Money Velocity with Interest Rate Stochastic Volatility and Exact Aggregation[1]

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Abstract

The determinants of money velocity are theoretically explored under various assumptions of interest rate uncertainty in a monetary general equilibrium model. Money is introduced by putting monetary services in the utility function. Monetary assets pay interest. When interest rates are uncertain, it is found that the degree of risk aversion in consumers' preferences and the risk in the return rates of the benchmark asset affect both the intercept and slope of the money velocity function, while the risk in return rates of monetary assets only affects the intercept of the money velocity function. The traditional money velocity function would become unstable if covariances change over time between interest rates and consumption growth rate or between interest rates and real money growth rate. We simulate the model developed in this paper and find that the coefficients of the money velocity function are volatile. The Swamy and Tinsley (1980) random coefficient model is then estimated with money velocity data to compare the results with those from model simulation. It is found that the estimated stochastic slope coefficient of the velocity function behaves in a manner that is approximately consistent with the simulation results.

1. **Introduction**

Explanations for the behaviour of money velocity go back to as early as the seventeenth century (see Humphrey (1993) for a review). But instability of the money velocity function since the late 1970's in the U.S. has called for a reexamination of the traditional views. See, e.g., Stone and Thornton (1987). One line of research focuses on the correct measurement of money and challenges the traditional practice of ignoring the aggregation problem in monetary economics research and policy design.[2] For example, Barnett *et al* (1984) found that the coefficients in a conventional demand-for-money equation using Divisia monetary aggregation are more stable than those in the same equation with simple sum monetary aggregation on quarterly data from 1959:1 to 1982:4. Another line of research focuses on the effects of institutional change on money velocity. See, e.g., Bordo and Jonung (1981, 1987, and 1989). Another hypothesis, proposed by Friedman (1983), attributes the several substantial declines of M1 velocity since 1981 to the increased money growth variability following the change of Federal Reserve operating procedures in October 1979. However, empirical tests of this variability hypothesis have not provided uniform evidence.[3] More recently, some research has investigated whether the observed variability of money velocity can be explained in monetary general equilibrium models since Lucas (1978), Svensson (1985), and Lucas and Nancy (1987). With the cash-in-advance specification, simulation results for money velocity in general equilibrium models have been generally disappointed. See, e.g., Hordrick *et al* (1991) and Giovannini and Labadie (1991).[4]

In this paper we explore the determinants of money velocity and the causes of the instability of a traditional money velocity function in a monetary general equilibrium model. One difference between our model and the previous ones in the monetary economics literature is that monetary assets in our model pay interest. Another feature of our model is that the principles of monetary aggregation theory are imposed. When nominal interest rates are certain, it is found that Barnett's (1978) user cost of monetary assets is the relevant determinant of money velocity. Under risk aversion, Barnett and Liu's (1994) generalized user cost of monetary assets is the determinant of money velocity. We find that the

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[2] For the measurement of money, see Barnett (1980, 1987), Barnett *et al* (1991), Belongia (1996), and Serletis (1995).

[3] See Belongia (1985), Fisher and Serletis (1989), Hall and Noble (1987), Mehra (1989), and Thornton (1995) for empirical evidence of the variability hypothesis.

[4] Hordrick *et al* (1991) and Giovannini and Labadie (1991) find an almost-constant money velocity in their simulation results. A constant velocity level is apparently counterfactual. It is, however, frequently assumed in a traditional quantity theory of money. Recently, a constant growth rate of velocity has been assumed in testing the quantity theory by Bullard (1994).

uncertainty of nominal interest rates and the degree of risk aversion play an important role in determining the stability of the money velocity function. If the covariances change between interest rates and the consumption growth rate, or between interest rates and the real money growth rate, the model predicts that money velocity will shift. Hence the coefficients of a money velocity function may change, if time-varying risk is present, as for example generated by an ARCH type process.[5] In fact, anything that causes covariances to change between interest rates and the consumption growth rate or between interest rates and the real money growth rate will contribute to a shift of the money velocity function. These causes could include financial innovations or money growth variability,[6] as previously investigated in the literature. In this sense, our study provides a general and coherent theoretical explanation for the instability of money velocity and nests many earlier explanations as special cases. We simulate the process of the slope coefficient of a simple traditional money velocity function based on U.S. quarterly data from 1960.1 to 1992.4, and find that the theoretical model generates a volatile slope coefficient when the degree of risk aversion in the model is moderately high, especially during the 1973-1976 and 1979-1982 periods. The Swamy and Tinsley (1980) random coefficient model for money velocity is estimated to compare the behaviour of the estimated stochastic coefficients with the simulated coefficients from the theoretical model. The estimated stochastic slope coefficient behaves in a manner that is approximately consistent with the simulation results.

The remainder of this paper is organized as follows. Section 2 develops a monetary general equilibrium model in which monetary assets provide monetary services as well as interest income. Section 3 derives the theoretical results for money velocity under the assumption that nominal interest rates are known. It is shown that Latane's equation (Latane (1954)) is a special case of the model developed in this paper. Section 4 generalises the result when the assumption of certain nominal interest rates is relaxed. The effect of risk aversion and interest rate uncertainty is investigated. Section 5 presents the results from model simulation and from estimation of a random coefficient model. The last section provides concluding remarks.

