An Empirical Model of Inventory Investment by Durable Commodity Intermediaries

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Abstract

We present a new detailed data set of high-frequency observations on inventory investment by a U.S. steel wholesaler. Our analysis of the data leads to six main conclusions: orders and sales are made infrequently; orders are more volatile than sales; order sizes vary considerably; there is considerable day-to-day variability in sales prices; inventory/sales ratios are unstable; and there are occasional stockouts. We model the firm generically as a *durable commodity intermediary.* We demonstrate that the firm's behavior at the product level is well approximated by an *optimal trading strategy* derived from a multi-dimensional nonlinear dynamic programming problem with continuous state and control variables which are subject to frequently binding inequality constraints. We show that the optimal trading strategy takes the form of a *generalized* (S, s) *rule,* in which the (S, s) bands are decreasing functions of the spot price. We simulate a calibrated version of this model, and show that the simulated data exhibit the key features of inventory investment we observe in our data.

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1 Introduction

This paper formulates and solves a dynamic model of optimal inventory investment by durable commodity intermediaries. Commodity intermediaries are companies whose business is to stockpile quantities of homogeneous durable goods such as steel, lumber, coal, etc. These firms do minimal production processing, and make profits via inter-temporal arbitrage, purchasing bulk quantities of durable commodities at competitive world spot market prices and subsequently selling their inventories to customers at a mark-up.

We study a new database from one such intermediary, a U.S. steel supplier. From this intermediary we have twenty months of confidential daily data on purchases, sales, and inventory holdings of over 2,000 separate products. Our analysis of these data yields six main conclusions:

1. Orders and sales are made infrequently.
2. Orders are more volatile than sales.
3. There is considerable variability in order levels.
4. There is considerable high-frequency variation in sales prices.
5. There is no stable inventory/sales relationship.
6. Inventory stockouts and near stockouts occur regularly, especially during regimes of low inventory holdings.

We observe all six facts at the individual product level. We observe facts 2, 3, and 4 at the firm level. To explain these facts we solve a dynamic programming model which treats each product as an independent "profit center". Using this model we ask whether the firm's behavior can be accurately approximated by the optimal trading strategy implied by the model's solution.

In the model, prices and quantity demanded are exogenous stochastic processes. Each period, the firm must decide how much steel to order to maximize the discounted present value of profits. The trading (order) strategy implied by our model takes the form of a *generalized* (S, s) *policy.* Given the current spot order price, p, the firm chooses an order limit point, $s(p)$, and a target inventory level, $S(p)$. The firm place no orders unless inventories fall to $s(p)$ or below, whereupon the firm places an order that sets the level of inventory to $S(p)$. We show that both $s(p)$ and $S(p)$ are decreasing functions of p.

Under the optimal policy the firm exploits low spot order price opportunities to make large purchases. The firm thus makes large capital gains on its inventory holdings once the price inevitably rises. Furthermore as the price rises, $S(p)$ and $s(p)$ fall, so the firm may make no new orders for many periods after a large order. When prices are high both $S(p)$ and $s(p)$ are small, so the firm maintains a relatively low level of inventories; in this case orders are relatively small and stockouts occasionally occur.

This opportunitistic purchasing by the firm implies that orders are infrequent, that order quantities are more variable than sales, that there is no stable inventory/sales ratio, that stock-outs occur when spot prices are high, and that inventory holdings follow "saw-tooth" trajectories. In contrast to standard (S, s) theory our model generates saw-tooth inventory holding patterns even in the absence of fixed costs to ordering.

While the main focus of this paper is to explain the high-frequency behavior of a single firm, the issues addressed may be of interest to economists studying movements of aggregates at lower, particularly business cycle, frequencies. In general, recessions can be characterized as periods of inventory liquidations. While in the U.S. inventory investment averages less than one-half of one percent of GDP, during a typical recession the reduction in inventory investment accounts arithmetically for about 50 percent of the reduction in GDP (Ramey and West, 1997). So if we want to understand business cycles, it is important to understand inventory investment behavior – and commodity intermediaries are in the inventory business.

Section 2 provides a brief review of the existing literature on inventory investment. Section 3 presents the steel inventory data and summarizes the five main conclusions from our empirical analysis that we will attempt to explain with a simple dynamic programming model of inventory investment. Section 4 presents the model and provides sufficient conditions for the optimal inventory investment policy to take the form of a generalized (S, s) rule. Section 5 displays numerically computed solutions and stochastic simulations of a calibrated example of the model. Section 6 summarizes our findings.

2 Background

There is an extensive literature on the role of commodity storage from an aggregate perspective (see, e.g. Working, 1949 and Williams and Wright, 1991); however we are unaware of more detailed micro-oriented studies of individual agents participating in these markets. Although the main ideas behind the role of commodity storage have been around for many years, only relatively recently have economists attempted to deduce the implications of this model for commodity prices. A stylized version of the dynamic rational expectations commodity storage model, (e.g. Deaton and Laroque, 1992 or Miranda and Rui, 1997) posits that the aggregate supply of a commodity is produced inelastically according to an *IID* process $\{z_t\}$. There is a stationary demand function $D(p)$, so in the absence of storage, equilibrium prices are also an *IID* process $\{D^{-1}(z_t)\}$. However if we assume a storage technology exists with a proportional cost of storage δ (i.e. one unit of the commodity stored at date t yields $(1-\delta)$ units of the commodity at date $t+1$), or an additive cost of storage c_t, then the equilibrium price must satisfy

$$p_t = \begin{cases} \frac{1}{(1+r)}(1-\delta)E_t p_{t+1} & \text{proportional cost of storage model} \\ c_t + \frac{1}{(1+r)}E_t p_{t+1} & \text{additive cost of storage model} \end{cases} \quad (1)$$

This literature has shown that many of the observed properties of commodity price series, particularly skewness in prices, occasional sharp price spikes (i.e. sharp price increases as opposed to price decreases), and high autocorrelations can be explained as a result of competitive storage even if the fundamental "forcing process" $\{z_t\}$ is *IID*. While the intertemporal equilibrium relationship (1) is not derived from "first principles", the argument is that if prices did not satisfy this relationship, speculators would buy or sell the commodity to equate current and expected future prices net of storage/carrying costs. Price spikes occur only during aggregate stockouts; otherwise speculators succeed in stabilizing prices, preventing sharp increases or decreases in commodity prices during times of production surpluses or shortages. The theory suggests that sudden crashes in commodity prices should not occur, since this would induce speculators to purchase and store the commodity for subsequent resale.

The steel intermediary we study is precisely one of the "speculators" implicit in the commodity storage model. However the recent collapse in commodity prices in the aftermath of the 1997 Asian financial crisis calls into question the power of inventory speculation in preventing sudden price declines. The physical costs of storing commodities such as steel are presumably very small and the rate of depreciation of steel is close to zero. However the interest opportunity costs of storing these commodities can be substantial, a fact that seems to have been overlooked in the commodity storage literature. It is reasonable to suppose that speculators will not buy large quantities of a commodity in the aftermath of a price crash if they expect it to be followed by a sustained recession that would limit their ability to resell the commodity at attractive prices in the future. This observation underscores the importance of extending the commodity storage model by building more detailed models of the speculators underlying these models — commodity intermediaries.

There is also an extensive literature of macro-level models of inventories which assume short-run increasing marginal costs to holding inventories. The workhorse model of this literature is the linear quadratic (LQ) model introduced by Holt, Modigliani, Muth and Simon (1960). The standard LQ model implies that that production (orders) should be smoother than sales. Since this implication is almost always rejected empirically, a variety modifications have been made. For example many authors augment these models with an "accelerator term" in the profit function which is essentially a quadratic penalty function from deviating from a fixed "target" inventory/sales ratio. This target is treated as an unknown parameter to be estimated (e.g. Blanchard, 1983, West, 1986, and Kashyap and Wilcox, 1993). Kahn (1987, 1992) justifies targeting an inventory/sales ratio by explicitly incorporating costly stock-outs. Bils and Kahn (1996) further justify targeting such a ratio by modeling sales as an increasing function of the available inventories. A second modification is to assume that firms operate on flat or even decreasing regions of their short-run marginal cost curves. Ramey (1991), Bresnahan and Ramey (1994), and Hall (1997) provide evidence that firms may often operate in such regions. Third, Blinder (1986) and Miron and Zeldes (1988) and others have added cost shocks in the form of input price shocks, while others such as Eichenbaum (1984, 1989) have added cost shocks in the form of unobservable technology shocks. Thus inventories are used to smooth production costs rather than the level of production. These modifications have improved the ability of the LQ model to explain aggregate inventory time series, although as we will show in the next section we have doubts about its ability to explain our product-level data.

Dynamic micro-level models of inventory investment incorporating a fixed cost to ordering were pioneered by Arrow, Harris, and Marschak (1951) and Scarf (1959). In these models, the optimal policy is of the (S, s) form. In the simplest case, the firm chooses an order limit point

s, and an upper inventory point S. The firm place no orders until inventories fall to s or below, whereupon the firm places an order to reset the inventory level to S. Blinder (1981), Caplin (1985), and Fisher and Hornstein (1998) argue that explicitly modeling fixed costs at the firm level helps explain inventory behavior at the aggregate level.

Despite extensive research in the area of inventory investment, a satisfactory model to explain this important time series has not yet been written down and solved. Even models which appear capable of explaining the basic features of the data have clear flaws. For example attempts to estimate macro models of inventory investment often yield parameter estimates of the wrong sign. Some of the problems may stem from a lack of high-quality data on production and inventories. Fair (1989) suggests that the observation that production is more volatile than sales is just a figment of poorly constructed data. Miron and Zeldes (1989) demonstrate that there is substantial measurement error in both the monthly manufacturing and inventory investment data. The absence of high quality inventory data at the macro-level motivates us to study this issue at the firm level. In their survey of the inventory literature for the *Handbook of Macroeconomics* Ramey and West (1997) "advocate more plant and firm-level studies, although gathering such data requires substantial work." (p. 47).

3 Data

A U.S. steel wholesaler (referred to below as the "firm") provided us detailed data on every transaction it undertook between July 1, 1997 to February 26, 1999 (434 business days) for its 2200+ individual products. For each transaction we observe the quantity (number of units and/or weight in pounds) of steel bought or sold, the sales price, the shipping costs, and the identity of the buyer or seller. The firm's records provide data on the level of inventories for each product at the beginning and end of each month. Using the inventory accumulation identity we can track the firm's inventory holdings (both in physical units and in dollar value) for each day within the month. Also since we observe the prices at which this firm buys and sells steel, we also have a near-perfect measure of the mark-ups charged to customers. Finally since we meet regularly with company executives, we are able to eliminate any discrepancies in the transaction and inventory data. This is an exceptionally clean dataset.

The firm records transactions on the day the steel either enters or leaves one of the warehouses.

