Monetary Equilibrium from an Initial State: the Case Without Discounting*

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Abstract

This paper studies the existence of single-price price equilibrium from a given initial distribution of money holdings in a search-theoretic model of money where agents have no time preference. The model is similar to recent models (Green and Zhou [2] and Zhou [5]) of search economies with no constraints on money inventories, except that here money is modeled as indivisible and traders are assumed to have overtaking-criterion preferences rather than discounting. The equilibrium concept is dynamic equilibrium from an initial distribution of money holdings rather than steady-state equilibrium (which possibly might not be reachable from an initial state) which was studied earlier. In this environment, under some mild conditions on the initial distribution, single-price equilibrium always exists. More precisely, there is an equilibrium path, along which agents trade at the same price, and the money-holdings distribution converges asymptotically to a unique geometric distribution.

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1. Introduction

Models of environments where traders are randomly matched in pairs, in which "double coincidence of wants" is absent, have recently undergone rapid development. They have already become an important class of models in the foundations of monetary economics.

Early versions of random-matching models specified that a trader can hold only a single unit of money, which is taken to come in indivisible units. This assumption makes the model highly artificial, and in particular it is responsible for some very peculiar results about welfare. In previous work (Green and Zhou [2], Zhou [5]), we have developed a random-matching model with divisible money and without inventory constraint. Surprisingly, we have found that the model possesses a continuum of steady-state, single-price, equilibria (for essentially any parameter values where equilibrium exists at all) that support distinct real allocations yielding different levels of welfare.

Our primary reason for undertaking the present research is to understand more clearly the significance of this indeterminacy of steady-state, single-price, equilibrium. On one view, the indeterminacy might be unimportant because possibly only one (or perhaps finitely many) of the equilibria could be reached from a given initial state of the economy. To examine this view, we need to define and analyze monetary equilibrium from an initial state. Moreover, to examine the possibility that several distinct equilibria from the same initial state might exist, we need to study nonstationary equilibria. This study is begun in the present paper, which reports research in progress.

In particular, in the current draft of this paper, we study equilibrium with indivisible money, although without an inventory constraint on money holdings. Also, we consider only the limiting case of an economy with perfectly patient traders, whose preferences we model in an overtaking-criterion framework. We restrict attention to this framework for two reasons. It is analytically more tractable than the analogous framework with discounting studied in Zhou [5], and it has the feature that all stationary Markov-perfect equilibria are single-price equilibria (which we establish in a companion paper, Green and Zhou [3]).

We find that single-price equilibrium from an initial state exists under a mild condition about the initial money-holdings distribution, and that this equilibrium is asymptotically stationary. Furthermore, for a given single-price steady state, we characterize a range of initial distributions from which single-price equilibria converge to that steady state. We conjecture that the results we obtain here can be generalized to a divisible-money environment. If that is so, then we will be able to show that single-price equilibrium from an initial state, as well as steady-state equilibrium, is indeterminate in this environment. The reason is that there would be a distinct equilibrium path corresponding to each alternative level of aggregate real money balances in the economy.

Even in its present interim state, our research has some intrinsic interest. The exclusive focus on stationary equilibria in study of random-matching monetary economies to date has made it difficult to draw comparisons between results about these models and results about models in a Walrasian spirit, in which equilibrium is defined with respect to an initial situation that generically is not stationary. Moreover, since adoption of a new monetary policy would typically be expected to move the economy from one steady state to another, the eventual usefulness of random-matching models to address policy questions in monetary economics is likely to be enhanced greatly by being able to characterize and analyze nonstationary equilibria beginning from an exogenous initial situation. For these reasons, the study of nonstationary equilibrium in this paper enhances the usefulness of the random-matching framework for money.

2. The Environment

Economic activity occurs at dates $0, 1, 2, \ldots$. Agents are infinitely lived, and they are nonatomic. For convenience, we assume that the measure of the set of all agents is one. Each agent has a type in $(0,1]$. The mapping from the agents to their types is a uniformly distributed random variable, independent of all other random variables in the model. Similarly, there is a continuum of differentiated goods, each indexed by a number $j \in (0,1]$. These goods are indivisible and nonstorable. Production is specialized such that only the agents of type i can produce one unit of "brand" i good at a cost c each period. Each agent consumes half of the brands in the economy; agent i consumes goods $j \in (i, \text{mod}(i+\frac{1}{2})]$ (for example, agent 0.3 consumes goods $j \in (0.3, 0.8]$, and agent 0.7 consumes goods $j \in (0.7,1] \cup (0,0.2]$). From consuming one unit of a desired good, an agent derives an instantaneous utility u. In addition to the production/consumption goods, there is a fiat money.[1] Money comes in indivisible units, and an agent can costless hold any number of units of money. The total nominal stock of money remains constant at M units per capita. We assume that agents do not discount. Their preferences are characterized by an overtaking criterion with respect to expected utility, which will be formalized below.

Agents randomly meet pairwise each period. By the assumed pattern of specialization in production and consumption, there is no double coincidence of wants in any pairwise meeting. Each agent meets a producer of his consumption goods with probability one-half, and a consumer of his production goods with probability one-half. So, every meeting is between a potential buyer and seller. Consumption goods cannot be used as a commodity money because they are nonstorable, so money is the only medium of exchange available. An agent is characterized by his type and the amount of money he holds. Within a pairwise meeting, each agent observes the

[1] Logically, fiat money is an economy-wide accounting system that satisfies restrictions such as we now describe. It is customary in the money/search literature, but not logically necessary, to interpret fiat money as some physical object.

other's type, but not the trading partner's money holding and trading history. They cannot communicate about this information either. However, the economy-wide money-holdings distribution is common knowledge. For simplicity, we assume that each transaction occurs according to the following simultaneous-move game. The potential buyer and seller submit a bid and offer respectively. Trade occurs if and only if the bid is at least as high as the offer, and in that case, the buyer pays the seller's offer price.[2]

3. The Definition of Equilibrium

The domain of agents' money holdings is $\mathbb{N}$. Let Δ be the space of probability measures on infinite-dimensional probability simplex, $\Delta = \{p \mid p = (p_0, p_1, \ldots), \forall k \in \mathbb{N}\ p_k \geq 0, \sum_{k=0}^{\infty} p_k = 1\}$. Suppose that the initial money-holdings distribution is given by p^0.

At each date, the set of agents is randomly partitioned into pairs. Within each pair, one of the agents desires to consume the good that the other is able to produce. Thus, a bid and offer are associated with each pair.

Now we provide an intuitive discussion of the distributions of bids and offers, and we state some formal assumptions about those distributions. Our assumptions are in the spirit if a "continuum law of large numbers."[3] For each random partition π of the agents into pairs at date t, there is a sample distribution B_t^π of bids and a sample distribution O_t^π of offers. We assume that these sample distributions do not depend on the partition. That is, there are bid and offer distributions B_t and O_t such that for all partitions π, $B_t^\pi = B_t$ and $O_t^\pi = O_t$. Moreover, because each agent has a trading partner assigned at random, the probability distribution of the trading partner's bid and offer should be identical to the sample distribution. That is, B_t and O_t are the probability distributions of bid and offer respectively that are received at date t by each individual agent, as well as being the sample distribution in each random pairing of the population of agents.

Now let the probability space (Ω, B, P) represent the stochastic process of encounters faced by a generic agent. This agent faces a sequence ω of random encounters, one at each date. His date-t encounter, with some agent of type j, is characterized by her trading type (buyer or seller) in the meeting and her bid/offer, denote it by $\omega_t = (\omega_{t1}, \omega_{t2})$,

$$\text{if } i \text{ meets a buyer,} \quad \omega_{t1} = b,\ \omega_{t2} \text{ is her bid}$$
$$\text{if } i \text{ meets a seller,} \quad \omega_{t1} = s,\ \omega_{t2} \text{ is her offer.}$$

[2] In Green and Zhou [2] and Zhou [4], the offer was assumed to be made before the bid. Equilibrium in that game corresponds exactly to equilibrium in the simultaneous-move game. The refinement of equilibrium in undominated strategies, introduced by Zhou [4], also applies straightforwardly to the simultaneous-move game.