2. Assumptions and Theoretical Specifications

In this section we outline an infinite-horizon, representative-agent model with a set of monetary assets which pay interest. Suppose that there exist *k* monetary assets. Monetary asset *i* pays nominal

[5] Time-varying coefficient models of money velocity are increasingly used in the literature. See, e.g., Dueker (1993 and 1995).

[6] For the effect of financial innovation on the economy, see recent work by Thornton (1994).

return rate R_{it} at the end of time period *t*. Money supply is assumed to be exogenous and serves as a moving endowment point in the consumer's budget constraint. There exists one nominal bond with holding period yield R_t, which is also paid at the end of period *t*. We assume that

$$R_t \geq \max \{R_{it}, i = 1,..., k\} \quad \text{for all } t.$$

This assumption says that monetary assets are dominated in holding period returns by the nominal bond, which is assumed to yield no monetary services. The price for the bond in period *t* is P_{bt}. There exists one equity asset, which is the exogenous endowment asset and yields resource flow $d_t > 0$. The price of one unit of the equity is P_{st}, and dividend d_t per unit is paid before the share is sold. The only consumption good is the resource flow d_t which is perishable. The price of that consumption good is P_t in period *t*. There are finitely many identical consumers with utility function $U(c_t, \mathbf{m}_t)$, which is continuous, increasing, and concave in all of its arguments, where c_t is the demand for consumption goods, and $\mathbf{m}_t = (m_{1t}, m_{2t}, ..., m_{kt})$ is a vector of real monetary assets held during period *t*.

The exogenous supply of monetary asset *i* in period *t* is X_{it}, and let $\sum_{i=1}^{k} X_{it} = X_t$ 1 be the simple sum aggregate of money supply. The representative consumer is assumed to maximize the infinite lifetime expected utility:

$$E_t \sum_{t=0}^{\infty} \beta^t U(c_t, \mathrm{m}_t) \tag{1}$$

where $\beta \in (0,1)$ is the subjective rate of time discount and E_t is the expectation operator, conditional on information at time period *t*. The budget constraint in each period is:

$$P_t c_t + P_{st}s_t + P_{bt} b_t + P_t \sum m_{it} \leq s_{t-1}(d_tP_t+P_{st})+ P_{b,t-1}b_{t-1}(1 + R_{t-1})$$
$$+ \sum m_{i,t-1}(1+R_{i,t-1})P_{t-1} + \sum [X_{it} - (1+R_{it-1})X_{i,t-1}], \tag{2}$$

where s_t is the quantity of equity, b_t is the quantity of bonds held during time period *t*, and $\sum$ is the summation from *i* = 1 to *k*. The last term of the budget constraint is different from that in a traditional model.[7] Since we assume that monetary assets pay interest, the supply of money must be adjusted for it.

[7]Regarding the traditional budget constraint, which here is only slightly modified, see Giovannini and Labadie (1991, eqs. (3) and (12)) and Hodrick, K. Lakota, and Lucas (1991, eq. (4)).

Money is introduced through the money-in-utility-function approach in this model. In this approach, the utility function must be viewed as the derived utility function that exists if money has positive value in equilibrium.[8] Alternatively, in a cash-in-advance model, it is difficult to justify the existence of a variety of monetary assets which pay different interest rates. In all the cash-in-advance models in the literature, there can exist only one monetary asset in the equilibrium of an economy.

Since monetary assets are dominated or stochastically dominated in returns by the nominal bond, a rational economic agent will not hold monetary assets if monetary assets do not yield monetary services. However, we cannot nest the simple sum monetary aggregator function in the utility function unless all the monetary assets are perfect substitutes for each other (see Barnett, 1987). To reduce the dimension of the vector of monetary assets to an aggregate index of ``money," we need to assume that the utility function $U(c_t, \mathbf{m}_t)$ is weakly separable in monetary assets and can be written as $F(c_t, f(\mathbf{m}_t))$, where $f(\mathbf{m}_t)$ is a linearly homogenous subutility function.[9] Under this assumption, it can be proved that the Divisia monetary aggregate can track the theoretical function $f(\mathbf{m}_t)$ exactly in continuous time or up to a third order remainder term in discrete time, if nominal yields R_t and R_{it} are known at the beginning of the time period.[10] If the agent is risk averse and nominal yields R_t and R_{it} are not known exactly to the agent at the beginning of time interval *t*, the Divisia aggregate's tracking ability is somewhat compromised. For the purpose of this paper, we first assume that R_t and R_{it} are known to consumers at the beginning of time interval *t*, and in Section 4 we will relax this assumption.

Let m_t be the value of the exact monetary aggregate over its components m_{it}, $i = 1, 2, ..., k$, so that $m_t = f(\mathbf{m}_t)$. The utility function $U(c_t, \mathbf{m}_t)$ consequently can be written as $F(c_t, m_t)$. In order to deal with the first order conditions in terms of the quantity aggregate rather than each individual monetary assets, we have to transform the budget constraint to replace the vector of monetary assets $\mathbf{m}_t$ with the exact monetary aggregate m_t. To do this, let π_t be the exact money price aggregate (or user cost aggregate) dual to m_t, and let π_{it} be the user cost of monetary asset $i = 1, 2, ..., k$ in period *t*. It can be shown that the exact aggregation-theoretic price aggregator function is the unit cost function. See Barnett (1987). According to Fisher's factor reversal test, expenditure on the aggregate must equal expenditure on the components, so that π_t must satisfy

$$\sum_{i=1}^{k} \pi_{it} m_{it} = \pi_t m_t \tag{3}$$

[8]See Blanchard and Fischer (1989, p192), Arrow and Hahn (1971), and Feenstra (1986).

