Continuous-Time Model of Business Fluctuations,

and Optimal Behavior of an Interest Rate

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Abstract

Presented here is the mathematical model with one commodity binding the commodity's demand, production, consumption, and savings values, and describing the economic system's reaction after increase of commodity's demand on market. It is also shown the formula for optimal behavior of an interest rate.

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1. Introduction

I continued here the research on the nature of business fluctuations (see Krouglov, 1997a, 1997b, and 1997c). I looked at the question how economy would react on the commodity's deficit on market (where I followed the approach developed by me in the discrete-time model for different circumstances - see Krouglov, 1997d), and how this reaction could contribute to the appearance of business fluctuations. Again I intentionally used the very simplified model, and eliminated some insignificant details to make the mechanism of business fluctuations look more prominent. As an additional result of the research I obtained an expression for the interest rate which depends on inflation's relative rate. Perhaps that formula may be of practical interest to some financial policy makers as for example central bankers.

2. Model Description

Consider model with one commodity, and denote its volume of production V_p , of consumption V_c , of savings V_s , and commodity's demand V_d . Designate r_p , r_c , r_s , and r_d be the production, consumption, savings, and demand rates respectively.

We look at the following task, and will try to describe it mathematically. Assume that at time $t < 0$ the commodity's demand was stable in time, and was satisfied by the production capacities. Then at time $t = 0$ this balance was broken, demand exceeded production on some constant value, and there appeared commodity's deficit on market. Economy would react by taking some part of commodity from consumption and redirecting it into production. This part would constitute economy's savings (otherwise investment into production), and would assist to the goal of production's increase. When production achieved the level of commodity's demand on market the process of commodity's redirecting from consumption into production would stop, and economy would remain in a new equilibrium point.

Mathematically it means for $t < 0$,

$$V_d = V_p = V_c = r^0 \cdot t + V^0 \ ,$$

$$V_s = 0 \ ,$$

$$r_d = r_p = r_c = r^0 \ ,$$

$$r_s = 0 \ ,$$

where r^0 and V^0 are constants.

We subtract the value r^0 from r_d , r_p , and r_c , and the value V^0 from V_d , V_p , and V_c to work further with normalized values for the sake of simplicity.

We can say that our economical system was in the rest until time $t = 0$, and the commodity's demand, production, and consumption all were at a zero level until that time.

At time $t = 0$ the commodity's demand was increased on constant rate $r_d = \Delta_r$, and constituted expression $V_d = \Delta_r \cdot t$. Therefore commodity's deficit on market could be expressed by the formula $V_{df} = V_d - V_p = \Delta_r \cdot t - V_p$, and savings going from consumption into production began to be proportional to that deficit,

$$\dot{V}_s = \lambda_s \cdot (V_d - V_p) \ , \tag{1}$$

where $\lambda_s > 0$.

Meanwhile these savings would aid to the production increase,

$$\dot{V}_p = \lambda_p \cdot V_s \, , \tag{2}$$

where $\lambda_p > 0$, and we could find the value V_p satisfying equations (1) and (2) and initial conditions for $t < 0$.

Often it requires to use the Laplace transform to solve similar differential equations (see Ditkin and Prudnikov, 1965) but here we can easily do it explicitly.

We assume that value λ_p is constant i.e. each new portion of investment would proportionally increase the production of commodity (that is happening when we deposit the money in a bank).

Therefore,

$$\ddot{V}_p = \lambda_p \cdot \dot{V}_s = \lambda_p \cdot \lambda_s \cdot (V_d - V_p) = \lambda_p \cdot \lambda_s \cdot (\Delta_r \cdot t - V_p) .$$

If we require also λ_s to be a constant we would get as a solution the equation of *harmonic oscillations* about the straight line $y = \Delta_r \cdot t$,

$$V_p = \Delta_r \cdot t - \Delta_r \cdot \sin(\sqrt{\lambda_p \cdot \lambda_s} \cdot t) , \qquad (3)$$

and economic system would never come into the equilibrium point.

To do so we will introduce into economic system so-called *damping ratio* (see Oppenheim et al., 1997),

$$\lambda_s = \lambda_s^0 + \gamma \cdot \frac{\dot{V}_{df}}{V_{df}} \qquad (4)$$

where $\lambda_s^0 > 0$ is a constant, and $\gamma > 0$.

Therefore,

$$\ddot{V}_{df} + \gamma \cdot \lambda_p \cdot \dot{V}_{df} + \lambda_s^0 \cdot \lambda_p \cdot V_{df} = 0 . \qquad (5)$$

Thus, for $0 < \gamma < 2 \cdot \sqrt{\frac{\lambda_s^0}{\lambda_p}}$, the second-order system described by the equation (5) is referred to as being *underdamped* (V_{df} and, respectively, V_p has a damped oscillatory behavior). If $\gamma > 2 \cdot \sqrt{\frac{\lambda_s^0}{\lambda_p}}$ the system is *overdamped* (the impulse response of the system is the difference between two decaying exponential curves). The case of

$\gamma = 2 \cdot \sqrt{\frac{\lambda_s^0}{\lambda_p}}$ is the *critically damped* case. Here the step response of the economic system has the shortest settling time (see Oppenheim et al., 1997).

