. Each agent knows his or her own realized value of T_i, denoted t_i, but only the distribution over the T_j's for $j \neq i$.

The value of the common parameter q is random as well, and described by the random variable Q, which is independent of the T_i's and described by the distribution function $G(\cdot)$. The distribution is non-degenerate so that $\text{var}(Q) > 0$.

Information Acquisition

Individuals have costly access to information concerning the realization of Q. For a cost $0 < c < 1$, an agent may observe the realization of a random signal S_i which provides information about the value of Q. The S_i's take values in [0,1] and are independently and identically distributed conditional on Q according to the distributions $G(\cdot|Q = q)$.

Represent an agent's interim information by $(t_i, s_i) \in [0, 1] \times ([0, 1] \cup \{\emptyset\})$, where $s_i = \emptyset$ indicates that i has not observed a signal.

The choice to acquire information to be made at an ex-ante stage before agent i has observed t_i. Largely, the results contained here will not be affected if instead information is acquired at an interim stage after agents observe their private type. I discuss this in the concluding remarks.

It is also assumed that the act of acquiring information is private. That is, when bidding in the auction agents will not have observed whether other agents have acquired information or not. Equilibrium implicitly provides a player with beliefs concerning the other agents' strategies to acquire information. Again, allowing for observation of who acquired information will not substantively change the results, and I discuss this in more detail in the concluding remarks.

Sealed Bid Auctions

Let $X = \{x \in \{0,1\}^n \mid k \geq \sum_i x_i\}$. Thus, $x_i = 1$ is interpreted as giving an object to i.

Let Δ denote the set of Borel probability distributions on $X \times [-1,1]^n$.

A sealed bid auction is a function $Y : [0,1]^n \to \Delta$, that provides a (possibly random) allocation of the objects and payment of each bidder as a function of the submitted bids. $b = (b_1, \ldots, b_n)$.

Uniform Price Auctions

A *uniform price auction* is a sealed bid auction in which all agents who obtain an object pay the same price, and no agent pays more than their bid (regardless of whether they get an object).

Formally, Y is such that for each $b \in [0,1]^n$ there is a set $Z_b \subset X \times [-1,1]^n$ such that $Y(b)$ places probability 1 on Z and such that $z = (x,w) \in Z_b$ satisfies: (i) $x_i = x_j = 1$ implies $w_i = w_j$ and (ii) $x_i = 1$ implies $b_i \geq w_i$.

In the above definition, (i) says that if two bidders are both allocated a good then they pay the same price and (ii) says that the price cannot exceed a player's bid. This is a very broad definition of uniform price auction and leaves wide open how the objects are allocated or how the price is selected. For most of the results, it is not necessary to be more specific.

Strategies

A strategy in the information acquisition stage is simply a probability $m_i \in [0,1]$ that bidder i becomes informed.

Second stage (behavioral) strategies for the auction are functions $b_i : [0,1] \times ([0,1] \cup \emptyset) \to [0,1]$. So, $b_i(t_i, s_i)$ is i's bid in the auction as a function of i's private type t_i and observed signal s_i (where $s_i = \emptyset$ indicates that no information was acquired).

Note that the strategies in the second stage of the game are pure strategies as a function of i's information set. The randomness in t_i provides sufficient mixing so that defining mixed strategies can be avoided in what follows. Mixed strategies can be defined in the obvious way following Milgrom and Weber's (1985) definition of distributional strategies. As mixed strategies would add nothing to the analysis which follows I avoid the complication in notation.

Equilibrium

Equilibrium refers to a sequential equilibrium of the two stage game.

3 Informational Efficiency in Common Value Uniform Price Auctions

In this section, let us specialize to the case of common values where $u(t_i, q)$ depends only on q, which will be written as $u(q)$. This provides the cleanest definition for informational efficiency, the best chance for it to be satisfied, and is allows for the easiest comparison to the previous literature.

Following Pesendorfer and Swinkels (1997), index a sequence of economies by r. Each economy in the sequence has a number k^r of objects to be sold and a number n^r of agents.

Informational Efficiency

A sequence $\{r\}$ of economies and uniform price auctions with corresponding equilibrium prices $\{P^r\}$, is *informationally efficient* if for all ε there exists r' such that for all $r \geq r'$

$$\text{Prob}(|u(Q) - P^r| > \varepsilon) < \varepsilon.$$

Pesendorfer and Swinkels (1997) show that if both k^r and $n^r - k^r$ go to infinity, then a Vickrey auction (with costless signals) is informationally efficient. The converse holds if signals are not too informative. Milgrom (1981) shows that if k^r is bounded and $n^r \to \infty$, then a necessary and sufficient condition for informational efficiency (with costless signals) is to have value distinction which roughly says that there are signals that are arbitrarily more likely under a higher value of Q compared to a lower value of Q (for each such higher and lower values).

What I show here is that regardless of the structure of information, informational efficiency cannot be achieved if there is any cost to information. Thus, it was critical to the previous literature that information was costlessly endowed, and informational efficiency results are not robust to even small information costs.

The theorem refutes informational efficiency by showing that there is a minimum probability that the price and value of the object differ by more than a fixed amount.

Theorem 1 *For any k and n and in any equilibrium of a uniform price auction, if some agent chooses to become informed (with positive probability), then*

$$\text{Prob}(|u(Q) - P| > \frac{c}{2}) \geq \frac{c}{2-c}.$$

The intuition behind the theorem is direct, and is closely related to the idea behind the Grossman-Stiglitz paradox: if an agent is willing to incur a cost to acquire information, then the price cannot already accurately approximate the value of an object. So there must be a minimum amount of noise in the equilibrium in order to sustain information aggregation. An important difference here is that Theorem 1 holds even for small economies and holds for a wide variety of price setting mechanisms, whereas Grossman and Stiglitz (1980) assume price taking behavior. This is an important distinction as discussed in Milgrom (1981). The formal proof is short and proceeds as follows.