[3] That is, we believe that they are logically consistent with the results from probability theory that we will apply in our analysis, although they cannot be derived from those results. See Green [1] and Gilboa and Matsui [4] for further discussion.

The encounters $\{\omega_t\}_{t=0}^{\infty} \equiv \omega$ are independent across time. Ω is the set of all possible sequences of encounters that an arbitrary agent in the economy faces.

At each date t, pairwise meetings are independent across the population. That is, for each agent, ω_{t1} follows a Bernoulli distribution, a potential buyer's bid price ω_{t2} is drawn from the bid distribution B_t, and a potential seller's offer price ω_{t2} is drawn from the offer distribution O_t. For $t \geq 1$, let B_t be the smallest σ-algebra on Ω that makes the vector of the first t coordinates, $\omega^t = (\omega_0, \omega_1, \ldots, \omega_{t-1})$, measurable, and $B_0 = \{\phi, \Omega\}$. Let P_t be the probability measure defined on B_t. Then, for all $t \geq 0$, and $k \in \mathbb{N}$,

$$P_t\{\omega_{t1} = b\} = P_t\{\omega_{t1} = s\} = \frac{1}{2} \tag{1}$$

$$P_t\Big\{\omega_{t2} = k \,|\, \omega_{t1} = b\Big\} = B_{tk} \tag{2}$$

$$P_t\Big\{\omega_{t2} = k \,|\, \omega_{t1} = s\Big\} = O_{tk}. \tag{3}$$

Define $B = B_\infty$ and $P = P_\infty$.

Let σ be the trading strategy of a generic agent of type i with initial money holding η_0. His date-t strategy σ_t specifies his bid σ_{t1}—his maximum willingness to pay if he is paired with a seller of his consumption goods—and his offer σ_{t2}—the price he is willing to sell if he meets a consumer of his production good—as a function of his initial money holding and his encounter history ω. The strategy σ_t is measurable with respect to B_t. As a buyer, the agent has to be able to pay his bid. Let η_t^σ denote the agent's money holding at the beginning of date t by adopting strategy σ. Then

$$\sigma_{t1}(\eta_0, \omega) \leq \eta_t^\sigma(\eta_0, \omega). \tag{4}$$

Given the agent's initial money holding η_0, encounter history ω, and strategy $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, his money holding evolves recursively as follows: $\eta_0^\sigma(\eta_0, \omega) = \eta_0$ and, for $t \geq 0$,

$$\eta_{t+1}^\sigma(\eta_0, \omega) = \begin{cases} \eta_t^\sigma(\eta_0, \omega) + \sigma_{t2}(\eta_0, \omega) & \text{if } \omega_{t1} = b \text{ and } \sigma_{t2}(\eta_0, \omega) \leq \omega_{t2} \\ \eta_t^\sigma(\eta_0, \omega) - \omega_{t2} & \text{if } \omega_{t1} = s \text{ and } \sigma_{t1}(\eta_0, \omega) \geq \omega_{t2} \\ \eta_t^\sigma(\eta_0, \omega) & \text{otherwise} \end{cases} \tag{5}$$

Let v_t^σ denote the agent's date-t utility from his date-t trading by adopting strategy σ. Then

$$v_t^\sigma(\eta_0, \omega) = \begin{cases} -c & \text{if } \omega_{t1} = b \text{ and } \sigma_{t2}(\eta_0, \omega) \leq \omega_{t2} \\ u & \text{if } \omega_{t1} = s \text{ and } \sigma_{t1}(\eta_0, \omega) \geq \omega_{t2} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Then, strategy σ *overtakes* another strategy $\hat{\sigma}$ if for all $\eta_0 \in \mathbb{N}$,

$$\liminf_{t\to\infty} \mathsf{E}\Big[\sum_{\tau=0}^{t} v_\tau^\sigma(\eta_0, \omega) - \sum_{\tau=0}^{t} v_\tau^{\hat{\sigma}}(\eta_0, \omega)\Big] > 0 \tag{7}$$

where E is the expectation operator with respect to the probability measure P.

We are going to focus on symmetric equilibrium at which an agent's strategy is only a function of his own trading history and initial money holding. In particular, the trading strategy does not depend on an agent's type.

At the beginning of date t, given all agents' trading strategy σ_t and the initial money-holdings distribution p^0, rational expectation requires that agents' belief regarding the bid distribution B_t and the offer distribution O_t that prevail during date-t trading confirm with the actual distributions implied by the strategy. That is, for all $k \in \mathbb{N}$,

$$B_{tk} = \sum_{l=0}^{\infty} p_l^0 P\left\{\sigma_{t1}(l,\omega) = k\right\} \tag{8}$$

$$O_{tk} = \sum_{l=0}^{\infty} p_l^0 P\left\{\sigma_{t2}(l,\omega) = k\right\} \tag{9}$$

where I is an indicator function; $I(\alpha,\beta)$ equals 1 if $\alpha = \beta$ and 0 otherwise.

The equilibrium concept that we adopt is Bayesian Nash equilibrium with respect to the overtaking criterion.

DEFINITION. A Bayesian Nash equilibrium is a four-tuple $\langle \sigma, p^0, \{B_t\}_{t=0}^{\infty}, \{O_t\}_{t=0}^{\infty} \rangle$ that satisfies

(i) p^0 is the initial money-holdings distribution in the environment.

(ii) Given the bid distributions $\{B_t\}_{t=0}^{\infty}$ and the offer distributions $\{O_t\}_{t=0}^{\infty}$, and given that all other agents adopt strategy σ, it is optimal for an arbitrary agent to adopt strategy σ as well, that is, there is no strategy that overtakes strategy σ.

(iii) Given that all agents adopt trading strategy σ, for each $t \geq 0$, B_t and O_t satisfy equations (8) and (9).

We are going to study one particular equilibrium at which at all dates, all traders offer to buy their desired consumption goods at price 1 as long as they have money, and accept price 1 in exchange for their production goods, hence, all trades occur at price 1. We call it price-1 equilibrium. This equilibrium is markovian in the sense that the dependence of agents' strategy on time and trading history is only through their own current money holdings to satisfy feasibility condition (4), despite the dynamic environment. Formally, define the strategy $\tilde{\sigma}$ as follows, for all $\eta_0 \in \mathbb{N}$, encounter history $\omega \in \Omega$, and $t \geq 0$,

$$\tilde{\sigma}_{t1}(\eta_0,\omega) = \min\{\eta_t^{\tilde{\sigma}}(\eta_0,\omega), 1\}, \qquad \tilde{\sigma}_{t2}(\eta_0,\omega) = 1. \tag{10}$$

Let $\tilde{p}^t \in \Delta$ denote the money-holdings distribution at the beginning of date t induced by strategy $\tilde{\sigma}$. The bid distribution implied by strategy $\tilde{\sigma}$ puts measure $1 - \tilde{p}_0^t$ on price 1 to reflect the bid of each agent with money, and it puts measure $\tilde{p}_0^t$ on price 0 to reflect the agents who have no money and cannot buy. The offer distribution implied by $\tilde{\sigma}$ is stationary and degenerate with mass at price 1. That is

$$\tilde{B}_{t0} = \tilde{p}_0^t, \qquad \tilde{B}_{t1} = 1 - \tilde{p}_0^t \tag{11}$$

$$\tilde{O}_{t0} = 0, \qquad \tilde{O}_{t1} = 1. \tag{12}$$

The evolution of the money-holdings distribution $\tilde{p}^t$ is specified in the next section. In the next two sections, we are going to show that $\langle \tilde{\sigma}, p^0, \{\tilde{B}_t\}_{t=0}^{\infty}, \{\tilde{O}_t\}_{t=0}^{\infty} \rangle$ is an equilibrium.

