Dynamics of Business Fluctuations in the Leontief-type Economy

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ABSTRACT

Presented here is the disequilibrium model of business fluctuations for quantity and prices, which is based on the Input-Output Model of Wassily Leontief. Disequilibrium states are described by the second-order differential equation system that interrelates commodities' demand, production, and prices. It is also analyzed the stability of equation system's solutions both for quantity and prices for different model scenarios. *Journal of Economic Literature* Classification Numbers: E 32; D 57.

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1. Introduction

It is well known that the classic Input-Output Model of Wassily Leontief (see Leontief, 1951, 1953) leads to the exponential growth of the production output derived as a solution of the first-order differential equation system. D. W. Jorgensen showed in 1960 that stability of production output and stability of corresponding prices does not coincide in the model.

In present paper I show how, on basis of Input-Output Model, obtain the disequilibrium model for quantity and prices, which is described by the second-order differential equation system. Such differential equation system entails the fluctuations of model's parameters. Latter model is based on the Continuous-Time Model of Business Fluctuations for economy with one commodity, which was developed by me earlier (see Krouglov, 1997a, 1997c, 1998). I also will investigate the stability of the obtained solutions in different model scenarios.

2. Model Assumptions

Beyond the usual for Input-Output Model assumptions the paper entertains some additional ones.

(a) The commodity's price is growing if demand on commodity exceeds its supply, and falling vice versa.

(b) The commodity's production rate is increasing if its price grows, and decreasing vice versa.

(c) The demand on commodity decreases if its price grows, and increases vice versa.

The first assumption is applied in the model with some variations.

Among different scenarios, it is investigated situations when commodity's price is changing due to the disequilibrium between commodity's production and whether its final demand only or both the intermediate and final demands. Also it is shown how the commodities' secondary market affects the solutions of differential equation system.

It is obtained that stability of the solutions is different in these cases.

3. First Model's Description

Let me consider model with n commodities, and denote V_d the vector of commodities' demand volumes on market, V_p the vector of commodities' production volumes, V_c the vector of commodities' consumption volumes (by the ultimate consumers). Designate r_d, r_p, and r_c to be the vectors of commodities' demand, production, and consumption rates respectively.

Demand, production, and consumption volumes and rates relate by the following expressions,

$$r_d = \dot{V}_d,\ r_p = \dot{V}_p, \text{ and } r_c = \dot{V}_c.$$

At the equilibrium point commodities' demand, production, and consumption rates are interrelated by the following matrix expression,

$$r_c^0 = r_p^0 - Ar_p^0 = r_d^0, \qquad (1)$$

where non-negative square matrix A consists of elements $a_{ij} \geq 0$ expressing the amount of commodity i needed for the production of one unit of commodity j (see Leontief, 1951, 1953).

It is known that for inverse matrix $(I - A)^{-1}$ to exist (I is an identity matrix) and to be non-negative the Frobenius eigenvalue of non-negative matrix A has to be less than one (see Gantmacher, 1959). This requirement is usually assumed in the Leontief model (see Gandolfo, 1996).

Here we consider another non-negative square matrix B, which consists of elements $b_{ij} \geq 0$ expressing the amount of commodity i needed for the production of commodity j with the unity rate (i.e. invested capital intensity). This matrix B is called the capital matrix in Leontief, 1953.

Therefore, to produce commodities with rate r_p we have to have investments in the amount of

$$V_i = Br_p, \tag{2}$$

where V_i is the vector of investment volumes into commodities' production. Depreciation is not included in (2) for the sake of simplicity.

So, at the equilibrium point investments are made in the amount of

$$V_i^0 = Br_p^0. \tag{3}$$

We will use later also the vector $r_i = \dot{V}_i$ showing the commodities' investment rates.

Now I require that at time $t = 0$ there is an exogenous change in commodities' demand rate

$$r_d(0) = r_d^0 + \Delta_r\,, \tag{4}$$

and show how system will come to its new equilibrium point (compare with Krouglov, 1998).