Proof of Theorem 1: Let i be an agent who in equilibrium places positive probability on becoming informed. Let $E(u_i)$ denote i's equilibrium expected continuation utility conditional on i acquiring information and before i observes the realization of S_i. Note that i can bid 0 and guarantee a non-positive expected payment and thus a non-negative expected utility, and so since i is acquiring information with positive probability in equilibrium it follows that $E(u_i) \geq 0$.

An absolute bound on i's expected utility is to suppose that i obtains an object whenever $u(Q) \geq P$ and does not whenever $u(Q) < P$. Thus

$$E[\max(u(Q) - P, 0)] - c \geq E(u_i).$$

Note that

$$\text{Prob}(|u(Q) - P| > \frac{c}{2}) + (1 - \text{Prob}(|u(Q) - P| > \frac{c}{2}))\frac{c}{2} - c \geq E[\max(u(Q) - P, 0)] - c.$$

Since $E(u_i) \geq 0$, it follows from the inequality above that

$$\text{Prob}(|u(Q) - P| > \frac{c}{2})(\frac{2-c}{2}) - \frac{c}{2} \geq 0,$$

which simplifies to the stated conclusion. ∎

Given that $\text{var}(Q) > 0$, informational efficiency requires a sequence of equilibria in which some agents are informed. However, Theorem 1 shows that informational efficiency is incompatible with information acquisition. Thus, we have the following corollary.

Corollary 1 *If $c > 0$, then no sequence of economies and corresponding equilibria where the second stage is a uniform price auction satisfies informational efficiency.*

Although Theorem 1 and its corollary have a very simple intuition and proof, its implications are that information aggregation in auctions is sensitive to the introduction of any cost of information. Thus, the intuition of Matthews (1984) is confirmed. It is clear that the results also extend to allow for heterogeneity in costs, provided there is a lower bound on costs for all but a fixed number of bidders.

As a final remark on informational efficiency, note that the results above do not preclude weak versions of the efficient markets hypothesis. The price is not precluded from revealing information. For instance, Milgrom (1981) examined costly information acquisition in k-object Vickrey auction, and showed an example where it is possible to have fully revealing "prices" in spite of costly information acquisition. There the price that a bidder faces is the k-th highest bid of the other agents which reveals a sufficient statistic for the relevant information of a bidder's opponents. This, however, does not imply informational efficiency, nor could it given Theorem 1 above. To understand the difference, note that Milgrom's result does not imply that enough information is gathered to accurately reflect the value of the good - and in fact Theorem 1 here implies that there must be a limit on how many bidders collect information and that the price could never come to approximate the value of the good.

4 Allocative Efficiency in Direct Mechanisms

Next, let us examine the possibility of allocative efficiency in large economies. While the intuition behind the results on informational efficiency is quite simple and consistent with that in the earlier literature, the issues behind allocative efficiency are more subtle.

For the remainder of the paper maintain the assumption that $u(t_i, q)$ is strictly increasing in each variable. This guarantees that the allocation problem is non-trivial as both t_i and q matter in agents' valuations. Also, for simplicity, in this section assume that Q has a finite support and that $G_{S_i}(\cdot|Q = q)$ has the same finite support $S \subset [0, 1]$ for each q in the support of the distribution of Q.

I work with the general class of direct mechanisms, so that any auction or market design is admitted.

Direct Mechanisms

Any auction procedure and equilibrium can be identified with a direct mechanism that corresponds to the interim stage (after agents have observed any acquired information). A direct mechanism cannot be explicitly dependent on who is informed or uninformed except through their reported types. The definitions below can be extended to the case where the act of acquiring information is publicly observed, with the obvious modifications.

A *direct mechanism* is a profile $\pi = (\pi_1, \ldots, \pi_n), \phi = (\phi_1, \ldots, \phi_n)$ of functions where $\pi_i : ([0,1] \times ([0,1] \cup \{\emptyset\}))^n \to [0,1]$ and $\phi_i : ([0,1] \times ([0,1] \cup \{\emptyset\}))^n \to [-1,1]$. So, given an announced profile of information $(t,s) = (t_1, \ldots, t_n, s_1, \ldots, s_n)$, $\pi_i(t,s)$ is the probability that i gets an object and $\phi_i(t,s)$ is the expected payment of i.

The definition of direct mechanisms does not need to allow for correlation between allocations of the objects or payments, given the risk neutrality of the agents. Also, no restrictions are put on how many objects are allocated as these are handled explicitly in the theorems that follow.

Interim Incentive Compatibility

Let

$$V_i(t_i, s_i, \widehat{t}_i, \widehat{s}_i, m_{-i}) = E[u_i(t_i, Q)\pi_i(T_{-i}, S_{-i}, \widehat{t}_i, \widehat{s}_i) - \phi_i(T_{-i}, S_{-i}, \widehat{t}_i, \widehat{s}_i) \mid t_i, s_i, m_{-i}].$$

V_i represents i's expected utility conditional on knowing his or her own type t_i, the information acquisition strategy m_{-i}, observing s_i, and reporting $\widehat{t}_i$ and $\widehat{s}_i$. This does not account for costs of information, which are handled separately below. This also takes other bidders' announcements to be truthful which is implied (as usual) by incentive compatibility, equilibrium, and the revelation principle as discussed below.

A direct mechanism (π, ϕ) is *interim incentive compatible* with respect to m if for all i, almost every (t,s) (given m)

$$V_i(t_i, s_i, t_i, s_i, m_{-i}) \geq V_i(t_i, s_i, \widehat{t}_i, \widehat{s}_i, m_{-i})$$

for any $(\widehat{t}_i, \widehat{s}_i) \in [0,1] \times (S \cup \{\emptyset\})$.

