PRIVATE MONEY AND RESERVE MANAGEMENT IN A RANDOM MATCHING MODEL[1]

Ricardo de O. Cavalcanti
Department of Economics, University of Miami
P.O. Box 248126, Coral Gables, FL 33124
E-mail: ricardoc@walleye.econ.umn.edu

Andres Erosa
Department of Economics, University of Western Ontario
London, Ontario, N6A 5C2 Canada
E-mail: EROSA@sscl.uwo.ca

Ted Temzelides
Department of Economics, University of Iowa
Iowa City, IA 52242
E-mail: tedt@blue.weeg.uiowa.edu

First version: September 1996; This version: September 1997.

[1]We wish to thank, without implicating in any way, Narayana Kocherlakota, Neil Wallace, Warren Weber, and Ruilin Zhu for helpful comments. All errors are ours. Part of this paper was completed while the third author held a position as an economist at the Federal Reserve Bank of Philadelphia. The views expressed here are those of the authors and do not necessarily reflect those of the Federal Reserve Bank of Philadelphia or the Federal Reserve System.

Abstract

We introduce an element of centralization in a random matching model of money that allows for private liabilities to circulate as media of exchange. Some agents, which we identify as banks, are endowed with the technology to issue notes and to record-keep reserves with a central clearinghouse, which we call the treasury. The liabilities are redeemed according to a stochastic process that depends on the endogenous trades. The treasury removes the banking technology from banks that are not able to meet the redemptions in a given period. This, together with the market incompleteness, gives rise to a reserve management problem for the issuing banks. We demonstrate that "sufficiently patient" banks will concentrate on improving their reserve position instead of pursuing additional issue. The model provides a first attempt to reconcile limited note issue with optimizing behavior by banks during the National Banking Era.

1 Introduction

While there are several interesting questions concerning private money, a useful framework for studying the operation of private monetary systems and the implications of interventions into such systems has not yet been developed.[2] Hayek has argued that private money would have a positive effect against sustained inflation by subjecting the government to the discipline of competition. However, other economists are skeptical about the stability of a competitive monetary system: How can a stable real-valued currency emerge if, having established a currency, the supplier can produce more at zero cost? A satisfactory answer to this question has not been offered. For this reason, the theory on the workings of a private monetary system is commonly seen as inherently difficult.

In this paper, we introduce a model of private money and show its potential usefulness by demonstrating that a stable monetary system can emerge in a way that resembles the conservative note issue by banks during the National Banking Era in the United States (1863-1913). This period provides a challenge for any model of private money. Standard theory suggests that if banks can issue their own currency, and if the public treats the private currency as a perfect substitute to lawful money, then banks will overissue notes unless they are obliged, by law, to back them by 100% reserves in lawful money. According to this argument, if the public does not distinguish between private bank notes and fiat money, a bank should always be able to exchange any amount of its redeemed notes with fiat money from the indifferent public, keeping them, in effect, outstanding. In other words, even if banks have to provide an amount of lawful money equal to the amount of their notes redeemed in any given period, this will not cause a problem for them as long as the amount of lawful money in the economy exceeds the amount of notes redeemed in any given period.

One problem with this conclusion is that, apparently, it is inconsistent with at least one important historical episode. When banks faced this opportunity during the National Banking Era, they did not issue as many notes as the collateral and other restrictions in place during that period would have permitted.[3] Champ, Wallace, and Weber (1994) carried out one of the recent studies on the question of

[2] See, for example, Fischer (1986) and King (1983) for a discussion of some of the issues and challenges in modelling private money.

[3] This has been identified as a puzzle. See, for example, Friedman and Schwartz (1963).

underissue during this period and concluded that the note-issuing banks were concerned about the demand-liability feature of their outstanding notes, which points to a reserve management problem faced by banks. Under a complete markets assumption, theory predicts overissue since banks can then always keep any amount of their notes outstanding. Thus, these models are not able to reconcile the public acceptance of the private notes with the facts about underissue during this period.

In this paper, we build a model of private money in the decentralized markets framework introduced by Kiyotaki and Wright (1989). Indeed, little needs to be added to the standard random matching model of money in order for inside money in the form of private bank notes to circulate as media of exchange and for a reserve management problem to arise for the note-issuing banks. Our model contains the following two features. First, the no-note-issuing public treats all notes and fiat money as perfect substitutes. Second, banks cannot keep the entire amount of their notes outstanding, since notes are redeemed as a result of the random and endogenous trades. The same market incompleteness that gives rise to money as a medium of exchange thus creates a reserve management problem for banks. In other words, banks are subject to frictions when raising the funds required to meet the random redemptions of their notes. We demonstrate that solving this problem makes it possible to reconcile optimizing behavior with limited note issue.

In our model, a subset of the agents is endowed with the ability to issue liabilities in order to finance consumption during encounters with producers of their preferred good. We will identify these agents with banks and the liabilities with bank notes. Banks also have the capability of recording with a clearinghouse, the treasury, their earnings from money accepted in decentralized trades. This technology enables them to build reserves by making deposits with the treasury, but it also creates a redemption process, since circulating notes might get redeemed in any given period if they are deposited as reserves of other banks. We assume that the treasury removes the banking technology from banks that hold an amount of reserves smaller than the amount of notes redeemed in a given period. Only a fraction of the outstanding notes of a bank is redeemed in any given period, so this rule is much weaker than, say, 100% reserve requirements. Since banks do not have the opportunity to always keep their issued notes outstanding, they must be concerned with the amount of reserves they hold, keeping in mind the probability of losing their note-issuing privilege if caught with negative reserves.

We demonstrate the existence of a monetary steady state equilibrium where liabilities circulate as private media of exchange, as a perfect substitute for fiat money. We characterize the optimal policy rules for banks and find that, for high discount factors, banks will limit their note issue. Indeed, concerned about the amount of their notes redeemed, banks in some states forgo consumption and do not issue new notes while, at the same time, they suffer the disutility of production in order to improve their reserve position. Finally, we demonstrate that if banks discount the future at a sufficiently low rate, they will never issue notes unless they can back them in reserves. Thus, 100% reserves may arise endogenously as a special case in our model. We also find that for low discount factors, most banks will be "illiquid" and may even display "wildcat behavior," i.e., they might find it optimal to concentrate on the short-run benefits of note issue and eventually exit the banking sector by failing to honor redemptions. Perhaps not surprisingly, our results on a stable monetary system depend on conditions guaranteeing the long-run profitability of banks. More precisely, three features of our model guarantee that the frequency of consumption for banks is higher than that of non-banks. First, there is limited entry in the banking sector. Second, since we do not impose 100% reserve requirements, banks have access to a form of borrowing through the "floating" of notes, which is not available to non-banks. Finally, the banking technology allows for the accumulation of reserves (in the form of record-keeping), while non-banks can hold at most one unit of money. Although the upper bound on money holdings greatly simplifies the analysis, we conjecture that it is not essential for generating a profitable bank sector, especially in the presence of the generous reserve requirement assumed here.

If we interpret private note issue as a form of credit, our model makes the methodological contribution of introducing credit in a lack-of-double-coincidence model of money. In the Kiyotaki-Wright model, market incompleteness, as modelled by a random matching technology, together with the assumption that agents' histories are not part of a public record, creates a role for money as a medium of exchange. However, in a world with fully decentralized markets and with agents' histories being private information, arrangements such as credit cannot exist. Diamond (1990), Shi (1996), and Aiyagari and Williamson (1997) provide alternative ways of introducing credit in a search framework, but in none of these models do liabilities circulate as private media of exchange, a necessary condition for them to be interpreted as private money.

The important question of whether this arrangement is part of an optimal one is left for future research. A standard Kiyotaki-Wright type model follows as a special case of ours. This leads us to the conjecture that the introduction of private liabilities might, at least in the cases where outside money is scarce, lead to improved welfare. We further speculate on this issue in our Conclusions section. The paper proceeds as follows: Section II describes the economic environment.

Section III contains the steady state value functions. Section IV characterizes the optimal policy rules and deals with the existence of a steady state monetary equilibrium. Section V concludes the paper. The appendix contains some of the proofs.

2 The Economic Environment

Time is discrete, t, measured over the positive integers. There is a $[0, \frac{1}{k}]$ continuum of each of k types of infinitely lived agents, and there are $k \geq 3$ indivisible perishable goods. The total measure of agents in the economy is 1. Agents are specialized in production and consumption of goods. Agents of type i consume good i only and produce good $i+1$ only (mod k). All agents are expected utility maximizers, and the "discount factor" is $\tilde{\beta}$, a positive number. Agents are randomly matched pairwise, once in every period. As is common in this literature, the assumptions on specialization rule out double coincidence meetings. The only storable assets are indivisible money objects that can be either government or private money. Each person has a storage capacity of one unit of money, government or private, and can produce at most one unit of good. We let m_{fiat} denote the fraction of each consumption type that is holding government money. Consumption of one unit of good gives utility u, and production of one unit gives disutility e. We assume that $u > e$.