[9]For a discussion of macroeconomic dimension reduction, see Barnett (1994).

[10]For the microeconomic theory of monetary aggregation, see Barnett (1987).

where

$$\pi_{it} = \frac{R_t - R_{it}}{1 + R_t} \tag{4}$$

as derived in Barnett (1978). We define R_{mt} such that

$$\sum_{i=1}^{k} (1 + R_{it}) m_{it} = (1 + R_{mt}) m_t \tag{5}$$

Note that R_{mt} can be interpreted as the aggregate rate of return dual to the exact monetary aggregate. Dividing equation (5) by $(1+R_t)$ and adding the resulting equation to equation (3), we have

$$\sum_{i=1}^{k} m_{it} = (\pi_t + \frac{1+R_{mt}}{1+R_t}) m_t \tag{6}$$

Let M_t^n be the nominal supply-side exact monetary aggregate. We assume that the monetary asset markets are in equilibrium when $M_t^n = m_t P_t$, where $m_t P_t$ is the nominal demand side exact monetary aggregate.[11] Under these assumptions the budget constraint can be written as

$$\begin{aligned} P_t c_t + P_{st} s_t + b_t P_{bt} (\pi_t + \frac{1+R_{mt}}{1+R_t}) \leq s_{t-1}(d_t P_t + P_{st}) + P_{t-1}(1+R_{m,t-1}) m_{t-1} + \\ b_{t-1}(1+R_{t-1}) P_{b,t-1} + M_t^n (\pi_t + \frac{1+R_{mt}}{1+R_t}) - M_{t-1}^n (1+R_{m,t-1}) \end{aligned} \tag{3}$$

The representative agent chooses controls $u_t = (c_t, m_t, s_t, b_t)$ for $t \geq 1$ to maximize expected lifetime utility

$$E_t \sum_{t=0}^{\infty} \beta^t F(c_t, m_t)$$

subject to constraint (7) with given m_0 and d_0. We have the exact monetary aggregate in both the utility function and budget constraint, and the macroeconomic `dimension reduction' is completely consistent with the microeconomic theory of monetary aggregation.

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[11] Actually there is a possible regulatory wedge between the supply and demand side aggregator functions, when required reserves pay no interest and thereby produce an implicit tax on the supply side. For more discussion on this issue, see Barnett (1987), who provides the formulas for both the demand and supply side exact monetary aggregator functions.

Let $z_t = (m_{t-1}, s_{t-1}, b_{t-1}, c_{t-1}, P_t, P_{st}, P_{bt}, R_t, M_t^n)$ be the set of state variables, and let

$$T(z_t) = \sum_{t=0}^{\infty} \beta^t F(c_t, m_t).$$

In equilibrium, $T(z_t)$ must satisfy the following Bellman's equation

$$T(z_t) = \max_{u_t} \{F(c_t, m_t) + \beta E_t T(z_{t+1})\}, \tag{8}$$

when the following market clearing conditions are satisfied:

$$c_t = d_t,$$
$$s_t = 1,$$
$$b_t = 0,$$

and

$$m_t = M_t^n / P_t.$$

The equilibrium condition on equities is a normalisation, while the equilibrium condition on bonds states that bonds are privately issued by some consumers and bought by others, and the net demand for bonds is zero in equilibrium. Recall that the representative agent is aggregated over consumers under Gorman's conditions for the existence of a representative consumer. Hence b_t is net per capita borrowing among consumers, where lending is negative borrowing. If interest rates are out of equilibrium, net borrowing need not be zero.[12]

3. Money Velocity With No Nominal Risk

In this section we derive the necessary first order conditions and the equations for money velocity, under the assumption that the nominal interest rates are known. The first order conditions of the maximization problem are

$$F_{ct} = \lambda_t P_t, \tag{9}$$

$$F_{mt} = \lambda_t P_t \left(\pi_t + \frac{1 + R_{mt}}{1 + R_t}\right) - \beta E_t[\lambda_{t+1}](1 + R_{mt}) P_t, \tag{10}$$

$$\lambda_t P_{st} = \beta E_t [\lambda_{t+1}(d_{t+1} P_{t+1} + P_{s,t+1})], \tag{11}$$

$$\lambda_t = \beta E_t[\lambda_{t+1}](1 + R_t), \tag{12}$$

where F_{ct} and F_{mt} are the marginal utilities of consumption goods and monetary services respectively, and λ_t is the Lagrange multiplier of the budget constraint (7). Equation (10) is the first order condition

[12] Similarly, see Marshall (1992, p.1321) and Boyle (1990, p.1042).

for monetary services. Equations (11) and (12) are standard Euler equations for stocks and bonds.