Information Incentive Compatibility

A direct mechanism (π, ϕ) is *information incentive compatible* if there exists an equilibrium (of the corresponding two stage mechanism with information acquisition) for which the direct mechanism is interim incentive compatible with respect to the equilibrium information choices m in the first stage.

The Revelation Principle

The revelation principle applies at the interim stage after information is observed, so that considering any equilibrium of the two stage (information acquisition game) with respect to a given auction, there is a corresponding direct mechanism that is information incentive compatible.

Allocative Efficiency

Consider a sequence of direct mechanisms $\{(\pi^r, \phi^r)\}$, on economies $\{(k^r, n^r)\}$, with corresponding information equilibria. Let $\underline{t}^r$ denote the lowest t_i of any i obtaining an object and $\overline{t}^r$ denote the highest t_i of any i *not* obtaining an object. These are random variables.

A sequence of information incentive compatible mechanisms $\{(\pi^r, \phi^r)\}$, with corresponding information equilibria and $(\underline{t}^r, \overline{t}^r)$, is *allocatively efficient* if

$$E[|\underline{t}^r - \overline{t}^r|] \to 0.$$

Allocative efficiency is a strong condition because it requires that in the limit, not even a single object that is sold is grossly misallocated.

Given a mechanism (π, ϕ), let

$$\overline{\pi}_i(t_i, s_i, \widehat{s}_i, m_{-i}) = E[\pi_i(T_{-i}, S_{-i}, t_i, \widehat{s}_i)|\ s_i, m_{-i}].$$

and

$$\overline{\phi}_i(t_i, s_i, \widehat{s}_i, m_{-i}) = E[\phi_i(T_{-i}, S_{-i}, t_i, \widehat{s}_i)|\ s_i, m_{-i}].$$

These represent i's expected probability of getting an object and expected payment when announcing $t_i, \widehat{s}_i$, conditional on s_i and m_{-i}.

Theorem 2 *Let $\{k^r, n^r\}$ be a sequence of economies such that $n^r \to \infty$ and $\frac{k^r}{n^r} \to a \in (0,1)$. Consider a sequence of information incentive compatible direct mechanisms, $\{(\pi^r, \phi^r)\}$, with corresponding information equilibria, such that $\sum_i \pi_i^r = k^r$ almost surely and $\overline{\phi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r)$ is nondecreasing in $\widehat{s}_i \in S$ for each i and almost every t_i, s_i.*[8] *If there is a non-vanishing probability that at least one agent acquires information,*[9] *then the sequence is not allocatively efficient.*

As the statement of the theorem above requires all k^r objects to be sold, it implicitly prohibits reserve prices that are non-trivially binding. However, the result is extendible to situations where the number of objects awarded is bounded below in probabilistic terms. This can be handled as the definition of allocative efficiency does not require that all the objects be allocated, only that the ones that are allocated be efficiently allocated.

The proof of Theorem 2 appears in the appendix. The intuition behind Theorem 2 is as follows. In the limit, applying an appropriate version of a strong law of large numbers, allocative efficiency implies that for an arbitrarily large proportion of the types (t_i's) above the critical level of $t^* = F^{-1}(1-a)$, the probability of obtaining an object approaches 1, regardless of a bidder's information (or lack thereof), and similarly for types below t^* the probability goes to zero, regardless of their information (or lack thereof). Then incentive compatibility implies that a bidder's expected payment must converge to be approximately independent of the signal and only dependent on t_i, in which case the bidder has no incentive to gather information. This implies that no bidders acquire information.

Thus, another way to see the theorem is to say that the only way in which allocative efficiency is possible is to have nobody acquire information in equilibrium. However, there cannot be an equilibrium which is allocatively efficient and has nobody acquiring information if costs are not prohibitively high. Thus, we find the following corollary.

Say that signals are nontrivially informative if $Prob\{E[u(t_i, Q)|S_i] \neq E[u(t_i, Q)]\} > 0$ for a positive measure set of t_i.

Corollary 2 *Let $\{k^r, n^r\}$ be a sequence of economies such that $n^r \to \infty$, $\frac{k^r}{n^r} \to a \in (0,1)$, and signals are nontrivially informative. Let $\{(\pi^r, \phi^r)\}$ be a sequence of information incentive compatible direct mechanisms with corresponding information equilibria such that $\sum_i \pi_i^r = k^r$*

[8] Note that the requirement is only that $\overline{\phi}$ be non-decreasing at $\widehat{s}_i \in S$ and does not say anything about $\widehat{s}_i = \emptyset$.

[9] If n_I^r is the equilibrium number of informed agents (which may be random), then the requirement is simply that there exists $b > 0$ such that $Prob\{n_I^r \geq 1\} > b$ infinitely often on the sequence.

almost surely and $\overline{\phi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r)$ *is nondecreasing in* $\widehat{s}_i \in S$ *for each* i *and almost every* t_i, s_i*. There exists* $\overline{c} > 0$ *such that if* $0 < c < \overline{c}$*, then the sequence is not allocatively efficient.*

To understand the corollary, note that from Theorem 2 we know that in such a sequence if allocative efficiency is satisfied, then the probability of having any information acquired is going to zero. Thus, with probability approaching 1 every agent reports $t_i, \emptyset$. Also, allocative efficiency implies that π_i is going to 0 for arbitrary proportions of $t_i < t^*$ and to 1 for $t_i > t^*$. Thus, incentive compatibility implies that $\overline{\phi}_i^r(t_i, \emptyset, \emptyset, m_{-i}^r) - \overline{\phi}_i^r(t_i', \emptyset, \emptyset, m_{-i}^r)$ is approximately $E[u(t^*, Q)]$ for an arbitrarily large proportion of the $t_i > t^*$ and $t_i' < t^*$. Thus, any sequence of allocatively efficient mechanisms must converge to effectively be a mechanism where agents simply have an option to buy at a fixed price of $E[u(t^*, Q)]$, and where agents are not choosing to become informed. However, for such a mechanism an informed agent has a higher expected utility than an uninformed agent.[10] So, if c is not prohibitively high, any sequence of equilibria must have the probability of information acquisition bounded below by a positive number and so allocative efficiency cannot be attained. If costs are prohibitively high, the it is possible to achieve allocative efficiency, as no bidder will acquire information and hence the auction is essentially of pure private values as information about Q plays no role.