Although the firm does receive some commitments for sales in advance, most of their sales are delivered within 24 hours of the commitment, and 95 percent of their orders are filled within five days. Indeed, the primary service this wholesaler provides is having the goods on hand and being able to deliver them to the customer on short notice. While back-orders do occasionally occur, we study products which customers can assume the firm will have on hand. We do not have data on when the firm makes an order to purchase steel. From discussion with company executives we know that some of their orders are made weeks in advance (up to 12 weeks when purchasing foreign steel), while some purchases are made with only a day or two notice. In this paper we assume the relevant time period is one business day.

Although this company offers over 2000 products, tables 2 and 1 provide summary statistics for prices and quantities for eighteen of their more important products which are considered baseline products within the industry. These products serve as key indicators from which the prices of other products are calculated, and display the characteristic features that we see for many other products. For reasons that will become clear subsequently, these products are also of interest because none involve any actual production processing beyond storage and redistribution. Finally, we chose relatively high volume products for which the firm made at least four orders during the sample period. Figure 1 plots an indicator of the firm's aggregate inventory holdings, the sum (in pounds) of the inventories for each of these eighteen products. Figure 2 plots the inventory/sales ratio measured as "days supply" which we define as the level of current inventories divided by the average daily sales rate for the previous 30 business days.[1] Figures 3 - 14 plot daily time series for inventories, days-supply, and spot order and sales prices, for products 2, 4 and 13 in tables 2 and 1. These figures also contain three dimensional scatterplots of purchase quantities as a function of current inventory and order prices.

Our analysis of these data can be summarized in six main conclusions:

1. *Orders and sales are made infrequently.* In the second column of table 1, we report the number of days in which each product enters one of the firm's warehouses. We have selected some of the highest volume products this firm deals in; nevertheless, orders are rarely made. Sales are made more frequently as can be seen from column (5) of table 1 and from the absence of long flat segments in the inventory graphs. However even for product 2, the

[1] Computing days-supply using future sales instead of past sales does not change the qualitative features of any of the graphs in this paper.

product with the most frequent sales, sales are made less than 3/4 of the days in the sample. Note also that the periodicity between successive orders is highly variable.

2. *Orders are more volatile than sales.* The last column in the bottom row of table 2 reports the ratio of the standard deviation of aggregate orders to the standard deviation of aggregate sales. This ratio is 9.2, which shows that for this firm orders are substantially more volatile than sales. Columns (2), (4), (6), and (8) of table 2 report the unconditional means and standard deviation of orders and sales. But since sales and orders and made infrequently, we also report in columns (3), (5), (7), and (9) the means and standard deviation conditional on an order or sale occurring. Not surprisingly, since orders are made less frequently than sales, the average order size is larger than the average sales size. As is found in many other studies, the standard deviation of orders is larger than the standard deviation of sales. This holds for all eighteen products; see column (10). Note that extremely large sales are relatively rare events as can be seen from the relatively small number of large discontinuous downward jumps in inventory levels in the time series plots.

3. *There is considerable variability in order levels.* In table 2, we can see that for all but four of the eighteen products, conditional on an order occurring, the standard deviation of the order size (column (5)) is larger than the mean order size (column (3)). This fact can also be seen graphically in our plots of the data for products 2, 4, and 13 in figures 3 - 14. Figures 3, 7, and 11 display the time path of the inventory holdings for these three products. These figures display a "saw-tooth" pattern for inventory holdings with intervals during which inventory levels steadily decrease due to sales to customers punctuated by periodic large orders that replenish inventory holdings. Thus, inventory holdings can be characterized as a jump process with a negative drift due to numerous small sales, and periodic discontinuous upward jumps due to a relatively small number of large orders.

   However the firm also makes many small orders. This is apparent in figures 4, 8, and 12, which display scatterplots of order size as a function of current inventory holding and the order price. In general, these three graphs illustrate that the lower the price and the lower the level of inventories, the larger the order. But a striking feature of these figures is the number of small orders – especially when inventories and the order price are high. Also note in figure 4 that most of the orders for product 2 lie in the *order price* band between

19.00 and 19.50. The tendency for order size to increase rapidly as a function of order price suggests that the firm's demand for product 2 is highly elastic. This suggests that inventory holdings are quite sensitive to the spot price of steel, a conclusion that is confirmed from an inspection of the time series for inventories and order prices in figures 3 and 5, 7 and 9, and 11 and 13, respectively. Comparing these graphs vertically, we see that the biggest upward jumps in inventories generally occur when the (interpolated) order price series hits historical lows. However our ability to make clear inferences about this is hampered by the fact that we only observe spot prices for these products on the days the firm places orders for steel. Thus we cannot be sure that the actual spot price series may actually have been even lower between the successive dates at which large purchases occurred.

4. *There is no stable inventory/sales relationship.* Figures 6, 10, and 14 display the inventory/sales ratio in terms of days-supply. As in the case of the aggregate days-supply series, these three inventory/sales ratios fluctuate widely and in the case of products 4 and 13 appear to have multiple "regimes" with high and low inventory/sales ratios.

5. *Inventory stockouts and near stockouts occur regularly, especially during regimes of low inventory holdings.* From figures 6, 10, and 14, we can see that the firm often allows inventories to fall to a level below 5 days worth of sales. Moreover, for product 13, the firm was completely stocked-out (i.e. had zero inventories) for 24 days during the time period.

6. *There is considerable high-frequency variation in the sales price, with large changes in sales prices on successive sale dates.* This firm is clearly charging some customers higher prices than others, a fact readily acknowledged by company executives. While we do not attempt to model the firm's pricing decisions in this paper, this feature of the data motivates our desire to do future work analyzing dynamic models of endogenous price setting and price discrimination.

We now consider whether any of the standard models of inventories outlined in section 2 are capable of explaining the six main facts listed above.

1. (S, s) **models.** The saw-tooth pattern of the inventory series is clearly reminiscent of an (S, s) policy, which also predicts intervals of steady declining inventories (due to sales to customers) interspersed by occasional upward jumps in inventories (due to new orders by

the firm). While the saw-tooth pattern of inventory holdings in figure 1 is suggestive of an (S, s) policy, closer analysis reveals that the firm's behavior cannot possibly be described by a standard (S, s) rule where S and s are fixed, time-invariant constants. Under such a policy the firm places an order of size $S - s$ when its current inventory q falls below the lower order threshold s. This implies that whenever the firm places an order we should see inventories replenished to the same target level S. However it is clear from figure 1 that the amount of inventory the firm holds after each order varies widely over time. Also, in the absence of large discontinuous downward jumps in inventories resulting from large sales (e.g. in limiting continuous-time versions of the (S, s) inventory model where sales follow a diffusion process), all orders should all be of of the same size $S - s$. It is clear from figure 1 that the size of the firm's orders vary widely over time. Finally, the frequent number of stockouts also casts doubt on the validity of the standard (S, s) policy, which predicts that (in the absence of jumps) that inventories should remain in the interval (s, S). When there are positive fixed costs of ordering, $s > 0$, so inventories should not fall below this level. On the other hand, if fixed costs of ordering inventories were 0, then the firm should place new orders each day to maintain the target inventory level S. In either case stockouts should not occur under the standard (S, s) model. Thus, we conclude that this firm's behavior is inconsistent with the predictions of the standard (S, s) inventory model.

2. **Production smoothing models.** Our finding that orders are on average 9 times more variable than sales shows that this firm's behavior is inconsistent with the predictions of standard production-smoothing models. These models imply that the variance of production should be lower than the variance of sales. Of course, one can question the relevance of the production smoothing model for studying the behavior of this firm since it does a minimal amount of actual production processing. Although this firm does have a small assembly line that "levels" steel coil (i.e. it unwinds the coil and chops it into rectangular sheets), the firm's main "production" activity for many of its other products such as heavy steel plate and pipe simply involves placing new orders to replace inventory at a time-varying "marginal cost" of p_t equal to the spot price of steel on day t. There are no costs of stopping, idling, and restarting an assembly line for these latter products, so it appears that in such case there is far less incentive to attempt to smooth production (which in this case simply amounts

to placing new orders for steel).[2] Indeed, to the extent that there are fixed costs to placing orders, it would appear that it is optimal for the firm to do the opposite of production-smoothing, namely to make relatively infrequent large orders rather than frequent small orders. We conclude that the standard versions of the production-smoothing model cannot provide a plausible empirical model for this firm.

3. *LQ* **models.** A particularly popular type of production smoothing model is the *LQ* model, which is the standard framework for modeling inventories in the macro literature. Unfortunately our analysis suggests that the *LQ* model has severe deficiencies at the micro level, particularly for describing the product level inventory holdings of this firm. The LQ model ignores the frequently binding constraint that orders must be non-negative and is therefore unable to explain the observation that orders are usually zero. Even if we were to interpret the LQ model's predictions of negative orders as representing "desired orders" and use Tobit-style censoring to map negative desired orders to zero, we believe that the linear laws of motion for the state variables in LQ models would have a hard time approximating the mass point at zero that we observe in the distributions of quantity ordered and sold.

4. *LQ* **models with inventory/sales ratio targets.** In order to explain the widely observed fact that production is more volatile than sales, the standard *LQ* production smoothing models have been augmented to include a target inventory/sales ratio and a quadratic penalty for deviating from this target (e.g. Blanchard, 1983). Although the assumption that the firm has a fixed target inventory/sales ratio is not derived from first principles, under certain circumstances tacking on such a term to the firm's cost function yields optimal policies for which production is more variable than sales. However our data provide little support for the hypothesis that the firm has a fixed inventory/sales target. A simple inspection of figure 2 shows that the inventory/sales ratio is extremely variable, beginning with a "low inventory regime" during which the firm has only a month's supply on hand, followed by a "high inventory regime" when it has more than 5 month's supply on hand. This variation does not appear to be due to non-stationarity in sales, but rather due to significant declines in the spot price of steel over this period. In simple terms, this firm appears to be engag-

[2] However Abel (1983) finds in a model with a production lag, stock-outs, and endogenous pricing the variance of sales exceeds the variance of production even if the cost of producing are linear.

ing in commodity price speculation, attempting to "buy low and sell high". This strategy implies that the firm should buy large quantities of steel when prices are low, holding it for subsequent resale when prices are higher. Such a strategy is inconsistent with maintaining a fixed inventory/sales ratio.

Our analysis of the firm's product level data suggests that cost shocks — which in this case are mainly due to changes in the spot price at which the firm acquires steel inventories — could be the key explanation for the observation that orders are more volatile than sales. A second explanation is the fact that this firm does not do any actual production processing for the products we have studied, and a third explanation is the existence of positive fixed costs associated with placing new orders for steel. We believe the first explanation is the key to understanding the large variation in inventory holdings over our sample period. The spot price of steel is clearly the most volatile of the cost shocks facing this firm, whereas the other production and storage costs are unlikely to have varied much over this period. Conversations with company executives do not give us any reason to believe that the fixed costs associated with ordering steel are large, and no reason to suppose that they should have changed over our sample period. Similarly, storage costs appear to have been nearly constant over our sample period. Besides the labor and depreciation costs associated with operating the factory building in which the steel inventories are stored, the main cost of storage is the opportunity cost of capital as measured by the short term interest rate. The interest rate has been fairly constant over our sample period, and there haven't been any changes in the physical production/storage technology that we are aware of. On the other hand the firm's major "cost of production", the spot price of steel, has declined fairly dramatically for many of its products. Many of these price declines are a consequence of reduced world-wide steel demand following the Asian crisis together with possible "dumping" of steel by foreign producers in Russia, Japan, Brazil, and other countries.

