Center Manifold, Stability, and Bifurcations in Continuous Time Macroeconometric Systems

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Abstract

In a recent paper, we studied bifurcation phenomena in continuous time macroeconometric models. The objective was to explore the relevancy of Grandmont's (1985) findings to models permitting more reasonable elasticities than were possible in Grandmont's Cobb Douglas overlapping generations model. Another objective was to explore the relevancy of his findings to a model in which some solution paths are not Pareto optimal, so that policy rules can serve a clearly positive purpose. We used the Bergstrom, Nowman, and Wymer (1992) UK continuous time second order differential equations macroeconometric model that permits closer connection with economic theory than is possible with most discrete time structural macroeconometric models. We do not yet have the ability to explore these phenomena in a comparably general Euler equations model having deep parameters, rather than structural parameters.

It was discovered that the UK model displays a rich set of bifurcations including transcritical bifurcations, Hopf bifurcations, and codimension two bifurcations. The point estimates of the parameters are in the unstable region. But we did not test the null hypothesis that the parameters are actually in the stable region. In addition, we did not investigate the dynamical properties on the bifurcation boundaries; and we did not investigate the relevancy of stabilization policy rules.

In this paper, we further examine the stability properties and bifurcation boundaries of the UK continuous time macroeconometric models by analyzing the stability of the model along center manifolds. The results of this paper show that the model is unstable on bifurcation boundaries for those cases we consider. Hence calibration of the model to operate on those bifurcation boundaries would produce no increase in the model's ability to explain observed data. However, we have not yet determined the dynamic properties of the model on the Hopf bifurcation boundaries, which sometimes do produce useful dynamical properties for some models. Of more immediate interest, it is also shown that bifurcations exist within the Cartesian product of 95% confidence intervals for the estimators of the individual parameters. This seems to suggest that we cannot reject the null hypothesis of stability, despite the fact that the point estimates are in the unstable region. However, when we decreased the confidence level to 90%, the intersection of the stable region and the Cartesian product of the confidence intervals became empty, thereby suggesting rejection of stability. But a formal sampling theoretic hypothesis test of that null would be very difficult to conduct, since some of the sampling distributions are truncated by boundaries, and since there are some corner solutions. A Bayesian approach might be possible, but would be very difficult to implement.

A new formula is also given for finding the closed forms of transcritical bifurcation boundaries. Finally, effects of fiscal policy on stability are considered. It is found that change in fiscal policy may affect the stability of the continuous time macroeconometric models. But we find that the selection of an advantageous stabilization policy is more difficult than expected. Augmentation of the model by feedback policy rules chosen from plausible economic reasoning can contract the stable region and thereby be counterproductive, even if the policy is time consistent and has insignificant effect on structural parameter values.

Keywords: Stability, bifurcation, macroeconometric systems

JEL Classification: C52, C32, E61

1 Introduction

1.1 Objectives

In Barnett and He (1998), we studied bifurcation phenomena in continuous time macroeconometric models to determine the degree to which Grandmont's (1985) findings with a simple Cobb Douglas overlapping generations model are relevant to less restrictive models. It has often been asked whether the complex dynamics attained by Grandmont in multiple bifurcation regimes could be attained with a more general model in a manner that would be consistent with more reasonable settings of elasticities for tastes and technology. In addition, since all solutions in Grandmont's model are Pareto optimal, the policy relevancy of complex dynamics in his model is not clear. It has sometimes been asked whether a policy relevant model might present a more important role for complex dynamics and for the existence of multiple bifurcation regimes. See, e.g., Woodford (1989), who speculates that the existence of complex dynamics may increase the potentially useful role of active policy, if imperfections exist in financial markets. Policy relevance also is implied by the recent paper by Goenka, Kelly, and Spear (1998).

Barnett and Chen (1988) and Barnett et al. (1997), among others, have tested for chaos and for other forms of nonlinearity in univariate time series. Their findings, however, do not condition upon an economic model, and hence cannot isolate the source of instability to be within the economy. If there were chaos in the weather, those chaotic shocks to the economy would be the source of the chaos observed in economic time series. Similarly the many findings of nonlinearity in univariate economic time series could have been caused by nonlinear shocks from the weather or other such sources external to the economy. Hence further progress in this areas requires the ability to condition upon an economic model. Mathematical solution for the boundary of the chaotic subset of the parameter space currently is not possible with models having more than three parameters, and at present we are not seeking to solve for that subset by numerical methods. But we do find that numerical solution for bifurcation boundaries between the stable subset and broader classes of nonlinear dynamic behavior can be accomplished. Although nonlinearity is central to this literature, our results currently are inherently local, since our approach is based upon a local analysis of a nonlinear model.

Extending Grandmont's results to more general and empirically plausible stochastic dynamic general equilibrium models is extremely difficult, and presents problems that currently cannot be solved analytically by methods available to mathematicians. With the implied systems of nonlinear Euler equations, even numerical methods have so far not been successfully applied to locating and characterizing bifurcation boundaries for such direct extensions of Grandmont's model. But with some policy relevant structural macroeconometric models, numerical methods for locating bifurcation boundaries currently are applicable. For that reason, we have chosen at present to work with a structural macroeconometric model. Although this approach does not permit us to access the deep parameters of tastes and technology, we nevertheless can produce closer connection to theory than would be possible with a more conventional discrete time macroeconometric model, by using a continuous time macroeconometric model. For that reason, in the prior paper and in this paper we currently are applying our numerical procedures to the Bergstrom, Nowman, and Wymer (1992) UK continuous time second order differential equations macroeconometric model. At a later date, we contemplate extending these procedures to apply to the Powell and Murphy (1997) model and eventually to a system of Euler equations derived from a reasonably plausible stochastic dynamic general equilibrium model. A first step in that direction is likely to be the application of these methods to the model in Leeper and Sims (1994).

In our prior paper, we demonstrated the existence of bifurcation boundaries within the economically feasible subset of the parameter space. Since the point estimates of the parameters are within the unstable region, it is natural to ask whether or not the null hypothesis of stability can be rejected. In this paper we have three objectives: (1) we explore the question of whether or not we can reject the hypothesis that the true values of the parameters are across a bifurcation boundary in the stable region. (2) We investigate the nature of the model's dynamics on bifurcation boundaries. (3) We explore the ability of policy control rules to move the bifurcation boundaries in such a manner as to include within the stable region the existing unchanged point estimates of the parameters.

The reason for the first objective is clear. But how to do it is less than obvious. The various subsets of the parameter space bounded by bifurcation boundaries and by economic feasibility constraints are defined by nonlinear inequality constraints. Such inequality constraints truncate sampling distributions. Not only were some of the parameter estimates close to boundaries, but in fact some of the parameter estimates were on boundaries. These facts violate regularity conditions for most sampling theoretic hypothesis testing procedures. While such methods as Kuhn-Tucker tests exist to deal with inequality constraints, those tests are available only for much simpler classes of models than we are using. While in principle, a Bayesian approach is possible, the application of Bayesian methods with such high dimensional irregular shaped sets is prohibitively challenging at the present time. In addition, the existing parameter estimates reported for the UK model by Bergstrom, Nowman, and Wymer (1992) were provided with standard errors but not with a full covariance matrix. Hence we do not have available the covariances between those estimators. Under these adverse circumstances, we limit our statistical inference to the use of the confidence intervals about the individual parameter estimates, and we produce the region defined by the Cartesian product of those intervals. The resulting Cartesian product region is centered about the point estimate, which is in the unstable region. The correct unknown multidimensional confidence region ellipsoid is likely to be a subset of that Cartesian product, and hence the resulting "test" is probably biased towards accepting stability when it is false. Clearly a rejection of stability by this approach is more convincing than an acceptance.

The second objective relates to the following fact: calibration of dynamic models such that the parameters are on a Hopf bifurcation boundary sometimes can produce a better match of model dynamics to observed data than is possible off bifurcation boundaries. Despite the fact that bifurcation boundaries are measure zero subsets of the parameter space, the economy's selection of parameter values from the parameter space may be far from random. In fact, there has been much recent interest in how the economy determines parameter values, perhaps through learning. Since the stable parameter subset typically is a small subset of the parameter space for dynamical models of physical systems, there is little reason in nature to expect to observe stable dynamics, unless human engineers have intentionally moved the parameters of a designed system into the stable region. Since human engineers do not set the parameters of atmospheric conditions and other natural phenomena, such natural systems do not typically converge to a stable solution. Similarly in economic systems, we should expect some mechanism to be at work in the setting of parameters, if they are observed to be in stable regions, rather than in any of the other infinite number of possible bifurcation regimes. The "uncertainty principle" suggested by Grandmont (1998) is a possibility. But if parameters are not drawn at random and can be expected to appear in unlikely subsets, then we cannot exclude the possibility that they might be on bifurcation boundaries through an unknown selection process. In addition, if the observed settings of parameters in the economy resulted from a currently unknown learning process, then what happens as bifurcation boundaries are crossed during that process becomes a nontrivial issue and could depend upon what happens on that measure zero boundary as it is being crossed. Finally the most important reason for concern about what happens when a bifurcation boundary is crossed is described in the third objective below: augmenting the model with a policy feedback law operates by intentionally moving bifurcation boundaries, even if all of the parameters elsewhere in the model are unaffected by the existence of the policy. Hence bifurcation need not result from parameters moving across fixed bifurcation boundaries, but rather from bifurcation boundaries themselves moving, while the parameters outside the policy feedback rule remain constant.