The Role of Non-decreasing Payments

A sufficient condition for $\overline{\phi}$ to be non-decreasing in $\widehat{s}_i$ is that $\phi(t, s)$ be non-decreasing in s_i. This applies to standard auctions, provided agents bid as non-decreasing functions of their information, which at least for some auctions follows naturally under the conditon that S_i satisfies the monotone likelihood ratio property relative to Q.[11]

Nevertheless, there are mechanisms which violate this condition and achieve allocative efficiency. Thus, the condition is important to theorem 2 and the corollary above. To see this most easily, consider an example where signals are perfectly informative, so that informed agents observe Q. Consider the following mechanism. Each agent announces t_i, s_i. Let n^* be the smallest integer greater than or equal to $\sqrt{n}$. If $\#\{i \ : \ s_i \neq \emptyset\} \neq n^*$, then no objects are allocated and no payments are made. Similarly, if there exists i and j such that $\emptyset \neq s_i \neq s_j \neq \emptyset$ then no objects are allocated. We are left with the case where $\#\{i \ : \ s_i \neq \emptyset\} = n^*$ and there is some q such that $s_i \neq \emptyset$ implies $s_i = q$. In this case

[10] An informed agent can purchase whenever $E[u(t_i, Q)|s_i] > E[u(t^*, Q)]$, which given the non-trivial information structure is a superior decision rule to purchasing whenever $E[(t_i, Q)] > E[u(t^*, Q)]$.

[11] See Pesendorfer and Swinkels (2000) for a proof that bidding functions are non-decreasing in symmetric equilibria of Vickrey Auctions.

objects are allocated to the k agents who announced the k highest t_i's (with ties broken by any method). Those agents pay $u(t^{k+1}, q)$ where t^{k+1} is the $k+1$-st highest announced t_i. Provided that information is not too costly, there is an equilibrium to this mechanism where exactly n^* agents acquire information and every agent announces truthfully. This mechanism is allocatively efficient.[12] Here $\overline{\phi}_i$ is not monotone in signals, illustrating the role of the condition in the theorem.

While this example keys off of the perfect signals, it is easily adaptable to work as long as signals are correlated with Q. Using methods similar to those in Crémer and McLean (1985), one can structure the payments of the n^* agents so that truthful announcement of s_i outweighs any potential gain in manipulating the price, and that those agents have incentives to become informed (rather than announce a guessed s_i).[13] This requires payments schemes that are not monotonic.

Weak Allocative Efficiency

Allocative efficiency is a strong condition in that it requires all allocated objects to be approximately efficiently allocated. Instead we may consider a definition which only requires a proportion approaching 1 of the allocated objects to be efficiently allocated. This turns out to be an important distinction.

Let $u_i(t_i, s) = E[u_i(t_i, Q)|T_i = t_i, (S_1 \ldots, S_n) = s]$.[14] Thus, $u_i(t_i, s)$ denotes i's expected utility given private type t_i and the complete vector of signals s.

Given a mechanism (π, ϕ), let

$$T(t, s) = \sum_i \pi_i(t, s) u(t_i, s)$$

and

$$\overline{T}(t, s) = \max_{\pi' : k \geq \sum_i \pi'_i(t,s) \forall (t,s)} \sum_i \pi'_i(t, s) u(t_i, s).$$

T and $\overline{T}$ are thus random variables (as they are functions of t, s), and the ratio $T/\overline{T}$ gives a measure of how well an auction allocates objects.

[12] This mechanism fails miserably when it comes to the multiplicity of equilibrium. However, given that it simply points out the importance of nondecreasing expected payments in Theorem 2, multiplicity is not an issue.

[13] The idea is that using the announced s_{-i}, one obtains information about Q. This leads to information about the distribution on s_i, which is different from the unconditional distribution. Payments are made to be higher conditional on s_i's that should be more likely given what others have announced, and lower conditional on s_i's that are less likely given what others have announced. This may require payments that exceed the maximum value of the object.

[14] This is a version of the conditional expectation, and so it is defined even for vectors s that would not be possible in equilibrium.

A sequence of direct mechanisms, $\{(\pi^r, \phi^r), m^r\}$, and corresponding information equilibria[15] is *weakly allocatively efficient* if for each ε there exists $\overline{r}$ such that for $r > \overline{r}$

$$Prob\{\frac{T^r}{\overline{T}^r} < 1 - \varepsilon\} < \varepsilon.$$

Theorem 3 *Let $\{k^r, n^r\}$ be a sequence of economies such that $n^r \to \infty$ and $\frac{k^r}{n^r} \to a \in (0,1)$. Consider a sequence of information incentive compatible direct mechanisms $\{(\pi^r, \phi^r)\}$ with corresponding information equilibria such that $k^r \geq \sum_i \pi_i$ almost surely and $\overline{\pi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r)$ is nondecreasing in $\widehat{s}_i \in S$ for each i and almost every t_i, s_i. If the sequence is weakly allocatively efficient, then the expected proportion of informed agents must tend to zero (i.e., $E[\frac{n_I^r}{n^r}] \to 0$).*

The proof of Theorem 3 appears in the appendix. The ideas are very similar to those behind the proof of Theorem 2. In order for an agent to pay to become informed, it must be that for a significant portion of private types, he or she expects to have some variation in the probability of getting an object as a function of the observed signal (as otherwise, by incentive compatibility, the expected utility would be roughly constant in announced signal which is inconsistent with information acquisition). So, if a non-trivial portion of the population is paying to become informed, then there is a positive probability that a non-trivial portion of the objects will be misallocated by a significant amount, which is inconsistent with weak allocative efficiency.