More sophisticated econometric and economic modeling is required in order to assess the relative importance of the different explanations of the observation that orders are more volatile than sales. A major problem is created by the fact that we only observe spot prices for the firm's products on the days it placed orders, resulting in infrequent observations of spot prices at irregular time intervals. Due to econometric problems arising from endogenous sampling of these spot price processes, we have been careful not to draw any conclusions about the high frequency behavior of steel prices by simple interpolations of our endogenously sampled spot price series. In future work we will develop estimators that correct for this endogenous sampling problem, but in the meantime we have focused our analysis on characterizing the main facts about inventory stocks, orders, and sales for which problems of endogenous sampling problems do not arise. Our analysis has lead us to reject all of the main models that have been used to model inventory behavior.

In the next section we formulate and solve a dynamic programming model in which the optimal policy is a generalization of the classic (S, s) policy in which the (S, s) bands are declining functions of the current spot price of steel. This suggests that many of the stylized facts we have observed for this firm, particularly the observation that orders are more variable than sales and the instability in inventory/sales ratios, could be a consequence of an optimal inventory speculation strategy on the part of the firm. We confirm this in section 5 by presenting simulations of a calibrated version of this model that show that the predicted behavior of this model is strikingly similar to the behavior of this firm. In particular simulated data from the model exhibits all of the main features that we have observed in the product level data for this firm.

4 The Model

Our model is in the tradition of the dynamic (S, s) model pioneered by Arrow *et. al.* (1951) and Scarf (1959). We extend their models to allow the spot market price at which the firm purchases the commodity to follow a Markov process. The uncertainty and serial correlation in spot prices imply that a simple (S, s) strategy with fixed S and s thresholds is generally no longer optimal. The optimal inventory investment strategy in our extended model is a function of the spot market price for the commodity p as well as inventory on hand q. However we show that under fairly general conditions a *generalized* (S, s) *rule* is optimal. The firm's optimal trading strategy consists of a pair of *functions* $S(p)$ and $s(p)$ satisfying $s(p) \leq S(p)$. The lower band $s(p)$ is the firm's *order threshold,* i.e. it is optimal for the firm to order inventory whenever $q \leq s(p)$. The upper band $S(p)$ is the firm's *target inventory level,* i.e. whenever the firm places an order to replenish its inventory, it orders an amount sufficient to insure that inventory on hand (the sum of the current inventory plus new orders) equals $S(p)$.

Furthermore, the (S, s) bands are generally monotonically declining functions of p, reflecting the simple logic of commodity speculation, namely to "buy low and sell high". Low spot prices present an opportunity for the intermediary to make an expected profit by purchasing the commodity when it is cheap, storing it, and subsequently selling it at a higher price. While we assume that the firm could sell the commodity immediately with a positive expected mark-up over the current spot price, most of its profits are obtained from selling the commodity in subsequent periods when the gross of markup prices at which the intermediary sells to its customers have "recovered". It follows that the firm's desired holdings of the commodity are larger when spot prices are low than when spot prices are high.

Under certain circumstances the generalized (S, s) rule takes the form of a "bang-bang" strategy with price "thresholds": whenever the spot price falls below a price threshold the firm makes a speculative "bet" by placing large orders for steel. This results in large, infrequent discontinuous increases in inventory levels during periods of unusually low "bargain prices" in the spot market, behavior. This behavior is consistent with the observed instabilities and "regime shifts" in the inventory/sales ratio that we observed in our steel intermediary data in section 3. It is suboptimal for the intermediary to set a fixed, time-invariant inventory/sales target as is typically assumed in LQ models since this impedes the firm's ability to profit from buying low and selling high. Indeed when spot prices are sufficiently high the firm's desired inventory holdings can fall to zero and the incidence of stockouts rises precipitously. The high sales revenues and high opportunity costs of inventory holding during high price "regimes" make it optimal for the firm to liquidate rather than replenish its inventory holdings. Once fully liquidated, the firm optimally chooses to forego inventory investment until spot prices revert to lower levels, even though this comes at a high cost in terms of sacrificed sales revenue and a steep increase in the incidence of stockouts.

We derive these results from a relatively simple dynamic programming model of a generic durable commodity intermediary. These intermediaries do not not undertake any physical production processing: their main function is to buy the durable good at spot prices, store it, and sell it subsequently at a markup. We make a number of strong simplifying assumptions about the operations of these intermediaries that we intend to relax in future work. Our first simplification is a *decentralization hypothesis* that allows us to model the inventory investment decisions for each product traded by the intermediary separately. This separation is formally justified under the assumption that there are no technological interdependencies (storage externalities or joint capacity constraints) for the different products the intermediary carries, and the strong assumption that the price processes for the different products are conditionally independent. Under these assumptions it is easy to show that the firm's multi-product inventory investment problem decomposes into independent subproblems: essentially each individual product becomes a separate sub-firm or "profit center" which can be modeled in isolation from the others.

We need this assumption to break the "curse of dimensionality" associated with the firm's dynamic programming problem. In the absence of decentralization, a "central planner" within the firm would have to solve a single 4000+ dimensional dynamic programming problem (since each of the firm's 2000+ products requires a minimum of two continuous state variables, p and q). Since the complexity of continuous-state and continuous-control DP problems increases exponentially fast in the number of state and control variables, it is clear that such a problem would is far too large to solve using current hardware and software. However under the decentralization hypothesis, the firm's problem decomposes into 2000+ lower dimensional DP problems, each of which is tractable. Thus under the decentralization hypothesis it becomes feasible for us to model the entire firm. There are interesting questions about how this firm decentralizes its operations in practice (many of the sales and pricing decisions for individual products are delegated to the firm's sales agents), but we do not have space here to offer more in depth consideration of these issues.

We abstract from difficult issues connected with modeling endogenous price setting and price discrimination and assume that the firm charges a fixed markup over the current spot price of the commodity. We also abstract from taxes and the details of the firm's financial policy: these issues will be discussed in more detail below. Finally, we abstract from delivery lags and assume that the firm cannot backlog unfilled orders. Thus, whenever demand exceeds quantity on hand, the residual unfilled demand is lost. This fundamental "opportunity cost" motivates the firm to incur inventory holding costs, even in the absence of any stockout penalty capturing the "goodwill costs" of lost future sales due to alienated customers.

We model the intermediary as making decisions about buying and selling a durable commodity in discrete time. For concreteness, we consider a model with daily time intervals, matching the intervals in our data set. The state variables for the firm are (p_t, q_t) where q_t denotes the inventory on hand at the start of day t, and p_t denotes the per unit spot price at which the intermediary can purchase the commodity at day t. We assume $\{p_t\}$ evolves according to an exogenous Markov process with transition density $g(p_{t+1}|p_t)$. At the start of day t the intermediary observes (p_t, q_t)

and places an order $q_t^o \geq 0$ for immediate delivery of the commodity at the current spot price p_t. We assume that the intermediary sets a uniform sales price to its customers, p_t^s, via an exogenously specified markup rule over the current spot price p_t:

$$p_t^s = f(p_t) + \epsilon_t, \quad E\{\epsilon_t | p_t\} = 0. \tag{2}$$

For concreteness, in our model below we assume a linear markup rule, $f(p_t) = \alpha_0 + \alpha_1 p_t$ where α_0 and α_1 are positive constants.

After receiving q_t^o and setting p_t^s, the intermediary observes the quantity demanded of the commodity by the intermediary's customers, q_t^d. We assume that the distribution of q_t^d depends on the spot price p_t, reflecting a stochastic form of downward sloping demand. Let $H(q_t^d | p_t)$ denote the distribution of realized customer demand. We assume that H has support on $[0, \infty)$ with at most one mass point at $q^d = 0$. H is regular in the sense that for any continuous, bounded function G, the function $EG(p, q)$ is a twice continuously differentiable function of its arguments where EG is given by:

$$EG(p, q) = \int G(p, q, q^d) H(dq^d | p). \tag{3}$$

We allow H to have a mass point at 0, reflecting the event that the intermediary receives no customer orders on a given day t. Let $h(q^d | p)$ be the conditional density of sales given that $q^d > 0$. This is a density with support on the interval $(0, \infty)$. Let $\eta(p) = H(0|p)$ be the probability that $q^d = 0$. Then we can write H as follows:

$$H(q^d | p) = \eta(p) + [1 - \eta(p)] \int_0^{q^d} h(q' | p) dq'. \tag{4}$$

As noted above, we assume that there are no delivery lags and unfilled orders are not backlogged. This eliminates the need to carry additional state variables describing the status of pending deliveries and backlogged orders. We also assume that the firm does not behave strategically with regard to its sales to its customers. In addition to charging an exogenously specified markup as in equation 2, the firm does not withhold any inventory for future sale when there is a current demand for it. Thus, we assume that the intermediary meets the entire demand for its product in day t subject to the constraint that it can not sell more that the quantity it has on hand, the sum of beginning period inventory q_t and new orders q_t^o, $q_t + q_t^o$. Thus the intermediary's realized sales to customers in day t, q_t^s, is given by

$$q_t^s = \min \left[q_t + q_t^o, q_t^d \right]. \tag{5}$$

We assume the durable commodity is not subject to physical depreciation. Therefore the law of motion for start of period inventory holdings $\{q_t\}$ is given by:

$$q_{t+1} = q_t + q_t^o - q_t^s. \tag{6}$$

Since the quantity demanded has support on the $[0, \infty)$ interval, equation (5) implies that there is always a positive probability of unfilled demand $q_t^s < q_t^d$. We let $\delta(p, q+q^o)$ denote the probability of this event:

$$\delta(p, q + q^o) = 1 - H(q + q^o|p). \tag{7}$$

Whenever $q_t^d > q_t^s$, equations (5) and (6) imply that a *stockout* occurs, i.e. $q_{t+1} = 0$. Of course, the firm can minimize the probability of a stockout by insuring that quantity on hand, $q + q^o$, is sufficiently high. It is interesting to ask whether it would ever be optimal for the firm to set $q + q^o = 0$, which *maximizes* the probability of a stockout. This can be optimal in our model if spot prices and holding costs are sufficiently high.

