Implementing Efficient Allocations in a Model of Financial Intermediation*

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Abstract

In a finite-trader version of the Diamond-Dybvig (1983) model, the symmetric, ex-ante efficient allocation is implementable by a direct mechanism (i.e., each trader announces the type of his own ex-post preference) in which truthful revelation is the strictly dominant strategy for each trader. When the model is modified by formalizing the sequential-service constraint (cf. Wallace, 1988), the truth-telling equilibrium implements the symmetric, ex-ante efficient allocation with respect to iterated elimination of strictly dominated strategies.

1 Introduction

This paper concerns the welfare analysis of maturity transformation in financial structure. Maturity transformation is the financing of an intermediary's assets by liabilities (demand deposits at a bank, in particular) that are callable, and that some traders do call in equilibrium, before the assets themselves mature. Bryant (1980) shows that such a portfolio structure is a means of insuring the depositors against unobservable risks, and he also identifies a multiplicity-of-equilibrium problem. He implicitly represents a bank as a rule or "allocation mechanism" that specifies the outcome, in each state of nature, of each possible profile of traders' decisions regarding whether or not to exercise the call options on their deposits. This rule constitutes a framework for strategic interaction among the traders. Bryant observes that maturity transformation is necessary in order to implement the symmetric, ex-ante efficient, allocation as a Bayesian Nash equilibrium. He shows also that some mechanisms that do implement that efficient allocation—notably the mechanism that most faithfully reflects the features of a bank-deposit contract in the context of his model—also can possess other equilibriums that are strictly Pareto dominated by the "intended" equilibrium.

Diamond and Dybvig (1983) address a related set of issues to Bryant. They study a model that brings the role of aggregate risk into sharp focus. They prove four main results.

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1. The phenomenon of Pareto-ranked bank-deposit-contract equilibriums can occur even in an environment where there is no aggregate risk.

2. However there is an allocation mechanism, suggested by historical banking regimes that have permitted suspension of convertability of deposits when a “run” occurs, that implements the symmetric, ex-ante optimal allocation in strictly dominant strategies. This is intuitively a particularly compelling notion of implementation that implies, among other things, that the Bayesian Nash equilibrium is unique. Obviously, then, there cannot be multiple, Pareto-ranked equilibriums.

3. In some environments with aggrate risk, a deposit scheme with suspension of payments cannot implement an ex-ante efficient allocation.

4. However, despite the presence of aggregate risk, it is possible to implement the symmetric, ex-ante efficient allocation in Bayesian Nash equilibrium. If deposit insurance is feasible, then it can provide one means to do so. Diamond and Dybvig’s analysis does not establish whether or not there is any allocation mechanism that implements the symmetric, efficient allocation as its unique Bayesian Nash equilibrium.

   Wallace (1988) provides a formalization of the sequential-service constraint to which previous researchers had appealed informally. He proves the following result that bears on Diamond and Dybvig’s fourth point.

5. If the provision of deposit insurance is genuinely regarded as a feature of the over-all allocation mechanism, and if it is this over-all mechanism to which the sequential-service constraint applies, then deposit insurance is not feasible to provide.

Taken together, the last three of these results raise the possibility that existence of multiple, Pareto-ranked equilibriums might be an unavoidable problem for any mechanism that implements the symmetric, ex-ante efficient allocation as a Bayesian Nash equilibrium in an environment with aggregate risk. Suppose that that were indeed the situation, and that one believed that traders’ strategic interactions were much more likely to proceed according to the Pareto-dominated equilibrium than according to the efficient one. If there were another mechanism that had a unique, “mediocre,” equilibrium that were situated strictly between the other two according to the Pareto relation, then one might be inclined to choose mechanism and to tolerate the inefficiency of its equilibrium rather than to incur the substantial risk of doing even worse, in order to have any chance of attaining efficiency. To the contrary, if there were a mechanism possesing a unique equilibrium, and if the symmetric efficient allocation were the outcome that would result from that equilibrium being played, then one would reject without hesitation the mechanism with the mediocre equilibrium if one were convinced that the Bayesian Nash equilibriums of both mechanisms would actually be played.

This tension between efficiency of outcome allocations and stability in the sense of uniqueness of Bayesian Nash equilibrium (and of characterization of equilibrium in terms of strategic dominance) is the specific topic of this paper. I review some basic concepts of implementation theory in section 2, and in section 3 I use this implementation framework to present a version of the Diamond-Dybvig environment with aggregate risk. (The environment that I study differs from Diamond and Dybvig’s in having only finitely many traders. I formulate this version both to introduce aggregate risk in a natural and explicit way, and also to clarify the formulation of the sequential-service constraint.) A naturally-defined mechanism makes it a dominant strategy for each trader to communicate his type truthfully, and via this dominant-strategy equilibrium it implements the symmetric, ex-ante efficient allocation. That is, in sharp contrast to Diamond and Dybvig's deposit-with-suspension mechanism, for this mechanism the distinction between environments with and without aggregate risk is immaterial. Finally, in section 4, I consider the analogous allocation mechanism in environments with aggregate risk and also a sequential-service constraint. I show that, under the assumption that traders' utility functions exhibit non-increasing absolute risk aversion, for traders truthfully to communicate their types remains the unique strategy profile that survives iterated elimination of strictly dominated strategies. Thus, again, the mechanism has a unique Bayesian Nash equilibrium that possesses an intuitively compelling stability property, and the outcome of that equilibrium being played is the symmetric, ex-ante efficient, equilibrium.