5 A Weakly Allocatively Efficient Mechanism

Theorem 3 leads one to doubt whether there exist sequences of auctions which satisfy weak allocative efficiency in the face of costly information acquisition. We know from Theorem 3 that satisfying weak allocative efficiency requires the equilibrium proportion of informed agents to vanish. I now show that there exists a simple mechanism for which there always exists a symmetric equilibrium and for which sequences of such equilibria in growing economies satisfy weak allocative efficiency.

In this section assume that there is a finite set $S \subset [0,1]$ such that $G(S_i|Q = q)$ has support in S for each q. This is weaker than the assumption maintained in the previous section as the support need not be the same for different q's nor are any assumptions placed on the distribution of Q.

[15] Note that m^r is important in determining the distribution over s, and thus the distribution of $T/\overline{T}$.

Given an economy k, n, consider the following mechanism. Agents announce s_i and either *yes* or *no*. Objects are awarded among the agents announcing *yes*. If more than k agents announce *yes*, then the k objects are randomly assigned (with equal probability) among the agents announcing yes. If k or fewer agents announce *yes*, then each receives an object and the remaining objects are not sold. If agent i is assigned an object, then he or she pays $E[u(t^*, Q)|S_{-i} = s_{-i}]$, where t^* is some fixed level.

Theorem 4 *Let $\{k^r, n^r\}$ be a sequence of economies such that $n^r \to \infty$, $\frac{k^r}{n^r} \to a \in (0,1)$. For any $c \geq 0$, there exists a sequence of symmetric equilibria of the mechanism described above (setting $t^* = F^{-1}(1-a)$) such that the sequence is weakly allocatively efficient.*

Note that in addition to proving weak allocative efficiency, Theorem 4 also provides for the existence of equilibria for any sequence of admissible economies. The existence is not merely a footnote, but an important issue in this setting. Pesendorfer and Swinkels (2001) have shown that a sequence of symmetric equilibria (if they exist) in a Vickrey style auction will satisfy a version of weak allocative efficiency with costs to acquiring information. However, that result's implications are then only interesting if we are sure that there exist (symmetric) equilibria to the Vickrey auction. Existence of equilibria to the Vickrey auction when individuals have a private and common signal turns out to be problematic. In fact, Jackson (1999) shows that existence can fail in the Vickrey auction even in very simple examples with nice monotonicity properties in each dimension of signal. The difficulty comes from the fact that high (or intermediate) bids of other bidders might indicate high private value rather than high common value. This can lead to discontinuities in the information learned as a function of winning conditional on a bid. Such discontinuities can lead to non-existence of equilibria. The extent to which existence is a problem for the Vickrey auction with multiple dimensions of signals is an open issue, with the only examples currently being of non-existence.[16] The mechanism defined here overcomes the existence problem as it greatly simplifies the decisions that players must make, as they cannot influence the price they face and must only declare whether they desire an object or not.

[16]Existence of equilibrium can be established if one augments the bidding space to allow for announcements of signals as well as bids, as shown by Jackson, Simon, Swinkels, and Zame (2001). However, those equilibria might not have the properties required in the Pesendorfer and Swinkels (2000) analysis. One possibility is to use the Jackson, Simon, Swinkels, and Zame result to demonstrate existence of equilibria with the larger strategy space, and then to show that under some conditions that the extra announcement of signals is not needed (which is shown to work in a private values setting in Jackson and Swinkels (1999)). It is not clear how well that approach will work outside of a private values setting, given the non-existence example in Jackson (1999).

The proof of Theorem 4 appears in the appendix. The ideas behind it are fairly straightforward. As the announced s_i does not affect an agent's payoff, it is a best response for each agent to announce s_i truthfully. An uninformed agent can then simply say *yes* if $t_i \geq t^*$ and *no* if $t_i < t^*$, as there is nothing they useful they can learn that is not already incorporated in the price they face. Informed agents use their knowledge of s_i in deciding whether or not to say *yes* or *no*, as it gives them information not reflected in the price that they face. They still, however, have a reasonably easily described strategy for saying *yes* which is monotone in t_i. After establishing existence, it is shown that weak allocative efficiency holds. Given the strategies of the uninformed, it is enough to show that the proportion of informed goes to 0. In fact, it is shown that either the number of informed is bounded (which happens whenever there is a positive cost to information), or else the strategies of the informed approximate saying *yes* when $t_i > t^*$ and *no* if $t_i < t^*$, in which case they would be better off not acquiring information. This follows from the fact that if there are a growing number of informed, then s_i provides little information that is not already in the price that i faces.

6 Concluding Remarks

The definitions of efficiency in this paper have ignored the costs of information acquisition. However, adding those costs explicitly into efficiency definitions makes no difference to the results, since the results imply that to achieve either type of allocative efficiency the number (or proportion) of agents acquiring information must go to zero. So, the proportion of society's resources wasted on information acquisition must go to zero if (weak) allocative efficiency is to be attained.