We define the intermediary's expected sales revenue $ES(p, q, q^o)$ by:

$$\begin{aligned} ES(p,q,q^o) &= E\{p^s q^s | p, q, q^o\} \\ &= E\{p^s|p\} E\{q^s|p,q,q^o\} \end{aligned} \tag{8}$$

where:

$$E\{p^s|p\} = f(p) \tag{9}$$

and:

$$E\{q^s|p,q,q^o\} = [1-\eta(p)] \left[\int_o^{q+q^o} q^d h(q^d|p) dq^d + \delta(p, q+q^o)[q+q^o] \right]. \tag{10}$$

A key property to notice about the function ES is that it is symmetric in its q and q^o arguments: from the definitions given above we see that ES can be written as $ES(p, q + q^o)$. Thus, expected sales revenue depends only on the total quantity on hand $q + q^o$, rather than upon the separate values of q and q^o. This symmetry is a key to the proof of the optimality of the generalized (S, s) policy.

We turn now to specifying the per period profit function, which requires some additional assumptions about taxes and the intermediary's financial policy. We appeal to the Modigliani-Miller Theorem to argue that, in the absence of taxes, borrowing constraints, and other capital market imperfections, the intermediary's inventory investment policy should be unaffected by its financial policy. This allows us to abstract from the details about the way the intermediary actually finances its inventory holdings and allows us to conclude that regardless of whether its inventory holdings are financed by debt or retained earnings, the intermediary incurs an interest (opportunity) cost of inventory holdings equal to $r_t p_t(q_t + q_t^o)$ in day t where r_t denotes the spot interest rate at date t. However in our model the intermediary is an entrepreneur whose personal intertemporal discount factor $\beta \in (0,1)$ may not equal the current market discount factor $1/(1+r_t)$. This implies that the owner would like to borrow when β is less than $1/(1+r_t)$ and lend otherwise. Thus, financial policy does affect the firm's expected discounted profits even in the absence of taxes, borrowing constraints, and other capital market imperfections. Since the steel company will not disclose information about their financial policy, we assume the intermediary finances its inventory holdings out of retained earnings, incurring an opportunity cost of maintaining inventory level q_t equal to $r_t p_t q_t$. Furthermore, we assume r_t is fixed; $r_t = r$ for all t.[3]

We assume the intermediary incurs a cost of ordering inventory given by a function $c^o(q^o)$ which may be discontinuous at $q^o = 0$ but is twice continuously differentiable for $q^o > 0$. The discontinuity in c^o at $q^o = 0$ reflects possible fixed costs of placing orders. For concreteness, we will assume a simple fixed order cost,

$$c^o(q^o) = \begin{cases} F & \text{if} \quad q^o \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

This specification can be easily generalized to account for per unit shipping costs and quantity discounts. However in order to derive the optimality of a generalized (S,s) policy we need to assume that the derivative of c^o is constant for $q^o > 0$. For simplicity we assume this derivative is 0 in what follows below.

We assume that the intermediary incurs a physical storage cost $c^h(q)$ of holding inventory level q, where c^h is nondecreasing and twice continuously differentiable. The intermediary perceives a "goodwill cost" $\gamma \geq 0$, where γ represents the present value of lost profits from customers who switch to alternative suppliers in the event that $q^d > q + q^o$. Finally the intermediary has a maximum storage capacity equal to $\overline{q} \leq \infty$. Thus the intermediary's single-period profits π is

[3] The assumption of constant interest rates can be easily relaxed as far as the theoretical presentation of the model is concerned, however it does lead to an extra state variable that complicates the numerical solution of the model. In future work we plan to include r_t as a state variable to study the sensitivity of inventories to interest rate fluctuations, a topic of interest in studies of the role of inventories in macroeconomic fluctuations.

given by:

$$\pi(p_t, p_t^s, q_t^s, q_t, q_t^o) = p_t^s q_t^s - p_t q_t^o - r p_t(q_t + q_t^o) - c^o(q_t^o) - c^h(q_t + q_t^0) - \gamma I\{q_t^s = q_t + q_t^o\}. \quad (12)$$

Notice that our assumptions imply that the profit function π is also symmetrical in its q_t and q_t^o arguments and can be written as $\pi(p_t, p_t^s, q_t^s, q_t + q_t^o)$.

The intermediary's inventory investment behavior is governed by the decision rule:

$$q_t^o = q^o(p_t, q_t), \quad (13)$$

where the function q^o is the solution to:

$$V(p_t, q_t) = \max_{q^o} E\left\{\sum_{j=t}^{\infty} \beta^{(j-t)} \pi(p_j, p_j^s, q_j^s, q_j^o + q_j^s) \Big| p_t, q_t\right\}, \quad (14)$$

The value function $V(p,q)$ is given by the unique solution to Bellman's equation:

$$V(p,q) = \max_{0 \le q^o \le \overline{q} - q} \Big[W(p, q + q^o) - pq^o - c^o(q^o)\Big], \quad (15)$$

where:

$$W(p,q) \equiv \Big[ES(p,q) - rpq - c^h(q) - \gamma\delta(p, q + q^o) + \beta EV(p,q)\Big]. \quad (16)$$

and EV denotes the conditional expectation of V given by:

$$\begin{aligned} EV(p,q) &= \eta(p) \int_{p'} V(p', q) g(p'|p) dp' \\ &+ [1 - \eta(p)]\delta(p,q) \int_{p'} V(p', 0) g(p'|p) dp' \\ &+ [1 - \eta(p)] \int_{p'} \int_0^q V(p', q - q') h(q'|p) g(p'|p) dq' dp'. \end{aligned} \quad (17)$$

The optimal decision rule $q^o(p,q)$ is given by:

$$q^o(p,q) = \underset{0 \le q^o \le \overline{q} - q}{argmax} \Big[W(p, q + q^o) - pq^o - c^o(q^o)\Big], \quad (18)$$

We now formally define a generalized (S,s) policy and provide sufficient conditions for it to be optimal.

Definition 0: *A generalized (S,s) policy is a decision rule of the form:*

$$q^o(p,q) = \begin{cases} 0 & \text{if } \ q \ge s(p) \\ S(p) - q & \text{otherwise} \end{cases} \quad (19)$$

where S and s are functions satisfying $S(p) \geq s(p)$ for all p.

Given a decision rule $q^o(p,q)$ we can define the (S,s) bands as follows:

Definition 1: *The upper band of the generalized (S,s) policy is the function $S(p)$ defined by:*

$$S(p) = q^o(p,0), \tag{20}$$

where q^o is the optimal decision rule for the inventory problem (15).

Definition 2: *The lower band of the generalized (S,s) policy is the function $s(p)$ defined by:*

$$s(p) = inf_{0 \leq q}\{q | q^o(p,q) = 0\}. \tag{21}$$

Although these definitions appear circular, the following theorem shows that these are equivalent characterizations of the optimal decision rule:

Proposition: *Consider the function $W(p, q + q^o)$ defined in equation (16), where W is defined in terms of the unique solution V to Bellman's equation (15). If $W(p, q+q^o)$ is a strictly concave function of q^o for any p, then the firm's optimal inventory investment policy $q^o(p,q)$ takes the form of a generalized (S,s) rule. That is, there exist a pair of functions (S,s) satisfying $S(p) \geq s(p)$ where $S(p)$ is the desired or target inventory level and $s(p)$ is the inventory order threshold, i.e.*

$$q^o(p,q) = \begin{cases} 0 & \text{if} \quad q \geq s(p) \\ S(p) - q & \text{otherwise} \end{cases} \tag{22}$$

where $S(p)$ is given by:

$$S(p) = argmax_{0 \leq q^o \leq \overline{q} - q}\Big[W(p, q^o) - pq^o\Big] \tag{23}$$

and the lower inventory order limit, $s(p)$ is the value of q that makes the firm indifferent between ordering and not ordering more inventory:

$$s(p) = \inf_{q \geq 0}\{q | W(p,q) - pq \geq W(p, S(p)) - pS(p) - F\}. \tag{24}$$

Proof: First note that if $W(p,q)$ is strictly concave in q, then $W(p,q) - pq$ is also strictly concave in q. Next note that $q = S(p)$ satisfies the inequality $W(p,q) - pq \geq W(p, S(p)) - pS(p) - F$. Since $s(p)$ is defined as the smallest such q satisfying this inequality, we have $s(p) \leq S(p)$.

Now consider the case where $S(p) = 0$. From equation (23) we see that $S(p) = 0$ if and only if $W(p,0) \geq W(p,q) - pq$ for all $q \in [0, \overline{q}]$. In this case we also have $W(p,0) \geq W(p,q) - pq - F$ for all $q \in [0, \overline{q}]$. Equation (24) that $s(p) = 0$, and equation (22) implies that $q^o(p,q) = 0$, i.e. it is never optimal for the firm to order new inventory. We now need to show that this is indeed the optimal policy. The fact that $S(p) = 0$ implies that $W(p,q) - pq$ is maximized at $q = 0$. Since $W(p,q) - pq$ is strictly concave in q, if it is maximized at $q = 0$ then it must be strictly decreasing for $q > 0$. This implies that for any $q^o > 0$ we have $W(p, q + q^o) - p(q + q^o) < W(p,q) - pq$. From the definition of $q^o(p,q)$ in equation (18) we see that the optimizing value of q^o is 0, i.e. $q^o(p,q) = 0$.

Now consider the case where $S(p) > 0$. First we show that it is never optimal to order if $q \geq S(p)$. Since $W(p,q) - pq$ is strictly concave and is maximized at $q = S(p)$, if follows that this function is strictly decreasing for $q \geq S(p)$. Thus if $S(p) \leq q < q'$ we have $W(p,q) + pq > W(p,q') - pq'$ or

$$W(p,q) > W(p,q') - p(q' - q). \tag{25}$$

From the definition of $q^o(p,q)$ in equation (18) it follows that $q^o(p,q) = 0$.

Next we show that when $q < S(p)$ if it is optimal to order, the optimal order quantity must be $S(p) - q$. By the definition of $S(p)$ we have $W(p, q + q') - p(q + q') \leq W(p, S(p)) - pS(p)$, for all $q' \in [0, \overline{q} - q]$. Rearranging we have

$$W(p, q + q') - pq' - F \leq W(p, q + [S(p) - q]) - p[S(p) - q] - F, \tag{26}$$

which shows that if the firm were to order, the optimal order quantity is $S(p) - q$. However we need to consider the possibility that the optimal order quantity is 0. This will be the case if and only if

$$W(p,q) \geq W(p, q + [S(p) - q]) - p[S(p) - q] - F. \tag{27}$$

By the definition of $s(p)$, if $q \in [s(p), S(p)]$ we have $W(p,q) \geq W(p, S(p)) - pS(p) - F$. This is equivalent to inequality (26), from which we conclude that $q^o(p,q) = 0$. If $q \in [0, s(p))$ we have

$$W(p,q) < W(p, S(p)) - pS(p) - F = \max_{q' \in [0, \overline{q} - q]} [W(p, q + q') - pq' - c^o(q')], \tag{28}$$

from which we conclude that $q^o(p,q) = S(p) - q$. `q.e.d.`

The key to the proof of Proposition 1 is the symmetry and concavity of the function W as a function of $q + q^o$. The symmetry property, the fact that W depends on q and q^o only via their sum $q + q^o$, follows directly from the definition of W. The concavity property is an assumption that appears to be satisfied in the numerical examples we have computed in the next section. Ideally we would like to provide lower level assumptions on the primitives of the problem in order to provide checkable sufficient conditions for this property to hold. To establish more general results we conjecture that we will need to employ a generalized form of concavity known as K*-concavity* defined analogously to the K-convexity property used by Scarf (1960) to establish the optimality of the (S, s) policy in a cost-minimization formulation of the problem. We leave these extensions to future work.