2 Intermediation as an allocation mechanism

One way of viewing a financial intermediary is as a trading club. People want to join such a club because features of the environment (including the informational features that engender problems of "adverse selection" and "moral hazard") make arms-length transactions infeasible or unsatisfactory. Instead, a trading club operates according to a charter that specifies which trades are to be made as function of information provided by members according to an explicitly defined protocol of communication and negotiation.

2.1 The environment of a Bayesian allocation mechanism

Consider a formal representation of an environment where such an intermediary would have a rationale. Let $\mathbf{I} = \{1, 2, \ldots, I\}$ be a set of traders who live in a risky environment. The possible *states* of this environment are the sample points of a probability space $(\Omega, \mathcal{B}, \Pr)$. There is a measurable space of *ex-post allocations* which will be denoted by $(\mathbf{A}, \mathcal{A})$. A *state-contingent allocation* is simply a $\mathcal{B}$-measurable function from Ω to $\mathbf{A}$. Denote the set of such $\mathcal{B}$-measurable functions by $\mathbf{A}^{\Omega}$. If $\vec{a} \in \mathbf{A}^{\Omega}$ and $\omega \in \Omega$, then $\vec{a}(\omega)$ is the ex-post allocation that the state-contingent allocation $\vec{a}$ specifies for state ω. (Henceforth a state-contingent allocation or an ex-post allocation will often be called simply an allocation, when it is clear from context which type of entity is being discussed.)

There is a set $\mathbf{F} \subseteq \mathbf{A}^{\Omega}$ of *feasible state-contingent allocations*. (That is, $\mathbf{F}$ is a set of $\mathcal{B}$-measurable functions $f\colon \Omega \to \mathbf{A}$.) The specification of F is supposed to reflect both individual restrictions such as nonnegativity of consumption and also aggregate restrictions such as materials balance.

This model will incorporate the *Harsanyi doctrine* that all traders are Bayesian utility maximizers, and that moreover Pr characterizes the prior beliefs common to all traders at "birth." Typically the model is used to understand the traders' behavior in "adulthood" after they have revised their beliefs in light of experience. The experience of trader i is called his *type*, and is represented as a sub σ-algebra $\mathcal{E}_i$ of $\mathcal{B}$. When an trader's type is described in terms of a $\mathcal{B}$-measurable random variable which the trader is assumed observe, $\mathcal{E}_i$ will be taken to be the smallest σ-algebra with respect to which the random variable observed by i is measurable. (Typically $\mathcal{E}_i$ is strictly smaller than $\mathcal{B}$ itself.)

In addition, assume that there is a sub σ-algebra $\mathcal{E}_0$ of $\mathcal{B}$ that represents information that is directly usable for allocation. That is, an allocation can be made contingent on this information without traders having to reveal it.

Each trader i has a state-dependent utility function $u_i\colon \mathbf{A} \times \mathbf{\Omega} \to \mathbb{R}$, and maximizes the expectation of this function conditional on his type. Denote this conditional expectation by the function by $U_i\colon \mathbf{A}^{\Omega} \times \mathbf{\Omega} \to \mathbb{R}$, which is defined by [1]

$$U_i(\vec{a}, \omega^*) = \mathsf{E}\left[u_i(\vec{a}(\omega), \omega) | \mathcal{E}_i\right](\omega^*). \tag{1}$$

2.2 Specification of an allocation mechanism

An allocation mechanism is specified in terms of two structures, a communication protocol and an allocation rule. The allocation rule is a function which determines an ex-post allocation on the basis of the data generated by traders' use of the communication protocol.

A communication protocol is described formally in terms of a finite *message space* M. Each trader i chooses, on the basis of his type, a message $m_i \in M$ to send. As a function of the state of the environment, then, trader i's message is an $\mathcal{E}_i$-measurable function $\mu_i\colon \mathbf{\Omega} \to M$. This function μ_i will be called i's *communication strategy*. When each trader follows his communication strategy in state ω, a profile $\mu(\omega) = (\mu_1(\omega), \ldots, \mu_I(\omega))$ is generated which can be used as an informational basis for allocation.

Thus the *allocation rule* of the mechanism is a measurable function

$$\alpha\colon \mathbf{\Omega} \times M^I \to \mathbf{A} \tag{2}$$

such that

$$\forall m \in M^{\mathbf{I}} \;\; \alpha(\omega, m) \text{ is } \mathcal{E}_0\text{-measurable} \quad \text{and} \quad \forall \mu \;\; \exists f \in \mathbf{F} \;\; \forall \omega \;\; f(\omega) = \alpha(\omega, \mu(\omega)). \tag{3}$$

(The domain of quantification of μ is the set of all communication-strategy profiles. The function α must be restricted in the way specified by (3), in order to guarantee that the mechanism will always determine a feasible allocation regardless of which communication strategies traders choose.) An allocation rule α and a communication-strategy profile μ together determine an allocation $\alpha \circ \mu \in \mathbf{F}$.