The treatment of information acquisition in this paper has focused on the case of ex-ante acquisition and non-observability of acquisition. All of the results can easily be extended to the case where information acquisition is publicly observed, with some modifications to the proofs. Whether an agent knows exactly how many others have acquired information, or only has an estimate, does not significantly change behavior and the intuition goes through. A change to interim information acquisition, however, does introduce some new aspects to the analysis, as incentives to acquire information will depend on an agent's realized private type. Theorems 1 and 3 and Theorem 4 will go through based on the same logic, again with some modifications to the proofs. However, Theorem 2 concerning the strong form of allocative efficiency must be weakened to claim that a vanishing proportion (rather than number) of agents must acquire information. The agents who choose to acquire information can be the marginal types (t_i's increasingly near t^*) and this can be consistent with allocative efficiency.

In fact, the mechanism in Theorem 4 would have this feature in the interim information acquisition case.

Throughout the paper it has been assumed that information concerning private values is free and only the common aspect is costly. The motivation for this is that information on private type (e.g., personal circumstances, capacity, etc.) is likely to be incidental or a by-product of other activities, while information on a common component is more likely to involve active research. Bergemann and Välimäki (1999) examine efficient allocations in settings where all information is costly.[17] They show that there are interesting differences in the possibility of reaching efficiency in private versus non-private value settings, with the former setting allowing for efficiency, but not always the latter. Interestingly, the costly information in the setting explored here concerns the common aspect, and yet approximate efficiency is still achievable. This suggests that more study is needed to see where the divide between costly aspects of information and costless aspects needs to fall in order to be able to reach efficient allocations, and also how much the consideration of approximate efficiency rather than exact efficiency makes a difference.

It also appears that part of the reason that the model examined here leads to efficiency (at least approximately) is that interdependencies in valuations arise only through the common Q. If interdependencies take other forms, efficiency can be impossible to achieve even with costless information. Examples of this can be found in Maskin (1992) and Dasgupta and Maskin (1997), and a general result showing problems in reaching efficient allocations with multi-dimensional signals can be found in Jehiel and Moldovanu (2001). The extent to which such difficulties in achieving (exactly) efficient allocations arise is well-outlined by Jehiel and Moldovanu. It may be interesting to explore in more detail what sorts of informational settings allow for versions of approximate efficiency studied here in large economies.

[17] In situations where both components are costly, the problem also begins to look more like (but not exactly like) auctions with an entry cost where many questions concerning efficiency are still open. See Gal, Landsberger, and Nemirovski (2001) for a recent examination of auctions with entry costs that could serve as a interesting basis for a further analysis of efficiency properties.

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Appendix

Proof of Theorem 2: Considering a subsequence if necessary, let $Prob\{n_I^r \geq 1\} > b$ for any r. Let t^* satisfy $F(t^*) = 1 - a$. Let t_r be the k^r-th out of a sample of n^r draws. By a strong law of large numbers for order statistics (e.g., the Glivenko-Cantelli theorem - see Billingsley (1968)),

$$Prob\left(\{t_r \to t^*\}\right) = 1.$$

Under the assumptions of Theorem 2, suppose that the sequence is asymptotically efficient, so that

$$E[|\underline{t}^r - \overline{t}^r|] \to 0.$$

It follows from Chebyshev's inequality that for any $\varepsilon > 0$ there exists r' such that for all $r > r'$,

$$Prob\left(\{|\underline{t}^r - \overline{t}^r| > \varepsilon\}\right) < \varepsilon.$$

Noting that $\max(\underline{t}^r, \overline{t}^r) \geq t_r \geq \min(\underline{t}^r, \overline{t}^r)$, it follows that for any ε there exists r^ε such that for all $r > r^\varepsilon$

$$Prob\left(\{|\underline{t}^r - t^*| > \varepsilon\}\right) < \varepsilon, \tag{1}$$

and

$$Prob\left(\{|\overline{t}^r - t^*| > \varepsilon\}\right) < \varepsilon. \tag{2}$$

Let A_ε^{ir} denote the event that either $t_i > t^* + \varepsilon$ and i does not get an object or $t_i < t^* - \varepsilon$ and i does get an object. Let B^{ir} denote the event that i is informed. (1) and (2) imply that

$$Prob\left(\cup_i [A_\varepsilon^{ir} \cap B^{ir}] \mid \cup_i B^{ir}\right) < \frac{2\varepsilon}{b}.$$

This implies that there is at least one i in any economy $r > r^\varepsilon$ such that $m_i^r > 0$ and

$$Prob\left(A_\varepsilon^{ir} \mid B^{ir}\right) < \frac{2\varepsilon}{b}. \tag{3}$$

Consider i, t_i, such that $m_i > 0$ and there exists δ such that

$$\min_{s_i, \widehat{s}_i \in S^2} \overline{\pi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r) > 1 - \delta. \tag{4}$$

Let $\underline{s}_i$ be the smallest $s_i \in S$. Incentive compatibility and the fact that $\overline{\phi}$ is nondecreasing and (4) imply that

$$\overline{\phi}_i^r(t_i, s_i, s_i, m_{-i}^r) - \overline{\phi}_i^r(t_i, s_i, \underline{s}_i, m_{-i}^r) < \delta, \tag{5}$$

for any $s_i \in S$. (4) and (5) then imply that

$$|\overline{V}_i^r(t_i, s_i, t_i, s_i, m_{-i}^r) - \overline{V}_i^r(t_i, s_i, t_i, \underline{s}_i, m_{-i}^r)| < 2\delta \tag{6}$$

for all $s_i, \in S$. Similarly, (6) holds for any i, t_i, and δ such that

$$\max_{s_i, \widehat{s}_i \in S^2} \overline{\pi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r) < \delta. \tag{7}$$