An uncountable subset of the $[0, \frac{1}{k}]$ continuum for each type is endowed with the ability to costlessly issue one unit of liability per period in exchange for purchases of goods. We will identify these agents with *banks* and will refer to these liabilities as *bank notes*. We assume that private notes are treated as a perfect substitute for each other and for outside money in all trades and demonstrate that this assumption is consistent with a steady state equilibrium. While there might exist other monetary equilibria where some or all notes are not accepted as a substitute of outside money, we thus concentrate on the steady state equilibrium that is consistent with historical observations from the National Banking Era.

Banks also have a capacity to hold, at most, one unit of outside money but have the ability to record-keep any notes or fiat money they earned from production with a central location. We will identify this storage as *reserves deposited with the treasury.* Given the production technology for goods, these assumptions imply that one unit of money (fiat or notes) is exchanged for one unit of goods in all decentralized trades, i.e., prices are exogenous. We let d denote the (integer) amount of reserves for a bank agent. Except for their ability to issue notes and record reserves, banks in this model are identical to non-banks. Notes issued circulate as private media of exchange until they are deposited by other banks. We let m be the (integer) amount of its own notes in circulation for a bank. We assume that the treasury acts according to the following exogenous rule: whenever a note issued by bank i is deposited by bank j, the account of the depositing bank j is credited and the account of the issuing bank i is debited by one. We identify this process with a *redemption process for the notes*. This note subsequently becomes a mere record-keeping device.[4] If a bank has a negative balance in a given period, i.e., if the amount of its own notes deposited by other banks with the treasury in that period exceeds its reserves, the treasury adopts the exogenous policy of depriving it of the banking technology. In this case, the bank becomes a non-bank in the model and is given one unit of fiat money upon exit.[5] We let $q \in [0,1]$ be the fraction of banks in the population and in each consumption type that exits the banking sector because of a negative balance in each period. Notes issued by banks that exit the industry might circulate in periods after the exit. We assume that the treasury and, therefore, all other agents in the model will honor such notes. Even though we do not model collateral explicitly in the model, this assumption captures the feature that bank notes during the National Banking Era had an implicit government guarantee, so that note holders were not facing any substantial risk from failures of note-issuing banks.[6]

Since exit from the banking sector is possible, for a steady state where the size of this sector is constant, we will also need entry as well as exit from the non-banking sector. We assume that with probability $\delta \in (0,1)$, each agent in the model dies and is replaced by a newborn. In the case where the exiting agent holds a note, the treasury redeems that note and adjusts the reserve balances.[7] We let $\mu_1 \in (0,1)$ be the fraction of newborns that are banks. Banks enter the economy

[4]Clearly, there are many ways to model the treasury's information in this model. We keep this information minimal, assuming that the treasury knows banks' reserves but not their number of notes in circulation.

[5]This is only a simplifying assumption. It guarantees that banks always deposit every unit of fiat money or notes they earn so that we need to keep track of one less state variable.

[6]During the National Banking Era, notes were fully backed by purchases of U.S. government bonds. The bonds were paying interest that, absent any hidden costs, made note issue a most profitable investment for banks. One question is why the entire amount of eligible bonds was not used in order to support note issue during most of that period (see Friedman and Schwartz (1963)). Here, we explore the implications of having the market for putting redeemed notes back in circulation *immediately* being "missing."

[7]Our results would go through if we assumed that agents' money holdings "disappear" after their deaths. We, however, find our current assumption more convenient to work with.

with $(d,m) = (0,0)$. We also assume that a fraction, μ_2, of the non-bank newborns enters with one unit of money holdings. We let $\mu = \mu_1 + \mu_2$. We let $\beta = (1-\delta)\tilde{\beta} \in (0,1)$ be the effective discount factor.[8] Banks and non-banks alike trade after the period starts. Each agent's type, including whether it is a bank or not, is private information. If a bank knew that it is dealing with another bank, then it might want to avoid issuing a note because this note would be redeemed in the current period if issued. With individual banking privileges being private information, we need to consider one fewer decision variable. We will concentrate on steady state equilibrium outcomes where all private notes are accepted as perfect substitutes. As mentioned above, the upper bound and indivisibility assumptions make prices exogenous, i.e., one unit of a good is always exchanged for one unit of fiat money or a note. For simplicity, we will rule out money-for-money trades.[9] After trades occur, and since we assume that banks that exit because of a negative balance are given one unit of fiat money, banks will always deposit their earnings with the treasury. The reserve balances are then adjusted. Finally, right before the new period starts, each bank is informed of its new reserve balance, from which each can infer the amount of its notes remaining in circulation. The timing of actions within a period is as follows:

$$t :\mapsto [v_{d,m}] \mapsto trade \mapsto deposits \mapsto deaths \mapsto [w_{d,m}] \mapsto balance\ adjustment \mapsto: t+1, \tag{1}$$

where $v_{d,m}$ is the steady state value function of a bank of type (d,m), and $w_{d,m}$ is the steady state value function after a bank has deposited its earnings for the period, but before the redemption process and the balance readjustment occur. For non-banks, the state is $m \in \{0,1\}$, and in each period, they choose the probability $\alpha \in [0,1]$ of accepting money (fiat or notes) in exchange for producing. For banks, the state is $(d,m) \in \mathbb{N} \times \mathbb{N}$, and in each period, they choose $(\gamma_{d,m}, \phi_{d,m}) \in [0,1] \times [0,1]$, where $\gamma_{d,m}$ is the probability of accepting money in exchange for producing and $\phi_{d,m}$ is the probability of issuing money in exchange for consuming, at

[8] Notice that this specification does not require that $\tilde{\beta}$ be less than 1.

[9] Clearly, a bank has an incentive to exchange one of its own notes for a note issued by another bank. However, since the non-bank is indifferent between the two, the case where all non-banks refuse such trades is consistent with an equilibrium. Including such money-for-money trades will not change the results, provided that the same upper bound conditions hold.

state (d,m).[10] We let $x_{d,m}$ be the measure of banks of type (d,m) within each consumption type and, therefore, in the population. We define $\gamma x = \sum_{d,m} \gamma_{d,m} x_{d,m}$ and $\phi x = \sum_{d,m} \phi_{d,m} x_{d,m}$ to be the fraction of banks that are willing to increase their reserves by producing and the fraction of banks that are willing to issue a new note in exchange for consumption, respectively, within a given period. Similarly, c_i stands for the measure of non-banks with 0 or 1 unit of money, respectively. Feasibility requires that $c_0 + c_1 + \sum_{d,m} x_{d,m} = 1$. In order to calculate the stationary measure of banks, observe that the fraction of banks tomorrow equals the fraction of banks today minus the exogenous fraction of banks that die, minus the endogenous fraction of banks that exit because of a negative balance, plus the fraction of newborns that are banks. In other words, $\Sigma x'_{d,m} = (1-\delta)(\Sigma x_{d,m}) - q + \delta\mu_1$. Thus, provided that $q \leq \delta\mu_1$, the steady state measure of banks is $\mu_1 - \frac{q}{\delta}$.

Since a bank will always find it optimal to deposit with the treasury every unit of money it earns, a note that is already in circulation is redeemed if it is earned by another bank in exchange for production or if it is in the hands of a non-bank that dies.[11] Therefore, π, the probability that any note is redeemed, is given by: $\pi = \delta + (1-\delta)\frac{\gamma x}{k}$. Hence π is bounded below by δ. Because there is a constant inflow of non-banks with and without money in the economy, c_0 and c_1 are also bounded away from zero. In what follows, we will concentrate on equilibria that are symmetric across agent types.

3 Steady State Value Functions

In this section, we describe the steady state value functions. We let s_i be the value of a non-bank with $i \in \{0,1\}$ units of money. First, for a non-bank with one unit of money holdings we have

$$s_1 = \frac{1}{k}(\alpha c_0 + \gamma x)(u + \beta s_0) + \left[1 - \frac{1}{k}(\alpha c_0 + \gamma x)\right]\beta s_1. \tag{2}$$

[10]Recall that, being consistent with observations from the National Banking Era, we concentrate on arrangements where all notes are treated symmetrically in decentralized trades. Thus, we do not index the agents' choice variables by the type of notes they are offered.

[11]Since there is a continuum of banks, we will ignore the case where a bank is presented with one of its own notes.