An equilibrium (specifically a *Bayesian Nash equilibrium*) of the allocation mechanism (M, α) is a communication-strategy profile μ^* such that, for any trader i and any profile μ that i can obtain by unilaterally changing his communication strategy while others' strategies remain the same, $U_i(\alpha \circ \mu, \omega) \leq U_i(\alpha \circ \mu^*, \omega)$ almost surely.

If μ^* is an equilibrium of (M, α) and $\vec{a} = \alpha \circ \mu^*$, then $\vec{a}$ will be called an *implementable allocation* of (M, α). Let $\mathcal{I}$ denote the set of every state-contingent allocation such that there exists a mechanism that implements it. An allocation $\vec{a} \in \mathcal{I}$ is *efficient* if

$$\begin{aligned} \forall\, \vec{c} \in \mathcal{I} \qquad & \Big\{ \exists\, i \in \mathbf{I} \;\; \Big\{ \mathsf{E}\left[u_i(\vec{c}(\omega), \omega)\right] < \mathsf{E}\left[u_i(\vec{a}(\omega), \omega)\right] \Big\} \\ & \qquad \text{or} \quad \forall\, i \in \mathbf{I} \;\; \Big\{ \mathsf{E}\left[u_i(\vec{c}(\omega), \omega)\right] = \mathsf{E}\left[u_i(\vec{a}(\omega), \omega)\right] \Big\} \Big\}. \end{aligned} \tag{4}$$

[1] See Breiman (1968) or another graduate-level textbook of probability theory for the definition of expectation conditional on a σ-algebra. In the following definition, an trader's conditional expected utility is written for notational convenience as depending on the entire ex-post allocation. Actually, in the model to be studied here, an trader i's own consumption will be the only aspect of the allocation that matters for the determination of i's utility.

This definition conforms to the usual definition of ex-ante Pareto efficiency (cf. Myerson, 1991).

Consider the allocation that, in each state of nature, maximizes the sum of traders' utilities. Typically this allocation will not be implementable, so it cannot be efficient. However, this maximizing allocation is necessarily efficient if it is implementable.

Lemma 1 *Suppose that $\vec{a}$ is implementable and satisfies*

$$\sum_{i \in \mathbf{I}} \mathsf{E}[u_i(\vec{a}(\omega), \omega)] = \max_{\vec{c} \in \mathbf{F}} \sum_{i \in \mathbf{I}} \mathsf{E}[u_i(\vec{c}(\omega), \omega)]. \tag{5}$$

Then $\vec{a}$ is efficient.

Proof It follows immediately from (5) that $\sum_{i \in \mathbf{I}} \mathsf{E}[u_i(\vec{a}(\omega), \omega)] \geq \sum_{i \in \mathbf{I}} \mathsf{E}[u_i(\vec{c}(\omega), \omega)]$ for every $\vec{c} \in \mathcal{I}$. This means that if $\mathsf{E}[u_i(\vec{a}(\omega), \omega)] \leq \mathsf{E}[u_i(\vec{c}(\omega), \omega)]$ for every $i \in \mathbf{I}$, then $\mathsf{E}[u_i(\vec{a}(\omega), \omega)] = \mathsf{E}[u_i(\vec{c}(\omega), \omega)]$ for every $i \in \mathbf{I}$. That is, the condition (4) defining efficiency must hold. ∎

Lemma 1 has the following, immediate corollary.

Lemma 2 *Suppose that $\vec{a}$ is implementable and satisfies*

$$\forall \omega \in \Omega \quad \sum_{i \in \mathbf{I}} u_i(\vec{a}(\omega), \omega) = \max_{a \in F(\omega)} \sum_{i \in \mathbf{I}} u_i(a, \omega). \tag{6}$$

Then $\vec{a}$ is efficient.

3 Banking—A schematic model

Bryant (1980) and Diamond and Dybvig (1983) introduce models of banking which Jacklin (1987) simplifies further to study capital-structure issues.[2] Now I formulate a finite-trader version of Jacklin's maturity-transformation model.

3.1 An environment where a maturity-transforming intermediary has a role

Define Ω and Pr by

$$\Omega = \{0, 1\}^{\mathbf{I}} \quad \text{and} \quad \forall \omega \quad \Pr(\omega) = 2^{-I}, \tag{7}$$

and define $\mathcal{E}_0$ and $\mathcal{E}_i$ by

$$\mathcal{E}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \forall i \in \mathbf{I} \;\; \mathcal{E}_i = \{\emptyset, \Omega, \{\omega | \omega_i = 0\}, \{\omega | \omega_i = 1\}\}. \tag{8}$$

Suppose that there is an aggregate endowment of one unit of a good per person, which can be transformed into a consumption good available at either date 1 or date 2. The transformation

[2] Jacklin deliberately neglects the *sequential service constraint*, which Diamond and Dybvig discuss informally and which Wallace (1988) formalizes and analyzes. Wallace emphasizes that a serious treatment of this constraint shows the institutional arrangement of deposit insurance as modelled by Diamond and Dybvig to be infeasible.