The third objective investigates the implications of augmenting the model's equations by a policy feedback rule. According to the Lucas critique, the augmentation of Euler equations with such a feedback rule can alter structural parameters. We ask whether the use of such feedback rules is as straightforward as previously believed, even if their use does not affect structural parameters. We find the selection of stabilization policy to be more easily counterproductive than previously believed, even without Lucas critique problems affecting the selection. When we consider the use of optimal control theory to choose feedback rules, the chance of success increases, but the risk of time inconsistency increases, and the existence of specification error in the model could undermine the appearance of policy success within the model.

1.2 Background

Much research effort has been devoted to analyzing economic dynamics. One particular area of interest is the analysis of bifurcation and chaotic phenomena in economic systems [see, for example, Barnett et al. (1996), Gandolfo (1996), Medio (1992), Wymer (1997)]. It has been demonstrated that economic systems exhibit many types of bifurcations such as pitchfork bifurcations in the tatonement process [see, Bala (1997), Scarf (1960)], transcritical bifurcations in the Bergstrom, Nowman and Wymer continuous time macroeconometric model [Barnett and He (1998)], Hopf bifurcations in growth models [e.g., Benhabib (1979), Boldrin and Woodford (1990), Dockner and Feichtinger (1991), Nishimura and Takahashi (1992)], and codimension two bifurcations in Barnett and He (1998). The theory of bifurcations and chaos in economics are described in several textbooks such as Gandolfo (1996), Medio (1992), and Puu (1991).

Bifurcations exist in both discrete time models [see, for instance, Boldrin and Woodford (1990), Gandolfo (1996)] and continuous time models [e.g., Barnett and He (1998), Gandolfo (1996), Medio (1992)]. Recently, there has been increasing interest in continuous time macroeconometric models since such models have several advantages, including higher modeling accuracy and the capability of forecasting the continuous time path of the variables [see, Bergstrom (1996), Bergstrom and Wymer (1976), Bergstrom et al. (1992), Nieuwenhuis and Schoonbeek (1997), Wymer (1997)]. Bergstrom (1996) provides an excellent survey of research advances in continuous time macroeconometric models. Continuous time models have been established for several countries including the UK and the US [see, for example, Bergstrom (1996)]. A typical feature of those models is that parameters are estimated to be in the unstable region, but close to the boundary with the stable region [see, for example, Bergstrom et al. (1992), Barnett and He (1998)]. In those models, parameter values are estimated based on real historical economic data. Since errors exist in parameter estimation, a natural and important task is to determine whether the inference of instability is statistically significant. Structural analysis of continuous time models has been considered in Gandolfo (1992), Nieuwenhuis and Schoonbeek (1997). Recently, Barnett and He (1998) investigated bifurcation phenomena in the Bergstrom, Nowman and Wymer continuous time macroeconometric model of the United Kingdom. We found that both transcritical bifurcations and Hopf bifurcations exist in that model. A detailed procedure was given in that paper for determining the bifurcation boundaries. The existence of an important class of codimension two bifurcations was also confirmed in that paper.

This paper is a continuation of Barnett and He (1998), aiming at further analysis of bifurcations in continuous time macroeconometric models. Contributions of this paper are stated as the follows:

- *Analysis of dynamic behavior on bifurcation boundaries.* A detailed analysis of the dynamic behavior of the continuous time model is provided when the parameters take their values on some of the bifurcation boundaries. In many cases we have tested, the Bergstrom, Nowman and Wymer model is unstable on bifurcation boundaries. But the nature of the dynamics at some points on the bifurcation boundaries remain unknown.

- *A new formula for determining transcritical bifurcation boundaries.* By utilizing the special structure of the linearized part of the system, a formula is provided for determining the analytical forms of the transcritical bifurcation boundaries.

- *The statistical significance of the instability inference.* Although the point estimates of the parameters are in the unstable region, those estimates appear to be relatively close to the bifurcation boundaries with the stable region. To determine whether the inference of instability is statistically significant, we first determine whether the bifurcation boundaries are within the economically feasible region of the parameter space. Those feasibility constraints are described in Table 2 of Bergstrom et al. (1992). We then produce the intersection of that feasible region with the Cartesian product of confidence intervals about each parameter estimate, to determine whether any of the bifurcation boundaries enter that intersection.

- *Effect of fiscal policy on stability.* To determine the potential usefulness of fiscal policy, we investigate the effect on bifurcation boundaries of augmenting the model with a fiscal policy feedback rule. We assume that the parameters of the other equations are not affected by the addition of the feedback fiscal policy rule, so that the source of the movement of the bifurcation boundaries is not the Lucas critique issue. The intent is to determine whether such rules can be expected to move the boundaries in such a manner as to include the point estimate of the parameters within the stable region. We investigate this matter both with a policy based upon heuristic economic reasoning and a policy derived from optimal control theory under the assumption of reputation equilibrium and intertemporal time consistency of the policy.

2 A continuous time macroeconometric model

Although the approach adopted in this paper is applicable to any continuous time macroeconometric systems, we will restrict our discussion to the well-regarded Bergstrom, Nowman and Wymer continuous time macroeconometric model of the United Kingdom given in Bergstrom et al. (1992) to ensure the relevance of the theory with practice. The model is described by the following 14 second-order differential equations.

$$D^2 \log C = \gamma_1(\lambda_1 + \lambda_2 - D\log C) + \gamma_2 \log\left[\frac{\beta_1 e^{-\{\beta_2(r - D\log p) + \beta_3 D\log p\}}(Q+P)}{T_1 C}\right] \tag{1}$$

$$D^2 \log L = \gamma_3(\lambda_2 - D\log L) + \gamma_4 \log\left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{L}\right] \tag{2}$$

$$D^2 \log K = \gamma_3(\lambda_1 + \lambda_2 - D\log K) + \gamma_6 \log\left[\frac{\beta_5 (Q/K)^{1+\beta_6}}{r - \beta_7 D\log p + \beta_8}\right] \tag{3}$$

$$D^2 \log Q = \gamma_7(\lambda_1 + \lambda_2 - D\log Q) + \gamma_8 \log\left[\frac{\{1 - \beta_9 (qp/p_i)^{\beta_{10}}\}(C + G_c + DK + E_n + E_o)}{Q}\right] \tag{4}$$

$$D^2 \log p = \gamma_9(D\log(w/p) - \lambda_1) + \gamma_{10} \log\left[\frac{\beta_{11}\beta_4 T_2 w e^{-\lambda_1 t}\{1 - \beta_5 (Q/K)^{\beta_6}\}^{-(1+\beta_6)/\beta_6}}{p}\right] \tag{5}$$

$$D^2 \log w = \gamma_{11}(\lambda_1 - D\log(w/p)) + \gamma_{12} D\log(p_i/qp) + \gamma_{13} \log\left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{\beta_{12} e^{\lambda_2 t}}\right] \tag{6}$$

$$D^2 r = -\gamma_{14} Dr + \gamma_{15}\left[\beta_{13} + r_f - \beta_{14} D\log q + \beta_{15}\frac{p(Q+P)}{M} - r\right] \tag{7}$$

$$D^2 \log I = \gamma_{16}(\lambda_1 + \lambda_2 - D\log(p_i I/qp))$$
$$+\gamma_{17}\log\left[\frac{\beta_9 (qp/p_i)^{\beta_{10}}(C + G_c + DK + E_n + E_o)}{(p_i/qp)I}\right] \tag{8}$$

$$D^2 \log E_n = \gamma_{18}(\lambda_1 + \lambda_2 - D\log E_n) + \gamma_{19}\log\left[\frac{\beta_{16} Y_f^{\beta_{17}}(p_f/qp)^{\beta_{18}}}{E_n}\right] \tag{9}$$

$$D^2 F = -\gamma_{20} DF + \gamma_{21}[\beta_{19}(Q+P) - F] \tag{10}$$

$$D^2 P = -\gamma_{22} DP + \gamma_{23}\{[\beta_{20} + \beta_{21}(r_f - D\log p_f)]K_a - P\} \tag{11}$$

$$D^2 K_a = -\gamma_{24} DK_a + \gamma_{25}\{[\beta_{22} + \beta_{23}(r_f - r) - \beta_{24} D\log q - \beta_{25} d_x](Q+P) - K_a\} \tag{12}$$

$$D^2 \log M = \gamma_{26}(\lambda_3 - D\log M) + \gamma_{27}\log\left[\frac{\beta_{26} e^{\lambda_3 t}}{M}\right]$$
$$+\gamma_{28} D\log\left[\frac{E_n + E_o + P - F}{(p_i/qp)I}\right] + \gamma_{29}\log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I}\right] \tag{13}$$

$$D^2 \log q = \gamma_{30} D\log(p_f/qp) + \gamma_{31}\log\left[\frac{\beta_{27} p_f}{qp}\right] + \gamma_{32} D\log\left[\frac{E_n + E_o + P - F}{(p_i/qp)I}\right]$$
$$+\gamma_{33}\log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I}\right] \tag{14}$$

where t is time, D is the derivative operator, $Dx = dx/dt$, $D^2 x = d^2x/dt^2$, and C, E_n, F, I, K, K_a,L, M, P, Q, q, r, w are endogenous variables whose definitions are listed below.