From equations (9), (10), and (12), we have:

$$\pi_t = \frac{F_{mt}}{F_{ct}}. \tag{13}$$

That is, the marginal rate of substitution between consumption goods and monetary services equals the aggregate user cost of the monetary services.[13] Assume that the utility function takes the constant relative risk aversion form

$$F(c_t, m_t) = \frac{1}{1-\phi}(c_t^s m_t^{1-s})^{1-\phi} \qquad if\ \phi \neq 1$$
$$= \ln(c_t^s m_t^{1-s}), \qquad if\ \phi = 1$$

where m_t is the real exact monetary aggregate, $s \in (0,1)$ is a constant, and $\phi \in (0, \infty)$ is the coefficient of relative risk aversion. We get the following relationship:

$$\pi_t = \frac{(1-s)}{s}\frac{c_t}{m_t}. \tag{14}$$

When solved for m_t, equation (14) is the equation of demand for the exact monetary aggregate. Given $c_t = d_t$ and the parameter s, the only determinant of the demand for monetary services in equilibrium is the user cost π_t. Although other factors, such as the inflation rate, are not in this equation directly, they may affect the demand for money through the user cost π_t, which is a function of the nominal interest rates R_t and R_{it}. Given these equilibrium conditions, we can examine the behaviour of money velocity.

Traditional money velocity is usually defined as the ratio of nominal income to the simple sum monetary aggregate

$$V_t = \frac{P_t d_t}{X_t}.$$

We define the aggregation theoretic exact money velocity by replacing the simple sum monetary aggregate with the exact monetary aggregate to get

$$V_t = \frac{d_t}{m_t}.$$

Using the definition $\pi_{it} = (R_t - R_{it})/(1+R_t)$, the identity $\pi_t m_t = \sum_{i=1}^{k} \pi_{it} m_{it}$, and the equilibrium condition $X_{it} = P_t m_{it}$, we have the following results from equation (14):

[13] We can get the same result if we start with the maximization problem in terms of the original disaggregated individual monetary assets. See appendix II.

$$v_t = \frac{s}{1 - s} \pi_t \qquad (15)$$

where

$$\Pi_t = \sum_{i=1}^{k} \frac{\pi_{it} X_{it}}{\sum_{i=1}^{k} X_{it}}$$

$$= (R_t - R_{smt})(1+R_t)^{-1}$$

with

$$R_{smt} = \sum_{i=1}^{k} \theta_{it} R_{it} ,$$

and

$$\theta_{it} = \frac{X_{it}}{\sum_{i=1}^{k} X_{it}} .$$

Observe that Π_t is a weighted average of the user costs π_{it}, which are the opportunity costs of holding monetary assets instead of the bond. The results of equations (15) and (16) both say that money velocity is a function of the user costs $\{\pi_{it}, i = 1, 2, ..., k\}$. If we define velocity in the tradition way (V_t), then the corresponding determinant should be a weighted average of the user costs π_{it}, with the weights being ratios of the X_{it} to the simple sum aggregate $X_t = \sum_{i=1,k} X_{it}$. If alternatively we define money velocity relative to the theoretic exact monetary aggregate, then the relevant determinant is the user cost aggregate π_t dual to the exact monetary quantity aggregate. Both velocity functions have the same form with the key elements being the user costs of monetary assets. The equivalence of the forms of the two velocity functions (15) and (16) depends upon our specification of the utility function.

Note that if all monetary assets yield no interest so that $R_{it} = 0$ for all *i*, then $\pi_t = \Pi_t = R_t/(1+R_t)$, as in Boyle (1990) and LeRoy (1984). In this special case, the inverse of money velocity is a linear function of the inverse of interest rates:

$$\frac{1}{v_t} = \frac{s}{1 - s} + \frac{s}{1 - s} \frac{1}{R_t} .$$

This equation was first estimated by Latane (1954) without rigorous derivation and reestimated by

Christ (1993) for M1 velocity.[14] According to this equation, variations in velocity are caused solely by fluctuations in the benchmark interest rates R_t. But when monetary assets themselves yield positive interest rates R_{it}, velocity could fluctuate even when the benchmark rate does not vary.

In short, money velocity is a variable rather than a constant in the model developed in this section. The opportunity cost and the taste parameters determine the stochastic behaviour of money velocity. Observe that, equations (15) or (16) have no intercepts and have constant slopes. These implications conflict with many published results. In the next section when interest rates uncertainty is introduced, we show that the intercept becomes nonzero, and the slope may be time-varying, if time-varying risk is present.

4. Money Velocity With Nominal Interest Risk

In this section the assumption that nominal interest rates are known is relaxed, and we keep the assumption that the economic agents are risk averse. We focus on the question of whether the model can explain the instability of money velocity reported in the literature through interest rate uncertainty.

We start with the monetary aggregation problem. In the previous section, the exact monetary quantity aggregator function $m_t = f(\mathbf{m}_t)$ can be tracked very accurately by the Divisia monetary aggregate, m_t^d, since that tracking ability is known under perfect certainty. However, when nominal interest rates are uncertain, the Divisia monetary aggregate's tracking ability is somewhat compromised. That compromise is eliminated by using the extended Divisia monetary aggregate derived by Barnett and Liu (1994) under risk. Let m_t^G denote the extended Divisia monetary aggregate over the monetary assets. The only difference between m_t^G and m_t^d is the user cost formula used to compute the prices in the Divisia index formula.