4. The Convergence of Money-Holdings Distribution at Price-1 Equilibrium

In this section, we assume that $\langle \tilde{\sigma}, p^0, \{\tilde{B}_t\}_{t=0}^{\infty}, \{\tilde{O}_t\}_{t=0}^{\infty} \rangle$ defined above is an equilibrium. We show that if all agents adopt the stationary strategy $\tilde{\sigma}$, and if the initial money-holdings distribution p^0 satisfies certain condition, then the economy asymptotically converges to a unique stationary equilibrium at which the money-holdings distribution is geometric. This is an important fact that will be used to show that the conjectured equilibrium is indeed an equilibrium.

From the specification of the conjectured price-1 equilibrium, for each agent, there are two decision-relevant objects at any date: the agent's own money holding which determines his feasible bid price, and the economywide money-holdings distribution which determines the bid distribution. Hence, the decision-relevant state at date t can be represented by his money holding η_t and the money holdings distribution p^t instead of initial money holding η_0, encounter history ω, bid distribution B_t and offer distribution O_t. Given that all agents adopt strategy $\tilde{\sigma}$, given the initial distribution p^0, the money-holdings distribution evolves as follows: for any $t \geq 0$,

$$p_0^{t+1} = \left(1 - \frac{m(p^t)}{2}\right) p_0^t + \frac{1}{2} p_1^t \tag{13}$$

$$\forall k \geq 1 \qquad p_k^{t+1} = \left(1 - \frac{m(p^t)}{2} - \frac{1}{2}\right) p_k^t + \frac{1}{2} p_{k+1}^t + \frac{m(p^t)}{2} p_{k-1}^t \tag{14}$$

where $m(p^t)$ is the measure of agents who have money, $m(p^t) = \sum_{k=1}^{\infty} p_k^t$. The sequence $\{p^t\}_{t=0}^{\infty}$ of money-holdings distributions can be obtained by applying (13) and (14) recursively. It is easy to check that at any point of time $t \geq 0$, the distribution satisfies the aggregation condition: the nominal money stock remains at M, that is, $\sum_{k=1}^{\infty} k p_k^t = M$.

For technical convenience, we work with a transformation of the probability measure p instead of p itself. Define a mapping $L\colon \Delta \to [0,1]^{\infty}$ as follows,

$$\forall p \in \Delta \quad \forall k \in \mathbb{N} \qquad L_k(p) = \sum_{j \geq k} p_j. \tag{15}$$

Obviously, $L_0(p) = 1$, $L_k(p) \in [0,1]$ and $L_k(p) \geq L_{k+1}(p)$ for all $k \in \mathbb{N}$. Let $\Gamma = L(\Delta)$. Then, for any $x \in \Gamma$, x satisfies that $x_0 = 1$, $x_k \in [0,1]$ and $x_k \geq x_{k+1}$ for all $k \in \mathbb{N}$. By definition, L is a one-to-one linear mapping from Δ to Γ. Therefore, to prove that the sequence of probability measure $\{p^t\}_{t=0}^{\infty}$ converges in distribution, we need only to show that the corresponding sequence $\{L(p^t)\}_{t=0}^{\infty}$ converges in the ℓ^1 metric. The aggregation condition for the distribution can be written as $\sum_{k=1}^{\infty} L_k(p^t) = M$, for any $t \geq 0$. Define S to be the space of all transformations of probability measure defined by (15) that satisfies the aggregation condition,

$$S = \left\{ x \,|\, x \in \Gamma,\ \sum_{k=1}^{\infty} x_k = M \right\}. \tag{16}$$

The set S is the space we are going to work with primarily in this section. It is easy to show the following.

LEMMA 1. *Both S and Γ are convex. That is, for $X = S$ or $X = \Gamma$, $\forall x, y \in X$ and $\forall \alpha \in [0,1]$, $\alpha x + (1-\alpha) y \in X$.*

By equations (13) and (14), the law of motion of the transformation of money-holdings distribution $L(p)$ is a mapping $T\colon S \to S$ such that for all $x \in S$, for all $k \geq 1$,

$$T_k(x) = \frac{1-x_1}{2} x_k + \frac{1}{2} x_{k+1} + \frac{x_1}{2} x_{k-1} \tag{17}$$

It is easy to check that $T_0(x) = 1$ and $T(x) \in S$. Given that $x^0 = L(p^0)$, $x^t = T(x^{t-1}) = L(p^t)$ for all $t \geq 1$. The following lemma states that the mapping T has a unique fixed point.

LEMMA 2. *The mapping T has a unique fixed point $\bar{x} \in S$ such that $\bar{x} = T(\bar{x})$:*

$$\forall k \in \mathbb{N} \quad \bar{x}_k = \bar{m}^k, \qquad \textit{where} \quad \bar{m} = \frac{M}{M+1}. \tag{18}$$

Proof. For all $x \in S$, by equation (17), $T(x) = x$ requires that for all $k \geq 1$,

$$x_k = \left(\frac{1-x_1}{2}\right) x_k + \frac{1}{2} x_{k+1} + \frac{x_1}{2} x_{k-1} \tag{19}$$

and $x_0 = 1$. This system of equations has a unique solution $\bar{x}$, $\forall k \in \mathbb{N}$ $\bar{x}_k = (\bar{x}_1)^k$. Since $\bar{x} \in S$, $M = \sum_{k=1}^{\infty} \bar{x}_k = \sum_{k=1}^{\infty} (\bar{x}_1)^k$, which implies that $\bar{x}_1 = \frac{M}{M+1} = \bar{m}$. ∎

The unique fixed point $\bar{x}$ of T given in Lemma 2 corresponds to the geometric money-holdings distribution with parameter $\bar{m}$: $\bar{p} = L^{-1}(\bar{x})$, $\bar{p}_k = (1-\bar{m})\bar{m}^k$ for all $k \in \mathbb{N}$. We want to show that starting from a given initial state x^0, the economy as a dynamic system evolving according to mapping T, converge asymptotically to the steady state characterized by $\bar{x}$. Toward this objective, we construct a Liapunov function which is a function of the state of the dynamic system. We show that the Liapunov function decreases over time and approaches to its minimum asymptotically. Therefore, by a standard argument of dynamical-systems theory, the economy asymptotically approaches a steady state, which is represented by the unique fixed point of T, $\bar{x}$, and does not depend on the initial state.

The Liapunov function we choose to use can be interpreted as the expected hazard rate for the corresponding distribution. Define $Z\colon \Gamma \to \mathbb{R}_+$, for all $x \in \Gamma$,

$$Z(x) = \sum_{k=0}^{\infty} \frac{(x_k - x_{k+1})^2}{x_k}. \tag{20}$$

For technical reasons, we define the function Z on the larger space Γ instead of on S. For Z to be a Liapunov function, it should be continuous in some metric, it should be decreasing along the trajectory of the system defined by T, and it should have a unique minimum on S where it is applied. We will show that Z has these properties one by one.