For the last case, the commodity's production behavior is described by the equation,

$$V_p = \Delta_r \cdot t \cdot (1 - e^{-\sqrt{\lambda_s^0 \cdot \lambda_p} \cdot t}) \tag{6}$$

Thus we found that optimal behavior of the production's proportion going for an investment can be expressed by the formula,

$$\lambda_s = \lambda_s^0 + 2 \cdot \sqrt{\frac{\lambda_s^0}{\lambda_p}} \cdot \frac{\dot{V}_{df}}{V_{df}} \ . \tag{7}$$

Hence the function of commodity's savings may be described by the following expression,

$$V_s = \Delta_r \cdot \left[(\sqrt{\frac{\lambda_s^0}{\lambda_p}} \cdot t - \frac{1}{\lambda_p}) \cdot e^{-\sqrt{\lambda_s^0 \cdot \lambda_p} \cdot t} + \frac{1}{\lambda_p} \right], \tag{8}$$

and we can see that this function monotone increases from $V_s = 0$ at time $t = 0$, achieves its maximum at the time $t = \frac{2}{\sqrt{\lambda_s^0 \cdot \lambda_p}}$,

$$\left(V_s\right)_{\max} = \frac{1}{\lambda_p} \cdot \Delta_r \cdot (1 + e^{-2}) \ ,$$

and then monotone decreases with $t \to +\infty$ approaching the stable level at infinity,

$$\lim_{t \to +\infty} V_s = \frac{1}{\lambda_p} \cdot \Delta_r \ .$$

Time $t = \frac{3}{\sqrt{\lambda_s^0 \cdot \lambda_p}}$ is the function's point of inflection.

Note that function of the commodity's consumption is equal to,

$$V_c = \Delta_r \cdot \left[t \cdot (1 - e^{-\sqrt{\lambda_s^0 \cdot \lambda_p} \cdot t}) - (\sqrt{\frac{\lambda_s^0}{\lambda_p}} \cdot t - \frac{1}{\lambda_p}) \cdot e^{-\sqrt{\lambda_s^0 \cdot \lambda_p} \cdot t} - \frac{1}{\lambda_p} \right]. \qquad (9)$$

It is also more convenient to rewrite formula (7) using the commodity's price. Since we use price's derivative (see Krouglov, 1997b) as an indicator of the necessity to adjust the production level (rather than prices themselves as mediums to optimize exchanges as for example in Fisher, 1972) and since price's change reflects the commodity's deficit on market,

$$\lambda_s = \lambda_s^0 + 2 \cdot \sqrt{\frac{\lambda_s^0}{\lambda_p}} \cdot \frac{\ddot{P}}{\dot{P}} , \qquad (10)$$

where commodity's price P is expressed by the equation ($\lambda_{df} > 0$ is a constant),

$$\dot{P} = \lambda_{df} \cdot V_{df} . \qquad (11)$$

If we notice that purpose of λ_s is to redirect resources from consumption into investment to increase the commodity's production, and the same is the aim of the interest rate's existence we can say that equation (10) shows that for its optimal behavior the interest rate has to be corrected in accordance with relative rate of the inflation's change (which is represented by the ratio $\frac{\ddot{P}}{\dot{P}}$ or otherwise $\frac{d(\ln \dot{P})}{dt}$).

As a last result, we see that the formula of commodity's price adjustment is equal to,

$$P = -\frac{\lambda_{df} \cdot \Delta_r}{\sqrt{\lambda_s^0 \cdot \lambda_p}} \cdot \left(t + \frac{1}{\sqrt{\lambda_s^0 \cdot \lambda_p}}\right) \cdot e^{-\sqrt{\lambda_s^0 \cdot \lambda_p} \cdot t} + \frac{\lambda_{df} \cdot \Delta_r}{\lambda_s^0 \cdot \lambda_p} + P^0, \qquad (12)$$

where P^0 is the initial value of the commodity's price at time $t = 0$. We see also that commodity's price monotone increases from P^0, and approaches the value

$$P^{\infty} = \frac{\lambda_{df} \cdot \Delta_r}{\lambda_s^0 \cdot \lambda_p} + P^0$$

at infinity. Moreover the function of commodity's price is *concave up* (its second derivative $\ddot{P} > 0$) within the time interval $0 \le t < \frac{1}{\sqrt{\lambda_s^0 \cdot \lambda_p}}$, and is *concave down* for time $t > \frac{1}{\sqrt{\lambda_s^0 \cdot \lambda_p}}$ (where the time $t = \frac{1}{\sqrt{\lambda_s^0 \cdot \lambda_p}}$ is its point of inflection).

3. Conclusion and Acknowledgments

In my research I looked for curiosity at the problem from a mathematician's point of view, and I would like to thank everyone who sent me comments via the Internet.

4. References

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