is simply storage until date 1, but whatever is not consumed at date 1 is augmented by a gross factor of $R > 1$ at date 2. Thus feasible ex-post allocations are the elements of the set

$$\mathbf{A} = \left\{ a \colon \mathbf{I} \to \mathbb{R}_+^2 \;\Big|\; \sum_{i \in \mathbf{I}} \left[a_1(i) + R^{-1} a_2(i) \right] \le I \right\}. \tag{9}$$

and

$$\mathbf{F} = \mathbf{A}^{\Omega}. \tag{10}$$

A trader's utility from allocation a in state ω is given by a function $v \colon \mathbb{R}_+ \to \mathbb{R}$ of a consumption aggregate which includes consumption at both dates if i is of type 1, but which consists of consumption at date 1 alone if i is of type 0.[3] That is,

$$\forall i \quad \forall \omega \quad u_i(a, \omega) = v\big(a_1(i) + \omega_i a_2(i)\big). \tag{11}$$

Assume that

v is strictly increasing, continuously twice differentiable and strictly concave;
v satisfies the Inada conditions $\lim_{\gamma \to 0} v'(\gamma) = \infty$ and $\lim_{\gamma \to \infty} v'(\gamma) = 0$;
$\forall \gamma \;\; \gamma v''(\gamma)/v'(\gamma) \le -1$ (Relative risk aversion ≥ 1 everywhere). (12)

Consider the problem of choosing $\vec{a} \in \mathbf{F}$ to maximize the sum of traders' expected utilities, if the allocation could be made measurable in the traders' types (that is, the "fully-informed utilitarian social planner's problem"). By strict concavity of v, Jensen's inequality, and the fact that $R > 1$ while consumption goods at the two dates are perfect substitutes for type-1 traders, the following conditions should hold. In each state ω, all type-0 traders should receive identical consumption bundles $(c_0(\omega), 0)$ and all type-1 traders should receive identical consumption bundles $(0, c_1(\omega))$. Letting $\theta(\omega) = \sum_{i \in I} \omega_i$ as in the preceding example, each ex-post allocation $a = \vec{a}(\omega)$ should satisfy the following two equations (a first-order condition and a feasibility condition derived from (9), respectively).

$$v'(c_0(\omega)) = R v'(c_1(\omega)) \tag{13}$$

and

$$\left[I - \theta(\omega) \right] c_0(\omega) + R^{-1} \theta(\omega) c_1(\omega) = I. \tag{14}$$

These two equations determine $\vec{a}(\omega)$ uniquely. It is evident that $c_0(\omega)$ and $c_1(\omega)$ depend on ω only through $\theta(\omega)$. The following lemma explains the significance of the assumption regarding relative risk aversion in (12).

Lemma 3 *Suppose that v satisfies the assumptions (12), including that* $\forall \gamma \;\; \gamma v''(\gamma)/v'(\gamma) \le -1$. *(Relative risk aversion ≥ 1 everywhere.) Then the allocation $\vec{a}$ defined from (13) and (14) by*

$$\left[\vec{a}(\omega) \right]_i = \left((1 - \omega_i)\, c_0(\omega),\; \omega_i\, c_1(\omega) \right) \tag{15}$$

[3] This formulation follows Diamond and Dybvig. Jacklin also considers a utility formulation in which both types of trader receive positive marginal utility from consumption of each date, but in which type-0 traders discount consumption at date 2 more heavily than type-1 traders do.

is efficient. The consumption level $c_1(\omega)$ of type-one traders is a nondecreasing function of $\theta(\omega)$. More generally, let η be a real variable taking values in $(0, I)$ and consider the problem of maximizing

$$(I - \eta)v\left(\frac{\gamma}{I - \eta}\right) + \eta v\left(\frac{R(I - \gamma)}{\eta}\right). \tag{16}$$

The solution, parametrized by η, is a function $\Gamma(\eta)$ that satisfies

$$\frac{d}{d\eta}\frac{R(I - \Gamma(\eta))}{\eta} \geq 0. \tag{17}$$

Proof By (13) and (14) and the concavity of v, $\vec{a}$ satisfies the optimality condition (6) of lemma 2. Therefore $\vec{a}$ is efficient by lemma 2.

To see that the more general monotonicity assertion of the present lemma implies the more specific assertion regarding c_1, note that if $0 < \theta(\omega) < I$, then $c_0(\omega) = \Gamma(\eta)/(I - \eta)$ and $c_1(\omega) = R(I - \Gamma(\eta))/\eta)$ by (13) and (14). This equivalence can be extended to $\theta(\omega) \in \{0, I\}$, in view of the Inada conditions on v. (That is, defining $\Gamma(0) = I$ and $\Gamma(I) = 0$ extends the definition of Γ on $(0, I)$ continuously.)