Let π_{it}^G denote the generalized user cost of monetary asset *i*. Barnett and Liu (1994, eq. (5.6)) prove that

$$\pi_{it}^G = \pi_{it}^e + \varphi_{it}$$

where

$$\pi_{it}^e = \frac{E_t(R_t - R_{it})}{E_t(1 + R_t)}$$

and

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[14] Also see Dickey (1993) and Laidler (1993) for a discussion.

$$\varphi_{it} = \frac{E_t(1+R_{it})}{E_t(1+R_t)} \frac{Cov(R_t, \frac{\partial T}{\partial C_{t+1}})}{\frac{\partial T}{\partial C_T}} - \frac{Cov(R_{it}, \frac{\partial T}{\partial C_{t+1}})}{\frac{\partial T}{\partial C_t}},$$

and where

$$T = E_t \sum_{t=0}^{\infty} \beta^t F(c_t, m_t^G).$$

Barnett and Liu (1994) show that π_{it} determine the risk premia in interest rates. Note that π_{it}^G reduces to equation (4) under perfect certainty.

We define the aggregate generalized user cost π_t^G dual to m_t^G by Fisher's factor reversal test:

$$\sum_{i=1}^{k} \pi_{it}^G m_{it} = \pi_t^G m_t^G,$$

and let

$$\Pi_t^G = \sum_{i=1}^{k} \frac{\pi_{it}^G X_{it}^G}{\sum_{i=1}^{k} X_{it}}$$

be the weighted average of the individual generalized user costs of monetary assets. We have the following proposition:

Proposition: When nominal interest rates (R_t, R_{it}) are not known with certainty at the beginning of period t, an equation analogous to equation (13) still holds, and money velocity is a function of the aggregate generalized user cost, which equals the marginal rate of substitution between consumption goods and monetary services, so that

$$\frac{F_{mt}}{F_{ct}} = \pi_t^G \quad (17)$$

$$v_t = \frac{s}{1-s} \pi_t^G \quad (18)$$

$$V_t = \frac{s}{1-s} \Pi_t^G \quad (19)$$

Proof: see Appendix I.

Equation (17) can also be proved from the maximization problem without prior aggregation over monetary assets. See Appendix II. Equation (18) and (19) are analogous to equations (15) and (16) respectively, although in equation (18) and (19), the generalized user cost becomes the determinant of money velocity.

We can simplify the generalized user cost π_{it}^G to get a more intuitive equation for money velocity V_t. Dropping the remainder term of a second order Taylor series approximation to T, we have the approximation:

$$\frac{\partial T}{\partial C_{t+1}} = \frac{\partial T}{\partial c_{t+1}}|_t + (c_{t+1} - c_t)(\frac{\partial^2 T}{\partial c_{t+1}^2}|_t) + (m_{t+1}^G - m_t^G)(\frac{\partial^2 T}{\partial c_{t+1}\partial m_{t+1}^e}|_t)$$

where $|_t$ denotes function values evaluated at the point (c_t, m_t^G). Taking the covariance of the left hand side with R_t, we get from the right hand side that

$$Cov(R_t, \frac{\partial T}{\partial C_{t+1}}) = (\frac{\partial^2 T}{\partial c_{t+1}^2}|_t)Cov(R_t, c_{t+1}) + (\frac{\partial^2 T}{\partial c_{t+1}\partial m_{t+1}^e}|_t)Cov(R_t, m_{t+1}^G).$$

Let k_1 be defined such that

$$k_1 = (-c_t \frac{\partial^2 T}{\partial C_{t+1}^2}|_t)/(\frac{\partial T}{\partial c_t}) = \beta(1-s-s\phi).$$

Then k_1 is the discounted relative risk aversion parameter, which in our specification is a constant. Similarly, define k_2 such that

$$k_2 = (m_t^e \frac{\partial^2 T}{\partial c_{t+1}\partial m_{t+1}^e}|_t)/(\frac{\partial T}{\partial c_t}) = \beta(1-s)(1-\phi).$$

Now assume that

$$\sigma_{rc} = Cov(R_t, c_{t+1}/c_t)$$

and

$$\sigma_{rm} = Cov(R_t, m_{t+1}^G/m_t^G)$$

are constants, so that the effect of R_t risk on the consumption of goods and monetary services is time-invariant. Let

$$\theta_1 = \frac{Cov(R_t, \partial T/\partial c_{t+1})}{\partial T/\partial c_t}$$

and

$$\theta_{2i} = \frac{Cov(R_{it}, \partial T/\partial c_{t+1})}{\partial T/\partial c_t}$$

Then it follows that

$$\theta_1 = k_2\sigma_{rm} - k_1\sigma_{rc}$$

and

$$\theta_{2i} = k_2\sigma_{im} - k_1\sigma_{ic},$$

where

$$\sigma_{im} = cov(R_{it}, m_{t+1}^G/m_t^G)$$

and

$$\sigma_{ic} = cov(R_{it}, c_{t+1}/c_t)$$

are assumed to be time-invariant. It follows that

$$\pi_{it}^G = \pi_{it}^e + (1-\pi_{it}^e)\theta_1 - \theta_{2i} = (1-\theta_1)\pi_{it}^e + (\theta_1 - \theta_{2i}).$$