We conclude this section by noting that the value function is linear in q with slope equal to p when $q < s(p)$. To see this, we simply substitute the form of the optimal decision rule (22) into the formula for V in Bellman's equation (15) to obtain:

$$V(p,q) = \begin{cases} W(p,S(p)) - p[S(p)-q] - F & \text{if} \quad q \le s(p) \\ W(p,q) & \text{otherwise} \end{cases} \tag{29}$$

Thus, V takes the form $V(p,q) = \gamma(p) + pq$ for $q \le s(p)$, which shows that the "shadow price" of an extra unit of inventory is p. The intuition for this simple result is straightforward: if the firm has an extra unit of q when $q \le s(p)$ then it needs to order one fewer unit in order to attain its target inventory level $S(p)$. The savings from ordering one fewer unit of inventory is simply the current spot price of the commodity, p. When $q > s(p)$ the shadow price of inventory is no longer equal to p. We do know that since $q = S(p)$ maximizes $W(p,q) - pq$, we must have $\partial W(p,q)/\partial q = p$ when $q = S(p)$. If W is strictly concave, this implies that $\partial W(p,q)/\partial q > p$ when $q \in (s(p), S(p)]$ and $\partial W(p,q)/\partial q < p$ when $q \in (S(p), \overline{q}]$. Thus, there is a kink in V function at the inventory order threshold, $q = s(p)$. Under certain conditions this kink may be inconsistent with our assumption that W is strictly concave in q. Thus we warn the reader that further work is required to establish sufficient conditions for the optimality of the generalized (S, s) policy. We simply note that the generalized (S, s) policy emerged from our numerical solutions in section 5 even though the generic policy iteration method we used to compute the optimal decision rule did not presume that a generalized (S, s) would be optimal. Further, as noted above, plots of our numerically computed $W(p,q)$ functions indicate that this function is strictly concave in q in all of the examples we have analyzed so far.

5 A Calibrated Example

To illustrate the behavior implied by our model we solved a discrete approximation of (15) numerically under the following assumptions. We assumed that the daily interest rate is time-invariant and equal to $r = .05/261$.[4] We assumed the firm uses the sales price markup rule $p_t^s = 0.9 + 1.06p_t$ and spot prices $\{p_t\}$ evolve according to a truncated lognormal $AR(1)$ process:

$$\log(p_{t+1}) = \mu_p + \lambda_p \log(p_t) + \epsilon_t \tag{30}$$

where $\mu_p = .06$, $\lambda_p = .98$, and $\{\epsilon_t\}$ is and *IID* $N(0, \sigma_p^2)$ sequence, with $\sigma_p^2 = 8.6510^{-5}$. The upper and lower truncation bounds on this process were chosen to be $(16, 25)$ which are beyond the minimum and maximum spot purchase prices observed in our sample or in long run simulations of the untruncated version of this process. These values yield a order price process with an invariant distribution with mean of 20.5 cents per pound and a standard deviation of 1.00 cents per pound. Given the markup, the mean and standard deviation of the sell price process are 22.6 and 1.06, respectively. The means of these price processes are in the range of means reported in table 1. The standard deviations are below those reported in table 1; but again, we are silent on the issue of price discrimination.

We assumed that quantity demanded, q_t^d, is a mixed truncated lognormal distribution conditional on p_t. That is, with probability .5 $q_t^d = 0$, and with probability .5 q_t^d is a draw from a truncated lognormal distribution with location parameter $\mu_q(p) = 4.43 - .7\log(p_t)$ and standard deviation parameter $\sigma_q = 1.081$. These parameters yield a stationary distribution for q_t^d (conditional on $q_t^d > 0$) with conditional mean equal to 18.3 and conditional standard deviation equal to 28.2. The units of the quantity variables are in 1,000's of pounds. The first two moments of the quantity demanded process are in the range of the moments reported in columns (7) and (9) in table 2.

We assumed that goodwill costs of stockouts γ and physical holding costs were zero, $c^h(q_t) = 0$, and that the fixed order cost is equal to \$50, i.e. $c^o(0) = 0$ and $c^o(q^o) = \$50$ if $q^o > 0$. Finally, we assumed that the firm owner's personal subjective discount factor was given (on a daily basis) by $\beta = 1/(1 + .05/261)$; so $\beta = 1/(1 + r)$.

We solved for the optimal inventory investment rule by the method of policy function iteration which computes a discrete approximation to the value function $V(p, q)$ as the unique fixed point to the Bellman equation, (15). We used a uniform discretization of the (p, q) state space to approximate the continuous DP problem by the solution to a finite state problem with 750 grid points (15 in the p dimension and 50 in the q dimension). The grid points are evenly spaced

[4] We assumed there are $365 - (2 \times 52)$ business days in a year.

along the p dimension. Along the q dimension, the distance between the grid points increases as q increases. Thus the grid point are more densely spaced in the region where there is more curvature in the decision rule. Although the state variables were discretized, we treated the control variable q^o as a continuous variable subject to the constraint that $0 \leq q^o \leq \bar{q} - q$. Policy iteration is not guaranteed to converge in continuous choice problems such as this one; but for this example, the algorithm converged in 39 iterations. Using the values computed at these 750 grid points we produced continuous approximations to the value function and decision rule via multi-linear interpolation.

As can be seen from Bellman's equation (15), the policy improvement step requires the solution of a constrained optimization problem involving the two functions $ES(p,q)$ and $EV(p,q)$, each of which is a conditional expectation of functions of two continuous variables (sales, $p^s q^s$, and the value function, $V(p,q)$). Since no analytic solutions to these conditional expectations exist, we resorted to numerical integration. We experimented with two different methods of numerical integration, a "quadrature" approach that approximates EV by a probability weighted sum:

$$\hat{EV}(p,q) = \frac{1}{N_p}\frac{1}{N_q}\sum_{i=1}^{N_p}\sum_{j=1}^{N_q} I\{q_j \leq q\}\hat{V}(p_i, q - q_j)h(p_j|p_i)g(p_i|p) \tag{31}$$

where $h(q_j|p_i)$ is a discretized approximation to the conditional probability density $h(q|p,q)$ and $g(p_i|p)$ is a discretized approximation to the transition probability density $g(p'|p)$. Further adjustments to this formula were made in order that $\hat{EV}(p,q)$ reflects that mass points on stockouts and zero sales as in equation (17). A second method of approximating EV was a "quasi monte carlo, probability integral transform method" (MC-PIT) given by

$$\hat{EV}(p,q) = \frac{1}{N}\sum_{i=1}^{N}\hat{V}(\tilde{p}_i, q - \tilde{q}_i) \tag{32}$$

where $\{\tilde{p}_i, \tilde{q}_i\}$ are draws from the density $h(q'|p',q)g(p'|p)$ computed from uniformly distributed draws $\{\tilde{u}_{1,i}, \tilde{u}_{2,i}\}$ from the unit square, $[0,1]^2$ via the probability integral transform method. Instead of using pseudo-random random draws for $\{\tilde{u}_{1,i}, \tilde{u}_{2,i}\}$ we obtained acceleration using *Generalized Faure sequences* (also known as *Tezuka sequences*). Using number theoretic methods (see, e.g. Neiderreiter 1992, or Tezuka, 1995), one can prove that for certain classes of integrands, the convergence of monte carlo methods based on deterministic *low discrepancy sequences* is $O(\log(N)^d/N)$ (where d is the dimension of the integrand and N is the number of points), whereas traditional monte carlo methods converge at rate $O_p(1/\sqrt{N})$. These favorable rates of convergence have been observed in practice, see e.g. Papageorgiu and Traub (1997).[5] The density $h(q'|p,q)$ is the conditional density of q' given that $q' \leq q$,

$$h(q'|p,q) = \frac{h(q'|p)}{1 - \delta(p,q)} \tag{33}$$

where $\delta(p,q) = \Pr\{q' > q|p\} = 1 - H(q|p)$. As in the quadrature method, we adjusted the MC-PIT formula (32) to account for mass points corresponding to stockouts and zero sales. We found that the optimizing solutions for q^o were sensitive to the way the functions ES and EV are approximated. It was critical to use methods that provide accurate approximations both their levels and their derivatives, since the latter determine the first order conditions for a constrained optimum for q^o. In regions where the value function is nearly flat in q^o, small inaccuracies in the estimated derivatives can create large instabilities in the estimated value of q^o. The solutions are also sensitive to the discretization of the p and q axes, and the number of points used in the discretization. Through a fair amount of experimentation we have developed numerical procedures that we trust. In particular different approximation methods for computing $\hat{ES}$ and $\hat{EV}$ produced nearly identical results.

Figures 15-18 present the optimal decision rule q^o as a function of p and q and the associated expected sales, value functions and (S,s) bands. Note that our solution technique does not exploit our prior knowledge about the form of the decision rule. The computed value function appear to be nearly linearly increasing in current inventory q. At low inventory levels (in regions the firm is expecting to buy steel), $V(p,q)$ is decreasing in p, whereas at high values of q, (in regions the firm is expecting to not buy but just sell steel) V is increasing in p. The kink at $s(p)$ is not apparent at this level of resolution. These results are consistent with the discussion in the previous section. The optimal decision rule is decreasing in both p and q, although it generally decreases faster in p than in q. In particular when $q^o(p,q) > 0$, $\partial q^o(p,q)/\partial q = -1$ which is consistent with the prediction of the generalized (S,s) rule that $q^o(p,q) = S(p) - q$.

Figure 17 shows the generalized $(S(p), s(p))$ bands implied by our model. The set of order limit points, $s(p)$, is the curve on the (q,p) plane where the $q^o(p,q)$ surface intersects the plane at $q^o = 0$. The set of target inventory points, $S(p)$, is the curve on the (q_o, p) plane where the $q^o(q,p)$ surface intersects the plane at $q = 0$. These bands are plotted in figure 18. Due to the

[5] We are grateful to Joseph Traub for providing the FINDER software co-authored with A.F. Papageorgiu that generated the low discrepancy sequences used in this study.

fixed costs of ordering ($50), the $S(p)$ band is strictly above the $s(p)$ band although the difference between the two bands decreases as the price increases. In other words, the order size at s is a decreasing function of the price.