Corresponding to (13), the first-order condition for (16) is

$$v'\left(\frac{\Gamma(\eta)}{I - \eta}\right) - Rv'\left(\frac{R(I - \Gamma(\eta))}{\eta}\right) = 0. \tag{18}$$

Taking the derivative of (18) with respect to η yields

$$\left[\Gamma'(\eta) + \frac{\Gamma(\eta)}{I - \eta}\right]\frac{v''(\Gamma(\eta)/(I - \eta))}{I - \eta} + R^2\left[\Gamma'(\eta) + \frac{I - \Gamma(\eta)}{\eta}\right]\frac{v''(R(I - \Gamma(\eta))/\eta)}{\eta} = 0. \tag{19}$$

Now consider the derivative in (17).

$$\frac{d}{d\eta}\frac{R(I - \Gamma(\eta))}{\eta} = \frac{-R}{\eta}\left[\Gamma'(\eta) + \frac{I - \Gamma(\eta)}{\eta}\right]. \tag{20}$$

In order to prove the lemma by establishing (17), then, it must be shown that the bracketed expression in (20) is negative. This expression is identical to one of the two bracketed expressions in (19), and clearly those two expressions must either have opposite sign or else both be zero, in order for (19) to hold. Thus the inequality

$$\Gamma'(\eta) + \frac{I - \Gamma(\eta)}{\eta} \leq 0, \tag{21}$$

which proves the lemma, is equivalent to

$$\frac{I - \Gamma(\eta)}{\eta} \leq \frac{\Gamma(\eta)}{I - \eta} \tag{22}$$

in view of (19).

Inequality (22) follows from the assumption that $\forall\gamma\ \gamma v''(\gamma)/v'(\gamma) \leq -1$. To see this, note that the assumption implies that

$$\frac{\partial}{\partial r}[rv'(rs)] \leq 0. \tag{23}$$

This inequality and equation (18) imply that

$$v'\left(\frac{I-\Gamma(\eta)}{\eta}\right) \geq v'\left(\frac{\Gamma(\eta)}{I-\eta}\right), \tag{24}$$

which implies (22) by the concavity of v. ∎

3.2 A mechanism with a unique, efficient equilibrium

Next I will show that conditions (13) and (14) imply that the efficient allocation can be implemented by a truth-telling equilibrium of an allocation mechanism analogous to that studied in the preceding model of the oasis. The mechanism here possesses a property that the mechanism in that other model lacks: that truth-telling is the *strictly dominant strategy* for each trader. By definition, this condition means that whether an trader is of type 0 or of type 1, he receives a higher utility level from revealing his type truthfully than from misrepresenting it—regardless of what reports other traders give. It follows (cf. Myerson, 1991) that the truth-telling equilibrium is the unique Bayesian Nash equilibrium of the mechanism. Therefore no alternative, inefficient, "run" equilibrium of this mechanism can exist.

The mechanism with this dominant-strategy property is constructed analogously to the mechanism that implements the efficient, symmetric allocation in the oasis economy. After having characterized the allocation that maximizes ex-ante expected utility (which has been done already by deriving conditions (13) and (14)), the mechanism is defined by depending on the truthfulness of traders' reports, and using them as the basis for assigning traders the ex-post consumption bundles determined by that allocation. Recall that, ordinarily, such a straightforward approach would be unsuccessful because truth-telling would not be a trader's equilibrium strategy. However, because of the particular form (11) of the state-contingent utility function and special features of the efficient, symmetric allocation, the approach does work in this case.

Theorem 1 *Let $M = \{0, 1\}$ be the set of signals for each trader. Define $x \colon M \times \{0, \ldots, I\} \to \mathbb{R}$ by the conditions (analogous to (13) and (14)) that*

$$v'(x(0, \eta)) = Rv'(x(1, \eta)) \tag{25}$$

and

$$[I - \eta]x(0, \eta) + R^{-1}\eta x(1, \eta) = I. \tag{26}$$

Define $\alpha \colon \Omega \times M^I \to \mathbf{A}$ by

$$\left[\alpha(\omega, m)\right]_i = \left((1 - m_i)\, x(m_i, \sum_{j \in \mathbf{I}} m_j),\ m_i\, x(m_i, \sum_{j \in \mathbf{I}} m_j)\right). \tag{27}$$

The truthful communication strategy $\hat{\mu}_i(\omega) = \omega_i$ is the strictly dominant strategy for each trader i. The mechanism thus implements the efficient, symmetric allocation in strictly dominant strategies, and consequently the profile of truthful communication strategies is its unique Bayesian Nash equilibrium

Proof If $\eta = \theta(\omega)$, then conditions (25) and (26) on $(x(0,\eta), x(1,\eta))$ are identical to conditions (13) and (14) on $(c_0(\omega), c_1(\omega))$. Lemma 3 therefore implies that the mechanism implements the efficient, symmetric allocation if the profile of truthful communication strategies is a Bayesian Nash equilibrium.

By Myerson (1991), a profile of strictly dominant strategies for a mechanism is the unique Bayesian Nash equilibrium of the mechanism. Therefore, to prove the lemma, it is sufficient to show that truthful communicatinon is the strictly dominant strategy for each trader. To verify this, consider separately each of the two possible values of ω_i. If $\omega_i = 0$, then by (25) and (27), i will receive a positive amount of consumption at date 1 if he sends message 0, but will receive 0 consumption at date 1 if he sends message 1. Because he has utility only for consumption at date 1 (by the definition (11) of his utility function), and because his utility is strictly increasing in the amount of this good that he consumes (by (11) and (12)), he strictly prefers to send message 0 rather than message 1 in state ω.