The first part of the above equation gives the expected payoff from a trade as a consumer in a single-coincidence meeting with a producer. The second part describes the expected payoff if no such trade occurs. For a non-bank with zero units of money holdings we have

$$s_0 = \frac{1}{k}(c_1 + \phi x)\max_{\alpha}[\alpha(-e + \beta s_1) + (1-\alpha)\beta s_0] + \left[1 - \frac{1}{k}(c_1 + \phi x)\right]\beta s_0. \quad (3)$$

The first part of the above equation gives the expected payoff of a non-bank from a trade as a producer in a single-coincidence meeting with a consumer. The second part gives the expected payoff if no such trade occurs. Next, the symmetric steady state value function for a bank of type (d,m) is given by

$$v_{d,m} = \frac{1}{k}(c_1 + \phi x)\max_{\gamma_{d,m}}\left[\gamma_{d,m}(-e + \beta w^0_{d+1,m}) + (1-\gamma_{d,m})\beta w^0_{d,m}\right]$$

$$+\frac{1}{k}(\alpha c_0 + \gamma x)\max_{\phi_{d,m}}\left\{\phi_{d,m}\left[u + p\beta w^0_{d-1,m} + (1-p)\beta w^1_{d,m}\right] + (1-\phi_{d,m})\beta w^0_{d,m}\right\}$$

$$+\left[1 - \frac{1}{k}(c_1 + \phi x) - \frac{1}{k}(\alpha c_0 + \gamma x)\right]\beta w^0_{d,m}. \quad (4)$$

The probability of redemption for a note in a given period depends on whether the note is issued in that period or whether it was already in circulation. A newly issued note is redeemed if it is accepted by a bank or if it is held by a non-bank that dies. We denote the probability of this event by p, where $p = \frac{\gamma x}{\gamma x + \alpha c_0} + \frac{\delta \alpha c_0}{\gamma x + \alpha c_0}$. For a note already in circulation the probability of redemption is π, as defined earlier. The first part of the value function describes the expected payoff of a bank of type (d,m) from a trade as a producer in a single coincidence meeting with a consumer. The second part of the equation gives the expected payoff of a bank of type (d,m) from a trade as a consumer in a single-coincidence meeting with a producer. The last part describes the expected payoff if no such trades occur. We let $w_{d,m}$ be the value function after the bank has deposited the earnings for the period with the treasury, but before the redemption process and the balance readjustment occur.

The index j takes value 1 if a bank issues a note in a given period, and value 0 otherwise. We then have:

$$w^j_{d,m} = \sum_{0 \le i \le \min\{d,m\}} p(i,m) v_{d-i,m+j-i} + \sum_{d<i\le m} I_{d,m} p(i,m) s_1, \tag{5}$$

where $p(i,m) = \binom{m}{i} \pi^i (1-\pi)^{m-i}$, $j = 1$ if a new note enters circulation, while $j = 0$ otherwise, and $I_{d,m} = \begin{cases} 1, & if \quad m > d; \\ 0, & if \quad m \le d. \end{cases}$ In the expression above, the first sum corresponds to the possibility that the notes redeemed do not exceed the bank's reserves and, therefore, the bank remains in business. The second sum corresponds to the possibility that the notes redeemed from the treasury in that period exceed the bank's recorded reserves and, therefore, the bank will exit the sector in the next period. The number of notes redeemed within a period is a random variable following a binomial distribution, and the bank takes the redemption probability, π, as given. In order to define a steady state equilibrium for this economy, we first need to consider the law of motion of $x_{d,m}$. In the appendix, we demonstrate how the optimal decision rules define an operator, Q, mapping the distribution of banks across states at the beginning of the period to distributions of banks across states at the end of the period. As we mentioned before, we assume that those banks that end the period with a negative balance will exit the industry. To capture this fact, we also define an operator T mapping the distribution of bank types at the beginning of a period to the distribution at the beginning of the next period. At a steady state equilibrium we have that $x = Tx$, and the measure of banks that exit the industry is given by $q = \sum_{\{d<0,\, m\ge 0\}} Qx_{(d,m)}$. We have the following:

Definition: *A symmetric steady state equilibrium is a set of value functions* $\{s, v\}$ *with* $s : \{0,1\} \to \mathbb{R}$ *and* $v : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, *together with a set of policy functions,* $\{\alpha, \gamma, \phi\}$, *a distribution over non-banks and banks,* $\{x, c_0, c_1\}$, *where* $x : \mathbb{N} \times \mathbb{N} \to [0,1]$ *and* $c_0, c_1 \in [0,1]$, *a measure of fiat money in circulation,* m_{fiat}, *and a probability of a note being redeemed,* $\pi \in [0,1]$, *such that*

(1) s_0, s_1 *are the solutions to the functional equation for non-banks when* α *is optimal.*

(2) v *is the solution to the functional equation for banks when* γ *and* ϕ *are optimal.*

(3) $x = Tx$.

(4) $c_1 = (1-\delta)\left[\frac{c_1+\phi x}{k}\alpha c_0 + \left(1-\frac{\gamma x+\alpha c_0}{k}\right)c_1\right] + q + \delta\mu_2$.
(5) $c_0 = (1-\delta)\left[\left(1-\frac{c_1+\phi x}{k}\right)c_0 + \frac{\gamma x+\alpha c_0}{k}c_1\right] + \delta(1-\mu)$.
(6) $\pi = \delta + (1-\delta)\frac{\gamma x}{k}$, $q = \sum_{\{(m,d):d<0,\ m\geq 0\}} Qx_{(d,m)}$.
(7) $q + \delta\mu_2 = \delta m_{fiat} + (1-\delta)m_{fiat}\frac{\gamma x}{k}$.
(8) $c_0 + c_1 + \sum_{d,m} x_{d,m} = 1$, $\sum_{d,m} m x_{d,m} + m_{fiat} = c_1$.

Conditions (1) and (2) above are self-explanatory. Conditions (3),(4) and (5) require that the agents' behavior is consistent with the steady state distribution over banks and non-banks. Condition (6) requires that the redemption probability and the measure of banks that exit are consistent with the agents' optimal behavior. Condition (7) equates the inflow to the outflow of fiat (outside) money in the economy. Finally, condition (8) describes aggregate feasibility. Next we move to the question of characterizing the banks' optimal policies at steady state.

4 Characterizing Policy Rules

In this section, we characterize the optimal policy rules for the agents in the economy and demonstrate the existence of a monetary steady state equilibrium. The first Lemma gives a sufficient condition for non-banks to accept money (fiat and private) in exchange for service in a symmetric steady state equilibrium. This will be helpful in showing that a monetary equilibrium exists.

Lemma 1: *Suppose that* $\frac{1}{k}\delta(1-\mu) \geq \left(\frac{1}{\beta}-1\right)\frac{1}{\frac{u}{e}-1}$. *Then, non-banks accept money if other non-banks do; so* $\alpha = 1$.

Recall that $\mu = \mu_1 + \mu_2$. Note that the condition of Lemma 1 will tend to be satisfied if the discount factor is large and if the fraction of newborns that are banks is small. If the fraction of agents that can issue notes is too high, a steady state equilibrium where notes are valued might not exist. It is also intuitive that non-banks will tend to accept money when the ratio u/e is sufficiently high. The next Lemma establishes the monotonicity properties of the value function for banks. It asserts that the value function is increasing in the amount of reserves and decreasing in the amount of notes in circulation.

Lemma 2: *The value function $v_{d,m}$, is (a) weakly increasing in d and (b) weakly decreasing in m.*

The next Lemma says that the value of a bank is always greater than the value of a non-bank with one unit of money holdings which, in turn, is greater than the value of a non-bank with no money holdings. Banks in our model can do everything that non-banks can, and, in addition, they can record earnings in reserves and "borrow" by costlessly issuing new liabilities. This additional value of remaining in business will make the reserve management problem meaningful.

Lemma 3: $s_0 < s_1 < v_{d,m}$ for all d and all m.

Our goal in this section is to characterize the behavior of banks at a monetary steady state. We build toward the characterization through a sequence of claims describing the optimal policy rules for banks throughout the state space. A summary of this exercise is presented in Figure 1. First, we find it convenient to prove a Lemma for the following special case.

Lemma 4: *If β is sufficiently large, then $\phi_{0,0} = 0$.*

The above Lemma provides a sufficient condition for a bank at state $(0,0)$ to choose not to issue a note in exchange for consumption if this note is not backed, in order to avoid a positive probability of having to exit. Recall that exit will occur if the producer in this meeting is another bank and thus the note gets redeemed. Although Lemma 4 is about a special case, we will find that its proof can be used in order to prove more general propositions later. Here is some intuition for the proof. By issuing a note at state $(0,0)$, a bank faces the possibility of having negative reserves and being forced to exit. The immediate gain of issuing a note is equal to the utility of consumption, u. As agents become more patient, the difference in utility between being a bank or a non-bank grows unboundedly. Thus, by issuing a note a bank faces an arbitrarily large utility loss with positive probability. This cost outweighs the short-run gain of printing a note today. A higher β can, therefore, reduce liquidity in the economy. Furthermore, as the discount factor approaches 1, a self-imposed 100% reserve requirement becomes the optimal policy for most banks in a steady state. The proof of Lemma 4 explores the fact that any small entry of non-banks with money in every period $(\mu_2 > 0)$ suffices to provide a positive lower bound on the liquidity in the economy and, as a result, on the probability that a newborn bank will accumulate enough reserves to enjoy consumption.

This behavior originates from assumptions guaranteeing that, on average, banks have a higher frequency of consumption than non-banks. First, there is limited entry in the banking sector. Second, since we do not impose 100% reserve requirements, banks have access to a form of borrowing through the floating of notes, which is not available to non-banks. Third, the banking technology allows for the accumulation of reserves (in the form of record-keeping), while non-banks can hold at most one unit of money. The next proposition provides a benchmark case. It says that if banks discount the future at a sufficiently high rate, they will always choose to borrow and consume today. The proof is trivial and, thus, omitted.

Proposition 1: *If* β *is sufficiently small, then* $\phi_{d,m} = 1$ *for all* d *and all* m.