Let θ_2 be the weighted average of the θ_{2i} such that

$$\theta_2 = \frac{\sum_{i=1}^{k}\theta_{2i}X_{it}}{\sum_{i=1}^{k}X_{it}}.$$

We find that

$$V_t = a_0 + a_1\Pi_t^e, \qquad (20)$$

where

$$a_0 = \frac{s}{1-s}(\theta_1 - \theta_2),$$

and

$$a_1 = \frac{s}{1-s}(1-\theta_1),$$

$$\Pi_t^e = \frac{\sum_{i=1}^{k}\pi_{it}^e X_{it}}{\sum_{i=1}^{k}X_{it}}.$$

Note that if the covariances σ_{ic}, σ_{im}, σ_{rc}, and σ_{rm} are time-varying, then a_0 and a_1 are also time-varying, and equation (20) is a time-varying or random coefficient model. Hence, a time-varying coefficient model of money velocity can be justified by the fact that interest rate uncertainty may change over time, as for example from an ARCH process.

It is worthwhile to look at the effect of risk aversion and interest rate uncertainty on money velocity in more detail. Note that under our assumptions about the parameters of the model, we have $k_1 > 0$. Also note that $k_2 > 0$, if $\phi < 1$ $k_2 < 0$, if $\phi > 1$. For expositional purpose, we call $\phi < 1$ low risk aversion and $\phi > 1$ high risk aversion.

In the low risk aversion case, if $\sigma_{rm} < 0$, then the larger the value of $|\sigma_{rm}|$, the larger the value of a_1, and the smaller the value of a_0. The net effect of an increase of $|\sigma_{rm}|$ in this case is to reduce the money velocity, since $\Pi_t^e < 1$ and the magnitude of the decrease of the intercept is larger than that of the increase of $a_1 \Pi_t^e$. Therefore, if increased money growth variability raises the value of $|\sigma_{rm}|$, and if the $|\sigma_{im}|$ are not affected, then money velocity will decline. In the high risk aversion case, the results are just the opposite. An increase of $|\sigma_{rm}|$ will lead to higher money velocity. It follows that Friedman's (1983) hypothesis that the increased money growth variability causes money velocity to decline can be justified in our model by either (1) $\phi < 1$ and $\sigma_{rm} < 0$ or (2) $\phi > 1$ and $\sigma_{rm} > 0$. Also note that the magnitude of the effect of money growth variability on money velocity depends upon other parameters of the model, such as *s*.

But there remains the effect of uncertainty of the individual monetary assets' own rates of return. Note that the covariances σ_{im} are not important in determining the magnitude of slope a_1, though those covariances are important in determining the value of the velocity function's intercept. Therefore, a shift in the values of the $|\sigma_{im}|$ will lead to a shift in the intercept of a money velocity function. If increased money growth variability raises both $|\sigma_{rm}|$ and $|\sigma_{im}|$, and if σ_{rm} and σ_{im} have the same sign, then the effect of the money growth variability on money velocity through σ_{rm} will be partially offset. The complicated nature of the effect of money growth variability on money velocity may partially explain the controversies in the empirical literature.

In short, if $\phi \neq 1$, the money growth variability will affect money velocity, but the direction and magnitude of the effect depend upon the degree of risk aversion and the correlation between interest rates and real money growth. If all the covariances are zero, as would be the case under perfect certainty, then (20) reduces to (16), as we would expect.

To further explore the economic interpretation of the coefficients in the money velocity function, note that using the first order condition on the bond price

$$\lambda_t = \beta\, E_t\, [\lambda_{t+1}(1+R_t)],$$

the parameter θ_1 can be written as

$$\theta_1 = 1 - E_t(1+R_t)E_t[\frac{T_{c,t+1}}{T_{ct}}],$$

where $E_t[T_{c,t+1}/T_{ct}]$ is the expected growth rate of the marginal utility of consumption goods. Therefore, the slope coefficient of the money velocity function is

$$a_1 = \frac{s}{1-s} E_t(1+R_t) E_t[\frac{T_{c,t+1}}{T_{ct}}].$$

If we use $E(R_t - R_{smt})$ as the independent variable in the money velocity function, rather than the user cost Π_t^e, we have

$$V_t = a_{0t} + b_t E(R_t - R_{smt}) \quad (21)$$

where

$$b_1 = \frac{s}{1-s} E_t[\frac{T_{c,t+1}}{T_{ct}}].$$

The subscript *t* in b_t and a_{0t} is used to indicate that the values of b_t and a_{0t} may not be time constant. In our model, given the specification of the utility function, we have

$$b_t = \frac{\beta s}{1-s} E_t[(\frac{c_{t+1}}{c_t})^{s(1-\phi)-1}(\frac{m^G_{t+1}}{m^G_t})^{(1-s)(1-\phi)}]. \quad (22)$$

From equation (22), it can be seen that the slope coefficient b_t depends upon both the growth rate of consumption and the growth rate of real money stock. If the conditional expectation operator depends upon the second or higher moments of the growth rate processes, the slope process b_t will depend on the variabilities of both the consumption growth rate and the real money growth rate. If $\phi = 1$, the expected growth rate of the real money stock will not affect the coefficient b_t, since in that case, the marginal utility of consumption goods does not depend on the real money stock. This section provides a theoretical avenue to examine the effects of money growth rate and consumption growth rate and their variability on the stochastic behaviour of money velocity. It is shown in this section that the expected real money growth rate or its variability will affect money velocity by shifting the coefficients of the traditional money velocity function. The magnitude of this effect is also determined by other parameters in the representative agent's preferences, such as risk aversion. If the representative agent in this model is risk neutral, the changes in money growth rate and its variability will not affect the stability of the money velocity function in equilibrium. On the other side, the higher the degree of risk aversion, the larger the effect of real money growth rate and consumption growth rate and their variabilities on the stability of the money velocity function.