Now consider the alternative case that $\omega_i = 1$. The strict concavity of v assumed in (12), together with (25), implies that

$$x(1, 0 + \sum_{j \neq i} \mu_j(\omega)) > x(0, 0 + \sum_{j \neq i} \mu_j(\omega)) \tag{28}$$

regardless of which communication strategies μ_j the other traders use. By (17) of lemma 3 and the fundamental theorem of calculus,

$$x(1, 1 + \sum_{j \neq i} \mu_j(\omega)) \geq x(1, 0 + \sum_{j \neq i} \mu_j(\omega)). \tag{29}$$

Therefore, given the functional form of i's utility function (equations (11) and (12)) and the specification of the mechanism (equation (27)), inequalities (28) and (29) together imply that trader i must strictly prefer to send message 1 rather than message 0. ∎

4 Banking in an environment with sequential service

The schematic model of banking studied above abstracts from an important feature of an actual bank: that traders do not all contract the bank at the same time, and that the bank must deal promptly with traders who contact it early. The bank therefore is constrained from making its treatment of those traders contingent on information yet to be provided by later traders, especially if the early traders wish to make withdrawals. This feature plays an important role in Diamond and Dybvig's (1983) intuitive discussion of their model, and it is formalized by Wallace (1988) who derives further consequences from it. In view of the striking discepancy between theorem 1 and Diamond and Dybvig's analyisis, and of the closer analogy between the theorem and Jacklin's (1987) analysis that also abstracts from the sequential-service constraint, it is a salient question whether or not theorem 1 can be extended to an environment with sequential service. Now I investigate this question and find an answer that is more or less in the affirmative. Specifically, if v satisfies non-increasing absolute risk aversion as well as the conditions specified in (12), then the profile of truthful communication strategies is the unique profile that survives iterated elimination of strictly dominated strategies. It follows that, as in theorem 1, this is the unique Bayesian Nash equilibrium of the natural mechanism that implements the efficient allocation.

In the present formalization of the sequential-service constraint, every trader contacts the bank at some time during date 0, these "arrival times" for different traders are stochastic and independently distributed, and each trader's arrival time is in his own information set. This last detail is crucial, for it implies that a trader who arrives very late can be almost certain that he is the last trader to arrive. Conditional on being last, truthful communication is the trader's unique utility-maximizing action. That is, any strategy that involves some untruthful communication by a trader when he arrives very late can be eliminated as being dominated by the strategy that agrees with it except at very late times, but that specifies truthful communication at those times. This result can then be "bootstrapped" to apply to communication at earlier times as well. One should note that, in Wallace's formalization of sequential service, a trader's time of arrival is not in his own information set. Under Wallace's assumption, it seems that iterated elimination of strictly dominated strategies may not lead necessarily to truthful communication.

4.1 Formalization of sequential service

Modelling sequential service requires that the maturity-transformation model must be modified by enlarging the state space Ω to represent information about arrival times, and by making corresponding changes in the definitions of agents' types and of feasible allocations.

To enlarge the state space, replace the definition (7) by

$$\begin{aligned}&\Omega = \{0,1\}^{\mathbf{I}} \times [0,1]^{\mathbf{I}}, \text{ and } p \in (0,1);\\ &\text{For all } i \leq I \quad \Pr(\omega_i = 1) = p;\\ &\text{For all } i \leq I \quad \omega_{I+i} \text{ is uniformly distributed;}\\ &\text{The projections of } \omega \text{ on its coordinates are independent r.v.'s.}\end{aligned} \tag{30}$$

Replace the definition in (8) of agent i's type by

$$\mathcal{E}_i = \{\{\omega|\omega_i = 0 \text{ and } \omega_{I+i} \in A\} \cup \{\omega|\omega_i = 1 \text{ and } \omega_{I+i} \in B\}|A \in \mathcal{F} \text{ and } B \in \mathcal{F}\}, \tag{31}$$

where $\mathcal{F}$ is the σ-algebra of Borel sets on $[0,1]$. That is, in each state a trader knows his own utility function and his own arrival time at the bank, but he knows nothing about the other traders.

Also replace the specification in (8) that the algebra $\mathcal{E}_0$ is trivial by the following definition, which intuitively specifies that information about all traders' arrival times may be used directly (that is, without having to be revealed by the traders' communication) as a basis for allocation.

$$\mathcal{E}_0 = \{\omega|\forall i \leq I \;\; \omega_{I+i} \in A_i|A_1 \in \mathcal{F}, \ldots, A_I \in \mathcal{F}\} \tag{32}$$

In order to formulate the sequential service constraint, define the arrival-order statistics by $\tau\colon \{0,\ldots,I\} \times \Omega \to \mathbf{I}$. That is, $\tau(1,\omega), \tau(2,\omega), \ldots \tau(I,\omega)$ are the first, second, $\ldots I$th traders in order of arrival determined by the coordinates $\omega_{I+1}, \omega_{I+2}, \ldots \omega_{2I}$ of ω. Ties can be assumed to be broken arbitrarily in the zero-probability event that several traders arrive simultaneously at the bank. Define the rank statistics $\rho\colon \mathbf{I} \times \{0,\ldots,I\} \times \Omega \to \{0,\ldots,I\}$, which are inverse to the order statistics in each state of nature, by $\rho(\tau(i,\omega),\omega) = i$.