Due to the demand's change we experience a commodities' deficit on market since time $t = 0$**,** and the deficit rate r_{df} at time t is as follows,

$$r_{df}(t) = r_d(t) - (I - A)r_p(t), \tag{5}$$

where $r_{df}(0) = r_d(0) - (I - A)r_p^0 = \Delta_r$**.**

Then the deficit volumes $V_{df}(t)$ integrated since time $t = 0$ are driving the changes in the commodities' prices,

$$\dot{P}(t) = \Lambda V_{df}(t), \tag{6}$$

where $P(t)$ is the vector of commodities' prices at time t, and Λ is a diagonal matrix with non-negative constant elements $\lambda_{ii} \geq 0$ on the main diagonal, which represent the coefficients of ***price inertia*** of particular commodity's price (i.e., how quickly commodity's price reacts to its deficit (surplus) on market).

Variations in the commodities' prices bring the changes both in the commodities' production and demand.

For the commodities' production, price changes entail the variations in production rate that have to be backed by the corresponding investments (or withdrawing the resources from production respectively),

$$\dot{r}_p(t) = \mathrm{M}\dot{P}(t), \tag{7}$$

$$r_i(t) = B\dot{r}_p(t), \tag{8}$$

where M is a diagonal matrix with non-negative elements $\mu_{ii} \geq 0$ on the main diagonal which represent the coefficients of ***positive price inductance*** of the particular commodity's price on its production (i.e., in what extent commodity's price increase (decrease) influences on the increase (decrease) of the commodity's production rate). Note that (8) is directly obtained from (2).

For the commodities' demand, the price changes directly affect the commodities' demand rates as follows,

$$r_d(t) = -\mathrm{N}\dot{P}(t), \tag{9}$$

where N is a diagonal matrix with non-negative constant elements $\nu_{ii} \geq 0$ on the main diagonal which represent the coefficients of ***negative price inductance*** of the particular commodity's price on its demand (i.e., in what extent commodity's price increase (decrease) influence on the decrease (increase) of the commodity's demand on market).

Therefore for the commodities' consumption rates, we can write the following expression, showing a decrease (or an increase) in the commodities' consumption rates due to investing some commodities into production (or withdrawing the commodities from production respectively),

$$r_c(t) = (I - A)r_p(t) - B\dot{r}_p(t). \tag{10}$$

I don't require $r_c(t) \geq 0$ in the (10). It can be explained by my intention to include the possibility for a credit line into model in the future.

4. Dynamics of Economic System

Let us start with the description of the commodities' deficit behavior.

Since the changes in production rates are proportional to the commodities' price rates,

$$\dot{r}_p(t) = \mathrm{M}\dot{P}(t),$$

and the changes in demand rates are proportional to the changes in the commodities' price rates,

$$\dot{r}_d(t) = -\mathrm{N}\ddot{P}(t),$$

the changes in deficit rates can be written as follows,

$$\begin{aligned} \dot{r}_{df}(t) &= \dot{r}_d(t) - (I - A)\dot{r}_p(t) \\ &= -\mathrm{N}\ddot{P}(t) - (I - A)\mathrm{M}\dot{P}(t) \\ &= -\mathrm{N}\Lambda r_{df}(t) - (I - A)\mathrm{M}\Lambda V_{df}(t). \end{aligned}$$

Therefore, the following matrix differential equation can be used to describe the dynamics of commodities' deficit on market,

$$\ddot{V}_{df}(t) + \mathrm{N}\Lambda\dot{V}_{df}(t) + (I - A)\mathrm{M}\Lambda V_{df}(t) = 0, \quad (11)$$

where $\dot{V}_{df}(0) = \Delta_r$, $V_{df}(0) = 0$.