LEMMA 3. *The function Z is strictly convex on Γ.*

Proof. Take arbitrary $x, y \in \Gamma$, $x \neq y$, and $\alpha \in (0,1)$. Let $w(\alpha) = (1-\alpha)x + \alpha y = x + \alpha(y-x)$. The set Γ is convex, hence $w(\alpha) \in \Gamma$. For all $k \in \mathbb{N}$, define $\delta_k \equiv y_k - x_k$ and

$$z_k(w(\alpha)) \equiv \frac{(w_k(\alpha) - w_{k+1}(\alpha))^2}{w_k(\alpha)} = \frac{(x_k - x_{k-1} + \alpha(\delta_k - \delta_{k+1}))^2}{x_k + \alpha\delta_k}.$$

Direct computation reveals that

$$\frac{d^2 Z(w(\alpha))}{d\alpha^2} = \sum_{k=0}^{\infty} \frac{d^2 z_k(w(\alpha))}{d\alpha^2} = \sum_{k=0}^{\infty} \frac{2}{x_k + \alpha\delta_k} \left(\delta_k - \delta_{k+1} - \frac{\delta_k (x_k - x_{k+1} + \alpha(\delta_k - \delta_{k+1}))}{x_k + \alpha\delta_k} \right)^2 \geq 0.$$

Moreover $\frac{d^2 Z(w(\alpha))}{d\alpha^2} = 0$ if an only if $\forall k \in \mathbb{N}$ $\frac{d^2 z_k(w(\alpha))}{d\alpha^2} = 0$, which is equivalent to $\forall k \in \mathbb{N}$ $y_{k+1}x_k = y_k x_{k+1}$, that is (since $x_0 = y_0 = 1$), $\forall k \in \mathbb{N}$ $x_k = y_k$. Given that $x \neq y$, $\frac{d^2 Z(w(\alpha))}{d\alpha^2} > 0$. Hence, Z is strictly convex on Γ. ∎

Using Lemma 3, the following proposition show that the function Z satisfies a crucial criterion of a Liapunov function: it is strictly decreasing along the trajectory of the system defined by T.

PROPOSITION 1. *For all $x \in S$, $Z(T(x)) < Z(x)$, unless $x = T(x)$.*

Proof. Define mappings $\lambda\colon S \to \Gamma$ and $\rho\colon S \to \Gamma$ as follows: for all $x \in S$, $k \in \mathbb{N}$,

$$\lambda_k(x) = \frac{x_{k+1}}{x_1}, \qquad \rho_0(x) = 1, \quad \rho_{k+1}(x) = x_1 x_k.$$

It is easy to check that $\lambda(x) \in \Gamma$ and $\rho(x) \in \Gamma$.[4] The measure λ is a normalized left-shift of x, and ρ is a normalized right-shift of x. Then by (17), we can rewrite $T(x)$ as convex combinations of x, $\lambda(x)$ and $\rho(x)$. That is,

$$T(x) = \frac{x_1}{2}\lambda(x) + \frac{1}{2}\rho(x) + \frac{1-x_1}{2}x.$$

Since Z is strictly convex on Γ by Lemma 3, unless $\lambda(x) = \rho(x) = x$,

$$\begin{aligned} Z(T(x)) &< \frac{x_1}{2}Z(\lambda(x)) + \frac{1}{2}Z(\rho(x)) + \frac{1-x_1}{2}Z(x) \\ &= \frac{x_1}{2}\frac{1}{x_1}\Big(Z(x) - (1-x_1)^2\Big) + \frac{1}{2}\Big((1-x_1)^2 + x_1 Z(x)\Big) + \frac{1-x_1}{2}Z(x) \\ &= Z(x). \end{aligned}$$

It is easy to verify that $\lambda(x) = x$ if and only if $x = T(x)$. Therefore, unless $x = T(x)$, we have $Z(T(x)) < Z(x)$. ∎

Because of the aggregation condition ($\sum_{k=1}^{\infty} x_k = M$), S is a subset of the complete metric space (X,d), where $X = \{x \in [0,1]^{\infty} \mid \sum_{k=0}^{\infty} |x_k| < \infty\}$ and d is the usual metric associated with X, for any $x,y \in X$,

$$d(x,y) = \sum_{k=0}^{\infty} |x_k - y_k|. \tag{21}$$

By standard argument, (S,d) is a complete metric subspace of (X,d). Next, we show that both Z and T are continuous mappings in metric d.

PROPOSITION 2. *The function Z is continuous on S.*

Proof. We need to show that for any given $\varepsilon > 0$, for any $x \in S$, there exists a δ-neighborhood of x such that for all y satisfying $d(x,y) < \delta$, $|Z(x) - Z(y)| < \varepsilon$.

Fix an arbitrary $\varepsilon > 0$, and an arbitrary $x \in S$. Since x_k is decreasing in k and $\sum_{k=1}^{\infty} x_k = M$, there exists an $I \geq 1$ such that

$$x_I < \varepsilon/8. \tag{22}$$

Let $J = \max\{j \mid j \leq I, x_j > 0\}$. So $x_J > 0$. Without loss of generality, assume $J \geq I - 1$. Let $\delta = (\varepsilon/8)x_J > 0$. (Otherwise $x_{J+1} = 0$, so $J+1$ satisfies $x_{J+1} < \varepsilon/8$.) Then for any y such that $d(x,y) < \delta$,

$$y_I \leq |y_I - x_I| + x_I \leq d(x,y) + x_I < (\varepsilon/8)x_J + \varepsilon/8 \leq \varepsilon/4. \tag{23}$$

For all $k \in \mathbb{N}$, define $\xi_k(x) \equiv (x_k - x_{k+1})/x_k \leq 1$, and $\xi_k(y) \equiv (y_k - y_{k+1})/y_k \leq 1$. Then, for $k \leq I-1$, $x_k \geq x_J$,

$$|\xi_k(x) - \xi_k(y)| \leq \frac{1}{x_k}\left(|x_{k+1} - y_{k+1}| + |x_k - y_k|\frac{y_{k+1}}{y_k}\right) < \frac{2}{x_J}\frac{\varepsilon}{8}x_J = \frac{\varepsilon}{4}. \tag{24}$$

[4] In general, for any $x \in S$, $\lambda(x) \notin S$ and $\rho(x) \notin S$. This is the reason we define Z on Γ instead of on S.

Now, applying (22)–(24), we have

$$
\begin{aligned}
& |Z(x) - Z(y)| \\
= & \sum_{k=0}^{I-1} \Big| (x_k - x_{k+1})\xi_k(x) - (y_k - y_{k+1})\xi_k(y) \Big| + \sum_{k \geq I} (x_k - x_{k+1})\xi_k(x) + \sum_{k \geq I} (y_k - y_{k+1})\xi_k(y) \\
\leq & \sum_{k=0}^{I-1} \Big((x_k - x_{k+1}) |\xi_k(x) - \xi_k(y)| + |x_k - y_k| \xi_k(y) + |x_{k+1} - y_{k+1}| \xi_k(y) \Big) + x_I + y_I \\
< & \sum_{k=0}^{I-1} (x_k - x_{k+1}) \frac{\varepsilon}{4} + \sum_{k=0}^{I-1} |x_k - y_k| + \sum_{k=0}^{I-1} |x_{k+1} - y_{k+1}| + \varepsilon/8 + \varepsilon/4 \\
< & \varepsilon/4 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/4 < \varepsilon.
\end{aligned}
$$

We have shown that for any given $\varepsilon > 0$, for any $x \in S$, there is $\delta > 0$ such that for all y satisfying $d(x,y) < \delta$, $|Z(y) - Z(x)| < \varepsilon$. Hence, Z is continuous on S. ∎

PROPOSITION 3. *The mapping T is continuous on S.*

Proof. We need to show that for any given $\varepsilon > 0$ and $x \in S$, there is a $\delta > 0$ such that for all y satisfying $d(x,y) < \delta$, $d(T(x), T(y)) < \varepsilon$.