Suppose that $\vec{a} = ((X_0^1, X_1^1), \ldots, (X_0^I, X_1^I)) \in \mathbf{A}^\Omega$ is an allocation.[4] The intuitive content of the sequential service constraint is that the mechanism represents a financial intermediary (call it a bank) operating at a specific location that the trader visits at some time during date 1. When trader i visits, he communicates a message $m \in M$ determined by a communication strategy μ_i that is measurable with respect to $\mathcal{E}_i$, and he then receives $X_0^i(\omega)$ immediately. This quantity thus must not depend on information from traders who arrive later in state ω than i does, since those traders have not yet communicated their information to the bank. Since all traders are envisioned to arrive at the bank at some time before date 2, when the consumption amounts $X_1^j(\omega)$ are distributed, those date-2 quantities are not analogously constrained.

That is, the amount $X_0^{\tau(1,\omega)}(\omega)$ of consumption given to trader $\tau(1,\omega)$ at date 1 must depend only on the identity of $\tau(1,\omega)$ and the time $\omega_{I+\tau(1,\omega)}$, both of which the bank observes, and on that trader's utility parameter $\omega_{\tau(1,\omega)}$, which he has the opportunity to communicate to the bank. (Whether or not he actually does communicate his utility parameter in equilibrium is irrelevant to the formulation of this constraint, which expresses the limitation imposed by the exogenous sequential nature of the *opportunities* for the bank to acquire information.) Next, the information that the bank can use to determine the date-1 consumption of the second trader to arrives consists of both this information about the first trader, which the bank remembers, and also the corresponding information about the second trader himself. And so forth. Formally, $((X_0^1, X_1^1), \ldots, (X_0^I, X_1^I))$ satisfies the *sequential service constraint* if

$$\forall i \quad X_0^{\tau(i,\omega)} = \mathsf{E}\left[X_0^{\tau(i,\omega)} | \tau(1,\omega), \ldots, \tau(i,\omega), \omega_{\tau(1,\omega)}, \ldots, \omega_{\tau(i,\omega)}, \omega_{I+\tau(1,\omega)}, \ldots, \omega_{I+\tau(i,\omega)}\right]. \quad (33)$$

In view of this constraint, the definition of $\mathbf{F}$ should be replaced by

$$\mathbf{F} = \left\{\vec{a} | \vec{a} \in \mathbf{A}^\Omega \text{ and } \vec{a} \text{ satisfies (33)}\right\}. \quad (34)$$

4.2 The efficient, symmetric, state-contingent allocation

In this section, I consider the solution of the optimization problem posed in equation (5), that is,

$$\text{Maximize } \sum_{i \in \mathbf{I}} \mathsf{E}[u_i(\vec{c}(\omega), \omega)] \text{ subject to } \vec{c} \in \mathbf{F},$$

with $\mathbf{F}$ defined by (34). Subsequently I will consider the problem of implementing this allocation, which is efficient by lemma 1.

The key to solving problem (36) is the observation, formalized below in lemma 4, that the arrival-order statistics $\tau(i,\omega)$ provide all of the relevant information about traders' arrival times. More precise arrival-time information is relevant neither to traders' enjoyment of utility nor to the technical feasibility of allocations in the sequential service environment.[5] In view of this observation, define mappings $\sigma^i \colon \Omega \to \{0,1\}^i$ for $1 \leq i \leq I$ by

$$\forall j \leq i \quad \sigma_j^i(\omega) = [j](\omega). \quad (35)$$

[4] In this section, since Ω is a continuum, $\mathbf{A}^\Omega$ denotes the set of Borel-measurable functions from Ω to $\mathbf{A}$.

[5] Each trader will use his information about his precise arrival time to make inference about his probable rank in the arrival queue (which he does not observe directly) though, so this information is relevant to implementation.

Define the set of 0–1 sequences of length at most I, including the null sequence, as $\mathcal{S}$. For $s \in \mathcal{S}$, let $\ell(s)$ denote the length of s. For $i \leq I$, define 0^i to be the sequence consisting of i consecutive zeros. Define the weak and strict extension-ordering relations on $\mathcal{S}$ by

$$\begin{aligned} r \leq s \quad &\Longleftrightarrow \quad \ell(r) \leq \ell(s) \text{ and } \forall i \leq \ell(r)\ \ [r_i = s_i]; \\ r < s \quad &\Longleftrightarrow \quad \ell(r) < \ell(s) \text{ and } \forall i \leq \ell(r)\ \ [r_i = s_i]. \end{aligned} \tag{36}$$

Lemma 4 *Suppose that $\vec{a} = ((X_0^1, X_1^1), \ldots, (X_0^I, X_1^I))$ solves problem (5) in the sequential-service environment. Then there exists a vector $x \in \mathbb{R}_+^{\mathcal{S}}$ such that*

$$\forall s \in \mathcal{S} \quad \left[s_{\ell(s)} = 1 \Longrightarrow x_s = 0\right] \tag{37}$$

and, almost surely for all i,

$$X_j^{\tau(i,\omega)} = \begin{cases} x_{\sigma^i(\omega)} & : \quad j = 0 \text{ and } \sigma_i^i(\omega) = \omega_{\tau(i,\omega)} = 0; \\ \frac{R}{\theta(\omega)}\left(I - \sum_{r \leq \sigma^I(\omega)} x_r\right) & : \quad j = 1 \text{ and } \sigma_i^i(\omega) = \omega_{\tau(i,\omega)} = 1; \\ 0 & : \quad \text{otherwise.} \end{cases} \tag{38}$$