To solve the last equation we write it in the normal form (see Petrovski, 1966),

$$\dot{r}_{df}(t) = -\mathrm{N}\Lambda r_{df}(t) - (I - A)\mathrm{M}\Lambda V_{df}(t) \quad (12)$$

$$\dot{V}_{df}(t) = r_{df}(t) \quad (13)$$

or otherwise

$$\dot{y}(t) = \Gamma y(t), \quad (14)$$

where vector

$$y(t) = \begin{pmatrix} r_{df}(t) \\ V_{df}(t) \end{pmatrix}$$

of size $2n$ has the initial value $y(0)=\begin{pmatrix}\Delta_r \\ 0\end{pmatrix}$,

and matrix

$$\Gamma=\begin{pmatrix}-\mathrm{N}\Lambda & -(I-A)\mathrm{M}\Lambda \\ I & 0\end{pmatrix}$$

is the square matrix of size $2n\times 2n$.

Solution of the (14) can be expressed (see Arnol'd, 1992) in the vector form as follows,

$$y(t)=y_0 e^{\Gamma t} \quad (t\geq 0), \tag{15}$$

where $y(0)=y_0$, and $e^{\Gamma t}$ is the Exponential of the linear operator specified by the matrix Γ of size $2n\times 2n$.

It is known that if every eigenvalue of matrix Γ has negative real part then there are such positive numbers α and r that

$$\|y(t)\|\leq r\|y_0\|e^{-\alpha t} \quad (t\geq 0), \tag{16}$$

where $\|y\|=\sqrt{\sum_{i=1}^{2n} y_i^2}$. It follows from (16) that solution $y(t)$ is Lyapunov stable and asymptotically stable (see Pontryagin, 1962).

Knowing $V_{df}(t)$, we can find $P(t)$ from the (6) and initial value $P(0)$. Then we can find $r_d(t)$ from the (9) and initial value $r_d(0)=r_d^0+\Delta_r$, and $r_p(t)$ from the (7) and initial value $r_p(0)=r_p^0$. Also we can find $V_i(t)$ from the (8) and initial value $V_i(0)=V_i^0$, and $r_c(t)$ from the (10).

5. Dynamics and Stability of Prices

Here I will follow the approach developed in Krouglov, 1997b for economy with one commodity.

Let me differentiate (6) and integrate (7),

$$\ddot{P}(t) = \Lambda r_{df}(t), \tag{17}$$

$$r_p(t) = \mathrm{M}P(t) + C_1 . \tag{18}$$

Together with (9) it gives us,

$$\begin{aligned}\ddot{P}(t) &= \Lambda\left(r_d(t) - (I - A)r_p(t)\right) \\ &= -\Lambda \mathrm{N}\dot{P}(t) - \Lambda(I - A)(\mathrm{M}P(t) + C_1),\end{aligned}$$

where an integration's constant C_1 serves for reconciliation between commodities' production rates and their prices at the initial time $t = 0$.

Therefore,

$$\ddot{P}(t) + \Lambda \mathrm{N}\dot{P}(t) + \Lambda(I - A)\mathrm{M}P(t) + C_2 = 0, \tag{19}$$

where $C_2 = \Lambda(I - A)C_1$.

We use the change of variables,

$$P_1(t) = P(t) + \mathrm{M}^{-1}C_1 ,$$

and rewrite the (19) as,

$$\dot{z}(t) = \mathrm{E}z(t), \tag{20}$$

where vector

$$z(t) = \begin{pmatrix} \dot{P}_1(t) \\ P_1(t) \end{pmatrix}$$

has the initial value $z(0) = \begin{pmatrix} 0 \\ \mathrm{M}^{-1}r_p^0 \end{pmatrix}$, and matrix

$$\mathrm{E} = \begin{pmatrix} -\Lambda \mathrm{N} & -\Lambda(I - A)\mathrm{M} \\ I & 0 \end{pmatrix}$$

is the square matrix of size $2n \times 2n$.