Consider any i such that $m_i > 0$. It is easily checked that for i to be willing to purchase information

$$Prob(\{t_i \ : \ \max_{s_i \in S} |\overline{V}_i^r(t_i, s_i, t_i, s_i, m_{-i}^r) - \overline{V}_i^r(t_i, s_i, t_i, \underline{s}_i, m_{-i}^r)| > 2\delta\}) \geq \frac{c - 2\delta}{3 - 2\delta},$$

for any $\frac{c}{2} > \delta > 0$ (noting that 3 is the maximum swing in utility from announcing s_i correctly versus simply announcing $\underline{s}_i$). Thus, from (4), (7), and (6) it follows that for i to be willing to purchase information

$$Prob(\{t_i \ : \ \min_{s_i, \widehat{s}_i \in S^2} \overline{\pi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r) < 1 - \delta \text{ and } \max_{s_i, \widehat{s^i} \in S} \overline{\pi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r) > \delta\}) \geq \frac{c}{3}, \tag{8}$$

for any $\frac{c}{2} > \delta > 0$. Given the same finite support on S_i for each q, it follows from (8) that for i to be willing to purchase information

$$Prob(\{t_i \ : \ \min_{s_i \in S} \overline{\pi}_i^r(t_i, s_i, s_i, m_{-i}^r) < 1 - \alpha\delta \text{ and } \max_{s_i \in S} \overline{\pi}_i^r(t_i, s_i, s_i, m_{-i}^r) > \alpha\delta\}) \geq \frac{c}{3}, \tag{9}$$

where $1 \geq \alpha > 0$ is a constant depending only on the underlying distributions of S_i and Q. To see this set $\alpha' = \min_{q, s_i', s_i''} \frac{Prob(Q=q|S_i=s_i')}{Prob(Q=q|S_i=s_i'')}$. Then $Prob(S_{-i}|S_i = \widehat{s}_i) \geq \sum_q Prob(S_{-i}|Q = q)\alpha' Prob(Q = q|s_i)$, and so the fact that $\overline{\pi}_i^r(t_i, \widehat{s}_i, \widehat{s}_i, m_{-i}^r) > \alpha' \overline{\pi}_i^r(t_i, s_i, \widehat{s}_i, m_{-i}^r)$ follows. A similar argument gives the other inequality for an α'' and take α to be the min of these two. Finally, set $\delta = \frac{c}{4}$ and then for a small enough ε, (3) and (9) lead to a contradiction. So our supposition was wrong and the theorem is established. ∎

Proof of Theorem 3: As in the proof of Theorem 2, it follows that (9) holds for any i with $m_i^r > 0$ and any $0 < \delta < \frac{c}{2}$.

Suppose that n_I^r/n^r does not converge to 0, and take a subsequence such that $n_I^r/n^r \to d > 0$. By a similar argument to that in the proof of Theorem 2, weak allocative efficiency then implies that for any ε and large enough r there are informed agents for whom $Prob\{A_\varepsilon^{ir} \mid B^{ir}\} < k\varepsilon$, where k is a constant depending on d. Again, set $\delta = \frac{c}{4}$ and then for small enough ε we reach a contradiction. ∎

Proof of Theorem 4: Let us first establish that there exists an equilibrium to the mechanism.

Fix any information strategy m. I show there exists an equilibrium in the subgame that follows. First, as s_i does not affect the probability that i is awarded an object, nor the price that i pays, it is a best response to announce s_i truthfully. Next, it is a best response for an uninformed i to say *yes* if $t_i \geq t^*$, and *no* otherwise. So we need only describe equilibrium strategies conditional on being informed.

Consider an informed i. Suppose that each other $j \neq i$'s strategy conditional on being informed and of type t_j, s_j can be described by saying *yes* if $t_j > \tau_j(s_j)$ and *no* otherwise, for some $\tau_j : S \rightarrow [0, 1]$. I first show that i's best response can then be characterized by such a function τ_i. It is a best response for agent i to say *yes* if

$$E[(u(t_i, Q) - E[u(t^*, Q)|S_{-i}])Z_i(S_{-i}, \tau_{-i})|s_i, m_{-i}] > 0, \tag{10}$$

and *no* otherwise, where Z_i is the probability of i getting an object conditional on saying *yes* when S_{-i} is observed by the other agents who are following strategy τ_{-i}, and where $S_j = \emptyset$ when j is not informed. Note that the left hand side of inequality (10) is continuous and strictly increasing in t_i, and so i's best response may be characterized by setting $\tau_i(s_i)$ to be the t_i that equates the left hand side to 0, setting $\tau_i(s_i) = 0$ if the left hand side is positive for all t_i and $\tau_i(s_i) = 1$ if the left hand side is negative for all t_i. Thus, we have a well defined τ_i as a function of each τ_{-i} and m_{-i}, and we can write $\tau_i = \psi_i(\tau_{-i}, m_{-i})$. Note that each τ_i may then be thought of as a $\#S$ dimensional vector, and that ψ_i is single valued and continuous in τ_{-i}, m_{-i}. Moreover, write ψ_i as a function of only a single pair τ_j, m_j, under the restriction that each $j \neq i$ plays the same strategy. Next, let $\nu_i(\tau_j, m_j)$ be the correspondence that takes value $\{1\}$ if the following expression is greater than 0, $[0, 1]$ if the following expression equals 0, and the value $\{0\}$ if the following expression is less than 0:

$$E[I_{T_i > \tau_i(S_i)} E[(u(T_i, Q) - E(u(t^*, Q)|S_{-i}))Z_i(S_{-i}, \tau_{-i})|S_i, m_{-i}]$$

$$-I_{T_i > t^*} E[(u(T_i, Q) - E(u(t^*, Q)|S_{-i}))Z_i(S_{-i}, \tau_{-i})|m_{-i}]] - c,$$

where t_{-i} and m_{-i} are the symmetric strategies where each $j \neq i$ plays τ_j and m_j, $\tau_i = \psi_i(\tau_j, m_j)$, and I is the indicator function. This expression is the ex-ante difference in expected payoffs from playing the second stage informed versus uninformed, and so ν_i represents the best response in information choices given τ_j and m_j. This correspondence is compact and convex valued. Also, as this expression is continuous in τ_j, m_j, it follows that ν_i is upper hemi-continuous. So we can apply Kakutani's fixed point theorem to the pair ψ_i, ν_i to find a fixed point which then constitutes a symmetric equilibrium.