C	real private consumption
E_n	real non-oil exports
F	real current transfers abroad
I	volume of imports
K	amount of fixed capital
K_a	cumulative net real investment abroad (excluding changes in official reserve)
L	employment
M	money supply
P	real profits, interest and dividends from abroad
p	price level
Q	real net output
q	exchange rate (price of sterling in foreign currency)
r	interest rate
w	wage rate

The variables $d_x, E_o, G_c, p_f, p_i, r_f, T_1, T_2, Y_f$ are exogenous variables with the following definitions:

d_x= dummy variable for exchange controls (d_x=1 for 1974-79, d_x=0 for 1980 onwards)
E_o= real oil exports
G_c= real government consumption
p_f= price level in leading foreign industrial countries
p_i= price of imports (in foreign currency)
r_f= foreign interest rate
T_1= total taxation policy variable defined by Bergstrom et al. (1992, p. 317)
T_2= indirect taxation policy variable defined by Bergstrom et al. (1992, p. 317)
Y_f= real income of leading foreign industrial countries,

The structural parameters $\beta_i, i = 1, 2, ..., 27$, $\gamma_j, j = 1, 2, ..., 33$, and $\lambda_k, k = 1, 2, 3$, can be estimated from historical data. A set of their estimates using quarterly data from 1974 to 1984 are given in Table 2 of Bergstrom et al. (1992). These equations are derived from economic theory. The exact interpretations of these 14 equations are available in Bergstrom et al. (1992).

Both endogenous and exogenous variables are time-varying quantities. The exogenous variables are assumed to satisfy the following conditions in equilibrium: $d_x = 0, E_o = 0, G_c = g^*(Q + P), p_f = p_f^* e^{\lambda_4 t}, p_i = p_i^* e^{\lambda_4 t}, r_f = r_f^*, T_1 = T_1^*, T_2 = T_2^*, Y_f = Y_f^* e^{((\lambda_1+\lambda_2)/\beta_{17})t}$, where g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^* and λ_4 are constants. As explained in Bergstrom et al. (1992), the assumptions are reasonable. Under such assumptions, it has been proven that $C(t)$, ..., $q(t)$ in (1)-(14) change at constant rates in equilibrium. Note that the system described by (1)-(14) is not autonomous, since time itself enters as an exogenous variable. To study the dynamics of the system around the equilibrium, we make a transformation by defining a set of new variables $y_1(t)$, $y_2(t)$, ..., $y_{14}(t)$:

$$y_1(t) = \log\{C(t)/C^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_2(t) = \log\{L(t)/L^* e^{\lambda_2 t}\}$$

$$y_3(t) = \log\{K(t)/K^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_4(t) = \log\{Q(t)/Q^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_5(t) = \log\{p(t)/p^* e^{(\lambda_3-\lambda_1-\lambda_2)t}\}$$

$$y_6(t) = \log\{w(t)/w^* e^{(\lambda_3-\lambda_2)t}\}$$

$$y_7(t) = r(t) - r^*$$

$$y_8(t) = \log\{I(t)/I^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_9(t) = \log\{E_n(t)/E_n^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{10}(t) = \log\{F(t)/F^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{11}(t) = \log\{P(t)/P^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{12}(t) = \log\{K_a(t)/K_a^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{13}(t) = \log\{M(t)/M^* e^{\lambda_3 t}\}$$

$$y_{14}(t) = \log\{q(t)/q^* e^{(\lambda_1+\lambda_2+\lambda_4-\lambda_3)t}\}$$

where $C^*, L^*, K^*, Q^*, p^*, w^*, r^*, I^*, E_n^*, F^*, P^*, K_a^*, M^*, q^*$ are functions of the vector (β, γ, λ) of 63 parameters in equations (1)-(14) and the additional parameters g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, λ_4. The following is a set of differential equations derived from (1)-(14):

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2 \{\log(Q^* e^{y_4} + P^* e^{y_{11}}) - \log(Q^* + P^*) - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1\} \tag{15}$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left\{ \frac{1}{\beta_6} \log\left[\frac{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}}{(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3}} \right] - y_2 \right\} \tag{16}$$

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \Big\{ (1+\beta_6)(y_4 - y_3) + \log[r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8] - \log[y_7 + r^* - \beta_7(D y_5 + \lambda_3 - \lambda_1 - \lambda_2) + \beta_8] \Big\} \tag{17}$$

$$D^2 y_4 = -\gamma_7 D y_4 + \gamma_8 \Big\{ \log\left[\frac{1 - \beta_9 (q^* p^*/p_i^*)^{\beta_{10}} e^{\beta_{10}(y_5 + y_{14})}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \right] + \log\left(C^* e^{y_1} + g^*(Q^* e^{y_4} + P^* e^{y_{11}}) + K^* e^{y_3}(D y_3 + \lambda_1 + \lambda_2) + E_n^* e^{y_9}\right) - \log\left(C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*\right) - y_4 \Big\} \tag{18}$$

$$D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \left\{ y_6 - y_5 - \frac{1+\beta_6}{\beta_6} \log\left[1 - \beta_5 \left(\frac{Q^*}{K^*} \right)^{\beta_6} e^{\beta_6 (y_4 - y_3)} \right] + \frac{1+\beta_6}{\beta_6} \log\left[1 - \beta_5 \left(\frac{Q^*}{K^*} \right)^{\beta_6} \right] \right\} \tag{19}$$

$$D^2 y_6 = \gamma_{11}(D y_5 - D y_6) - \gamma_{12}(D y_5 + D y_{14}) + \gamma_{13} \left\{ \frac{1}{\beta_6} \log[(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}] - \frac{1}{\beta_6} \log[(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3}] \right\} \tag{20}$$

$$D^2y_7 = -\gamma_{14}Dy_7 + \gamma_{15}[[\beta_{15}\frac{p^*e^{y_5}(Q^*e^{y_4} + P^*e^{y_{11}})}{M^*e^{y_{13}}} - \beta_{15}\frac{p^*(Q^* + P^*)}{M^*} - \beta_{14}Dy_{14} - y_7] \quad (21)$$

$$D^2y_8 = \gamma_{16}(Dy_5 + Dy_{14} - Dy_8) + \gamma_{17}\Big\{(1 + \beta_{10})(y_5 + y_{14}) - y_8$$
$$+ \log[C^*e^{y_1} + g^*(Q^*e^{y_4} + P^*e^{y_{11}}) + K^*e^{y_3}(Dy_3 + \lambda_1 + \lambda_2) + E_n^*e^{y_9}]$$
$$- \log[C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*]\Big\} \quad (22)$$

$$D^2y_9 = -\gamma_{18}Dy_9 - \gamma_{19}\{\beta_{18}(y_5 + y_{14}) + y_9\} \quad (23)$$

$$D^2y_{10} = -\{\gamma_{20}+2(\lambda_1+\lambda_2)\}Dy_{10}-(Dy_{10})^2+\gamma_{21}\beta_{19}\Big\{\frac{Q^*e^{y_4}+P^*e^{y_{11}}}{F^*e^{y_{10}}} - \frac{Q^*+P^*}{F^*}\Big\} \quad (24)$$

$$D^2y_{11} = -\{\gamma_{22} + 2(\lambda_1 + \lambda_2)\}Dy_{11} - (Dy_{11})^2$$
$$+\gamma_{23}\{\beta_{20} + \beta_{21}(r_f^* - \lambda_4)\}\left[\frac{K_a^*e^{y_{12}}}{P^*e^{y_{11}}} - \frac{K_a^*}{P^*}\right] \quad (25)$$

$$D^2y_{12} = -\{\gamma_{24} + 2(\lambda_1 + \lambda_2)\}Dy_{12} - (Dy_{12})^2 + \gamma_{25}\Big\{[\beta_{22} + \beta_{23}(r_f^* - r^* - y_7)$$
$$-\beta_{24}(Dy_{14} + \lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^*e^{y_4} + P^*e^{y_{11}}}{K_a^*e^{y_{12}}} - [\beta_{22} + \beta_{23}(r_f^* - r^*)$$
$$-\beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^* + P^*}{K_a^*}\Big\} \quad (26)$$

$$D^2y_{13} = -\gamma_{26}Dy_{13} - \gamma_{27}y_{13} + \gamma_{28}\Big\{\frac{E_n^*e^{y_9}Dy_9 + P^*e^{y_{11}}Dy_{11} - F^*e^{y_{10}}Dy_{10}}{E_n^*e^{y_9} + P^*e^{y_{11}} - F^*e^{y_{10}}}$$
$$+Dy_5 + Dy_{14} - Dy_8\Big\} + \gamma_{29}\Big\{\log[E_n^*e^{y_9} + P^*e^{y_{11}} - F^*e^{y_{10}}$$
$$-K_a^*e^{y_{12}}(Dy_{12} + \lambda_1 + \lambda_2)] - \log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)]$$
$$+y_5 + y_{14} - y_8\Big\} \quad (27)$$

$$D^2y_{14} = -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14})$$
$$+\gamma_{32}\Big\{\frac{E_n^*e^{y_9}Dy_9 + P^*e^{y_{11}}Dy_{11} - F^*e^{y_{10}}Dy_{10}}{E_n^*e^{y_9} + P^*e^{y_{11}} - F^*e^{y_{10}}} + Dy_5 + Dy_{14} - Dy_8\Big\}$$
$$+\gamma_{33}\Big\{\log[E_n^*e^{y_9} + P^*e^{y_{11}} - F^*e^{y_{10}} - K_a^*e^{y_{12}}(Dy_{12} + \lambda_1 + \lambda_2)]$$

$$-\log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)] + y_5 + y_{14} - y_8 \Big\} \tag{28}$$