Solution of the (20) is

$$z(t) = z_0 e^{\mathrm{E}t} \quad (t \geq 0), \tag{21}$$

where $z(0) = z_0$, and this solution is Lyapunov stable when every eigenvalue of matrix E has the negative real part.

Last statement contradicts to the so-called *dual stability theorem* (see Jorgensen, 1960) that tells (see Gandolfo, 1996), if the output system is relatively stable, the price system is relatively unstable, and vice versa.

For example, we can see when,

$$\Lambda = \lambda I \text{ and } \mathrm{M} = \mu I,$$

all eigenvalues of both matrices Γ and E are the same. Thus, for this example, both the solutions $y(t)$ and $z(t)$ are whether stable or unstable.

6. Introducing the Impact of the Secondary Market

Let me assume that resources invested into commodities' production are demanded, and resources withdrawn from production are supplied on the market that regulates the commodities' prices.

Then we have to take into account the commodities' investment rates $r_i(t)$, and (5) becomes

$$r_{df}(t) = r_d(t) - (I - A)r_p(t) + B\dot{r}_p(t). \tag{22}$$

Therefore we can write

$$\begin{aligned}\dot{r}_{df}(t) &= \dot{r}_d(t) - (I - A)\dot{r}_p(t) + B\ddot{r}_p(t) \\ &= -\mathrm{N}\ddot{P}(t) - (I - A)\mathrm{M}\dot{P}(t) + B\mathrm{M}\ddot{P}(t) \\ &= (B\mathrm{M} - \mathrm{N})\Lambda r_{df}(t) - (I - A)\mathrm{M}\Lambda V_{df}(t),\end{aligned}$$

and dynamics of the commodities' deficit on market can be defined by the following differential equation system,

$$\ddot{V}_{df}(t) + (\mathrm{N} - B\mathrm{M})\Lambda\dot{V}_{df}(t) + (I - A)\mathrm{M}\Lambda V_{df}(t) = 0, \tag{23}$$

where $\dot{V}_{df}(0) = \Delta_r$ and $V_{df}(0) = 0$.

The same equation can be written in the normal form as

$$\dot{y}(t) = \Gamma_1 y(t), \tag{24}$$

where vector

$$y(t) = \begin{pmatrix} r_{df}(t) \\ V_{df}(t) \end{pmatrix}$$

has the initial value $y(0) = \begin{pmatrix} \Delta_r \\ 0 \end{pmatrix}$,

and matrix Γ_1 is as follows,

$$\Gamma_1 = \begin{pmatrix} (B\mathrm{M} - \mathrm{N})\Lambda & -(I - A)\mathrm{M}\Lambda \\ I & 0 \end{pmatrix}.$$

Solution of the (19) is

$$y(t) = y_0 e^{\Gamma_1 t} \quad (t \geq 0), \tag{25}$$

where $y(0) = y_0$, and this solution is Lyapunov stable when every eigenvalue of matrix Γ_1 has the negative real part.

In the situation with secondary market, the price equation (19) becomes,

$$\ddot{P}(t) + \Lambda(\mathrm{N} - B\mathrm{M})\dot{P}(t) + \Lambda(I - A)\mathrm{M}P(t) + C_2 = 0, \tag{26}$$

and we rewrite it again in the normal form,

$$\dot{z}(t) = \mathrm{E}_1 z(t), \tag{27}$$

where vector

$$z(t) = \begin{pmatrix} \dot{P}_1(t) \\ P_1(t) \end{pmatrix}$$

has the initial value $z(0) = \begin{pmatrix} 0 \\ \mathrm{M}^{-1} r_p^0 \end{pmatrix}$,

and matrix E_1 is as follows,

$$\mathrm{E}_1 = \begin{pmatrix} \Lambda(B\mathrm{M} - \mathrm{N}) & -\Lambda(I - A)\mathrm{M} \\ I & 0 \end{pmatrix}.$$

Solution of (27) is

$$z(t) = z_0 e^{\mathrm{E}_1 t} \quad (t \geq 0), \tag{28}$$

where $z(0) = z_0$, and the solution is Lyapunov stable when every eigenvalue of matrix E_1 has the negative real part.