Fix an arbitrary $\varepsilon > 0$ and an arbitrary $x \in S$. By (17), for all $y \in S$, for all $k \geq 1$,

$$
T_k(y) = \frac{1 - y_1}{2} y_k + \frac{1}{2} y_{k+1} + \frac{y_1}{2} y_{k-1}.
$$

Take $\delta = \varepsilon/3 > 0$, and let y be such that $d(y,x) < \delta$. Then,

$$
\begin{aligned}
d(T(x), T(y)) \ = & \ \sum_{k=1}^{\infty} |T_k(y) - T_k(x)| \\
= & \sum_{k=1}^{\infty} \frac{1}{2} \Big(|x_k - y_k| + |x_{k+1} - y_{k+1}| + x_1 |x_k - y_k| + x_1 |x_{k+1} - y_{k+1}| + (y_k + y_{k+1}) |x_1 - y_1| \Big) \\
< & \ \frac{1}{2} \Big(\varepsilon/3 + \varepsilon/3 + \varepsilon/3 + \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \Big) = \varepsilon
\end{aligned}
$$

Therefore, T is continuous on S. ∎

The set S we have been working with is unfortunately not compact. To insure the convergence of the system from some initial state, we introduce a subset of S that is compact. Define an ordering relation between two vectors x and y: y dominates x, denote $x \preceq y$, if and only if for all $k \in \mathbb{N}$ $x_k \leq y_k$. For a given strictly positive vector $\pi \in X$, let S_π be the set of vectors in S that are dominated by π,

$$
S_\pi = \{ x \in S \,|\, x \preceq \pi \}. \tag{25}
$$

PROPOSITION 4. *For a given strictly positive vector* $\pi \in X$*, the set* S_π *is compact.*

Proof. To prove S_π is compact, we need to show that S_π is complete and totally bounded subset of X. The completeness of S_π is trivial given that S is complete, and the proof is omitted here. To show that S_π is totally bounded, we need to show that there exist a finite ε-net for S_π in X for any $\varepsilon > 0$.

Fix an arbitrary $\varepsilon > 0$. Since π is strictly positive and $\pi \in X$, $\sum_{k=0}^{\infty} \pi_k < \infty$. Hence, there exists an $I > 0$ such that $\sum_{k>I} \pi_k < \varepsilon/2$. For any $x \in S_\pi$, let $\hat{x}$ be the vector of x truncated at I, $\hat{x} = (x_0, x_1, \ldots, x_I, 0, 0, \ldots)$. Then $d(x, \hat{x}) = \sum_{k>I} x_k \leq \sum_{k>I} \pi_k < \varepsilon/2$. Let $\hat{S}_\pi$ be the set of $\hat{x}$ associated with $x \in S_\pi$. The set $\hat{S}_\pi$ is a totally bounded in I-dimensional Euclidean space (with the usual metric). Let A be a finite $\varepsilon/2$-net for $\hat{S}_\pi$. Then A is a finite ε-net for S_π. ∎

The vector π can be any strictly positive element of X. In particular, let π^θ denote the geometric vector defined by some $\theta \in (0,1)$: for all $k \in \mathbb{N}$,

$$\pi_k^\theta = \theta^k. \tag{26}$$

The vector π^θ as defined above is an element of X as well as Γ. Also, it is a fixed point of T. The following lemma states that for π^θ, S_{π^θ} is closed under T.

LEMMA 4. *For any* $x \in S$ *and any* $\theta \in (0,1)$*, if* $x \preceq \pi^\theta$*, then* $T(x) \preceq \pi^\theta$*.*

Proof. Suppose that there exists a $\theta \in (0,1)$ such that $x \preceq \pi^\theta$. By definition, $x \preceq \pi^\theta$ implies that $x_k \leq \theta^k$ for all $k \in \mathbb{N}$. By equation (17), for all $k \geq 1$,

$$T_k(x) = \frac{1}{2}\Big((1-x_1)x_k + x_{k+1} + x_1 x_{k-1}\Big) \leq \frac{1}{2}\Big((1-x_1)\theta^k + \theta^{k+1} + x_1\theta^{k-1}\Big).$$

Since the expression in the righthand side of the above inequality is an increasing function of x_1 and by assumption, $x_1 \leq \theta$,

$$T_k(x) \leq \frac{1}{2}\Big((1-\theta)\theta^k + \theta^{k+1} + \theta\theta^{k-1}\Big) = \theta^k.$$

By definition, $T_0(x) = 1 = \pi_0^\theta$. Therefore, $T(x) \preceq \pi^\theta$. ∎

By Lemma 4, if a given initial state x^0 satisfies the following condition,

($\star$) *there exists a* $\theta \in (0,1)$ *and a* $t > 0$ *such that* $T^t(x^0) \preceq \pi^\theta$

then all the subsequent states of the dynamic system $T^n(x^0)$, $n \geq t$, are dominated by π^θ as well, hence, they are elements of S_{π^θ}.

PROPOSITION 5. *If the initial state x^0 satisfies condition $(\star)$, then the economy as a dynamic system evolving from x^0 according to mapping T converges asymptotically to the steady state characterized by distribution $\bar{x}$ which uniquely satisfies $T(\bar{x}) = \bar{x}$ and $d(\bar{x}, 0) = d(x^0, 0)$.*

Proof. Suppose that condition $(\star)$ holds, that is, there exists $\theta \in (0,1)$ and a $t > 0$ such that $T^t(x^0) \preceq \pi^\theta$, then $T^n(x^0) \in S_{\pi^\theta}$ for all $n \geq t$. By Proposition 4, S_{π^θ} is a compact set, and by Proposition 2, the function Z is continuous on S, hence on S_{π^θ}, Z achieves its minimum on S_{π^θ}. Furthermore, by Lemma 3, Z is strictly convex on S, hence on S_{π^θ}, Z has a unique minimum on S_{π^θ}. Last, Z is strictly decreasing along the trajectory of the system defined by T by Proposition 1. Therefore, Z is a Liapunov function. With this Liapunov function, we show next the convergence of the system from the initial state x^0.

From the given x^0, construct a sequence $\{x^n\}_{n=1}^\infty$ by applying T recursively, $x^n = T^n(x^0)$. Consider the sequence excluding the first t elements, $\{x^n\}_{n=t}^\infty$, which is in S_{π^θ} by assumption. By Proposition 1, the corresponding sequence $\{Z(x^n)\}_{n=t}^\infty$ is monotonically decreasing. Since S_{π^θ} is compact, there exists a subsequence $\{x^{n_k}\}$ that converges to some $\hat{x} \in S_{\pi^\theta}$. Suppose that $\hat{x}$ is not a fixed point of T. Then by Proposition 1, $Z(T(\hat{x})) < Z(\hat{x})$. Since Z is continuous and T is continuous, there exists a $\delta > 0$ such that for all y satisfying $d(\hat{x}, y) < \delta$, $Z(T(y)) < Z(\hat{x})$. Since $\{x^{n_k}\}$ converges to $\hat{x}$, there exists an K such that for all $k \geq K$, $d(\hat{x}, x^{n_k}) < \delta$, hence, $Z(T(x^{n_k})) < Z(\hat{x})$, or,

$$Z(x^{n_k+1}) < Z(\hat{x}). \tag{27}$$

But since $\{Z(x^n)\}_{n=t}^\infty$ is monotonically decreasing, and since $\hat{x}$ is the limit of x^{n_k}, regardless of the arrangement of the subsequence,

$$Z(x^{n_k+1}) \geq Z(\hat{x}) \tag{28}$$

which contradicts (27). Therefore, the limit $\hat{x}$ has to be a fixed point of T. Since T has a unique fixed point $\bar{x}$ in S by Lemma 2, $\bar{x} = \hat{x} \in S_{\pi^\theta}$. Hence, for the given initial state x^0, $T^n(x^0) \to \bar{x}$ as $n \to \infty$. This strengthened statement, that the entire sequence (rather than only the subsequence selected above) converges to $\bar{x}$, follows from a standard argument involving the Liapunov function Z. ∎

The assumption made in Proposition 5, that the initial state x^0, a linear transformation of the initial distribution p^0, satisfies condition $(\star)$, is not a strong assumption as it seems. The following proposition states a class of initial state that satisfies the condition.