The equilibrium of the original system (1)-(14) corresponds to the equilibrium $y_i = 0, i = 1, 2, ..., 14$ of (15)-(28). The major advantage of the new system described by (15)-(28) is that it is autonomous, but still retains all the dynamic properties of the original system (1)-(14). Autonomous systems are the main subject of nonlinear systems theory. Generally speaking, it is difficult to analyze non-autonomous systems. In this paper, we will analyze the local dynamics of (15)-(28) in a local neighborhood of the equilibrium $y_i = 0, i = 1, 2, ..., 14$. For simplicity, the system (15)-(28) is denoted as

$$Dx = f(x, \theta), \tag{29}$$

where

$$x = [y_1 \;\; Dy_1 \;\; y_2 \;\; Dy_2 \;\; ... \;\; y_{14} \;\; Dy_{14}]' \in R^{28}$$

is the state vector, while

$$\theta = [\beta_1, ..., \beta_{27}, \gamma_1, ..., \gamma_{33}, \lambda_1, \lambda_2, \lambda_3]' \in R^{63}$$

is the parameter vector, and $f(x, \theta)$ is a vector of functions of x and θ obtained from (15)-(28). Every component of $f(x, \theta)$ is smooth (infinitely differentiable) in a neighborhood of the origin. Note that (29) is a first-order system. The point $x^* = 0$ is an equilibrium of (29). Since θ represents physical quantities, its entries are bounded by theoretical and a priori feasibility constraints [see, Table 2 of Bergstrom et al. (1992)]. Let Θ denote the feasible region determined by those bounds. Θ is a bounded region.

3 Stability of the Equilibrium

In this section, we examine the local stability of the system (29) around the equilibrium $x^* = 0$. For this purpose, write (29) as

$$Dx = A(\theta)x + F(x, \theta), \tag{30}$$

where

$$A(\theta) = \frac{\partial f(x, \theta)}{\partial x}|_{x=x^*}$$

is the Jacobian of $f(x, \theta)$ evaluated at the equilibrium $x^* = 0$, $A(\theta) \in R^{28\times 28}$, and

$$F(x, \theta) = f(x, \theta) - A(\theta)x = o(x)$$

is the terms of higher order. In nonlinear systems theory, the local stability of (29) can be studied by examining the eigenvalues of the coefficient matrix $A(\theta)$. Briefly,

(a) if all eigenvalues of $A(\theta)$ have strictly negative real parts, then (29) is locally asymptotically stable in the neighborhood of x^*.

(b) If at least one of the eigenvalues of $A(\theta)$ has positive real part, then (29) is locally asymptotically unstable in the neighborhood of x^*.

(c) If all eigenvalues of $A(\theta)$ have nonpositive real parts and at least one has zero real part, the stability of (29) usually cannot be determined from the matrix $A(\theta)$. One needs to analyze higher order terms in order to determine the stability of the system. In most cases, one needs to examine the system behavior along a certain manifold to determine the stability [see, e.g., Khalil (1992)].

Since $A(\theta)$ is a function of θ, stability of (29) could be dependent on θ, which we shall verify is indeed true. For the set of estimated values of $\{\beta_i\}$, $\{\gamma_j\}$, and $\{\lambda_k\}$ given in Table 2 of Bergstrom et al. (1992), all the eigenvalues of $A(\theta)$ are stable except three of them:

$$s_1 = 0.0033, s_2 = 0.0090 + 0.0453i, s_3 = 0.0090 - 0.0453i,$$

where $i = \sqrt{-1}$ is the imaginary unit. However, the real parts of these unstable eigenvalues are (relatively) so small that it is unclear that their signs are statistically significant. Next, we shall examine the statistical significance of the inference of instability.

For each $\theta_i, i = 1, 2, ..., 63$, its estimate and the corresponding standard deviation are provided in Table 2 of Bergstrom et al. (1992). For a given confidence level p, a confidence interval can be obtained for any θ_i:

$$[\hat{\theta}_i - \epsilon_p\sigma_i, \quad \hat{\theta}_i + \epsilon_p\sigma_i]$$

where $\hat{\theta}_i$ and σ_i are respectively the estimate and standard deviation for parameter θ_i, and ϵ_p is the standard normal percentile, as is consistent with the distributional assumptions in Bergstrom et al. (1992). Both $\hat{\theta}_i$ and σ_i are available in Table 2 of Bergstrom et al. (1992). For example, the 95% confidence interval for $\theta_1 = \beta_1$ is $[0.9324, 0.9476]$. Let $[\underline{\theta}_i, \bar{\theta}_i]$ denote the confidence interval for parameter θ_i, $i = 1, 2, ..., 63$. For several parameters, the estimates were on the boundaries of the theoretical feasible intervals. In this case, we assume $\sigma_i = 0$ and the confidence interval becomes one point, the estimate. There are 8 such parameters: $\theta_8, \theta_{11}, \theta_{13}, \theta_{28}, \theta_{33}, \theta_{48}, \theta_{60}, \theta_{63}$. Hence our inferences condition upon their corner solution values. Define

$$\Theta_1 = \{\theta \in \Theta \mid \theta_i \in [\underline{\theta}_i, \bar{\theta}_i], i = 1, 2, ..., 63\}.$$

Then Θ_1, $\Theta_1 \subset \Theta$, denotes the region of θ determined by the Cartesian product of those confidence intervals. Any change in the stability of (29) over Θ_1 implies that we cannot reject the hypothesis that the parameters are on the other side of the bifurcation boundary from the side on which their points estimates lie.

Consider the following problem of minimizing the maximum real parts of eigenvalues of matrix $A(\theta)$:

$$\min_{\theta \in \Theta} R_{\max}(A(\theta)) \tag{31}$$

where

$$R_{\max}(A(\theta)) = \max_i \{\text{real}(\lambda_i) : \lambda_1, \lambda_2, ..., \lambda_{28} \text{ are eigenvalues of } A(\theta)\}.$$

Since the dimension of $A(\theta)$ is 28×28, which is relatively high, we cannot acquire a closed-form expression for $R_{\max}(A(\theta))$. We use the gradient method to solve the minimization problem (31). More precisely, let $\theta^{(0)}$ be the estimated set of parameter values given in Table 2 of Bergstrom et al. (1992). At step n, $n \geq 0$, with $\theta^{(n)}$, let

$$\theta^{(n+1)} = \pi_{\Theta_1}[\theta^{(n)} - a_n \frac{\partial R_{\max}(A(\theta))}{\partial \theta}|_{\theta=\theta^{(n)}}],$$

where $\{a_n, n = 0, 1, 2, ...\}$ is a sequence of (positive) step sizes and $\pi_{\Theta_1}[\theta]$ is the projection onto Θ_1. The algorithm found the following point, $\theta^* \in \Theta_1$,

$$\begin{aligned}\theta^* = [&0.9400, 0.2936, 2.6871, 0.2030, 0.2562, 0.1961, 0.1345, 0.0000, 0.2440,\\ &-0.2577, 1.0000, 23.5000, -0.0100, 0.0473, 0.0288, 13.5460, 0.4562, 1.0678,\\ &0.0100, 0.0061, 0.2763, 0.2948, 44.8543, 0.1173, 0.0004, 71.4241, 0.8213,\\ &4.0000, 1.0289, 0.6698, 0.0697, 0.1311, 0.0010, 3.7078, 0.4860, 1.0537,\\ &0.0042, 3.4562, 0.4858, 0.1300, 1.0044, 0.0379, 1.3839, 0.3777, 3.9947,\\ &3.6534, 3.9995, 4.0000, 4.0000, 3.9400, 0.4775, 0.0071, 0.6114, 0.0574,\\ &0.1718, 0.1227, 2.2845, 0.1489, 0.0035, 0.0000, 0.0042, 0.0036, 0.0100].\end{aligned}$$

The corresponding $R_{\max}(A(\theta^*)) = -0.0017$, implying that all eigenvalues of $A(\theta^*)$ have strictly negative real parts and the system (29) is locally asymptotically stable around $x^* = 0$ for θ^*. This suggests that we cannot reject the hypothesis of stability.