We see again that for

$$\Lambda = \lambda I \text{ and } \mathrm{M} = \mu I,$$

all eigenvalues of both matrices Γ_1 and E_1 are the same.

7. Second Model's Description

Here I assume that all commodities (required both as intermediate input and final demand) are traded on the market and the market forces determine their prices. Otherwise the prices for commodities required only as intermediate input would be non-determinable.

Since I eliminated from consideration the distinction between cases whether commodities are used to satisfy intermediate or final demand with respect to their impact on prices, the analysis becomes simpler.

Instead of (1) we can write for the equilibrium state,

$$r_c^0 = r_p^0 = r_d^0 \,, \tag{29}$$

and the deficit rate is defined by the following equation rather than (5),

$$r_{df}(t) = r_d(t) - r_p(t), \tag{30}$$

where $r_{df}(0) = r_d(0) - r_p^0 = \Delta_r$.

Then the dynamics of commodities' deficit on the market is described by the following differential equation system,

$$\ddot{V}_{df}(t) + \mathrm{N}\Lambda\dot{V}_{df}(t) + \mathrm{M}\Lambda V_{df}(t) = 0\,, \tag{31}$$

where $\dot{V}_{df}(0) = \Delta_r$, $V_{df}(0) = 0$.

Since all matrices Λ, M and N are diagonal with non-negative elements, the oscillatory processes for each commodity are developing independently, and solutions are always guaranteed to be stable.

Indeed, the solution's stability requires for every root τ of characteristic equation,

$$\left|\tau^2 I + \tau\mathrm{N}\Lambda + \mathrm{M}\Lambda\right| = \prod_{i=1}^{n}\left(\tau^2 + \lambda_{ii}\nu_{ii}\tau + \lambda_{ii}\mu_{ii}\right) = 0\,, \tag{32}$$

to have the negative real part, which is fulfilled.

Obviously, the fluctuations of commodities' prices also become stable in this scenario.

If we look at the changes, which the secondary market brings in the situation, we see that stability of the solutions may be violated.

Here the dynamics of commodities' deficit is described by the equation,

$$\ddot{V}_{df}(t) + (\mathrm{N} - B\mathrm{M})\Lambda\dot{V}_{df}(t) + \mathrm{M}\Lambda V_{df}(t) = 0\,, \tag{33}$$

where $\dot{V}_{df}(0) = \Delta_r$, $V_{df}(0) = 0$.

The stability of solutions is secured only when every root τ of the characteristic equation,

$$\left|\tau^2 I + \tau(\mathrm{N} - B\mathrm{M})\Lambda + \mathrm{M}\Lambda\right| = 0\,, \tag{34}$$

has the negative real part.

The similar statement can be drawn for the stability of commodities' prices in this case.

8. Conclusion

Thus the paper describes a model of business fluctuations based on Leontief's Input-Output Model. The quantity side of the model takes a standard dynamic input-output model. Latter model is extended by adding disequilibrium price and quantity dynamics, which follows an exogenous change in the demands for commodities. This result in a second-order linear differential equation system whose stability properties are analyzed. The price dynamics is defined as dual to the one for quantity, and described by the second-order linear differential equation system also.

It is obtained if the model reflects the variation of prices due to disbalance between commodities' both intermediate and final demands and their production, the solutions of system are always stable (within the limits of the model's assumptions).

If commodities' prices vary due to the disbalance between the commodities' final demand and production only, the stability of the system may be violated because of interdependency among production of various commodities.

On the other hand, if we consider the impact of commodities' secondary market, the stability of the system can also be violated owing to Leontief production function defined by the linear operator.

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