If $v(0) = 0$ and $\theta^*(s) = \sum_{i<\ell(s)} s_i$ and $\pi(s) = p^{\theta^*(s)}(1-p)^{\ell(s)-\theta^*(s)}$, and if $\vec{a}$ and x are related according to (38), then

$$\sum_{i \in \mathbf{I}} \mathsf{E}\left[u_i(\vec{a}(\omega), \omega)\right] = \sum_{\substack{\ell(s) = I \\ \theta^*(s) > 0}} \pi(s) \left[\left(\sum_{r \leq s} v(x_r)\right) + \theta^*(s) v\left(\frac{R}{\theta^*(s)}\left(I - \sum_{q \leq s} x_q\right)\right)\right] + \pi(0^I) v(x_{0^I}). \tag{39}$$

Proof One can alternatively characterize $\vec{a}$ in terms of a vector of random variables $((Y_0^1, Y_1^1), \ldots, (Y_0^I, Y_1^I))$, where $Y_j^i(\omega) = X_j^{\tau(i,\omega)}(\omega)$ a.s. for each i and j. Consider the state-contingent allocation $\vec{c} = ((Z_0^1, Z_1^1), \ldots, (Z_0^I, Z_1^I))$, defined by $Z_j^{\tau(i,\omega)}(\omega) = \mathsf{E}[Y_j^i(\omega)|\sigma^i]$ a.s. It is easily verified that $\vec{c} \in \mathbf{F}$, and for every i, $\mathsf{E}[u_i(\vec{a}(\omega), \omega)] \leq \mathsf{E}[u_i(\vec{c}(\omega), \omega)]$, with strict inequality for at least one i if $\vec{a} \neq \vec{c}$. (This inequality must hold because v is strictly concave and $\vec{c}$ is obtained by taking conditional expectation with respect to $\vec{a}$.) That is, $\vec{a} \neq \vec{c}$ would contradict the hypothesis of the lemma. By construction, $\vec{c}$—that is to say, $\vec{a}$—can be characterized in terms of a vector $x \in \mathbb{R}_+^{\mathcal{S}}$. This vector must actually satisfy (38), by the same considerations that prove the efficiency assertion in lemma 3. (Note that, in the context of (38), condition (37) states that traders of type 1 consume exclusively at date 2.) Condition (39) is verified by straightforward computation. ∎

By lemma 4, a solution to optimization problem (5) can be found by optimizing over a set of vectors in $\mathbb{R}_+^{\mathcal{S}}$. Specifically, given the strict concavity of the right side of (39), a solution is characterized by a vector that satisfies the first-order conditions for optimization of (39) subject to the constraint (37). That is, the following lemma holds.

Lemma 5 *A necessary and sufficient condition for a state-contingent allocation $\vec{a} = ((X_0^1, X_1^1), \ldots, (X_0^I, X_1^I))$ to solve problem (5) in the sequential-service environment is that there should exist a vector*

$x \in \mathbb{R}_+^{\mathcal{S}}$ that satisfies (37), (38), and for all $r \in \mathcal{S}$ such that $r_{\ell(r)} = 0$,

$$\pi(r)v'(x_r) - R \left[\sum_{\substack{\ell(s) = I \\ \theta^*(s) > 0 \\ r \leq s}} \pi(s)v' \left(\frac{R}{\theta^*(s)} \left(I - \sum_{q \leq s} x_q \right) \right) \right] + 0^{\theta^*(r)} \pi(0^I)v'(x_{0^I}) = 0. \qquad (40)$$

4.3 A mechanism with a unique, efficient equilibrium

The first-order condition (40) just derived for the sequential-service environment has analogous structure to the first-order condition (13) in the simultaneous-communication environment studied in section 3. Lemma 1, which provides the key to establishing dominant-strategy implementability of the symmetric, efficient allocation in that environment, is proved by examining condition (13). An analogous result is provable on the basis of condition (40), and it also leads to an implementability result.

Theorem 2 *Suppose that v satisfies condition (12) and also the condition that*

$$\forall \gamma \;\; \frac{d}{d\gamma} \frac{v''(\gamma)}{v'(\gamma)} \geq 0 \qquad \textit{(Absolute risk aversion non-increasing everywhere).} \qquad (41)$$

Let $M = \{0, 1\}$ be the set of signals for each trader. Let $x \colon \mathcal{S} \to \mathbb{R}_+$ be the vector satisfying the optimality conditions (37) and (39). Define $\alpha \colon \Omega \times M^I \to \mathbf{A}$ by

$$[\alpha(\omega, m)]_i = \left((1 - m_i)\, x(m_{\tau(1,\omega)}, \ldots, m_{\tau(\rho(i,\omega),\omega)}),\; m_i \frac{R}{\sum_{j \leq I} m_j} \left(\sum_{j \leq I} x(m_{\tau(1,\omega)}, \ldots, m_{\tau(j,\omega)}) \right) \right). \qquad (42)$$

Then the profile of truthful-communication strateies $\hat{\mu}_i(\omega) = \omega_i$ is the unique profile that survives iterated elimination of strictly dominated strategies. The mechanism thus implements the symmetric, ex-ante efficient by a unique Bayesian Nash equilibrium.

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