One interesting fact is that if we reduce the confidence level to 90%, which results in smaller confidence intervals, the algorithm failed to find a value of θ under which the system (29) is stable. This seems to suggest that, with 90% confidence level, the system (29) is unstable for all parameter $\theta \in \Theta_1$, and we cannot accept the hypothesis of stability.

4 Transcritical bifurcations

On one hand, we have seen in the previous section that $A(\theta)$ has three eigenvalues with strictly positive real parts for the set of parameter values given in Table 2 of Bergstrom et al. (1992). On the other hand, all eigenvalues of $A(\theta)$ have strictly negative real parts for $\theta = \theta^*$. Since eigenvalues are continuous functions of entries of $A(\theta)$, there must exist parameter values of θ such that the (29) becomes unstable from stable (or stable from unstable) when θ crosses such values. Those parameter values correspond to bifurcation points at which the system (29) has structural changes. Different types of bifurcations may arise according to the way unstable eigenvalues are created. In this section, we analyze the occurrence of transcritical bifurcations. Another important class of bifurcations, the Hopf bifurcations, will be considered in the next section.

An equilibrium point x^* of (29) is called *hyperbolic* if the coefficient matrix $A(\theta)$ has no eigenvalues with zero real parts. For a hyperbolic equilibrium x^*, the asymptotic behavior of (29) is determined by the eigenvalues of $A(\theta)$ according to (a)-(b) in the previous section. For small perturbations of parameters, there are no structural changes in the stability of a hyperbolic equilibrium, provided that the perturbations are sufficiently small. Therefore, bifurcations occur at non-hyperbolic equilibria only.

Transcritical bifurcation:

A transcritical bifurcation occurs when a system has a non-hyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and additional transversality conditions are satisfied [given by the Sotomayor's Theorem in Sotomayor (1973)].

For a one-dimension system,

$$Dx = G(x, \theta),$$

the transversality conditions for a transcritical bifurcation at $(x, \theta) = (0, 0)$ are

$$G(0,0) = G_x(0,0) = 0, G_\theta(0,0) = 0, G_{xx}(0,0) \neq 0, \text{ and } G^2_{\theta x} - G_{xx}G_{\theta\theta}(0,0) > 0. \qquad (32)$$

Transversality conditions for higher-order dimension systems are given in Sotomayor (1973). The canonical form of such systems is

$$Dx = \theta x - x^2.$$

The bifurcation diagram of a transcritical bifurcation is

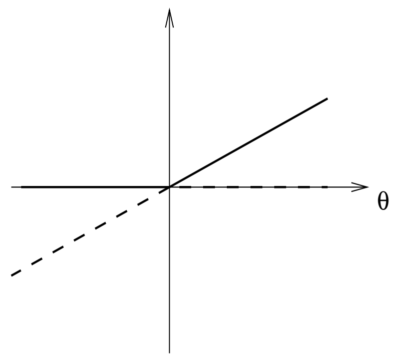


Figure 1. Transcritical bifurcation diagram.

When $det(A(\theta)) = 0$, $A(\theta)$ has at least one zero eigenvalue. If $A(\theta)$ has exactly one simple zero eigenvalue under the transversality conditions in (32), this θ corresponds to a transcritical bifurcation. So the first condition we are going to use to find the bifurcation boundary is

$$det(A(\theta)) = 0. \tag{33}$$

Analytical forms of bifurcation boundaries can be obtained for most parameters. For example, if we are interested in bifurcations when two parameters θ_i, θ_j change, while others are kept at θ^*, the matrix $A(\theta)$ may be rewritten as

$$A(\theta) = A(\theta^*) + B(\theta^*)D(\mu)C(\theta^*), \tag{34}$$

where $\mu = [\theta_i, \theta_j]$, and $D(\mu)$ is a matrix of appropriate dimension. The dimension of $D(\mu)$ is usually much smaller than that of $A(\theta)$. In this case, the following proposition is helpful for simplifying the determination of transcritical bifurcation boundaries.

Proposition 1. Assume that $A(\theta)$ has structure (34) and that all eigenvalues of $A(\theta^*)$ have strictly negative real parts. Then $det(A(\theta)) = 0$ if and only if

$$det(I + D(\mu)C(\theta^*)A^{-1}(\theta^*)B(\theta^*)) = 0. \tag{35}$$

Proof. Consider the matrix

$$\begin{bmatrix} A(\theta^*) & -B(\theta^*) \\ D(\mu)C(\theta^*) & I \end{bmatrix}.$$

Then

$$\begin{bmatrix} A(\theta^*) & -B(\theta^*) \\ D(\mu)C(\theta^*) & I \end{bmatrix} \begin{bmatrix} I & A^{-1}(\theta^*)B(\theta^*) \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A(\theta^*) & 0 \\ D(\mu)C(\theta^*) & I + D(\mu)C(\theta^*)A^{-1}(\theta^*)B(\theta^*) \end{bmatrix}.$$

Hence,

$$det(\begin{bmatrix} A(\theta^*) & -B(\theta^*) \\ D(\mu)C(\theta^*) & I \end{bmatrix}) = det(A(\theta^*))det(I + D(\mu)C(\theta^*)A^{-1}(\theta^*)B(\theta^*)). \tag{36}$$

On the other hand,

$$\begin{bmatrix} I & B(\theta^*) \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\theta^*) & -B(\theta^*) \\ D(\mu)C(\theta^*) & I \end{bmatrix} = \begin{bmatrix} A(\theta^*) + B(\theta^*)D(\mu)C(\theta^*) & 0 \\ D(\mu)C(\theta^*) & I \end{bmatrix},$$

which implies that

$$det(\begin{bmatrix} A(\theta^*) & -B(\theta^*) \\ D(\mu)C(\theta^*) & I \end{bmatrix}) = det(A(\theta^*) + B(\theta^*)D(\mu)C(\theta^*)) = det(A(\theta)). \tag{37}$$

The combination of (36) and (37) results in

$$det(A(\theta)) = det(A(\theta^*))det(I + D(\mu)C(\theta^*)A^{-1}(\theta^*)B(\theta^*)).$$

Since all eigenvalues of $A(\theta^*)$ have strictly negative real parts, $det(A(\theta^*)) \neq 0$. Therefore, the preceding equation implies that $det(A(\theta)) = 0$ if and only if (35) holds. □

Proposition 1 is useful for simplifying the calculation of $\det(A(\theta))$. To demonstrate the usefulness of this approach, consider finding the bifurcation boundary for $\mu = [\theta_2, \theta_{23}] = [\beta_2, \beta_{23}]$. Only the following entries of $A(\theta)$ are functions of μ.

$$A_{2,10}(\mu) = \gamma_2(\beta_2 - \beta_3), A_{2,13}(\mu) = -\gamma_2\beta_2$$

$$A_{24,7}(\mu) = \gamma_{25}\delta Q^*/K_a^*, A_{24,13}(\mu) = -\gamma_{25}\beta_{23}(Q^* + P^*)/K_a^*$$

$$A_{24,21}(\mu) = \gamma_{25}\delta P^*/K_a^*, A_{24,23}(\mu) = -\gamma_{25}\delta(Q^* + P^*)/K_a^*$$

where $\delta = \beta_{22} + \beta_{23}(r_f - r^*) - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)$. In this case, $B(\theta^*) \in R^{28\times 2}$ has all zero entries except that its $(2,1)$ entry is 1 and its $(24,2)$ entry is 1. $C(\theta^*) \in R^{5\times 28}$ also has zero entries, except the entries are 1 at the following locations: (1,7), (2,10), (3,13), (4,21), (5,23), and $D(\mu)$ is

$$D(\mu) = d(\mu) - d(\theta^*)$$

where

$$d(\mu) = \begin{bmatrix} 0 & A_{2,10}(\mu) & A_{2,13}(\mu) & 0 & 0 \\ A_{24,7}(\mu) & 0 & A_{24,13}(\mu) & A_{24,21}(\mu) & A_{24,23}(\mu) \end{bmatrix}$$

Direct calculation yields

$$C(\theta^*)A^{-1}(\theta^*)B(\theta^*) = \begin{bmatrix} 13.7090 & -17.1187 \\ 0 & 0 \\ -1.7276 & 2.1573 \\ -616.4935 & 389.2039 \\ -616.4935 & 389.2039 \end{bmatrix}.$$

Using Proposition 1, we know that $det(A) = 0$ is equivalent to

$$\det(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + D(\mu)\begin{bmatrix} 13.7090 & -17.1187 \\ 0 & 0 \\ -1.7276 & 2.1573 \\ -616.4935 & 389.2039 \\ -616.4935 & 389.2039 \end{bmatrix}) = 0,$$

or equivalently,

$$-14.23 + 15.91\theta_2 + 0.28\theta_{23} - 0.50\theta_2\theta_{23} = 0.$$

The following diagram depicts the bifurcation boundary when μ varies inside Θ_1.