To complete the proof, let us verify that a sequence of such equilibria is weakly allocatively efficient for any $c \geq 0$. If $E[n_I^r/n^r] \to 0$, then by Markov's inequality it follows that that for any $\varepsilon > 0$, $Prob(n_I^r/n^r \geq \varepsilon) \to 0$. Then weak allocative efficiency follows from the strategies of the uninformed (and the fact that the potential number of bidders saying *yes* over the number of objects k^r is going to one (again by the Glivenko-Cantelli Theorem)). So, divide the sequence of enconomies into those for which $E[n_I^r] \geq \sqrt{n^r}$, and those for which $E[n_I^r] < \sqrt{n^r}$.[18] Along the second subsequence, we know that weak allocative efficiency holds.

So consider the first subsequence, where $E[n_I^r] \geq \sqrt{n^r}$. It follows from the martingale convergence theorem that for any ε there exists N_ε such that for $N \geq N_\varepsilon$ and any i

$$Prob\{|E[u(t^*,Q)|S_{-i}^N] - E[u(t^*,Q)|S^N]| > \varepsilon\} < \varepsilon. \tag{11}$$

where S^N is a vector of N observations of (informed) signals S_j. Fixing any $\gamma > 0$, again by Markov's inequality there exists r_γ such that if $r > r_\gamma$, then for any i

$$Prob(n_I^r \geq N_\varepsilon \mid m_{-i}^r) > 1 - \gamma.$$

Thus, for any γ, setting $\varepsilon = \gamma$ in (11) implies that there exists r_γ such that if $r > r_\gamma$ then for any i

$$Prob\{|E[u(t^*,Q)|S_{-i}^r] - E[u(t^*,Q)|S^r]| > \gamma \mid m_{-i}^r\} < 2\gamma. \tag{12}$$

Let $\nu = \min_{s_i} Prob\{S_i = s_i\}$. It then follows from (12) that for $r > r_\gamma$ and any i and s_i

$$Prob\{|E[u(t^*,Q)|S_{-i}^r] - E[u(t^*,Q)|S^r]| > \gamma \mid S_i = s_i, m_{-i}^r\} < \frac{2\gamma}{\nu}. \tag{13}$$

(13) then implies that

$$E[(u(t_i,Q) - E[u(t^*,Q)|S_{-i}^r])Z_i^r(S_{-i}^r,\tau_{-i}^r)|S_i = s_i, m_{-i}^r]$$
$$\geq E[(u(t_i,Q) - E[u(t^*,Q)|S^r])Z_i^r(S_{-i}^r,\tau_{-i}^r)|S_i = s_i, m_{-i}^r] - \gamma - \frac{2\gamma}{\nu}.$$

This implies that

$$E[(u(t_i,Q) - E[u(t^*,Q)|S_{-i}^r])Z_i^r(S_{-i}^r,\tau_{-i}^r)|S_i = s_i, m_{-i}^r]$$
$$\geq E[(E[u(t_i,Q)|S] - E[u(t^*,Q)|S^r])Z_i^r(S_{-i}^r,\tau_{-i}^r)|S_i = s_i, m_{-i}^r] - \gamma - \frac{2\gamma}{\nu},$$

[18] As for the claim made following theorem 4 that the number of informed is bounded if $c > 0$, in the place of $\sqrt{n^r}$ we could use any N^r such that $N^r \to \infty$ and $\frac{N^r}{n^r} \to 0$ and the following arguments would still hold. These imply that the informed strategies converge to the uninformed strategies regardless of cost, and so it cannot be that $n_I^r \to \infty$ if information is costly.

which in turn implies that
$E[(u(t_i,Q) - E[u(t^*,Q)|S_{-i}])Z_i^r(S_{-i}^r,\tau_{-i}^r)|S_i = s_i, m_{-i}^r]$

$$\geq E[E[u(t_i,Q) - u(t^*,Q)|S^r]Z_i^r(S_{-i}^r,\tau_{-i}^r)|S_i = s_i, m_{-i}^r] - \gamma - \frac{2\gamma}{\nu}. \tag{14}$$

So, fix any $t_i > t^*$ and let $w = \min_q[u(t_i,q) - u(t^*,q)]$, noting that $w > 0$ by the continuity and monotonicity of u. Find γ so that $\gamma + \frac{2\gamma}{\nu} < \frac{wa}{2}$. It follows from (14) that for $r > r_\gamma$ and any i

$$E[(u(t_i,Q) - E[u(t^*,Q)|S_{-i}])Z_i(S_{-i},\tau_{-i}^r)|s_i, m_{-i}^r] > 0.$$

For any $t_i > t^*$, there exists $\overline{r}$ such that for all $r > \overline{r}$ and any i, $\tau_i^r < t_i$. A similar argument holds for $t_i' < t^*$. Thus, fixing any ε there exists r_ε such that for all $r > r_\varepsilon$ and any i, $t^* - \varepsilon < \tau_i^r < t^* + \varepsilon$ for all i. Thus, on the selected subsequence we can conclude that weak allocative efficiency holds. As it also holds on the other subsequence, the conclusion of the theorem follows. ∎