PROPOSITION 6. *If the initial money-holdings distribution p^0 has a thin tail that is dominated by the tail of a geometric distribution, that is, there is a $J > 0$ and an $\alpha \in (0,1)$ such that $p_j^0 \leq (1-\alpha)\alpha^j$ for all $j > J$, then $x^0 = L(p^0)$ satisfies condition $(\star)$.*

Proof. The condition states that there is a $J > 0$ and an $\alpha \in (0,1)$ such that $p_j^0 \leq (1-\alpha)\alpha^j$ for all $i > J$, or equivalently, $x_j^0 \leq \alpha^j$ for all $j > J$. Let K be the minimum of the support of p^0, $K = \min\{j \,|\, p_j^0 \neq 0\}$. Then by definition, $x_j^0 = 1$ for all $j \leq K$.

First, suppose that $x_1^0 < 1$ ($p_0^0 > 0$ and $K = 0$). Then, there exists a $\theta \in [\alpha, 1)$ such that $x_1^0 \leq \theta^J$. Since x_j^0 is decreasing in j, for $j \leq J$, $x_j^0 \leq x_1^0 \leq \theta^J \leq \theta^j$. For $j > J$, $x_j^0 \leq \alpha^j \leq \theta^j$ since $\alpha \leq \theta$. Therefore, $x^0 \preceq \pi^\theta$. That is, condition $(\star)$ holds for x^0.

Next, consider the case where $x_1^0 = 1$ ($p_0^0 = 0$ and $K \geq 1$). Then, $x_j^0 = 1$ for all $j \leq K$ and $x_{K+1}^0 < 1$. By equation (17), $T_K^1(x^0) = (x_{K+1}^0 + 1)/2 < 1$, and $T_{j+1}^1(x^0) \leq \alpha^j$ for all $j > J$. Similarly, after K repeated operations of T on x^0, we have $T_1^K(x^0) < 1$, $T_{j+K}^K(x^0) \leq \alpha^j$ for all $j > J$. Now we can treat $T^K(x^0)$ as the x^0 in the case above. Specifically, there exists a $\theta \in [\alpha^{\frac{J}{J+K}}, 1)$ such that $T_1^K(x^0) < \theta^{J+K}$. For $j \leq J+K$, $T_j^K(x^0) \leq T_1^K(x^0) \leq \theta^{J+K} \leq \theta^j$. For $j > J+K$, $x_j^0 \leq \alpha^{j-K} \leq \theta^j$ given $\theta \geq \alpha^{\frac{J}{J+K}}$. Therefore, $T^K(x^0) \preceq \pi^\theta$, which implies that condition $(\star)$ holds. ■

As a practical matter, economists are not likely to find that the condition in Proposition 6 is a restrictive one. An initial money-holdings distribution p^0 with finite support (that is, there is a $J > 0$ such that $p_j^0 = 0$ for all $j > J$) satisfies the condition. Distributions with finite support are dense in the space of probability simplex Δ. The condition is also satisfied if one is to increase the nominal money stock in an economy from a steady-state geometric distribution by distributing a finite amount of money to people whose money holdings are less than certain finite amount (e.g. "poor" people), in other words, if the money injection has finite support. We can conclude now that if the initial money-holdings distribution satisfies the condition given in Proposition 6, and if all agents adopt strategy $\tilde{\sigma}$, then by Proposition 5, the economy asymptotically converges to a unique stationary equilibrium at with the money-holdings distribution is geometric.

5. The Existence of Price-1 Equilibrium

In this section, we show that from an initial distribution p^0 such that the assumption in Proposition 6 is satisfied, the price-1 equilibrium defined in Section 3 is a Bayesian Nash equilibrium. In particular, we show that for an arbitrary agent, given that all other agents in the economy adopt the strategy $\tilde{\sigma}$ defined in (10) (hence the bid and offer distributions are given by $\{\tilde{B}_t\}_{t=0}^\infty$ and $\{\tilde{O}_t\}_{t=0}^\infty$ defined in (11)–(12)), it is optimal for the agent in question to adopt strategy $\tilde{\sigma}$ as well, that is, no strategy overtakes $\tilde{\sigma}$.

Consider an arbitrary agent of type i. Suppose that the agent's initial money holdings is η_0. Since η_0 is fixed and is taken as given when we compare different strategies, for notational convenience, we will suppress η_0 as an argument of all functions such as σ and η_t^s in the rest of the section, and write them as functions of ω alone. Also note that given all the other agents adopt strategy $\tilde{\sigma}$ and the agent in question has measure 0, although his trading history will be determined by his strategy σ, his encounter history ω is independent of the strategy he adopts.

Let $\eta_t^\sigma(\omega)$ denote the agent's money holdings at the beginning of date t in encounter history ω if he adopts strategy σ, $\eta_0^\sigma(\omega) = \eta_0$. Define the agent's *achievement function* at the beginning of date t if he adopts strategy σ, $A_t^\sigma\colon \Omega \to \mathbb{R}_+$, to be the sum of his total utility up to date t and the future utility that will be brought by the money accumulated up to date t, η_t^σ, given that the agent buys his future consumption goods at price 1. That is, for any encounter history $\omega \in \Omega$,

$$A_t^\sigma(\omega) = \sum_{\tau=0}^{t-1} v_\tau^\sigma(\omega) + \eta_t^\sigma(\omega) u \tag{29}$$

where $v_\tau^\sigma(\omega)$ is defined in (6), and $\eta_t^\sigma(\omega)$ is defined recursively by (5). For notational convenience, define for all $t \geq 0$,

$$\tilde{A}_t = A_t^{\tilde{\sigma}}, \qquad \tilde{v}_t = v_t^{\tilde{\sigma}}, \qquad \tilde{\eta}_t = \eta_t^{\tilde{\sigma}}. \tag{30}$$

Note that by the definition (7) of the overtaking criterion, given that all other agents adopt strategy $\tilde{\sigma}$, any strategy that specifies at any time to offer to sell at price 0 is obviously overtaken by some strategy since the seller in transaction gains nothing but suffers a loss in production cost c. In the rest of the paper when we compare strategies with $\tilde{\sigma}$, we exclude those strategies with 0 offer price at any time. The following lemma shows that strategy $\tilde{\sigma}$ is associated with the highest achievement function of any strategy.

LEMMA 5. *If all other agents adopt strategy $\tilde{\sigma}$, then for an arbitrary agent facing any encounter history $\omega \in \Omega$, adopting a strategy σ, for all $t \geq 0$, $A_t^\sigma(\omega) \leq \tilde{A}_t(\omega)$.*

Proof. Consider an agent of type i with a history $\omega \in \Omega$. Obviously, $A_0^\sigma(\omega) = \tilde{A}_0(\omega) = \eta_0 u$. We compare an arbitrary strategy σ with $\tilde{\sigma}$ at the begining of date $t+1$, $t \geq 0$.

Case (1). $\omega_{t1} = s$ and $\omega_{t2} = 1$. In this case, regardless the agent's strategy (including $\tilde{\sigma}$), $A_{t+1}^\sigma(\omega) = A_t^\sigma(\omega)$.

Case (2). $\omega_{t1} = b$ and $\omega_{t2} = 0$. This is a case that the buyer encountered has no money, hence, has bid price 0. By remark above, $\sigma_{t2}(\omega) > 0$. So regardless of the strategy (including $\tilde{\sigma}$), no trade can take place, $A_{t+1}^\sigma(\omega) = A_t^\sigma(\omega)$.