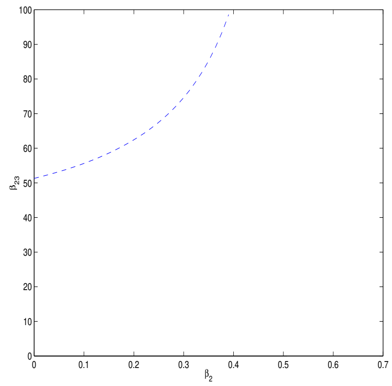


Figure 2. Candidate of transcritical bifurcation boundary for β_2, β_{23} within Θ_1.

Stability of the system (29) when parameters take values on the bifurcation boundary needs to be determined by examining the higher order terms in (30). This is usually done with the help of center manifold theory. After appropriate coordinate transformation, it is possible to write (30) as [see, for example, Glendinning (1994), Guckenheimer and Holmes (1983)]:

$$Dx_1 = A_1(\theta)x_1 + F_1(x_1, x_2, \theta) \tag{38}$$

$$Dx_2 = A_2(\theta)x_2 + F_2(x_1, x_2, \theta) \tag{39}$$

where all eigenvalues of $A_1(\theta)$ have zero real parts and all eigenvalues of $A_2(\theta)$ have strictly negative real parts. Center manifold theory says that there exists a center manifold $x_2 = h(x_1)$ such that

$$h(0) = 0 \text{ and } Dh(0) = 0.$$

Substituting $x_2 = h(x_1)$ into (38), we obtain

$$Dx_1 = A_1(\theta)x_1 + F_1(x_1, h(x_1), \theta). \tag{40}$$

The stability of (29) is connected to that of (40) through the following theorem.

Theorem 1 [Henry (1981), Carr (1981)] If the origin of (40) is locally asymptotically stable (respectively unstable) then the origin of (29) is also locally asymptotically stable (respectively unstable).

Substituting $x_2 = h(x_1)$ into (39), we have

$$Dx_2 = Dh(x_1)Dx_1 = Dh(x_1)[A_1(\theta)x_1 + F_1(x_1, h(x_1), \theta)] = A_2(\theta)h(x_1) + F_2(x_1, h(x_1), \theta)$$

or $h(x_1)$ satisfies

$$Dh(x_1)[A_1(\theta)x_1 + F_1(x_1, h(x_1), \theta)] = A_2(\theta)h(x_1) + F_2(x_1, h(x_1), \theta), \tag{41}$$

$$h(0) = 0, Dh(0) = 0. \tag{42}$$

The equations (41) and (42) can be used to solve or approximate, at least in principle, $h(x_1)$. In practice, solving (41) and (42) would be difficult. One usually uses a Taylor series approximation of $h(x_1)$ with several terms to determine the local asymptotic stability (instability) of (40). For most cases, especially codimension one bifurcations, the dimension of (40) is usually one or two. In the case of transcritical bifurcations, the dimension of (40) is one. In this case, let

$$F_1(x_1, x_2, \theta) = a_1\frac{x_1^2}{2!} + x_1 a_2 x_2 + a_3\frac{x_1^3}{3!} + ...$$

$$F_2(x_1, x_2, \theta) = b_1\frac{x_1^2}{2!} + x_1 b_2 x_2 + b_3\frac{x_1^3}{3!} + ...$$

Assume that $h(x_1)$ has the following Taylor expansion

$$h(x_1) = \alpha\frac{x_1^2}{2!} + \beta\frac{x_1^3}{3!} + ...$$

Then (41) becomes

$$(\alpha x_1 + \beta\frac{x_1^2}{2!} + ...)[A_1(\theta)x_1 + a_1\frac{x_1^2}{2!} + x_1 a_2(\alpha\frac{x_1^2}{2!} + \beta\frac{x_1^3}{3!} + ...) + a_3\frac{x_1^3}{3!} + ...]$$

$$= A_2(\theta)(\alpha\frac{x_1^2}{2!} + \beta\frac{x_1^3}{3!} + ...) + b_1\frac{x_1^2}{2!} + x_1 b_2(\alpha\frac{x_1^2}{2!} + \beta\frac{x_1^3}{3!} + ...) + b_3\frac{x_1^3}{3!} + ... \ .$$

By comparing coefficients of the same order terms and also observing that $A_1(\theta) = 0$ at a bifurcation point, we know that

$$\alpha = -A_2^{-1}(\theta)b_1, \quad \beta = A_2^{-1}(\theta)(\alpha a_1 - b_2\alpha).$$

Therefore, (40) becomes

$$Dx_1 = A_1(\theta)x_1 + a_1\frac{x_1^2}{2!} + (\frac{a_2\alpha}{2!} + \frac{a_3}{3!})x_1^3 + ... \tag{43}$$

The stability analysis of (43) determines the stability information of (30).

As an example, consider the stability of (30) on the transcritical bifurcation boundary for parameters β_2, β_{23}. On the bifurcation boundary shown in Figure 2, the stability of the system (29) could be determined using the previously described approach. For example, consider the point $(\beta_2, \beta_{23}) = (0.1068, 55.9866)$ on the boundary. We found that (40) becomes

$$Dx_1 = 0.1308x_1^2 + o(x_1^2),$$

which is locally asymptotically unstable at $x_1 = 0$. Therefore, we know from center manifold theory that the system (29) is locally asymptotically unstable at this transcritical bifurcation point.

5 Hopf bifurcations

In this section, we examine the existence of Hopf bifurcations in the system (29).

Hopf bifurcations

Hopf bifurcations occur at points at which the system has a non-hyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues, and additional transversality conditions also are satisfied [Hopf Theorem in Guckenheimer and Holmes (1983)].

The dimension of a system needs to be at least two in order for a Hopf bifurcation to occur. The transversality conditions are rather lengthy and given in Glendinning (1994). The canonical form of such systems is

$$Dx = -y + x(\theta - (x^2 + y^2)),$$

$$Dy = x + y(\theta - (x^2 + y^2)).$$

The bifurcation diagram of a Hopf bifurcation is

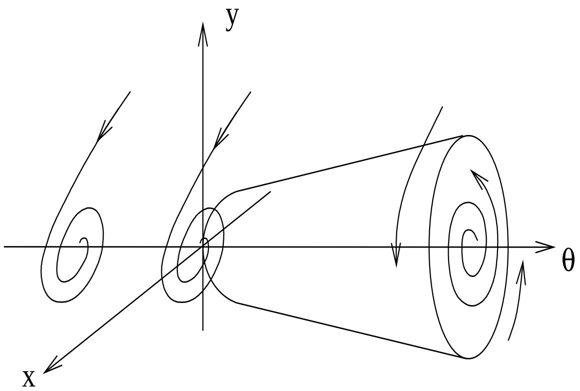


Figure 3. Hopf bifurcation diagram.

We next determine the boundaries of Hopf bifurcations. Consider the case of $det(A(\theta)) \neq 0$, but $A(\theta)$ has at least one pair of pure imaginary eigenvalues (with zero real parts and non-zero imaginary parts.) If $A(\theta)$ has exactly one such pair, and some additional transversality conditions hold, this point is on a Hopf bifurcation boundary.

To find Hopf bifurcation points, let $p(s) = det(sI - A)$ be the characteristic polynomial of A and express it as

$$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + ... + c_{n-1} s^{n-1} + s^n$$

where $n = 28$ for the system (29). Construct the following $(n-1)$ by $(n-1)$ matrix

$$S = \left[\begin{array}{ccccccccc}
c_0 & c_2 & \dots & c_{n-2} & 1 & 0 & 0 & \dots & 0 \\
0 & c_0 & c_2 & \dots & c_{n-2} & 1 & 0 & \dots & 0 \\
 & \dots & & & & & & \dots & \\
0 & 0 & \dots & 0 & c_0 & c_2 & c_4 & \dots & 1 \\
c_1 & c_3 & \dots & c_{n-1} & 0 & 0 & & \dots & 0 \\
0 & c_1 & c_3 & \dots & c_{n-1} & 0 & 0 & \dots & 0 \\
 & \dots & & & & & & \dots & \\
0 & 0 & & \dots & 0 & c_1 & c_3 & \dots & c_{n-1}
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \frac{n-2}{2} \text{ rows} \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \frac{n}{2} \text{ rows}
\end{array}$$

Let S_0 be obtained by deleting rows 1 and n/2 and columns 1 and 2, and let S_1 be obtained by deleting rows 1 and n/2 and columns 1 and 3. Then the matrix A has exact one pair of pure imaginary eigenvalues if [see, e.g., Guckenheimer et al. (1997)]

$$det(S) = 0, \quad det(S_0)det(S_1) > 0.$$

If $det(S) \neq 0$ or if $det(S_0)det(S_1) < 0$, then A has no pure imaginary eigenvalues. If $det(S) = 0$ and $det(S_0)det(S_1) = 0$, then A may have more than one pair of pure imaginary eigenvalues. Therefore, the second condition for a bifurcation boundary is

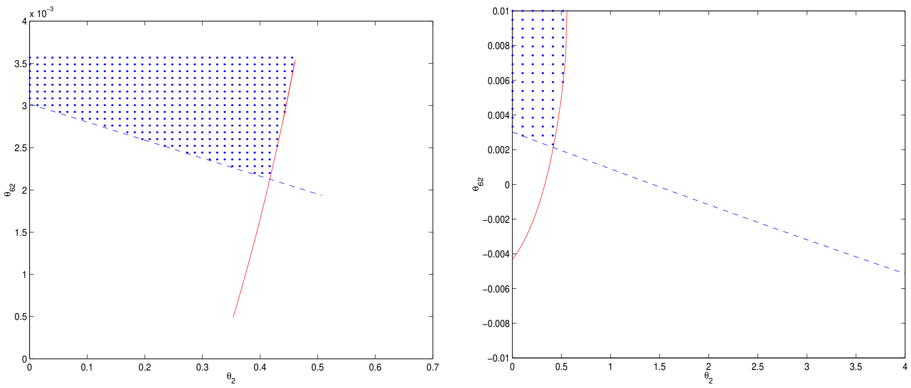
$$det(S) = 0, \quad det(S_0)det(S_1) \geq 0. \tag{44}$$

We will use (44) to find candidates for bifurcation boundaries and then check which segments are true boundaries. Since solving (44) analytically is impossible for most problems, a numerical procedure was given in Barnett and He (1998) to find bifurcation boundaries. The stability of (29) at parameter values on the bifurcation boundary can be analyzed in the same manner as for transcritical bifurcations.