5. Some Empirical Results

In this section, we first simulate the model using quarterly data over the period of 1960.1 to 1992.4

for some specifications of parameters to examine the stability of the coefficients in the traditional money velocity function implied by our theoretical model. We then estimate a random coefficient model of money velocity to examine the stability of the coefficients empirically. The random coefficient model approach we follow is Swamy and Tinsley's (1980). The results from both the empirical estimation and the theoretical simulation are compared to see whether the empirical behaviour of the money velocity can be explained by the model developed in this paper.

The data on monetary assets and their corresponding holding period yields were provided by the Federal Reserve Bank of St. Louis. Output data are GNP. The inflation rate is the growth rate of the price deflator for GNP. The benchmark asset return path is approximated by the upper envelope of the three month Treasury bill rate path and the time paths of each individual monetary asset's own rate of return.[15] The growth rate of consumption is replaced by the real GNP growth rate. With M1, which includes no assets having highly risky rates of return, the regular Divisia monetary aggregate closely tracks the generalized Divisia monetary aggregate. Hence we use ordinary Divisia M1 to measure the theoretical monetary quantity aggregate.

To simulate the process b_t in equation (21), we first set $\beta = 0.99$, $s=0.972$, and $\phi \in \{0.5, 2, 5\}$. The three different values of ϕ are chosen to capture the influence of different degrees of risk aversion on the stability of the coefficient b_t. The parameter *s* mainly affects the sample mean of b_t. We estimate a VAR (vector autoregressive) model of real money and GNP growth rates using quarterly data from 1960.1 to 1992.4. The estimated VAR model is then used to estimate the conditional expectation

$$E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{s(1-\phi)-1}\left(\frac{m_{t+1}}{m_t}\right)^{(1-\phi)(1-s)}\right].$$

The simulated b_t process is plotted in Figure 1. From Figure 1 (Appendix III (1)), it can be seen that when $\phi = 0.5$ (low risk aversion case), the simulated slope process b_t is almost constant. When the value of ϕ increases, the variability in b_t also increases. When $\phi = 5$, the process b_t shows a lot of variability. There are two periods during which b_t is extremely volatile. One is from 1972 to 1976 and the other is from 1979-1982. The latter period approximately corresponds with the episode of the `monetarist experiment' of the Federal Reserve System. The b_t process also shows some variability in the recent years of 1991-1992. These simulation results confirm the theoretical prediction that if the degree of risk aversion is higher, the traditional money velocity function will be less stable.

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[15]Even the upper envelope is too low, since the theoretical benchmark asset is completely illiquid and therefore must have higher expected yield than the upper envelope over any expected yield-curve-adjusted rates of return on monetary assets providing any monetary services.

These theoretical results from model simulation can be compared with empirical estimation. We estimate equation (21) with stochastically varying coefficients. We use the Swamy and Tinsley (1980) asymptotically efficient estimation procedure. Letting $\alpha_t = (a_{0t}, b_t)$, we assume, as in Swamy and Tinsley (1980):

$$\alpha_t = \alpha^0 + e_t$$

where α^0 is a vector of constants, and

$$e_t = \Phi_1 e_{t-1} + \Phi_2 e_{t-2} + u_t,$$

where Φ_1 and Φ_2 are matrices of parameters to be estimated, and u_t is a random vector with mean zero and covariance matrix Ω. The estimated b_t process for M1 money velocity is plotted in Figure 2 (Appendix III (2)). We estimate the b_t process with both prior information incorporated and with no prior information (i.e., with diffuse prior information) about b_t. The two processes show some significant important similarities. From Figure 2, it can be seen that b_t is very volatile during the periods from 1972 to 1974 and from 1979 to 1982. This is approximately coincident with the simulation result with moderate risk aversion.

Overall, the results from model simulation and from model estimation of a random coefficient model show that our model can capture the main features of the coefficient process b_t in the traditional money velocity equation.

6. Conclusion

In this paper we have theoretically explored the determinants of money velocity. The effects of risk aversion and interest rate uncertainty on money velocity are examined within a monetary general equilibrium model. This paper indicates that if covariances between interest rates and consumption growth or between interest rates and money growth are generated by an ARCH type process, the traditional money velocity function will become unstable. Both model simulation and estimation produce significant variability in the slope of the traditional money velocity function, especially during the 1972-1974 and 1979-1982 periods. This study sheds some new light on the nature of the instability of traditional money velocity functions.

Appendix I

Proof of Proposition 1

We define Δ_t by setting

$$\sum_{i=1}^{k} (1-\pi_{it}^{G}) m_{it} = \Delta_t m_t^{G} ,$$

so that

$$\sum_{i=1}^{k} m_{it} = (\pi_t^{G} + \Delta_t) m_t^{G}$$

and we define R_{mt}^{G} such that

If we write the budget constraint in real terms, the first order conditions (9), (10), and (12), become

From equation (25), we have $\sum_{i=1}^{k} (1+R_{it}) m_{it} = (1+R_{mt}^{G}) m_t^{G}$.