Case (3). $\omega_{t1} = b$ and $\omega_{t2} = 1$. If $\sigma_{t2}(\omega) = 1 = \tilde{\sigma}(\omega)$, $A_{t+1}^\sigma(\omega) - A_t^\sigma(\omega) = -c + u = \tilde{A}_{t+1}(\omega) - \tilde{A}_t(\omega)$. If $\sigma_{t2}(\omega) > 1$, the encountered buyer is not able to buy, hence trade does not take place with σ, but it does take place with $\tilde{\sigma}$, $A_{t+1}^\sigma(\omega) - A_t^\sigma(\omega) = 0 < -c + u = \tilde{A}_{t+1}(\omega) - \tilde{A}_t(\omega)$.

Combine the above three cases, we conclude that for any strategy σ, for all history $\omega \in \Omega$, $A_0^\sigma(\omega) - \tilde{A}_0(\omega) = 0$, and for all $t \geq 0$,

$$A_{t+1}^\sigma(\omega) - \tilde{A}_{t+1}(\omega) \leq A_t^\sigma(\omega) - \tilde{A}_t(\omega).$$

Hence, by induction for all $t \geq 0$, $A_t^\sigma(\omega) \leq \tilde{A}_t(\omega)$. ∎

In Section 4, we have shown that if all agents adopt strategy $\tilde{\sigma}$, for a given initial distribution p^0, if the assumption in Proposition 6 is satisfied, the economy conveges to a unique stationary equilibrium at which the money-holdings distribution is the geometric $\bar{p}$. For each agent using strategy $\tilde{\sigma}$, when all other agents adopt $\tilde{\sigma}$ also, the expected money holdings at this limit is M.[5] In other words,

$$\lim_{t\to\infty} \mathsf{E}[\tilde{\eta}_t(\omega)] = M. \tag{31}$$

The next lemma is about the expected money holdings if an agent adopt some other strategy σ.

LEMMA 6. *Under the assumption in Proposition 6, given that an arbitrary agent adopts strategy σ while all other agents adopt strategy $\tilde{\sigma}$, if $\mathsf{E}[A_t^\sigma(\omega) - \tilde{A}_t(\omega)] \not\to -\infty$ as $t \to \infty$, then* $\liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma(\omega)] \geq M$.

Proof. For strategy σ, for all $\omega \in \Omega$, define $\delta^\sigma(\omega)$ to be the set of dates at which the agent who adopts strategy σ meets a buyer, but his offer price is above 1, $\delta^\sigma(\omega) = \{t \,|\, \omega_{t1} = b, \sigma_{t2}(\omega) > 1\}$. For any set D, let $\#D$ denote the cardinality of D.

Claim 1. If $\mathsf{E}[A_t^\sigma(\omega) - \tilde{A}_t(\omega)] \not\to -\infty$ as $t \to \infty$, $\#\delta^\sigma < \infty$ a.s.
To prove this, consider an arbitrary encounter sequence $\omega \in \Omega$. For $t \in \delta^\sigma(\omega)$, given that the agent adopts strategy σ, there is no trade takes place ($\sigma_{t2}(\omega) > 1 = \omega_{t2}$), hence, $A_{t+1}^\sigma(\omega) - A_t^\sigma(\omega) = 0$, while if the agent adopts strategy $\tilde{\sigma}$, then trade takes place at price 1, $\tilde{A}_{t+1}(\omega) - \tilde{A}_t(\omega) = u - c > 0$. Therefore,

$$\forall t \in \delta^\sigma(\omega) \quad A_{t+1}^\sigma(\omega) - \tilde{A}_{t+1}(\omega) = A_t^\sigma(\omega) - \tilde{A}_t(\omega) - (u-c). \tag{32}$$

For $t \notin \delta^\sigma(\omega)$, it is easy to check, for cases (1)–(3) as in the proof of Lemma 5, that $A_{t+1}^\sigma(\omega) - \tilde{A}_{t+1}(\omega) = A_t^\sigma(\omega) - \tilde{A}_t(\omega)$. Hence, by (32), if $\#\delta^\sigma(\omega) = \infty$, $\lim_{t\to\infty}[A_t^\sigma(\omega) - \tilde{A}_t(\omega)] = - - \infty$, which implies that if $P\{\omega \,|\, \#\delta^\sigma(\omega) = \infty\} > 0$, $\lim_{t\to\infty} \mathsf{E}[A_t^\sigma(\omega) - \tilde{A}_t(\omega)] = -\infty$, which contradicts to the assumption. That is, the claim holds.

Given claim 1, for any $\varepsilon > 0$, there exists a $t_\varepsilon > 0$ such that $P\{\omega \,|\, \max \delta^\sigma(\omega) \leq t_\varepsilon\} > 1 - \varepsilon/2$. Recall that for all $t \in \delta^\sigma(\omega)$, $\tilde{\sigma}_{t2}(\omega) = 1$. Define $\mu_\varepsilon(\omega) \equiv \min\{t \,|\, t \geq t_\varepsilon, \tilde{\eta}_t(\omega) = 0\}$.

Claim 2. For all $\omega \in \Omega$ such that $\max \delta^\sigma(\omega) \leq t_\varepsilon$, *for all $t \geq \mu_\varepsilon(\omega)$, $\tilde{\eta}_t(\omega) \leq \eta_t^\sigma(\omega)$.*
This claim can be proved by induction. For $t = \mu_\varepsilon(\omega)$, the claim holds automatically since $\tilde{\eta}_t(\omega) = 0 \leq \eta_t^\sigma(\omega)$. Suppose that it holds for some $t \geq \mu_\varepsilon(\omega)$, consider date-$(t+1)$ transaction.

if $\omega_{t1} = b, \ \omega_{t2} = 1,$ then $\tilde{\eta}_{t+1}(\omega) = \tilde{\eta}_t(\omega) + 1 \leq \eta_t^\sigma(\omega) + 1 = \eta_{t+1}^\sigma(\omega)$

[5] This is a continuum-law-of-large-number type of assertion, so (31) must be assumed as an axiom.

$$
\begin{array}{ll}
\text{if } \omega_{t1} = b,\ \omega_{t2} = 0, & \text{then } \tilde{\eta}_{t+1}(\omega) = \tilde{\eta}_t(\omega) \le \eta_t^{\sigma}(\omega) = \eta_{t+1}^{\sigma}(\omega) \\
\text{if } \omega_{t1} = s,\ \tilde{\eta}_t(\omega) = 0, & \text{then } \tilde{\eta}_{t+1}(\omega) = 0 \le \eta_{t+1}^{\sigma}(\omega) \\
\text{if } \omega_{t1} = s,\ \tilde{\eta}_t(\omega) \ge 1,\ \sigma_{t1}(\omega) \ge 1, & \text{then } \tilde{\eta}_{t+1}(\omega) = \tilde{\eta}_t(\omega) - 1 \le \eta_t^{\sigma}(\omega) - 1 = \eta_{t+1}^{\sigma}(\omega) \\
\text{if } \omega_{t1} = s,\ \tilde{\eta}_t(\omega) \ge 1,\ \sigma_{t1}(\omega) = 0, & \text{then } \tilde{\eta}_{t+1}(\omega) = \tilde{\eta}_t(\omega) - 1 < \eta_t^{\sigma}(\omega) = \eta_{t+1}^{\sigma}(\omega)
\end{array}
$$

That is, $\tilde{\eta}_{t+1}(\omega) \le \eta_{t+1}^{\sigma}(\omega)$. Hence, the claim holds for all $t \ge \mu_\varepsilon(\omega)$.