6 Numerical Examples

Example 1. Figure 4 (a) and (b) show bifurcation boundaries for $\mu = [\theta_2, \theta_{62}]$. Figure 4 (a) illustrates bifurcation boundaries when μ varies within Θ_1, which is the Cartesian product of the 95% confidence intervals of the estimates, while Figure 4 (b) describes bifurcations within Θ. All other parameters are kept at θ^*. In both (a) and (b), the dashed lines are determined by (33) and the solid lines are calculated according to (44). The shaded area is the parameter region for which the system (29) is stable. Therefore, the dashed lines along the shaded regions are the transcritical bifurcation boundaries and the solid lines along the shaded regions represent the Hopf bifurcation boundaries.

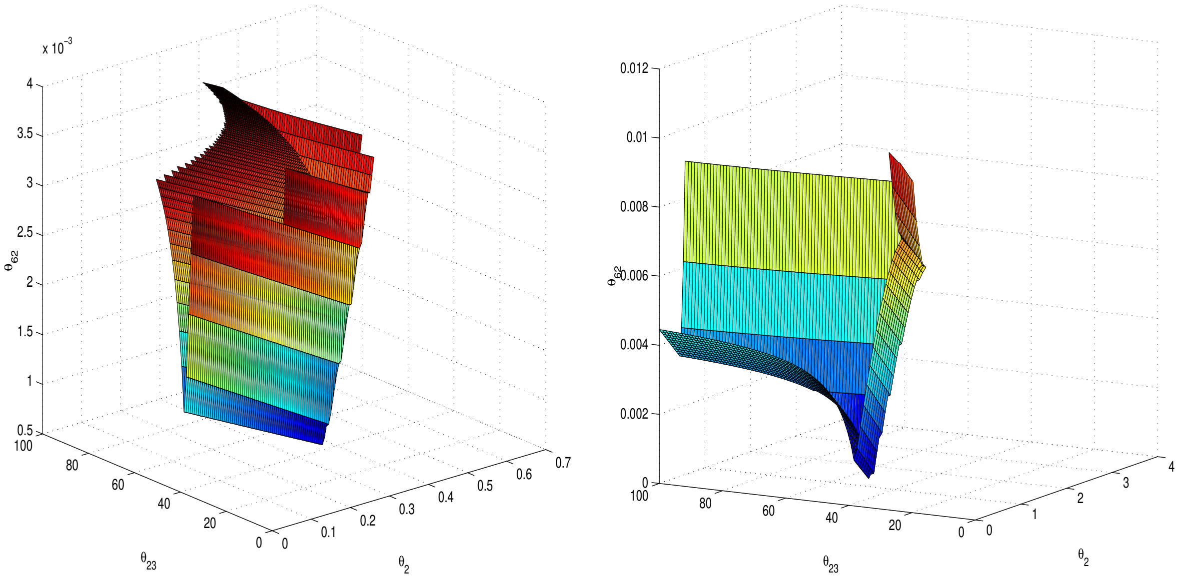
Of special interest is the intersection point of the transcritical bifurcation boundary and the Hopf bifurcation boundary. This point corresponds to a codimension two bifurcation. The properties of (29) near this point deserve further investigation.



(a) bifurcation within Θ_1 (b) bifurcation within Θ

Figure 4. Bifurcation boundaries for θ_2, θ_{62}.

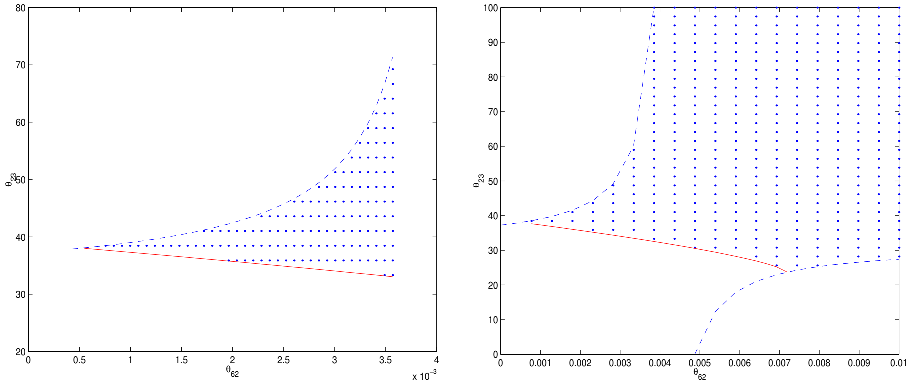
Example 2. If we add another parameter, θ_{23}, to our consideration, a bifurcation surface in 3-dimensional space could be obtained. Figure 5 shows the bifurcation boundaries for $\mu = [\theta_2, \theta_{23}, \theta_{62}]$. Both transcritical and Hopf bifurcation boundaries are shown.



(a) bifurcation within Θ_1 (b) bifurcation within Θ

Figure 5. Bifurcation boundaries for $\theta_2, \theta_{23}, \theta_{62}$.

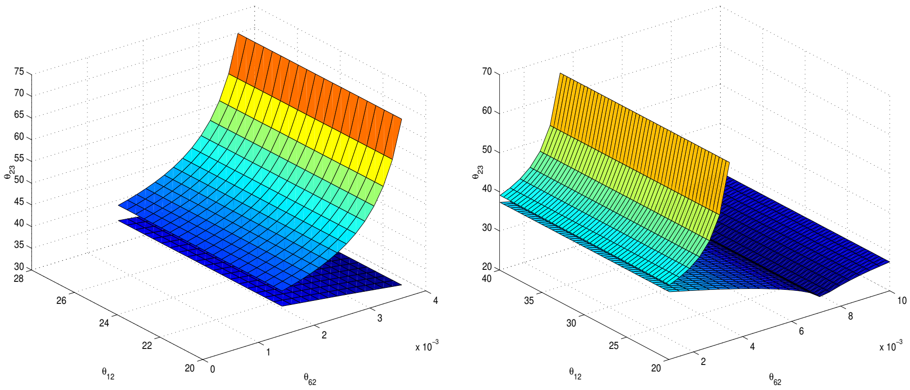
Example 3. Figure 6 shows bifurcation boundaries for θ_{23}, θ_{62}.



(a) bifurcation within Θ_1 $\qquad$ (b) bifurcation within Θ

Figure 6. Bifurcation boundaries for θ_{23}, θ_{62}.

Example 4. Figure 7 shows bifurcation boundaries for $\theta_{12}, \theta_{23}, \theta_{62}$.



(a) bifurcation within Θ_1 $\qquad$ (b) bifurcation within Θ

Figure 7. Bifurcation boundaries for $\theta_{12}, \theta_{23}, \theta_{62}$.

7 Control of bifurcations

We have seen in the previous section that both transcritical and Hopf bifurcations exist in the continuous time macroeconometric model. In this section, we shall investigate the control of bifurcations using fiscal feedback laws.

We first consider the effect of a heuristically plausible fiscal policy of the following form as suggested in Bergstrom et al. (1994):

$$D \log T_1 = \gamma \left[\beta \log\{ \frac{Q}{Q^* e^{(\lambda_1 + \lambda_2)t}} \} - \log \left\{ \frac{T_1}{T_1^*} \right\} \right]. \tag{45}$$

The control feedback rule (45) adjusts the fiscal policy instrument, T_1, towards a partial equilibrium level, which is an increasing function of the ratio of output to its stead state level.

In (45), β is a measure of the strength of the feedback, and γ governs the speed of adjustment. By choosing appropriate parameters β, γ, it was found in Bergstrom et al. (1994) that the control law (45) can reduce the positive real parts of unstable eigenvalues, implying that the policy might be stabilizing. However, we find that the control law (45) is unlikely to stabilize the systems (1)-(14).