As can be seen in figure 18, when the price is near the lower truncation price bound (16), the firm wishes to hold the upper bound of inventories (5 million pounds).[6] This makes sense because the firm knows prices cannot go any lower. The firm cannot make a capital loss on any steel purchased at the lower bound. Nevertheless, this boundary issue is not a major concern since the firm very rarely ever observes prices in this region; this region is over 4 standard deviations away from the mean of the price process.

Figures 19- 22 present the results from a single stochastic simulation of the DP model for 434 periods. At first glance, the simulated series look quite similar to the actual data. Figure 19 shows the time series for inventory levels, and there appears to be multiple regimes. During the first 275 days of the simulation, inventory levels are centered around 200,000 pounds. This average level matches the average level of inventory holdings for product 13 and the first 200 days of product 4. Starting around day 275, the firm enters a "high inventory regime" with the simulated inventory levels reaching a peak over 1,500,000 pounds. This peak is consistent with observed levels of inventories for products 2 and 4. During this high inventory regime, days supply reaches almost 350. The transition from the low to high inventory regime occurs when the order price falls below a threshold value. Later, as prices begin to rise (from day 360 to the end of the simulation) the firm lets its inventory holdings gradually fall.

The high- and low-regime property of the optimal inventory holdings can be seen from the decision rule, $q^o(q,p)$. In figure 17, $q^o(q,p)$ is sharply decreasing in p when $q^o(q,p) > 0$. This occurs for two reasons. First, the firm takes advantage of low order prices to build up inventories knowing that it will be able to capture a capital gain on its inventory holdings when prices rise. Second, the firm faces a downward sloping demand curve for its product; so when the price falls, quantity demanded, q^d, rises and the firm will hold more inventories to accommodate the increase in demand.

The simulation results are consistent with this intuition. Figure 21 presents the censored and uncensored order and sales price series. In this graph, the solid line is the analogue of what we observe in our dataset, we linearly interpolate between the prices at which transactions took place;

[6] In figure 17 we plot the decision rule for prices between 17 and 24 to make the graph more easily readable.

the dotted line includes the unobserved prices at which no transactions occurred. During the high inventory regimes (e.g. days 260-434) the firm opportunisticly bought at the troughs. This result is of course sensitive to our specification of the law of motion of the price process, equation (30).

The exogenous price and quantity demanded processes implied that the firm sold steel on 210 days at average price of 22.67 during the simulation period. The decision rule dictated that the firm purchased steel on 26 days at an average price of 20.04. The average order size was $116,000$ pounds. And the conditional standard deviation of the order size was 62.3. These implied moments from the model are consistent with the moments we observe in the data. Finally the ratio of the standard deviation of orders to the standard deviation of sales for this simulation is 2.4. So the model does imply that orders are more volatile than sales. Longer simulations generate similar results.

These results are also qualitatively similar to the actual inventory time series for our firm in figures 3-14. Our DP model display regime shifts in the inventory levels and days supply of inventory with little evidence of a single fixed inventory/sales target; however, we have not systematically searched over the parameter space to ensure that our DP model captures the full volatility and magnitude in these regime shifts. In our individual product data, we also see very large orders occurring when prices hit what appear to be record lows. However comparing figures 4, 8, and 12 with figure 20, we see that the DP model generates fewer small size orders than we observe in the data. This suggests that perhaps the fixed order cost is too large; however when we set the fixed cost to zero, we get the counterfactual result that with prices are high, the firm tightly matches orders to sales, setting orders almost every period equal to last period's sales. Finally the model does imply occasional stockouts; in the simulation presented, the firm stocks out on day 108 when quantity demanded was unusually large (over 1 million pounds) and current inventories were relatively low (around 250,000 pounds).

We conclude that cost shocks in the form of serially correlated spot prices in the steel market is the principal explanation for the observed volatility in inventory/sales ratios and the fact that orders are more volatile than sales. We believe this simple model provides a promising starting point for more rigorous estimation and testing using more advanced econometric methods.

6 Concluding Remarks

This paper has presented a new data set containing high quality, high frequency observations on product-level inventory investment by a U.S. steel wholesaler. Our empirical analysis yielded six conclusions about inventory investment and price setting by this firm: 1) orders are more volatile than sales, 2) orders are made infrequently, 3) there is considerable volatility in order levels, 4) there is no stable inventory/sale relationship, 5) there is considerable volatility in sales prices consistent with price discrimination, and 6) inventory stockouts occur relatively frequently, especially during periods of high commodity prices when inventory holdings are low. We showed that the standard versions of the (S, s) model, production smoothing models, and LQ models with target inventory/sales ratios are incapable of explaining these facts. We introduced a generic model of optimal inventory speculation by durable commodity intermediaries and showed that the optimal inventory investment strategy takes the form of a generalized (S, s) policy where the S and s bands are declining functions of the spot price of the commodity. Via simulations of a calibrated version of our DP model, we demonstrated that the firm's behavior at the product level can be well approximated by an optimal trading strategy. We employed a novel continuous version of Howard's policy iteration algorithm to solve a two-dimensional nonlinear infinite horizon dynamic programming problem with continuous state and control variables that are subject to frequently binding inequality constraints. The predicted behavior from the generalized (S, s) rule appears to explain a number of different aspects of inventory investment behavior by our steel wholesaler, including highly variable inventory/sales ratios and occasional stockouts during low inventory regimes when the spot price for steel is relatively high.

In future work we plan to undertake more rigorous econometric estimation and testing of our generalized (S, s) model which will account for difficult problems of "dynamic selectivity bias" arising from endogenous sampling of the prices at which the firm purchases inventory. We also plan to extend the model to allow for additional state and control variables such as the firm's sales price p_t and the interest rate r_t. The former will allow us to study endogenous price determination and price discrimination, whereas the latter will allow us to study the impact of monetary policy on inventory investment as a potential propagating mechanism in business cycles. In doing so, we will need to address some difficult issues connected with the curse of dimensionality underlying the solution of high dimensional DP problems such as the one considered in our paper. Recent progress in this area by Rust (1997, 1998) and Rust, Traub, and Woźniakowski (1998) make us optimistic about the prospect for solving these larger and more realistic models. Finally, we plan to study aggregate our micro model to study endogenous price determination for the commodity market as a whole, in order to determine whether the no arbitrage conditions assumed in the rational expectations commodity price model of Williams and Wright can be derived from microfoundations in a market where there is considerable price dispersion and frictions despite the apparent homogeneity of the product.

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product (1)	# order days (2)	mean order price (3)	std order price (4)	# sell days (5)	mean sell price (6)	std sell price (7)	std(order price)/ std(sell price) (8)
1	46	20.23	2.80	213	22.24	1.99	1.40
2	61	19.54	1.27	314	22.12	1.01	1.27
3	4	19.27	0.22	114	21.93	1.43	0.16
4	60	19.83	1.41	286	22.11	1.23	1.15
5	26	20.20	1.58	88	22.21	1.21	1.30
6	38	20.05	1.63	190	22.64	1.38	1.19
7	13	20.57	3.48	46	22.36	1.58	2.21
8	9	21.01	3.20	38	23.53	1.01	3.17
9	23	21.25	2.52	95	23.63	1.05	2.41
10	47	21.96	2.88	176	23.86	1.16	2.49
11	8	21.98	2.84	11	23.69	0.75	3.76
12	21	21.82	2.99	66	24.14	1.03	2.90
13	31	21.58	3.10	97	24.17	1.19	2.61
14	21	21.44	2.19	40	24.36	1.47	1.49
15	24	21.66	2.48	45	24.53	1.93	1.29
16	11	20.90	2.56	15	25.22	1.01	2.52
17	4	24.78	2.90	7	25.34	0.68	4.24
18	5	23.99	0.24	9	26.71	1.10	0.22

Table 1: First and Second Moments of Prices

There are 434 business days in the sample period. Column (2) reports the number of days the firm made one or more orders. Likewise column (5) reports the number of days one or more sales were made. Columns (3), (4), (6), and (7) are in cents per pound.

product (1)	mean order (2)	mean (o|o>0) (3)	std order (4)	std (o|o>0) (5)	mean sale (6)	mean (s|s>0) (7)	std sale (8)	std (s|s>0) (9)	std(o|o>0)/ std(s|s>0) (10)
1	8.61	81.43	45.78	118.95	5.99	12.29	10.29	11.82	10.06
2	24.34	173.54	122.75	287.52	19.78	27.23	23.80	24.01	11.97
3	1.70	184.41	18.31	51.44	3.19	12.19	10.04	16.64	3.09
4	25.63	185.78	149.33	365.74	21.49	33.04	39.71	45.21	8.09
5	2.99	50.05	21.12	72.78	2.52	12.45	7.15	11.39	6.39
6	9.33	106.83	47.45	125.36	8.34	19.19	14.21	16.02	7.83
7	1.53	51.15	12.63	54.95	1.55	14.65	5.49	9.73	5.65
8	1.12	54.27	10.26	49.65	1.07	12.98	4.13	7.23	6.87
9	5.05	95.46	30.73	98.04	3.18	14.56	7.43	9.35	10.48
10	14.33	132.64	65.96	158.15	11.06	27.33	19.90	23.14	6.83
11	0.51	27.77	4.08	12.86	0.41	16.41	2.72	5.62	2.29
12	4.33	89.61	28.55	98.28	3.14	21.04	9.23	13.99	7.03
13	6.68	93.80	36.42	103.66	5.78	26.21	17.66	29.74	3.48
14	3.64	75.30	24.00	82.67	2.23	24.26	9.18	19.75	4.19
15	5.50	99.65	35.03	115.60	3.47	33.54	15.90	38.22	3.02
16	2.83	111.98	23.53	92.61	1.03	29.73	6.28	17.59	5.27
17	0.95	102.92	13.94	118.84	0.32	19.60	3.46	20.58	5.78
18	1.56	135.91	22.11	173.77	0.54	26.14	3.72	0.00	∞
aggregate	120.62	274.70	507.52	738.79	95.10	101.90	81.63	80.30	9.20

Table 2: First and Second Moments of Quantities

Columns (2)-(9) are in 1,000's of pounds.

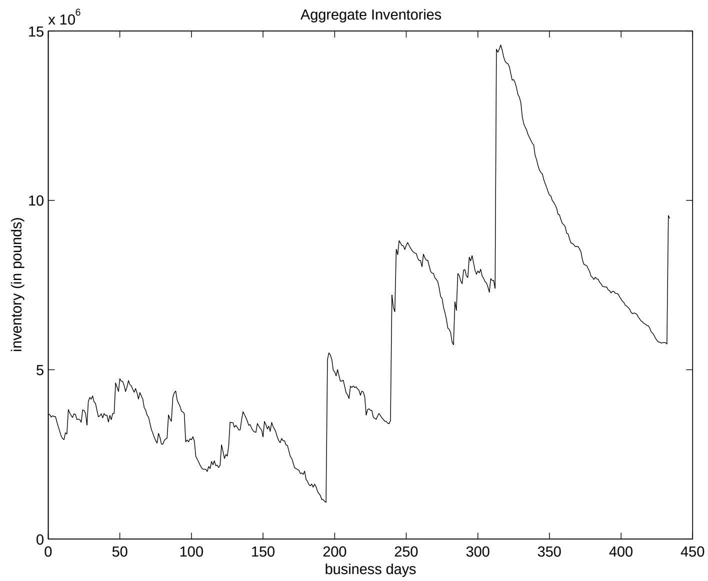

Figure 1: Aggregate inventory holdings for the eighteen products studied.