By the definition of μ_ε, $\mu_\varepsilon < \infty$ a.s. (since $\tilde{\eta}_t(\omega)$ is a random walk with nonpositive drift, and with reflecting barrier at 0, $\tilde{\eta}_t(\omega)$ hits 0 in finite time a.s.). Then for the ε chosen above, there exists a $\xi_\varepsilon > 0$ such that $P\{\omega \,|\, \mu_\varepsilon(\omega) \le \xi_\varepsilon\} > 1 - \varepsilon/2$. Define

$$\Omega_1(\varepsilon) = \{\omega \,|\, \max \delta^{\sigma}(\omega) \le t_\varepsilon \text{ and } \mu_\varepsilon(\omega) \le \xi_\varepsilon\}$$

$$\Omega_2(\varepsilon) = \{\omega \,|\, \max \delta^{\sigma}(\omega) > t_\varepsilon \text{ or } \mu_\varepsilon(\omega) > \xi_\varepsilon\}.$$

Then $\Omega = \Omega_1(\varepsilon) \cup \Omega_2(\varepsilon)$, $P(\Omega_1(\varepsilon)) > 1 - \varepsilon$ and $P(\Omega_2(\varepsilon)) \le \varepsilon$. Take $\varepsilon = 1/n^2$. For $\omega \in \Omega_1(1/n^2)$, $t_{1/n^2} \le \mu_{1/n^2}(\omega) \le \xi_{1/n^2}$. For a fixed n, consider the sequence $\{\eta_t^{\sigma}(\omega) - \tilde{\eta}_t(\omega)\}$ for $t \ge \xi_{1/n^2}$. Let $\Omega_{1n} = \Omega_1(1/n^2)$ and $\Omega_{2n} = \Omega_2(1/n^2)$.

$$
\begin{aligned}
\liminf_{t\to\infty} \mathsf{E}[\eta_t^{\sigma}(\omega) - \tilde{\eta}_t(\omega)] \;\ge\; & \liminf_{t\to\infty} \int_{\Omega_{1n}} (\eta_t^{\sigma}(\omega) - \tilde{\eta}_t(\omega))\, dP(\omega) \\
& + \liminf_{t\to\infty} \left(\int_{\Omega_{2n}} \eta_t^{\sigma}(\omega)\, dP(\omega) - \int_{\Omega_{2n}} \tilde{\eta}_t(\omega)\, dP(\omega) \right). \qquad (33)
\end{aligned}
$$

The first term of the right hand side of (33) is nonnegative by claim 2 since $\omega \in \Omega_{1n}$. Since $\tilde{\eta}_t$ is ergodic, Ω_{2n} is a fixed set for a given n, and the limit of $\mathsf{E}[\tilde{\eta}_t(\omega)]$ exists and equals M by (31), the second lim inf on the right hand side of (33) can be broken down to two terms,

$$\liminf_{t\to\infty} \left(\int_{\Omega_{2n}} \eta_t^{\sigma}(\omega)\, dP(\omega) - \int_{\Omega_{2n}} \tilde{\eta}_t(\omega)\, dP(\omega) \right) = \liminf_{t\to\infty} \int_{\Omega_{2n}} \eta_t^{\sigma}(\omega)\, dP(\omega) - MP(\Omega_{2n}). \quad (34)$$

The first term of the right hand side of (34) is nonnegative. Combine (33) and (34), we have

$$\liminf_{t\to\infty} \mathsf{E}[\eta_t^{\sigma}(\omega) - \tilde{\eta}_t(\omega)] \ge -MP(\Omega_{2n}). \qquad (35)$$

Take limit of $n \to \infty$ for inequality (35), the left hand side is unrelated to n, hence not affected, and the right hand side goes to 0 since $P(\Omega_{2n}) \le 1/n^2 \to 0$. Therefore, $\liminf_{t\to\infty} \mathsf{E}[\eta_t^{\sigma}(\omega)] \ge \lim_{t\to\infty} \mathsf{E}[\tilde{\eta}_t(\omega)] = M$. ∎

Now, we are ready to prove the main proposition of the paper.

PROPOSITION 7. *Under the assumption in Proposition 6, given that all other agents adopt strategy $\tilde{\sigma}$, it is optimal for an arbitrary agent to take strategy $\tilde{\sigma}$ as well. That is, there is no strategy σ that overtakes $\tilde{\sigma}$.*

Proof. For an arbitrary strategy σ, consider the following two cases.

Case 1. $\limsup_{t\to\infty}\mathsf{E}[\eta_t^\sigma(\omega)] \geq M$. Then, for any $\varepsilon > 0$, there exists an inifinite set $G_\varepsilon^\sigma = \{t \,|\, \mathsf{E}[\eta_t^\sigma(\omega)] \geq M - \varepsilon/2\}$. Since $\lim_{t\to\infty}\mathsf{E}[\tilde{\eta}_t(\omega)] = M$ by (31), the set $J_\varepsilon^\sigma = \{t \,|\, \mathsf{E}[\tilde{\eta}_t(\omega)] < M + \varepsilon/2\}$ is also infinite. For all $t \in G_\varepsilon^\sigma \cap J_\varepsilon^\sigma$, by Lemma 5,

$$0 \geq \mathsf{E}[A_t^\sigma - \tilde{A}_t] = \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] + \mathsf{E}[\eta_t^\sigma - \tilde{\eta}_t]u \geq \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] - u\varepsilon$$

(the ω is suppressed for convenience). Since ε can be arbitrarily small, the above inequality implies that

$$\liminf_{t\to\infty} \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] \leq 0.$$

By definition of overtaking criterion (7), strategy σ does not overtake $\tilde{\sigma}$.

Case 2. $\limsup_{t\to\infty}\mathsf{E}[\eta_t^\sigma(\omega)] < M$. By the proof of Lemma 5, for all $\omega \in \Omega$, $\{A_t^\sigma(\omega) - \tilde{A}_t(\omega)\}_{t=0}^\infty$ is a weakly decreasing sequence. If $\mathsf{E}[A_t^\sigma(\omega) - \tilde{A}_t(\omega)] \not\to -\infty$ as $t \to \infty$, by Lemma 6, $\liminf_{t\to\infty}\mathsf{E}[\eta_t^\sigma(\omega)] \geq M$, which contradicts to the assumption. If $\mathsf{E}[A_t^\sigma - \tilde{A}_t] \to -\infty$ as $t \to \infty$, and since

$$\mathsf{E}[A_t^\sigma - \tilde{A}_t] = \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] + \mathsf{E}[\eta_t^\sigma - \tilde{\eta}_t]u$$

we have

$$\liminf_{t\to\infty} \mathsf{E}[\sum_{\tau=0}^{t-1} v_\tau^\sigma - \sum_{\tau=0}^{t-1} \tilde{v}_\tau] \leq \liminf_{t\to\infty} \mathsf{E}[A_t^\sigma - \tilde{A}_t] + uM - u\liminf_{t\to\infty} \mathsf{E}[\eta_t^\sigma] = -\infty.$$

Again by the definition (7), strategy σ does not overtake $\tilde{\sigma}$. ∎

By Proposition 7, strategy $\tilde{\sigma}$ is a Bayesian Nash strategy according to the overtaking criterion. This proves (ii) of the equilibrium definition, and (iii) is evidently satisfied. Hence, the price-1 equilibrium always exists.

6. Conclusion

We have shown that a price-1 equilibrium from an initial state exists (under a mild assumption) and is asymptotically stationary. In fact, there can be other equilibria as well. For example, suppose that the initial distribution is distributed on the lattice in multiples of six. Then there would be equilibria with asymptotic distributions on the the lattices in multiples of one, two, three and six. Because the asymptotic distributions differ, the equilibria clearly are distinct. It will be apparent from our earlier paper (Green and Zhou [2]) that the equilibrium asymptotically distributed on the finest lattice is the one that achieves the highest level of welfare for the economy, since it facilitates the greatest amount of trade.

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