Define

$$y_{15} = \log \left\{ \frac{T_1}{T_1^*} \right\}.$$

Then it is easy to verify that y_{15} satisfies

$$Dy_{15} = \gamma\beta y_4 - \gamma y_{15}.$$

Adding this equation to the system (29), we obtain

$$Dw = A'(\theta)w + F'(x, \theta) \tag{46}$$

where

$$w = \begin{bmatrix} x \\ y_{15} \end{bmatrix}, \quad F'(x, \theta) = \begin{bmatrix} F(x, \theta) \\ 0 \end{bmatrix}$$

and $A'(\theta)$ is the corresponding coefficient matrix. Figure 8 shows the effect of the simple fiscal policy on the bifurcation boundaries for β_2 and β_5. Three sets of parameter values of β, γ are considered. The case $\beta = 0, \gamma = 0$ corresponds to the original system (1)-(14), in which no fiscal policy control is applied.

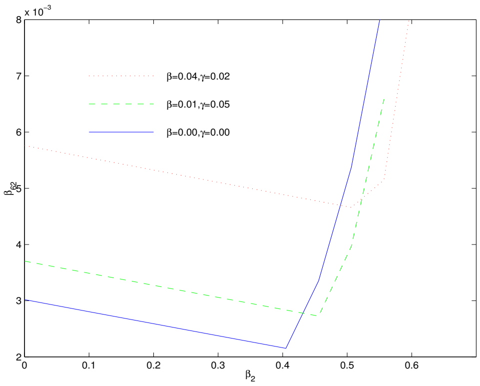


Figure 8. Effect of a simple fiscal policy.

Figure 8 clearly indicates that some stable regions could be destabilized and some unstable regions could be stabilized. Since the feasible stable region is smaller under control than without control, the policy is not likely to succeed.

Next we consider a more sophisticated fiscal control policy, based upon optimum control theory. Let the control be

$$u = \log\left\{\frac{T_1}{T_1^*}\right\}. \tag{47}$$

Under the control (47), the system (29) becomes

$$Dx = A(\theta)x + Bu + F(x,\theta) \tag{48}$$

where $B = [0 \ \ -\gamma_2 \ 0 \ ... \ 0]^T \in R^{28}$. Direct verification yields that the controllability matrix $[B \ \ AB \ \ ... \ \ A^{27}B]$ has rank 7, implying that the pair (A, B) is not controllable. Therefore, it is not possible to set the closed-loop eigenvalues of the coefficient matrix of (48) arbitrarily. However, a numerical analysis shows that there exists a linear transformation $z = Tx$ such that

$$Dz = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u$$

where $A_{11} \in R^{21\times 21}, A_{21} \in R^{7\times 21}, A_{22} \in R^{7\times 7}$, $B_2 = [0 \ ... \ 0 \ 1] \in R^7$,

$$TA(\theta)T^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

and (A_{22}, B_2) is controllable. The exact numerical procedure for this decomposition can be found, for example, in Khalil (1992). Further, all eigenvalues of A_{11} have negative real parts, implying that $(A(\theta), B)$ is stabilizable.

To obtain a feedback control law stabilizing (48), we consider solving the problem of minimizing

$$J = \int_0^\infty [x^T U x + V u^2] dt$$

where $U \in R^{28\times 28}$ and $V \in R^1$ are positive definite. It is known from linear system theory that the optimal feedback control law is given by

$$u = Kx, \quad K = -V^{-1}B^T P$$

where P is positive definite and solves the algebraic Ricatti equation

$$PA + A^T P - PBV^{-1}B^T P + U = 0.$$

Choose $U = I$, $V = 1$. Then we get

$$K = [1.5036, 0.4754, 0.0178, 0.0307, -1.1897, 18.5851, 7.2979,$$

$$1.9063, 2.3147, 23.2392, 0.7488, 7.2091, 38.9965, 39.4000,$$

$$0.1841, 0.2129, 0.3061, 0.0494, -0.0027, 0.0000, -0.0013,$$

$$-0.0002, 0.9550, 1.8482, -0.3329, -0.5475, 0.9369, -1.0402]. \tag{49}$$

Under the control $u = Kx$, (48) becomes

$$Dx = [A(\theta) + BK]x + F(x, \theta). \tag{50}$$

The choice of K ensures that all the eigenvalues of $A + BK$ have strictly negative real parts. Therefore, the state feedback law $u = Kx$ indeed stabilizes the system (50). Direct verification confirms that there exist no bifurcations under the control law (50) for (β_2, β_5).

We further check the stability of (50) under the control law (50) for all parameter $\theta \in \Theta$. Our purpose is to see if there is a parameter $\theta' \in \Theta$ at which the system (50) is unstable. Such a parameter can be found by replacing (31) with maximizing the maximum real parts of eigenvalues of matrix $A(\theta)$. The following $\theta' \in \Theta_1$ has been found

$$\begin{aligned}\theta' = [&0.9400, 0.5074, 2.0913, 0.2030, 0.2612, 0.1933, 0.2309, 0.0000, 0.2510,\\ &-0.3423, 1.0000, 23.5000, -0.0100, 0.2086, 0.0332, 13.5460, 0.4562, 0.9322,\\ &0.0100, 0.0034, 0.1324, -0.5006, 100.0000, 0.0000, 0.0004, 71.4241, 0.8213,\\ &4.0000, 1.0289, 0.3631, 0.1201, 0.1000, 0.0010, 3.7015, 0.4860, 1.1270,\\ &0.0042, 3.3994, 0.4802, 0.1300, 0.6851, 0.0620, 1.2134, 0.3830, 4.0000,\\ &3.2535, 3.8592, 4.0000, 4.0000, 3.5723, 0.4775, 0.0071, 0.6104, 0.0143,\\ &0.1718, 0.1227, 2.5551, 0.1833, 0.0035, 0.0000, 0.0018, 0.0004, 0.0100].\end{aligned}$$

The corresponding $R_{\max}(A(\theta')) = 0.4971$. Therefore, there indeed exists a parameter $\theta' \in \Theta_1$ at which (50) is unstable.

Because of the Lucas critique, the problems associated with using structural models for policy simulations are well known. In addition, the possibility of time inconsistency of optimal control policy conditionally upon a structural model is well known. While the use of Euler equation models having deep parameters is to be preferred for policy simulations, we are not yet able to investigate bifurcation with a sufficiently rich Euler equation model. Nevertheless, it is interesting to ask whether the use of control feedback policy with a structural model would be easily implemented, if the Lucas critique and time inconsistency issues did not exist. It seems often to be assumed that such active policy easily could be designed, if it were not for the problems produced by the Lucas critique parameter instability to policy variation and the time inconsistency of optimal control.

But our results above indicate that even without those problems, the design of a successful feedback policy can be difficult. Even when the structural parameters of the other equations remain constant, adjoining a policy feedback rule to a system causes bifurcation boundaries to shift. The policy is successful, if those shifts cause the stable region to move towards the actual values of the parameters sufficiently to include the parameters within the stable region. We find that with the UK continuous time model, the selection of a fiscal policy feedback rule from heuristic economic reasoning is counterproductive. While the use of optimal control theory is successful conditionally upon the model, the negative results from the heuristic nonoptimal policy raise serious questions about the robustness of that conclusion to specification error, and hence the relevancy of the conclusion to real world policy. Furthermore, this abstracts from the problems of possible time inconsistency of optimal control policy, which further complicates the matter.

In short, the effects of policy feedback rules depend upon the complicated geometry of bifurcation boundaries and how they are moved by augmentation of the model by the feedback rule. It is not at all unlikely that such policies, when applied in the real world, could prove to be counterproductive.

8 Conclusions

In this paper, we continue our investigation of bifurcation phenomena in continuous time macroeconometric models, using the Bergstrom, Nowman, and Wymer continuous time second order differential equations macroeconometric model of the United Kingdom. We have obtained a new formula for determining bifurcation boundary candidates for transcritical bifurcations, and we find that the dynamics of the model are locally asymptotically unstable on those bifurcation boundaries. While the dynamical properties on the Hopf bifurcation boundary are likely to be more interesting, we have not yet successfully solved the more difficult problem of numerically investigating dynamics on that boundary.

We found the intersection of the stable region with the feasible subset of the Cartesian product of the individual parameters' confidence intervals. Although the point estimates of the parameters are in the unstable region, we find that intersection is nonempty when the confidence level is set at 95%, but becomes empty when the confidence level is decreased to 90%. These results suggest that an inference of instability for this model may be reasonable, at least in the 90% confidence level case. However, the problems associated with conducting a rigorous hypothesis test of the null hypothesis of stability in this model are prohibitively difficult.

We also investigate the effects of fiscal policy on stabilization, and we find that conducting a successful active countercyclical policy may be more difficult than previously believed, as a result of the manner in which the bifurcation boundaries are affected by policy and the complex geometry of those boundaries.

The obvious priority for further advances in this area would be application of these methods to a stochastic dynamic general equilibrium model that could be viewed as an empirically plausible and policy relevant extension of the model investigated by Grandmont (1985). But the use of these methods with the implied system of Euler equations poses significant challenges that are likely to motivate our future research in this area. One possibility, as a next step in the direction, would be to apply the methods in this paper to the model in Leeper and Sims (1994), while progress in a related direction could be acquired by applying these methods to the model in Powell and Murphy (1997).

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