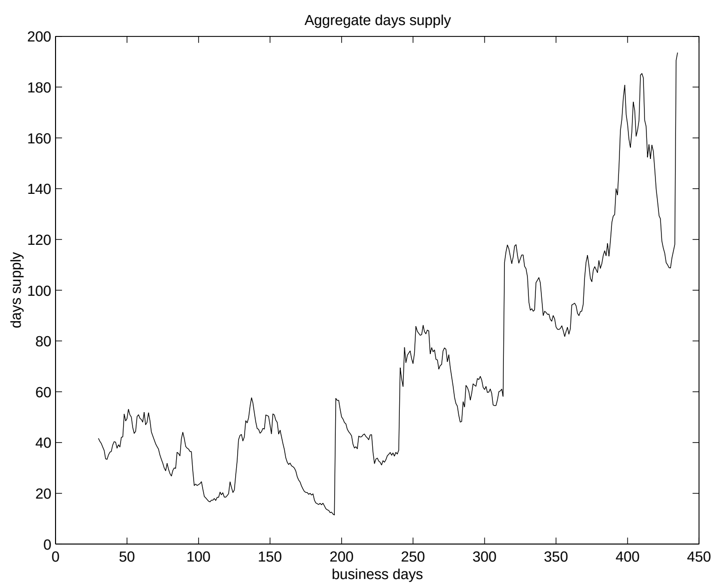

Figure 2: Aggregate days-supply for the eighteen products studied (in business days).

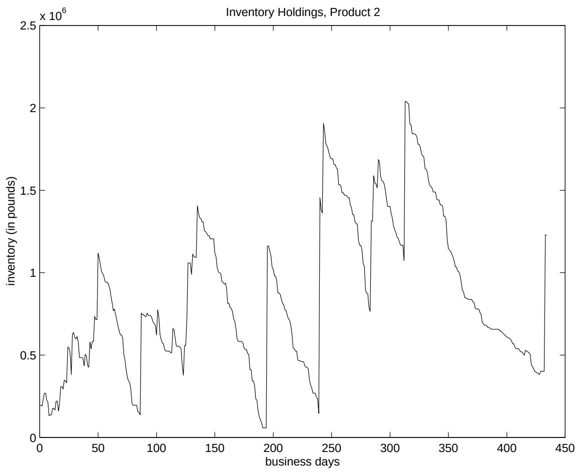

Figure 3: Times series plot of the inventory for product 2.

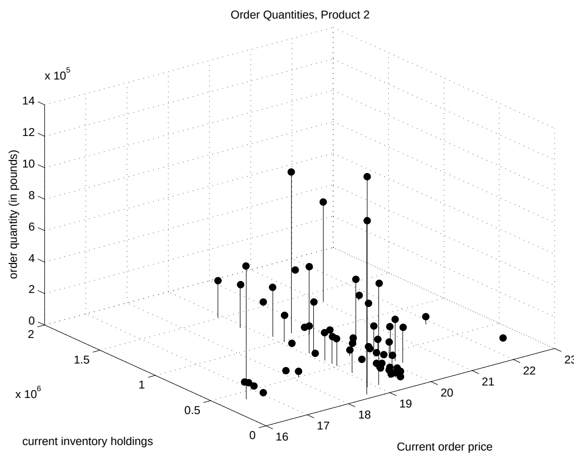

Figure 4: Size of purchases for product 2 as a function current inventory holdings and the buy price.

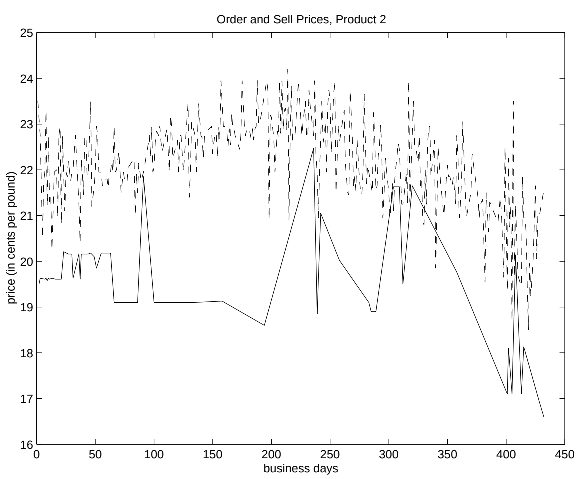

Figure 5: Order prices (solid line) and sell prices (dashed line) for product 2.

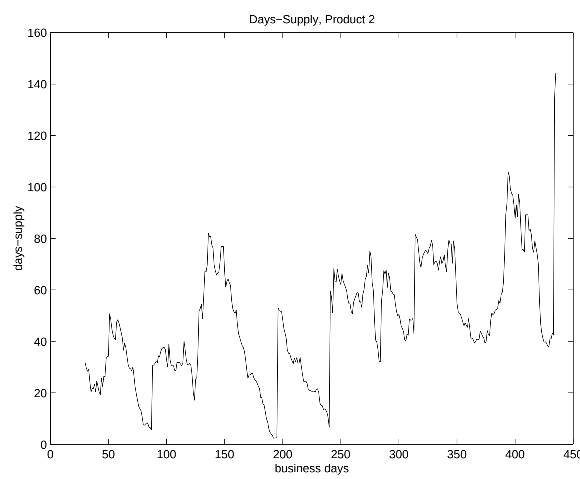

Figure 6: Days-supply of inventory for product 2 (in business days).

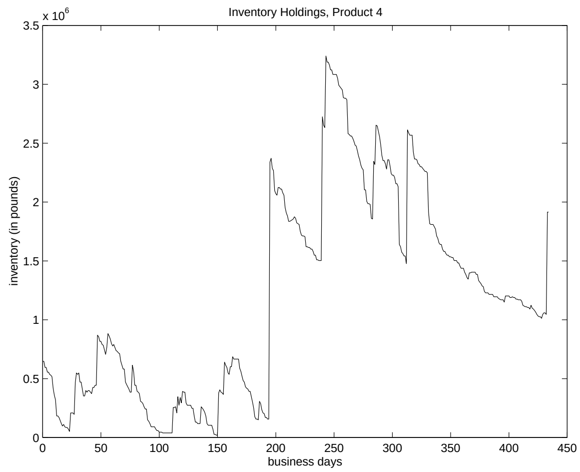

Figure 7: Times series plot of the inventory for product 4.

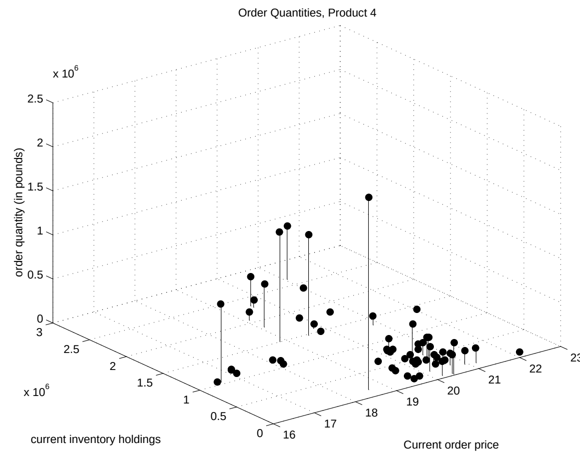

Figure 8: Size of purchases for product 4 as a function current inventory holdings and the buy price.

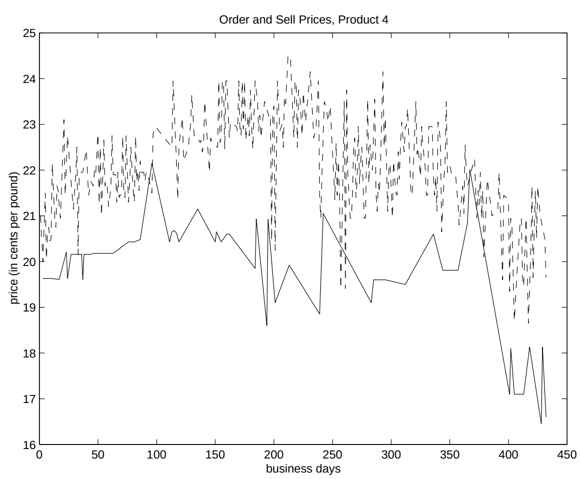

Figure 9: Order prices (solid line) and sell prices (dashed line) for product 4.

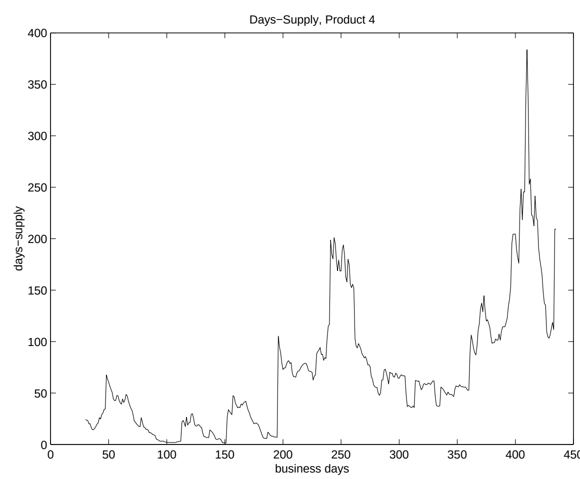

Figure 10: Days-supply of inventory for product 4 (in business days).

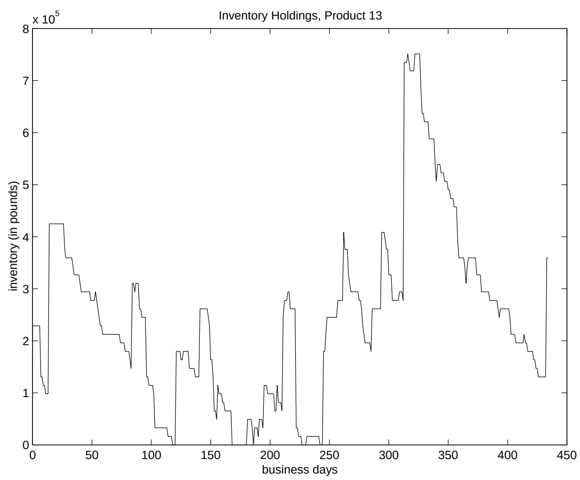

Figure 11: Times series plot of the inventory for product 13.