The next proposition characterizes the optimal behavior of banks around the 45^0 line in the (d,m) space for different values of the underlying parameters (see Figure 1). States above this line correspond to "illiquid" banks in the sense that for banks in this region the amount of notes outstanding is greater than the amount of reserves. States below this line have the opposite implication. Generally, a bank will choose to cross into the illiquid zone provided that it is large enough, since, in that case, the probability of a resulting negative balance in this region is small. At the same time, if the discount factor is high enough, banks around the 45^0 line will also choose to produce in order to improve their reserve position.

Proposition 2: *(a) Fix* d. *Then as* $\beta \to 1$, $\phi_{d,d} = 0$.
(b) Fix $\beta \in (0,1)$. *There exists* D *such that* $\phi_{d,d} = 1$ *if* $d \geq D$.
(c) Fix (d,m) *such that* $m < d$. *Then* $\phi_{d,m} = 1$.

The proof of part (a) follows from Lemma 1. For fixed d, a bank issuing a note at state (d,d) faces a probability of being caught with negative reserves that is bounded away from zero. Thus, for beta large enough, $\phi_{d,d} = 0$. The proof of part (b) relies on two facts. First, for β less than one, the cost of exiting the industry is finite. Second, as d grows, a bank issuing a note at state (d,d) faces a probability of being caught with negative reserves that is close to zero. Part (c) is established by comparing the immediate gain of issuing an additional note (i.e., the utility of consumption, u) with the loss from having one extra note in circulation. It is shown that when m is less than d, this loss is less than u in present value terms. This is true since, when m is less than d, the value function satisfies the condition that $v_{d,m} - v_{d,m+1} < u/\beta$.

The next proposition characterizes the behavior of banks that are "too liquid" or "too illiquid" in the sense that either $d - m$ or $m - d$ are positive and large, respectively. Banks that have too many reserves compared with the amount of their notes outstanding will concentrate on issuing more notes instead of building additional reserves. Perhaps not surprisingly, banks that have too few reserves compared with the amount of their notes outstanding will do the same, since they expect to exit the banking sector with probability one in the near future.

Proposition 3: *(a) Fix m. There exists a D_m such that for all $d \geq D_m$, $\phi_{d,m} = 1$.*
(b) Fix m. There exists a D_m such that for all $d \geq D_m$, $\gamma_{d,m} = 0$.
(c) Fix d. There exists an M_d such that for all $m \geq M_d$, $\phi_{d,m} = 1$.
(d) Fix d. There exists an M_d such that for all $m \geq M_d$, $\gamma_{d,m} = 0$.

The next proposition asserts that if β is high enough and $d < m$, banks will always work toward improving their reserve position. The proof follows from a similar argument to that in the proofs of Propositions 2 and 3. An increase in the reserve position implies a discrete reduction in the probability of being caught with negative reserves. Since, as β approaches 1, the loss of exiting the banking industry grows unboundedly, it is worth it for a bank to suffer the disutility of production (e) and to forfeit the utility of consumption (u) in order to improve its reserve balance.

Proposition 4: *Fix (d,m), such that $d < m$. As β approaches 1, we have*
(a) $\gamma_{d,m} = 1$.
(b) $\phi_{d,m} = 0$.

The statements in the above propositions are summarized in Figure 1. As the values of d and m vary, the optimal policy rules give rise to four regions in the (d,m) space. In region I, the bank's reserves are high compared to the number of its notes in circulation. In that case, a bank will find it optimal to issue a note when faced with the opportunity, thus increasing the number of its notes in circulation. At the same time, such a bank will reject opportunities to increase its reserves. Banks in region II will still find it optimal to put a new note in circulation, given the opportunity, but being less liquid, they will now also accept trades that increase their reserves. Banks in region III are becoming alarmingly illiquid and will find it optimal to both improve their reserves and stop issuing new notes. In other words, concerned about the possibility of having to give up their note issuing privilege, these banks will concentrate on improving their reserve position. Finally, banks in region IV have too few reserves compared to the number of their notes in circulation and will thus have to exit the banking sector with high probability in the near future. These banks would not benefit from increasing their reserves since redemptions will arrive at a faster rate, making them even less liquid than before. They will, therefore, exhibit "wildcat banking" behavior and will only issue new notes until they are forced to exit due to a negative balance. Notice that in all four cases, the redemption process reduces both reserves and the notes in circulation by the same number and thus always moves a bank southwest, along a 45^0 line. While banks in our model will not enter region IV voluntarily, we cannot rule out the possibility that the redemption process might bring them there from another region. In that case, they will never exit this region before they exit the banking sector.

Given our assumptions on the technology and the treasury policy, and given the optimal policy rules for banks, the next Lemma asserts that at each point in time, all but an arbitrarily small number of banks will have a bounded number of notes in circulation and, therefore, a bounded number of notes in reserves. Recall that a bank can issue at most, one note per period, and the probability of a note's being redeemed is bounded away from zero. As the number of notes in circulation becomes large, the fraction of these notes redeemed approaches a constant, and the number of notes redeemed becomes greater than 1 for all but an arbitrarily small fraction of banks, which we will ignore. Let χ be an invariant distribution of banks across states (d,m).

Lemma 5: *For all* $\varepsilon > 0$, *there exists an M large enough such that for all* d, $\sum_{d,m\geq M}\chi_{d,m} < \varepsilon$ *and for all* m, $\sum_{d\geq M,m}\chi_{d,m} < \varepsilon$.

Next, Proposition 5 says that there exists a unique invariant distribution on the state space, associated with a symmetric monetary equilibrium. An arbitrarily small probability of deaths guarantees that transition probabilities to any state are bounded away from zero. This, in turn, implies that the mapping from the distribution of banks across (d,m) types today to the distribution tomorrow satisfies a contraction property. For the proof of this proposition we impose an upper bound on the state space for banks. The previous Lemma suggests that this bound, if large enough, will not bind but for an arbitrarily small number of banks. In addition, Proposition 5 asserts that in a monetary equilibrium, a positive fraction of banks will both issue and accept private liabilities.

Proposition 5: *Suppose that a monetary equilibrium exists. This equilibrium is characterized by a unique invariant distribution,* χ*, of banks across states* (d,m)*. Furthermore, if* β *is sufficiently large and* e *is sufficiently small,* $\gamma x > 0$ *and* $\phi x > 0$.

Proposition 6 asserts the existence of a steady state monetary equilibrium with a nontrivial distribution over bank states. The proof is similar to the one in Aiyagari and Wallace (1991) and, thus, omitted.

Proposition 6: *For the parameters satisfying the condition of Lemma 1, there exists a monetary steady state equilibrium with trade and a non-degenerate asset distribution for the banks.*

Let $NB = \sum_{\{(d,m):d<m\}}(m-d)x_{d,m}$ be the total amount of circulating notes that are not backed by reserves in the economy. The following proposition suggests that for low enough discount factors, banks will issue notes not backed in reserves, while if the discount factor is sufficiently high, they will remain completely liquid, in which case there will be no exit from the sector because of negative reserves. It is worth mentioning that the last case is consistent with 100% reserves arising endogenously as part of an equilibrium outcome.

Corollary: *(a) If* β *is high enough, there exists a steady state equilibrium with* $NB = 0$*.*
(b) If β *and* e are *low enough, there exists a steady state equilibrium with* $NB > 0$*.*

Figure 1 provides the intuition for part (a). Proposition 2(a) implies that for high enough discount factors, banks will never cross the 45^0 line in the (d,m) space (with d and m less than a fixed upper bound M). In this case, there are no illiquid banks in equilibrium. To see why the claim in part (b) holds, consider the limit case where $e = 0$. Notice that the condition in Lemma 1 is satisfied for all $\beta \in (0,1)$. Then Proposition 6 guarantees the existence of a monetary steady state. Using our characterization of the decision rules, we can conclude that, for β sufficiently low, there is a positive probability that a newborn bank will become illiquid in some finite time. Since there is a positive measure of banks at state $(0,0)$, we can then conclude that there will be a positive number of illiquid banks. By a continuity argument, we can then argue that there exists an equilibrium with illiquid banks for e close to zero.

5 Conclusions

We studied private money issue and redemption in a version of the framework of the Kiyotaki-Wright model. In our model, a departure from the extreme decentralized markets allows for private liabilities to circulate as media of exchange in a symmetric steady state equilibrium. The optimal policy rules, as summarized in Figure 1, suggest that liability issuers in certain states forgo consumption and suffer the disutility of production in order to build a better reserve position. The market incompleteness leads liability-issuing banks to be concerned about the amount of their outstanding notes, even though the public views all notes and currency as perfect substitutes. This suggests that rules less restrictive than 100% reserve requirements can be consistent with a money-issuing banking system that is stable. Admittedly, the model is extreme in many respects. For example, the market frictions of the matching model that we studied rule out the possibility of redeemed notes being put back in circulation immediately, or that borrowing is used in order to prevent a negative balance. Certainly, the (limited) ability of banks to borrow or to keep issued notes outstanding is not described by either extreme. However, as long as banks are not able to keep the entire amount of their notes in circulation, they will be concerned about their reserve position even when the public views all notes and currency as perfect substitutes. In such cases, the reserve management problem will be relevant.