$F_{mt} = T \ldots + R_{mt}^{G})]$ (23)(24)(25)

Substituting this equation into (24), we have

Note that $\lambda_t = F_{\ldots} = T_{\ldots}$ and $\beta\lambda_{\ldots} = \ldots$

and hence

Recall that

therefore

and

therefore

Appendix II

The results in section 3 and 4 are in terms of monetary quantity aggregates and their dual user cost price aggregates. In this appendix, we show that if we aggregate over the first order conditions of each individual disaggregated monetary asset, we can get the same result.

To see this, let us first consider the case of riskless interest rates. The decision problem is to maximize

(26)

subject to

$$P_t c_t + P_{st}s_t + P_{bt} b_t + P_t \sum m_{it} \leq s_{t-1}(d_t P_t + P_{st}) + P_{b,t-1} b_{t-1}(1 + R_{t-1})$$
$$+ \sum m_{i,t-1}(1+R_{i,t-1})P_{t-1} + \sum [X_{it} - (1+R_{it-1})X_{i,t-1}], \qquad (27)$$

where the summation is from i=1 to k. The first order conditions are

(29) (28)

(30)

(31)

where U_{it} is the partial derivative of U with respect to m_{it}.

From equations (28), (29), and (31), we have

(33)

Recall that we have defined $m_t = f(\mathbf{m}_t)$. The utility function $U(c_t, \mathbf{m}_t) = F(c_t, f(\mathbf{m}_t))$ consequently can be written as $F(c_t, m_t)$. From Fisher's factor reversal test, we have

and hence

Taking the summation of equation (33) over $i = 1, 2, \ldots, k$, we have

or (34)

so that

Substituting equation (34) into the above equation, we have

so that

We have the same result as that in section 3.

Now consider uncertain interest rates. In th … tion (29) becomes

and equation (31) becomes (35)

while equation (28) remains unchanged (36)

From equations (35) and (28), we have

From equations (37) and (28), we have (37)

where r_t is the real interest rate corresponding to R_t. So we have

Equation (37) can be w …

where r_{it} is the real interest ra … into (39), we have (38) (39)

or

Hence we h …

where π_{it}^G … (40)

Note that we use π_t^G as the aggregate monetary price index dual to the generalized Divisia quantity aggregate m_t^G. By taking derivatives with respect to m_{it} on both sides of Fisher's factor reversal test condition,

we have

Taking the summation of (40) over i = 1, … tion (41), we have (41)

since in this case

and equation (42) becomes

Therefore (42)

This is the same result as in sectio …

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Figure 1: Slope Coefficient of Money Velocity Function by Random Coefficient Estimation

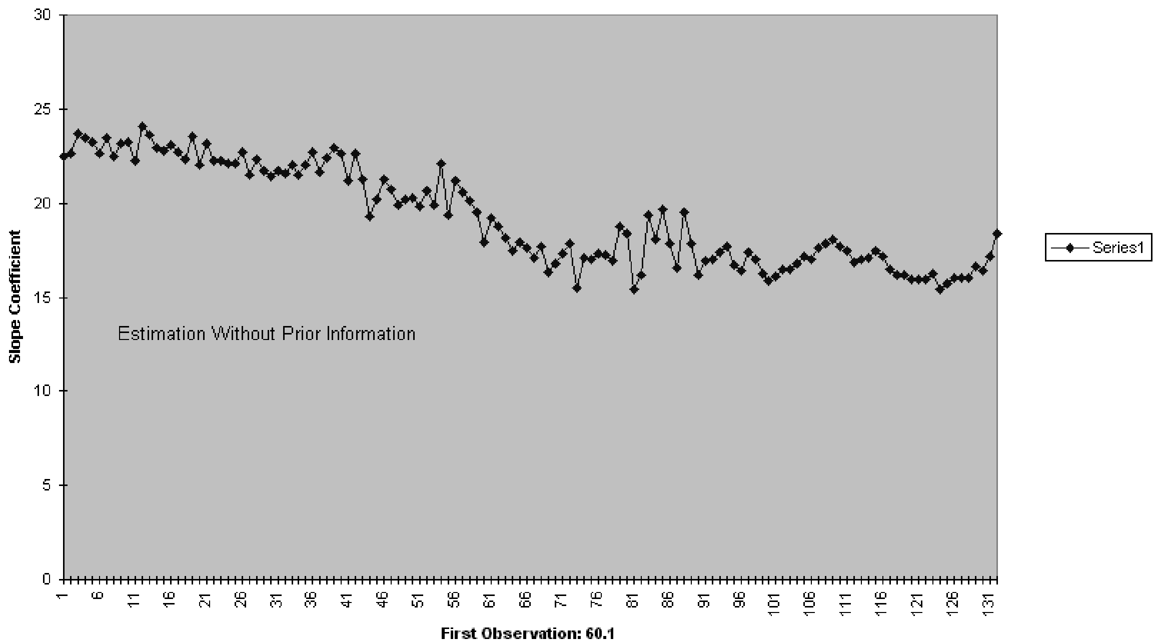

Figure 2: Simulated Slope Coefficient of Velocity Function