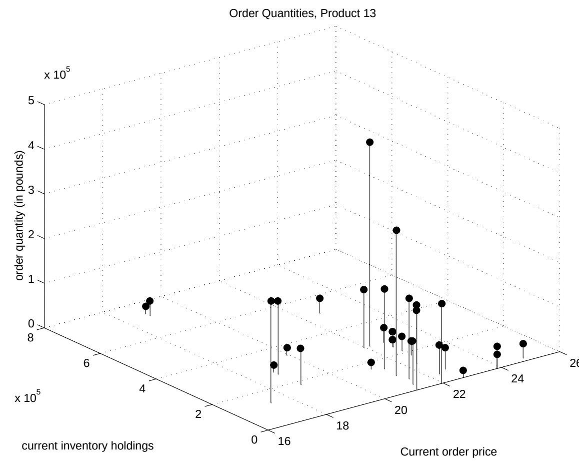

Figure 12: Size of purchases for product 13 as a function current inventory holdings and the buy price.

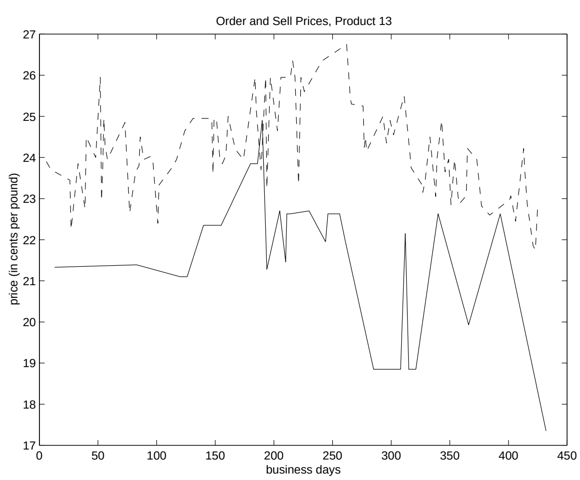

Figure 13: Order prices (solid line) and sell prices (dashed line) for product 13.

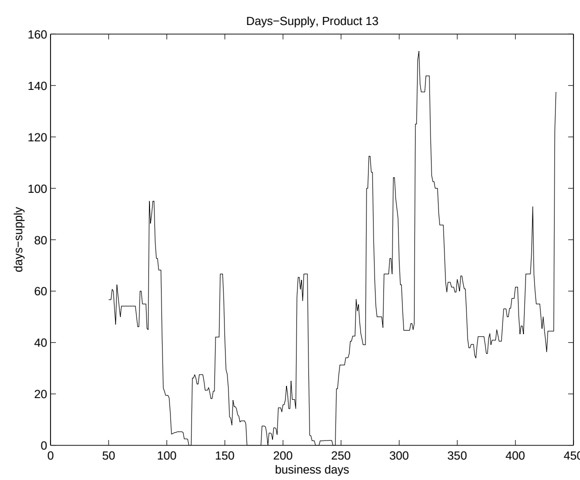

Figure 14: Days-supply of inventory for product 13 (in business days).

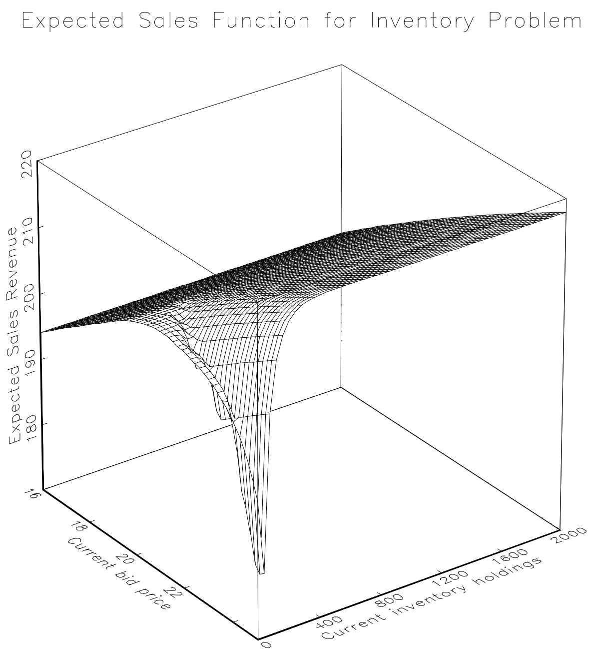

Figure 15: Expected sales revenue, ES for the calibrated example.

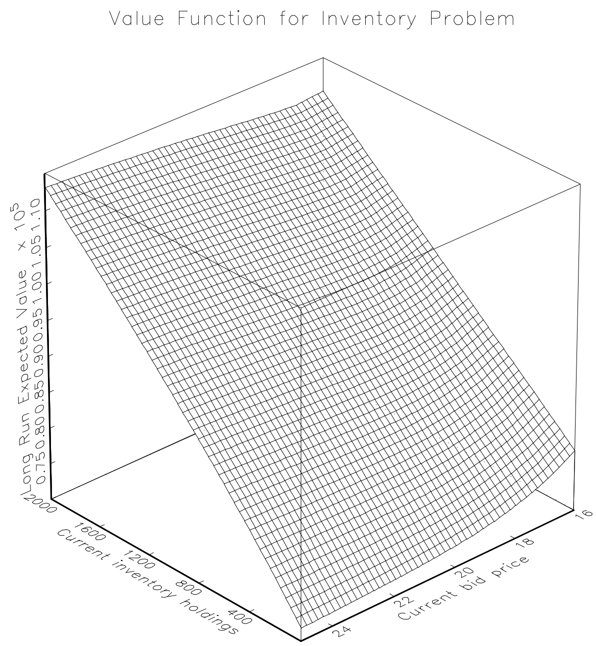

Figure 16: The value function, $V(q,p)$ for the calibrated example.

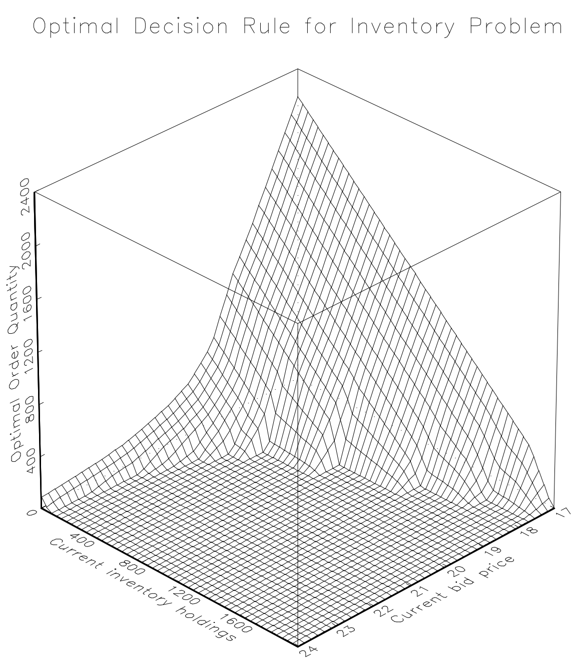

Figure 17: Decision rule, $q^o(q,p)$, for the calibrated example.

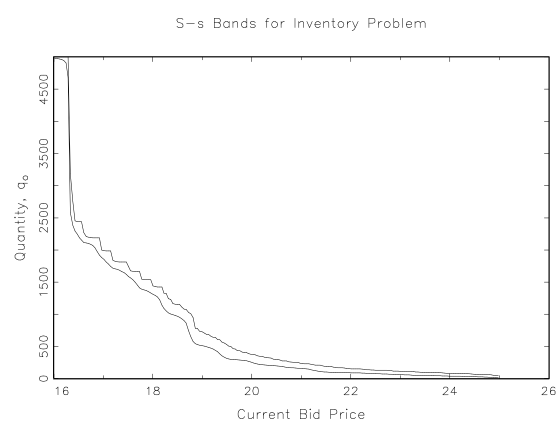

Figure 18: $S(p)$ and $s(p)$ for the calibrated example.

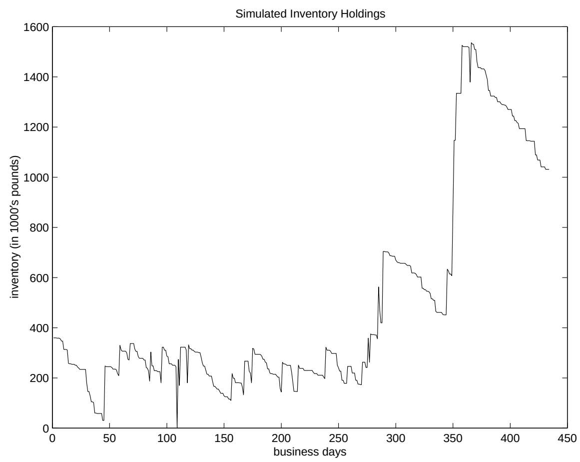

Figure 19: Simulated inventory holdings

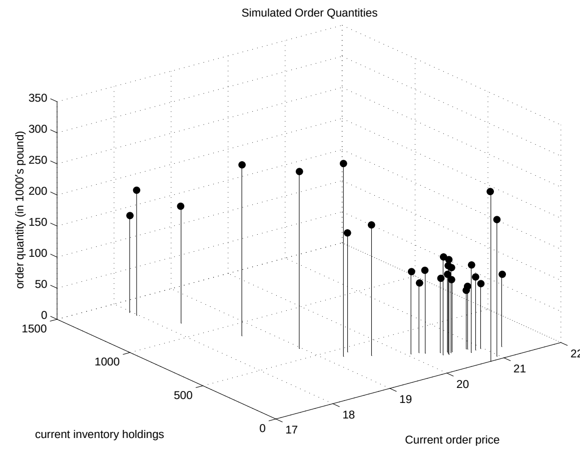

Figure 20: Simulated orders as a function current inventory holdings and buy price.

Figure 21: Censored (solid line) and Uncensored (dotted line) order and sales prices from the simulation.

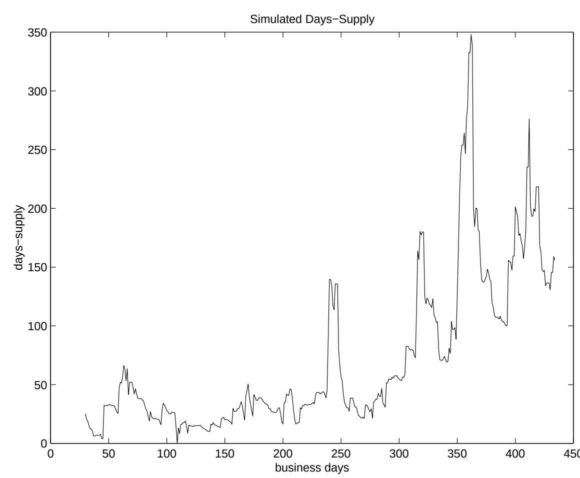

Figure 22: Simulated days-supply of inventory (in business days).