Extensions of the basic model studied here could include studying the case where all agents can have access to the note-issuing technology, as well as the case where agents' types are publicly known in decentralized trades. In the latter case, a bank might refuse to trade with another bank, knowing that in this case redemption will occur instantly and, instead, might wait for a meeting with a non-bank producer in order to consume. By introducing divisible production, we could study endogenous price formation via a bargaining protocol. This would complicate the model significantly, but it should not affect any of our results. Another extension would be to allow for the case where notes issued by banks that exit the industry cannot be used as reserves and, therefore, there is some risk to the note holders. We believe that most of our results will also hold true in that case.

One question is that of a welfare comparison between our model and, say, a standard Kiyotaki-Wright model with the same steady state amount of outside money. Although we do not explicitly study this question in this paper, it is worth speculating. If the stock of outside money in the economy is small, i.e., if outside money is scarce, the ability to issue inside money will almost certainly enhance trade and, therefore, improve welfare. Interestingly, this is consistent with historical observations about scarcity of money and the introduction of private money in colonial America.[12]

Potentially important issues to study are the optimality properties of our steady

[12] See, for example, Hanson (1979).

state equilibrium and alternative policy rules for the treasury. For example, this framework seems suitable for a welfare comparison between inelastic and elastic currency regimes in the presence of periodic cycles in the demand for currency. We leave such questions for future research.

6 References

Aiyagari R., Wallace, N. and Wright R. (1996): Coexistence of Money and Interest-Bearing Securities. *Journal of Monetary Economics* **37**.

Aiyagari R. and S. Williamson (1997): Credit in a Random Matching Model with Private Information. University of Iowa Working Paper #97-03.

Aiyagari R. and N. Wallace (1991): Existence of Steady States with Positive Consumption in the Kiyotaki-Wright Model. *Review of Economic Studies* **58,** 901-916.

Champ B., N. Wallace, and W. Weber (1994): Interest Rates Under the U.S. National Banking System. *Journal of Monetary Economics* **34**, 101-115.

Diamond P. (1990): Pairwise Credit in a Search Equilibrium. *Quarterly Journal of Economics* **105**, 285-320.

Freeman S. (1996): Clearinghouse Banks and Banknote Over-issue. *Journal of Monetary Economics* **38**, 101-115.

Fischer S. (1986): Friedman Versus Hayek on Private Money. Review Essay. *Journal of Monetary Economics* **17**, 433-439.

Friedman M. and A. Schwartz (1963): A Monetary History of the United States 1867-1960. Princeton, N.J.: Princeton University Press.

Green E. (1996): Money and Debt in the Structure of Payments. Manuscript. Federal Reserve Bank of Minneapolis.

Hanson J.R. (1979): Money in the Colonial American Economy: An Extension. *Economic Inquiry* **XVII**, 281-286.

Hayek F.A. (1990): Denationalization of Money - The Argument Refined. Third Edition. Institute for Economic Affairs, London.

King R.G. (1983): On the Economics of Private Money. *Journal of Monetary Economics* **12**, 127-158.

Kiyotaki N. and R. Wright (1989): On Money as a Medium of Exchange. *Journal of Political Economy* **97(4),** 927-954.

Salin P. (Ed.) (1984): Currency Competition and Monetary Union. Financial and Monetary Policy Studies 8. Martinus Nijhoff Publishers.

Shi S. (1996): Credit and Money in a Search Model with Divisible Commodities. *Review of Economic Studies* **63,** 627-652.
Taub Bart (1985): Private Money with Many Suppliers. *Journal of Monetary Economics* **16**, 195-208.

7 Appendix

The law of motion of $x_{d,m}$: The optimal decision rules define an operator, Q, mapping the distribution of banks across states at the beginning of the period to distributions of banks across states at the end of the period. Formally, let $G = \{f : \mathbb{N}\times\mathbb{N}\to\mathbb{R}\}$ and $G^{'} = \{g : \mathbb{Z}\times\mathbb{N}\to\mathbb{R}\}$. Then we define the operator $Q : G \to G^{'}$ by

$$Qx_{d',m'} = \sum_{d,m}(1-\delta)\frac{c_1+\phi x}{k}\left[\gamma_{d,m}P^0_{d+1,m}(d^{'},m^{'}) + (1-\gamma_{d,m})P^0_{d,m}(d^{'},m^{'})\right]x_{d,m}$$

$$+\sum_{d,m}(1-\delta)\frac{\gamma x+\alpha c_0}{k}\left\{\phi_{d,m}\left[pP^0_{d,m}(d^{'},m^{'}) + (1-p)P^1_{d,m}(d^{'},m^{'})\right] + (1-\phi_{d,m})\,P^0_{d,m}(d^{'},m^{'})\right\}x_{d,m}$$

$$+\sum_{d,m}(1-\delta)\left[1-\frac{c_1+\phi x}{k}-\frac{\gamma x+\alpha c_0}{k}\right]P^0_{d,m}(d^{'},m^{'})x_{d,m}$$

$$+\delta\mu x_{0,0}I_{\{(d^{'},m^{'})=(0,0)\}}. \tag{6}$$

The first sum in the above expression describes the transition to state $(d^{'},m^{'})$, provided the bank trades as a producer. The second and third sums describe the transition to state $(d^{'},m^{'})$, provided the bank trades as a consumer or does not trade at all, respectively. Finally, the last expression refers to the entry to state $(0,0)$ by a newborn bank. More precisely, for the transition probabilities we have

$$P^j_{d,m}(d^{'},m^{'}) = \begin{cases} p(i,m), & if\ (i=m-m^{'}\geq 0,\ j=0)\ or\ (i=m+1-m^{'}\geq 0,\ j=1); \\ 0, & otherwise. \end{cases} \tag{7}$$

The above expression describes the respective steady state transition probabilities, $P^j_{d,m}(d^{'},m^{'})$, from any state (d,m) at the beginning of the period, to state $(d^{'},m^{'})$, after the balance adjustments take place. The index j reflects the fact that the transition probabilities depend on whether a new note was issued in the current period. The number of notes redeemed within a period is a random variable following a binomial distribution, and the bank takes the redemption probability, π, as given.

As we mentioned before, we assume that those banks that end the period with a negative balance will exit the industry. To capture this fact we define the operator $T : G \to G$, mapping the distribution of bank agent types at the beginning of a period to the distribution at the beginning of the next period. Formally, for all $(d^{'}, m^{'})$, T is defined as the restriction of Q to the set of banks that stay in business: $Tx_{d',m'} = Qx_{d',m'}$.

Proof of Lemma 1: We have that $\alpha = 1$ if $s_1 - s_0 \geq \frac{e}{\beta}$ or, by substituting, if $[\frac{1}{k}(c_0 + \gamma x)u + \frac{1}{k}(c_1 + \phi x)e][1 - \beta(1 - \frac{1}{k}(c_0 + \gamma x) - \frac{1}{k}(c_1 + \phi x)]^{-1} \geq \frac{e}{\beta}$ or, by rearranging, if $\frac{1}{k}(c_0 + \gamma x) \geq \left(\frac{1}{\beta} - 1\right)\frac{1}{\frac{u}{e} - 1}$. Since $c_0 \geq \delta(1 - \mu)$, the above condition will be satisfied if $\frac{1}{k}\delta(1 - \mu) \geq \left(\frac{1}{\beta} - 1\right)\frac{1}{\frac{u}{e} - 1}$. This condition holds if either $u - e$ or β is high enough. ■

Proof of Lemma 2: (a) We first demonstrate that v is weakly increasing in d. Let $G = \{f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}\}$. Define the operator $T : G \to G$ by $Tv = \{$*left hand side of the functional equation for a bank*$\}$. We know that T has a unique fixed point, v, in the space of bounded continuous functions. If T also maps the space of weakly increasing continuous functions into itself, and since this space is complete, this fixed point will also be a weakly increasing continuous function. We need to show that T preserves monotonicity, i.e., for any fixed m, $d_1 \leq d_2$ implies that $Tv(d_1, m) \leq Tv(d_2, m)$. We first show that for all m and any j, $w^j_{d_1,m} \leq w^j_{d_2,m}$. First, consider the case where $d_1 \geq m$. Then $w^j_{d_h,m} = \sum_{0 \leq i \leq m} \binom{m}{i} \pi^i (1 - \pi)^{m-i} v_{d_h - i, m+j-i}$, where $h = 1, 2$. Notice that the terms multiplying v are the same for $h = 1, 2$. Since v is monotone, it follows that $w^j_{d,m}$ is monotone. Next, consider the case where $d_2 < m$. Then

$$w^j_{d_h,m} = \sum_{0 \leq i \leq m} \binom{m}{i} \pi^i (1 - \pi)^{m-i} v_{d_h - i, m+j-i} + \sum_{d_h \leq i \leq m} \binom{m}{i} \pi^i (1 - \pi)^{m-i} s_1.$$

Note that the first d_1 terms in the first sum are weakly greater when $d_h = d_2$ than when $d_h = d_1$, given that v is monotonically increasing in d. The result then follows, since, for all (d, m), $v_{d,m} > s_1$. By the same reasoning, the desired inequality follows for the case where $d_1 < m < d_2$, since, once again, v is monotonically increasing in d and, for all (d,m), $v_{d,m} > s_1$.

(b) Now we show that v is weakly decreasing in m. Using the argument in (a), we need to show that for any fixed d, $m_1 \leq m_2$ implies that $Tv(d,m_1) \geq Tv(d,m_2)$. As before, we first show that for all d and any j, $w^j_{d,m_1} \geq w^j_{d,m_2}$. We consider the case where $m_2 < d$ first. In that case, $w^j_{d,m_h} = \sum_{0 \leq i \leq m_h} \binom{m_h}{i} \pi^i (1-\pi)^{m_h - i} v_{d-i,m_h+j-i}$. Define $p(i,m)$ by $p(i,m) = \binom{m}{i} \pi^i (1-\pi)^{m-i}$. We then have $\frac{p(i,m)}{p(i,m+1)} = \frac{\frac{m!}{(m-i)!i!}}{\frac{(m+1)!}{(m+1-i)!i!}} \frac{\pi^i}{\pi^i} \frac{(1-\pi)^{m-i}}{(1-\pi)^{m+1-i}} = \frac{m+1-i}{m+1} \frac{1}{1-\pi}$. This expression is less than 1 if and only if $i > \pi(m+1)$. Therefore, $p(i,m)$ is greater than $p(i,m+1)$ for low values of i and is lower for high values of i. The result then follows, since $w^j_{d,m}$ is a convex combination of decreasing functions of m. Therefore, $w^j_{d,m_1} \geq w^j_{d,m_2}$. Since $v_{d,m} > s_1$, for all (d,m), the same argument provides the result for the cases where $d < m_1$ and the case where $m_1 < d < m_2$. ■

Proof of Lemma 3: For the first inequality, notice that $\alpha > 0$ implies that $-e + \beta s_1 \geq \beta s_0$, which, in turn, implies that $s_1 \geq s_0 + \frac{e}{\beta}$ and, therefore, $s_1 > s_0$. Since v is increasing in d and decreasing in m, for the second inequality it is sufficient to show that $v_{0,m} > s_1$, for m large. In this case, the bank will have a negative balance with probability 1 at the end of the period and, therefore, will exit the sector. The bank can issue one more note this period, and in the next period it will still have the same value function as a non-bank with one unit of money, s_1. So $v_{0,m} > s_1$, even for an arbitrarily high m. ■

Proof of Lemma 4: We have that $\phi_{0,0} = 0$ if $u < p\beta(v_{0,0} - s_1) + (1-p)\beta(v_{0,0} - v_{0,1})$, where p is the probability that a newly issued note is redeemed. Since $v_{0,0} > v_{0,1}$, it is sufficient to show that $v_{0,0} - s_1 > \frac{u}{p\beta}$. Consider a voluntary 100%-reserves rule for a bank. According to this arbitrary decision rule, banks issue notes only if they can fully back them in reserves. We shall deal with a lower bound for the bank's value attained by such a rule under a worst-case scenario: that each bank faces a redemption probability equal to 1, so that notes never stay in circulation. Let $\tilde{v}$ denote the expected discounted utility for banks under this scenario. Notice that $\tilde{v}$ depends only on available reserves, d. To complete the description of $\tilde{v}$, we choose the production decision rule for banks as follows.

Banks with $d = 0$ and $d = 1$ always produce to acquire reserves. Banks with $d \geq 2$ produce if and only if $\beta(\tilde{v}_{d+1} - \tilde{v}_d) \geq e$. Given these restrictions, $\tilde{v}$ is a lower bound on the optimal value function v. Therefore, it will be sufficient to show that $\tilde{v}_0 - s_1 > \frac{u}{p\beta}$, for β large enough. Let $q_1 = \frac{1}{k}(c_0 + \gamma x)$ and $q_2 = \frac{1}{k}(c_1 + \phi x)$ be the probabilities of a single coincidence meeting as a consumer and as a producer respectively for a bank. Since both μ_1 and μ_2 are strictly positive, c_0 and c_1 are bounded away from zero, and so are q_1 and q_2. For $d = 0, 1$, the value function $\tilde{v}$ satisfies:

$$\tilde{v}_0 = \beta\tilde{v}_0 + q_2[-e + \beta(\tilde{v}_1 - \tilde{v}_0)] \tag{8}$$

and

$$\tilde{v}_1 = \beta\tilde{v}_1 + q_1[u + \beta(\tilde{v}_0 - \tilde{v}_1)] + q_2[-e + \beta(\tilde{v}_2 - \tilde{v}_1)]. \tag{9}$$

Regarding non-banks, we shall work with their optimal values which satisfy $s_0 = \beta s_0 + q_2[-e + \beta(s_1 - s_0)]$ and $s_1 = \beta s_1 + q_1[u + \beta(s_0 - s_1)]$. Given that q_2 is bounded away from zero, a straightforward calculation reveals that $\tilde{v}_0 \to \tilde{v}_1 - e$ as $\beta \to 1$. Since p is bounded away from zero, because δ is strictly positive, the assertion that $\tilde{v}_0 - s_1 > \frac{u}{p\beta}$ holds for β sufficiently high, now follows from showing that $\tilde{v}_1 - s_1 \to +\infty$ as $\beta \to 1$. This limit is computed as follows. Given the production decision rule attaining $\tilde{v}$,

$$(1-\beta)\tilde{v}_2 = q_1[u + \beta(\tilde{v}_1 - \tilde{v}_2)] + q_2 \max\{0, -e + \beta(\tilde{v}_3 - \tilde{v}_2)\} \geq q_1[u + \beta(\tilde{v}_1 - \tilde{v}_2)].$$

Since $(1-\beta)\tilde{v}_1 = q_1[u + \beta(\tilde{v}_0 - \tilde{v}_1)] + q_2[-e + \beta(\tilde{v}_2 - \tilde{v}_1)]$, we can work with $\tilde{v}_2 - \tilde{v}_1$ in order to obtain the inequality $[1 - \beta(1 - q_1 - q_2)](\tilde{v}_2 - \tilde{v}_1) \geq q_2 e + q_1\beta(\tilde{v}_1 - \tilde{v}_0)$. Now (9) implies

$$\begin{aligned}[1-\beta(1-q_1-q_2)](1-\beta)\tilde{v}_1 \quad \geq \quad & (1-\beta(1-q_1-q_2))q_1 u - (1-\beta(1-q_1))q_2 e \\ & +(1-\beta(1-q_1))q_1\beta(\tilde{v}_0 - \tilde{v}_1),\end{aligned}$$

or, rearranging terms,

$$\begin{aligned}[1-\beta(1-q_1-q_2)](1-\beta)\tilde{v}_1 \quad \geq \quad & (1-\beta)(q_1 u - q_2 e) + q_1^2\beta u + \\ & q_1 q_2 \beta(u-e) + (1-\beta(1-q_1))q_1\beta(\tilde{v}_0 - \tilde{v}_1).\end{aligned} \tag{10}$$

Similarly, expressions for s_0, s_1 and $s_1 - s_0$ promptly imply

$$[1-\beta(1-q_1-q_2)](1-\beta)s_1 = (1-\beta)q_1 u + q_1 q_2 \beta(u-e). \tag{11}$$

Because $\tilde{v}_1 - \tilde{v}_0 \to e$ as $\beta \to 1$, we can use (10) and (11) to notice that

$$\lim_{\beta\to 1}\{[1-\beta(1-q_1-q_2)](1-\beta)(\tilde{v}_1 - s_1)\} \geq \lim_{\beta\to 1}\{q_1^2\beta(u-e) - (1-\beta)(q_1+q_2)e\}.$$

Since $u > e$ and q_1 is bounded away from zero, we conclude that $\tilde{v}_1 - s_1 \to +\infty$ as desired. ■

Proof of Proposition 2: (a) $\phi_{d,d} = 0$ if and only if $u + p\beta w^0_{d-1,d} + (1-p)\beta w^1_{d,d} < \beta w^0_{d,d}$, where p is the probability that the note is redeemed instantly. This, in turn, is true if $u < p\beta(w^0_{d,d} - w^0_{d-1,d}) + (1-p)\beta(w^0_{d,d} - w^1_{d,d})$. Since $w^0_{d,d} \geq w^1_{d,d}$, the last inequality follows if $u < p\beta(w^0_{d,d} - w^0_{d-1,d})$. Note that $w^0_{d,d} - w^0_{d-1,d} = \sum_{i=0}^{d-1} p(i,d)(v_{d-i,d-i} - v_{d-1-i,d-i}) + p(d,d)(v_{0,0} - s_1) \geq p(d,d)(v_{0,0} - s_1)$. Therefore, it suffices to show that for β close to 1, $u < p\beta p(d,d)(v_{0,0} - s_1)$. This follows since $v_{0,0} - s_1 \to \infty$ as $\beta \to 1$, and since $p \geq \mu_1 \delta > 0$ and $p(d,d) > \delta^d > 0$. ■

(b) Consider the case where $m = d$ and d is large. We have that $\phi_{d,d} = 1$ if and only if $u + p\beta w^0_{d-1,d} + (1-p)\beta w^1_{d,d} > \beta w^0_{d,d}$. This is true if and only if $u > p\beta(w^0_{d,d} - \beta w^0_{d-1,d}) + (1-p)\beta(w^0_{d,d} - \beta w^1_{d,d})$. Note that $w^0_{d,d}$ is bounded, since it belongs to the interval $(0, \frac{u}{1-\beta})$, and an increasing function of d. Therefore, $\lim_{d\to\infty} w^0_{d,d} = \lim_{d\to\infty} w^0_{d-1,d} = K$, for some finite constant K. Also, $w^0_{d-1,d} < w^1_{d,d} < w^1_{d,d+1}$, which implies that $\lim_{d\to\infty} w^0_{d,d} = \lim_{d\to\infty} w^1_{d,d} = K$. Therefore, the above inequality holds for d large.

(c) Fix (d,m) with $d > m$. For a bank that can issue a note we have that $\phi_{d,m} = 1$ if and only if $u + p\beta w^0_{d-1,m} + (1-p)\beta w^1_{d,m} \geq \beta w^0_{d,m}$. We know that $w^0_{d-1,m} \leq w^1_{d,m}$. Thus, it is sufficient to show that $u + \beta w^0_{d-1,m} \geq \beta w^0_{d,m}$. This, in turn, is true if $u + \beta \sum_{i=0}^{m} p(m,i) v_{d-1-i,m-i} \geq \beta \sum_{i=0}^{m} p(m,i) v_{d-i,m-i}$ which holds if $v_{d-1-i,m-i} + \frac{u}{\beta} \geq v_{d-i,m-i}$, for all $0 \leq i \leq m$. The proof then reduces to showing that for all (d,m) such that $d > m$, $v_{d,m} - v_{d-1,m} \leq \frac{u}{\beta}$. Let C be the set of functions $f : [0,\bar{M}]^2 \to [0, \frac{u}{\frac{1}{\beta}-1}]$ that satisfy this property, where $\bar{M}$ is an upper bound on the state space (see Lemma 5 below). For all (d,m) in $[0,\bar{M}]^2$, define an operator $T : C \to C$ by

$$T\bar{v}_{d,m} = \frac{1}{k}(c_1 + \phi x) \max_{\gamma_{d,m}} [\gamma_{d,m}(-e + \beta \bar{w}^0_{d+1,m}) + (1-\gamma_{d,m})\beta \bar{w}^0_{d,m}]$$
$$+ \frac{1}{k}(\alpha c_0 + \gamma x) \max_{\phi_{d,m}} \left\{ \phi_{d,m}[u + p\beta \bar{w}^0_{d-1,m} + \ (1-p)\beta \bar{w}^1_{d,m}] + (1-\phi_{d,m})\beta \bar{w}^0_{d,m} \right\}$$
$$+ [1 - \frac{1}{k}(c_1 + \phi x) - \frac{1}{k}(\alpha c_0 + \gamma x)]\beta \bar{w}^0_{d,m},$$

where

$$\bar{w}^j_{d,m} = \sum_{0\le i\le \min\{d,m\}} \binom{m}{i}\pi^i(1-\pi)^{m-i}\bar{v}_{d-i,m+j-i} + \sum_{d<i\le m} I_{d,m}\binom{m}{i}\pi^i(1-\pi)^{m-i}s_1.$$

We know that T has a unique fixed point, v, in the space of bounded continuous functions. If T also maps C into itself, and since this space is complete, the unique fixed point will also satisfy the desirable property. Therefore, we need to show that if $\bar{v}_{d,m}$ satisfies $\bar{v}_{d,m} - \bar{v}_{d-1,m} \le \frac{u}{\beta}$ for all (d,m) with $d<m$, then so does $T\bar{v}_{d,m}$. We have that $T\bar{v}_{d,m} - T\bar{v}_{d-1,m} = p_1q_1 + p_2q_2 + p_3q_3$, where $p_1 = \frac{1}{k}(c_1+\phi x)$, $p_2 = \frac{1}{k}(\alpha c_0 + \gamma x)$, $p_3 = 1-p_1-p_2$, and

$$\begin{gathered} q_1 = \max_{\gamma_{d,m}}[\gamma_{d,m}(-e+\beta\bar{w}^0_{d+1,m}) + (1-\gamma_{d,m})\beta\bar{w}^0_{d,m}] \\ -\max_{\gamma_{d,m}}[\gamma_{d,m}(-e+\beta\bar{w}^0_{d,m}) + (1-\gamma_{d,m})\beta\bar{w}^0_{d-1,m}], \\ q_2 = \max_{\phi_{d,m}}\left\{\phi_{d,m}[u+p\beta\bar{w}^0_{d-1,m} + (1-p)\beta\bar{w}^1_{d,m}] + (1-\phi_{d,m})\beta\bar{w}^0_{d,m}\right\} \\ -\max_{\phi_{d,m}}\left\{\phi_{d,m}[u+p\beta\bar{w}^0_{d-2,m} + (1-p)\beta\bar{w}^1_{d-1,m}] + (1-\phi_{d,m})\beta\bar{w}^0_{d-1,m}\right\}, \\ q_3 = \beta(\bar{w}^0_{d,m} - \bar{w}^0_{d-1,m}). \end{gathered}$$

It is then sufficient to show that $\max\{q_1,q_2,q_3\} \le \frac{u}{\beta}$.
Step 1: $q_3 \le \frac{u}{\beta}$.
We have that $q_3 = \beta(\bar{w}^0_{d,m} - \bar{w}^0_{d-1,m})$. By the definition of w, and since $d<m$, this expression equals $\beta\sum_{i=0}^m p(m,i)[\bar{v}_{d-i,m-i} - \bar{v}_{d-1-i,m-i}]$. Since $\bar{v}$ satisfies the desirable property, this expression is less than or equal to $\beta\frac{u}{\beta} = u < \frac{u}{\beta}$.
Step 2: $q_2 \le \frac{u}{\beta}$.
By the same argument as in step 1, we have that $\bar{w}^0_{d-1,m} - \bar{w}^0_{d-2,m}$, $\bar{w}^1_{d,m} - \bar{w}^1_{d-1,m}$, and $\bar{w}^0_{d,m} - \bar{w}^0_{d-1,m}$ are all less than $\frac{u}{\beta}$. It is then straightforward to show that $q_2 \le \frac{u}{\beta}$, for all possible combinations of $\phi_{d,m}$ and $\phi_{d-1,m}$. For example, suppose that $\phi_{d,m} = 1$ and $\phi_{d-1,m} = 0$. Then

$$\begin{gathered} q_3 = u + p\beta\bar{w}^0_{d-1,m} + (1-p)\beta\bar{w}^1_{d,m} - \beta\bar{w}^0_{d-1,m} \\ \le p\beta(\bar{w}^0_{d-1,m} - \bar{w}^0_{d-2,m}) + (1-p)\beta(\bar{w}^1_{d,m} - \bar{w}^1_{d-1,m}) \\ \le p\beta\frac{u}{\beta} + (1-p)\beta\frac{u}{\beta} = u \le \frac{u}{\beta}. \end{gathered}$$

This implies that $T\bar{v}_{d,m}$ satisfies the desirable property and the proof is complete. ■

Proof of Proposition 3: (a) A bank with an opportunity to consume faces

$$\max_{\phi_{d,m}} \left\{ \phi_{d,m}[u + p\beta w^0_{d-1,m} + (1-p)\beta w^1_{d,m}] + (1-\phi_{d,m})\beta w^0_{d,m} \right\}.$$

We have that $\phi_{d,m} = 1$ if and only if $u + p\beta w^0_{d-1,m} + (1-p)\beta w^1_{d,m} > \beta w^0_{d,m}$. This is true if and only if $u > p\beta(w^0_{d,m} - w^0_{d-1,m}) + (1-p)\beta(w^0_{d,m} - w^1_{d,m})$. Since $w^0_{d,m} \geq w^1_{d,m}$, it is sufficient to show that $u > p\beta(w^0_{d,m} - w^0_{d-1,m})$. Note that $\lim_{d\to\infty} w^0_{d,m} = \lim_{d\to\infty} \sum_{i=0}^{m} p(m,i)v_{d-i,m-i} = \sum_{i=0}^{m} \lim_{d\to\infty}\{p(m,i)v_{d-i,m-i}\}$. Also, v is increasing in d and bounded. Thus, the above limit exists and equals a constant, i.e., $0 < \lim_{d\to\infty} v_{d-i,m-i} = K_{m-i} < \frac{u}{1-\beta}$. Therefore, $\lim_{d\to\infty} w^0_{d,m} = \lim_{d\to\infty} w^0_{d-1,m} = \sum_{i=0}^{m} p(m,i)K_{m-i} = \bar{K}_m$, a constant, thus, $\phi_{d,m} = 1$, for d large.

(b) Consider a bank facing an opportunity to increase reserves. We have that $\gamma_{d,m} = 0$ if and only if $-e + \beta w^0_{d+1,m} \leq \beta w^0_{d,m}$. We know that $\lim_{d\to\infty} w^0_{d,m} = \lim_{d\to\infty} w^0_{d-1,m} = \bar{K}_m$. The result then follows for d large.

(c) A bank that is given the opportunity to issue a note faces the following problem:

$$\max_{\phi_{d,m}} \left\{ \phi_{d,m}[u + p\beta w^0_{d-1,m} + (1-p)\beta w^1_{d,m}] + (1-\phi_{d,m})\beta w^0_{d,m} \right\}.$$

We have that for any fixed d, $\lim_{m\to\infty} w^j_{d,m} = \lim_{m\to\infty} \left\{ \sum_{i=0}^{d} p(m,i)v_{d-i,m+j-i} + \sum_{i=d+1}^{m} p(m,i)s_1 \right\}$. The first sum in this expression is finite, so we have $\lim_{m\to\infty} p(m,i) = \lim_{m\to\infty} \left\{ \frac{m!}{(m-i)!i!}\pi^i(1-\pi)^{m-i} \right\} = \lim_{m\to\infty} \pi^i(1-\pi)^{m-i} = 0$. Thus, $\lim_{m\to\infty} w^j_{d,m} = \lim_{m\to\infty} \left\{ 0 + s_1 \sum_{i=d+1}^{m} p(m,i) \right\} = s_1$, and, for any fixed d, there exists an M_d large enough such that $u + ps_1 + (1-p)\beta s_1 > \beta s_1$ and, therefore, $\phi_{d,m} = 1$.

(d) We know that $\gamma_{d,m} = 0$ if and only if $-e + \beta w^0_{d+1,m} \leq \beta w^0_{d,m}$. Fix $d \geq 0$. We have $\lim_{m\to\infty} w^j_{d+1,m} = \lim_{m\to\infty} w^j_{d,m} = s_1$. So given d, there exists a large enough M_d such that $-e + \beta w^0_{d+1,m} = -e + \beta s_1 < \beta s_1 = \beta w^0_{d,m}$, for $m \geq M_d$. Therefore, $\gamma_{d,m} = 0$ for all $m \geq M_d$. ■

Proof of Proposition 4: (a) We will prove the claim for the case where $m = d+1$. The proof can be generalized for any state (d,m), such that $d < m$. Fix d.

Then $\phi_{d,d+1} = 0$ if and only if $u + p\beta w^0_{d-1,d+1} + (1-p)\beta w^1_{d,d=1} < p\beta w^0_{d,d+1} + \beta p w^0_{d,d+1}$. Since $w^0_{d,d+1} \geq w^1_{d,d+1}$, it is sufficient to show that $u + p\beta w^0_{d-1,d+1} < \beta p w^0_{d,d+1}$. We have $w^0_{d-1,d+1} = \sum_{i=0}^{d-1} p(i,d+1) v_{d-1-i,d+1-i} + [p(d,d+1) + p(d+1,d+1)]s_1$ and $w^0_{d,d+1} = \sum_{i=0}^{d} p(i,d+1) v_{d-i,d+1-i} + p(d+1,d+1)s_1$. Therefore, $w^0_{d,d+1} - w^0_{d-1,d+1} = \sum_{i=0}^{d-1} p(i,d+1)(v_{d-i,d+1-i} - v_{d-1-i,d+1-i}) + p(d,d+1)(v_{0,1} - s_1)$. The result then follows, since $w^0_{d,d+1} - w^0_{d-1,d+1} \to \infty$ as $\beta \to 1$. To see why this is true, notice that $v_{0,1} = \frac{1}{k}(c_1 + \phi x)\beta w_{1,1} + A$ and $s_1 = \frac{1}{k}(c_1 + \phi x)\beta s_1 + B$, where for the constant terms we have that $A \geq B$. In addition, $v_{d-i,d+1-i} - v_{d-1-i,d+1-i} \geq 0$, $p(d,d+1) > 0$, and $w_{1,1} \geq v_{0,0}$. The implication then follows, since $v_{0,0} - s_1 \to \infty$ as $\beta \to 1$.

(b) We have that $\gamma_{d,m} = 1$ if and only if $-e + \beta w^0_{d+1,d+1} \geq \beta w^0_{d,d+1}$. In addition, $w^0_{d+1,d+1} - w^0_{d,d+1} = \sum_{i=0}^{d} p(i,d+1)(v_{d+1-i,d+1-i} - v_{d-i,d+1-i}) + p(d+1,d+1)(v_{0,0} - s_1)$. Again, $v_{d+1-i,d+1-i} - v_{d-i,d+1-i} \geq 0$, and, since $\pi > 0$, $p(d+1,d+1) > 0$. Therefore, $v_{0,0} - s_1 \to \infty$ as $\beta \to 1$. Thus, the result follows. ■

Proof of Lemma 5: Since at a monetary equilibrium we have that $\pi \geq \delta > 0$, there exists a large enough value of m, say M, such that the number of notes redeemed is greater than 1, with probability arbitrarily close to 1. Given that banks can issue, at most, one unit of money per period, we conclude that $m \leq M$ for all banks, i.e., for any $\varepsilon > 0$, M can be chosen such that $\sum_{m>M} x_{d,m} < \varepsilon$. ■

Proof of Proposition 5: For mathematical convenience, we impose an exogenous upper bound, M, on m. Lemma 5 shows that if this bound is large enough, it will not bind with probability 1. This also implies that d is bounded above by M. We then have that $d \in \{0,1,...,M\} \equiv Z_D$ and $m \in \{0,1,...,M\} \equiv Z_M$. Let $Z = Z_D \times Z_M$. The agents' optimal policies together with the matching technology define a Markov chain on the finite state space: $Z \cup \{0,1\}$, where $\{0,1\}$ represents the two possible states for the non-banks. Let Π denote the Markov matrix associated with the Markov chain. Let l denote the cardinality of $Z \cup \{0,1\}$. Define a mapping $T : S^l \to S^l$ by $Ts = s\Pi$, for all $s \in S^l$, the l-dimensional unit simplex. Consider the following labeling of the state space: Let 0 denote the state of a non-bank with no money, 1 be the state of a non-bank with one unit of money, 2 be the state of a bank with $(d,m) = (0,0)$, 3 be the state of a bank with $(d,m) = (1,0)$,... . For any such state, j, we have the following lower bounds on the transition probabilities: $\pi_{j,0} \geq \delta(1-\mu_1)$, $\pi_{j,1} \geq q$, $\pi_{j,2} \geq \delta\mu_1$, $\pi_{j,3} \geq \delta\mu_1 \frac{1}{k}(c_1 + \phi x)$, ... For

$j=0,...,l$, let $\varepsilon_j = \min_i \pi_{i,j}$. Then $\sum_{j=0}^{l} \varepsilon_j \geq \delta(1-\mu_1)+q+\delta\mu_1+\delta\mu_1\frac{1}{k}(c_1+\phi x)+ ... \geq \delta+q > 0$. Therefore, T is a contraction of modulus at least $1-\delta-q$. By the contraction mapping theorem, T has a unique fixed point. In addition, from any initial distribution across states, the process converges to the invariant distribution at a geometric rate and there are no cyclically moving subsets. To show that $\gamma x > 0$, and since $x_{0,0} > 0$, it is sufficient to show that $\gamma_{0,0} = 1$. This is true if $-e+\beta v_{1,0} > \beta v_{0,0.}$ Using $\phi_{1,0} = 1$ (see proposition 2c), we have $v_{1,0} = A+\rho\{u+p\beta v_{0,0}+(1-p)\beta w_{1,0}^1\}$ and $v_{0,0} = B+\rho[\max\{u+p\beta s_1+(1-p)\beta w_{0,0}^1, \beta w_{0,0}^0\}]$, where A and B represent the payoffs when there is no opportunity to consume and ρ is the probability of facing the opportunity to issue a note in exchange for consumption. Note that $u+p\beta v_{0,0}+(1-p)\beta w_{1,0}^1 > u+p\beta s_1+(1-p)\beta w_{0,0}^1$ and, since $\phi_{1,0} = 1$, we also have that $u+p\beta v_{0,0}+(1-p)\beta w_{1,0}^1 > u+p\beta s_1+(1-p)\beta w_{0,0}^1 > \beta w_{1,0}^0 \geq \beta w_{0,0}^0$. We conclude that $v_{1,0} > v_{0,0}$ and, therefore, that $\gamma_{0,0} = 1$, for e small enough. This, in turn, implies that $\gamma x > 0$. To show that $\phi x > 0$, notice that $\gamma_{0,0} > 0$ implies that $x_{1,0} > 0$. But since $\phi_{1,0} = 1$, this implies that $\phi x > 0$. ■

Proof of Corollary: (a) By Lemma 5, $\sum_{m>M} x_{d,m} = 0$. By proposition 3(b), there exists a D such that $\gamma_{d,m} = 0$, for all $d \geq D$. Then, in a steady state equilibrium, we have: $\sum_{\{(d,m):m>\bar{M} \text{ or } d>\bar{M}\}} = 0$, where $\bar{M} = \max\{D, M\}$. By proposition 2(a), as $\beta \to 1$, we have that $\phi_{d,d} = 0$ for all $d \in [0, \bar{M}]$. This, in turn, implies that $NB = 0$.

(b) Follows from Proposition 2(b). ■

Figure 1: