Mathematical Description of Business Fluctuations

Alexei Krouglov

Typhoon Technologies Inc. -- Matrox R & D Centre

10 Allstate Parkway, Toronto, Ontario L3R 5P8, Canada

Abstract

Presented here is the mathematical model with one commodity that includes differential equations relating commodity's production, consumption, and price values. Shown conditions for the fluctuations of these values.

JEL classification: E 32

Keywords: Business fluctuations

1. Introduction

In this article author incorporated some results indicated in his previous models (see Krouglov, 1997a and 1997b). Here he described the closed model for interrelationship among production, consumption, and price values for *external* economical system (distinctive from *internal* economical system where there are shown *internal* sources of economical development as in Krouglov, 1997c). Also presented conditions of commodity's deficit (surplus) fluctuations that, in turn, force the fluctuations of production, consumption, and price values.

2. Mathematical Model

Consider model with one commodity, and denote its price P, the volume of production V_p, and the volume of consumption V_c. Designate r_p and r_c to be a production and consumption rates. Assume that the introduced above values at the initial time were P^0, V_p^0, V_c^0, r_p^0, and r_c^0 respectively.

For the values V_p, V_c, and P we assume the fulfillment of the following equations,

$$\frac{d^2V_p}{dt^2} = \lambda_1 \cdot \frac{dP}{dt} \tag{1}$$

$$\frac{dP}{dt} = \lambda_2 \cdot \left(V_c - V_p\right) \tag{2}$$

$$\frac{d^2V_c}{dt^2} = -\lambda_3 \cdot \frac{d^2P}{dt^2} \tag{3}$$

where λ_1, λ_2, $\lambda_3 > 0$ are constants.

If we subtract Eq. (3) from Eq. (1) we get

$$\frac{d^2\left(V_p - V_c\right)}{dt^2} + \lambda_2 \cdot \lambda_3 \cdot \frac{d\left(V_p - V_c\right)}{dt} + \lambda_1 \cdot \lambda_2 \cdot \left(V_p - V_c\right) = 0 \tag{4}$$

that is *the equation of free oscillations* for the variable $V_p - V_c$ (e.g. see Piskunov, 1965).

Eq. (4) has the following roots of its characteristic equation,

$$k_1 = \lambda_2 \cdot \left(-\frac{\lambda_3}{2} + \sqrt{\left(\frac{\lambda_3}{2}\right)^2 - \frac{\lambda_1}{\lambda_2}}\right),$$

$$k_2 = \lambda_2 \cdot \left(-\frac{\lambda_3}{2} - \sqrt{\left(\frac{\lambda_3}{2}\right)^2 - \frac{\lambda_1}{\lambda_2}} \right).$$

Case one

If $\frac{\lambda_3}{2} > \sqrt{\frac{\lambda_1}{\lambda_2}}$ the solution of Eq. (4) is

$$V_p - V_c = C_1 \cdot e^{k_1 \cdot t} + C_2 \cdot e^{k_2 \cdot t} \tag{5}$$

where

$$C_1 = \frac{\left(V_p^0 - V_c^0\right) \cdot k_2 - \left(r_p^0 - r_c^0\right)}{k_2 - k_1} ,$$

$$C_2 = \frac{\left(V_p^0 - V_c^0\right) \cdot k_1 - \left(r_p^0 - r_c^0\right)}{k_1 - k_2} .$$

Because k_1, $k_2 < 0$ and k_1, $k_2 \in \Re$, fluctuations of the variable $V_p - V_c$ do not take place. Deviation $V_p - V_c$ asymptotically approaches zero with $t \to +\infty$. It may be said that *inconsistency* on market in this case is regulated rather by the adaptation of consumers than by the one of producers.

We can obtain from Eq. (2)

$$P = -\lambda_2 \cdot C_1 \cdot \frac{1}{k_1} \cdot e^{k_1 \cdot t} - \lambda_2 \cdot C_2 \cdot \frac{1}{k_2} \cdot e^{k_2 \cdot t} + P_{sys}^0$$

where

$$P_{sys}^0 = P_0 - \frac{\lambda_2}{\lambda_1} \cdot \lambda_3 \cdot \left(V_p^0 - V_c^0\right) - \frac{1}{\lambda_1} \cdot \left(r_p^0 - r_c^0\right) .$$

Eq. (1) gives us the following,

$$V_p = -\lambda_1 \cdot \lambda_2 \cdot C_1 \cdot \frac{1}{k_1^2} \cdot e^{k_1 \cdot t} - \lambda_1 \cdot \lambda_2 \cdot C_2 \cdot \frac{1}{k_2^2} \cdot e^{k_2 \cdot t} + r_{sys} \cdot t + V_{sys}^0$$

where

$$r_{sys} = r_c^0 - \lambda_2 \cdot \lambda_3 \cdot \left(V_p^0 - V_c^0\right),$$

$$V_{sys}^0 = V_c^0 + \frac{\lambda_2}{\lambda_1} \cdot \lambda_3^2 \cdot \left(V_p^0 - V_c^0\right) + \frac{\lambda_3}{\lambda_1} \cdot \left(r_p^0 - r_c^0\right).$$

We can conclude from Eq. (3) that

$$V_c = \lambda_2 \cdot \lambda_3 \cdot C_1 \cdot \frac{1}{k_1} \cdot e^{k_1 \cdot t} + \lambda_2 \cdot \lambda_3 \cdot C_2 \cdot \frac{1}{k_2} \cdot e^{k_2 \cdot t} + r_{sys} \cdot t + V_{sys}^0 .$$

We see that V_p and V_c asymptotically approach the linear function $V_{sys} = r_{sys} \cdot t + V_{sys}^0$ with $t \to +\infty$.

Case two

If $\frac{\lambda_3}{2} = \sqrt{\frac{\lambda_1}{\lambda_2}}$ then

$$V_p - V_c = \left(C_1 + C_2 \cdot t\right) \cdot e^{-\frac{\lambda_2 \cdot \lambda_3 \cdot t}{2}} \tag{6}$$

where

$$C_1 = V_p^0 - V_c^0 ,$$

$$C_2 = \frac{1}{2} \cdot \lambda_2 \cdot \lambda_3 \cdot \left(V_p^0 - V_c^0\right) + \left(r_p^0 - r_c^0\right) .$$

Here again, deviation $V_p - V_c$ approaches zero with $t \to +\infty$ (slowly than in case one), and process of fluctuations doesn't take place (we may only have one peak). Again the consumers' influence prevails the producers' efforts.

We can write from Eq. (2)

$$P=\frac{2}{\lambda_3}\cdot\left(C_1+C_2\cdot t\right)\cdot e^{-\frac{\lambda_2\cdot\lambda_3\cdot t}{2}}+\frac{1}{\lambda_1}\cdot C_2\cdot e^{-\frac{\lambda_2\cdot\lambda_3\cdot t}{2}}+P_{sys}^0$$

where P_{sys}^0 is simplified to

$$P_{sys}^0=P_0-\frac{4}{\lambda_3}\cdot\left(V_p^0-V_c^0\right)-\frac{1}{\lambda_1}\cdot\left(r_p^0-r_c^0\right).$$

Eq. (1) give us the following,

$$V_p=-\left(C_1+C_2\cdot t\right)\cdot e^{-\frac{\lambda_2\cdot\lambda_3\cdot t}{2}}-\frac{\lambda_3}{\lambda_1}\cdot C_2\cdot e^{-\frac{\lambda_2\cdot\lambda_3\cdot t}{2}}+r_{sys}\cdot t+V_{sys}^0,$$

and similarly from Eq. (3)

$$V_c=-2\cdot\left(C_1+C_2\cdot t\right)\cdot e^{-\frac{\lambda_2\cdot\lambda_3\cdot t}{2}}-\frac{\lambda_3}{\lambda_1}\cdot C_2\cdot e^{-\frac{\lambda_2\cdot\lambda_3\cdot t}{2}}+r_{sys}\cdot t+V_{sys}^0$$

where r_{sys} is the same as in case one, and V_{sys}^0 is simplified to

$$V_{sys}^0=4\cdot V_p^0-3\cdot V_c^0+\frac{\lambda_3}{\lambda_1}\cdot\left(r_p^0-r_c^0\right).$$

Case three

If $0<\frac{\lambda_3}{2}<\sqrt{\frac{\lambda_1}{\lambda_2}}$ then the roots of characteristic equation are complex values

$$k_1=\alpha+j\cdot\beta\ ,\quad k_2=\alpha-j\cdot\beta$$

where $\alpha=-\frac{\lambda_2\cdot\lambda_3}{2}<0$, $\beta=\lambda_2\cdot\sqrt{\frac{\lambda_1}{\lambda_2}-\left(\frac{\lambda_3}{2}\right)^2}$, and the solution of Eq. (4) is

$$V_p-V_c=A\cdot e^{\alpha\cdot t}\cdot\sin(\beta\cdot t+\vartheta_0) \qquad (7)$$

where

$$A=\sqrt{\left(V_p^o-V_c^o\right)^2+\left[\left(\left(r_p^o-r_c^0\right)-\alpha\cdot\left(V_p^o-V_c^o\right)\right)/\beta\right]^2}\ ,$$

$$\vartheta_0=\arctan\left[\left(V_p^o-V_c^o\right)/\left[\left(\left(r_p^o-r_c^0\right)-\alpha\cdot\left(V_p^o-V_c^o\right)\right)/\beta\right]\right].$$

Since $\alpha<0$ the value $A\cdot e^{\alpha\cdot t}$ approaches zero with $t\rightarrow+\infty$, and we have *the equation of damped oscillations* for V_p-V_c (see Piskunov, 1965).

To simplify subsequent calculations to find values P, V_p, and V_p we move to complex numbers, make the necessary differentiation and integration, and then return back to real numbers.

Due to proportionality V_p-V_c to a sinusoid we consider the complex variable $\overline{V_p-V_c}$ where V_p-V_c is its imaginary part. Hence we get from Euler's relation,

$$\overline{V_p-V_c}=A\cdot e^{\alpha\cdot t}\cdot e^{j\cdot(\beta\cdot t+\vartheta_0)}=A\cdot e^{j\cdot\vartheta_0}\cdot e^{(\alpha+j\cdot\beta)\cdot t},$$

and from Eq. (2) we have the following,

$$\overline{P}=-\lambda_2\cdot A\cdot e^{j\cdot\vartheta_0}\cdot\frac{1}{(\alpha+j\cdot\beta)}\cdot e^{(\alpha+j\cdot\beta)\cdot t}+\overline{C_1}\ .$$

Therefore

$$P=\mathrm{Im}\{\overline{P}\}=\frac{\lambda_2}{\lambda_1}\cdot\frac{\lambda_3}{2}\cdot A\cdot e^{\alpha\cdot t}\cdot\sin(\beta\cdot t+\vartheta_0)+\sqrt{\frac{\lambda_2}{\lambda_1}-\left(\frac{\lambda_2}{\lambda_1}\right)^2\cdot\left(\frac{\lambda_3}{2}\right)^2}\cdot A\cdot e^{\alpha\cdot t}\cdot\cos(\beta\cdot t+\vartheta_0)+P_{sys}^0$$

Similarly, we have from Eq. (1) the following,

$$\frac{d^2\overline{V_p}}{dt^2}=-\lambda_1\cdot\lambda_2\cdot A\cdot e^{j\cdot\vartheta_0}\cdot e^{(\alpha+j\cdot\beta)\cdot t}$$

or

$$\frac{d\overline{V_p}}{dt} = -\lambda_1 \cdot \lambda_2 \cdot A \cdot e^{j\cdot\vartheta_0} \cdot \frac{1}{(\alpha + j\cdot\beta)} \cdot e^{(\alpha + j\cdot\beta)\cdot t} + \overline{C_2}$$

and

$$V_p = \left(1 - \frac{\lambda_2}{\lambda_1} \cdot \frac{\lambda_3^2}{2}\right) \cdot A \cdot e^{\alpha\cdot t} \cdot \sin(\beta \cdot t + \vartheta_0) - \lambda_3 \cdot \sqrt{\frac{\lambda_2}{\lambda_1} - \left(\frac{\lambda_2}{\lambda_1}\right)^2 \cdot \left(\frac{\lambda_3}{2}\right)^2} \cdot A \cdot e^{\alpha\cdot t} \cdot \cos(\beta \cdot t + \vartheta_0) + r_{sys} \cdot t + V_{sys}^0$$

We can draw from Eq. (3)

$$\frac{d^2\overline{V_c}}{dt^2} = \lambda_2 \cdot \lambda_3 \cdot A \cdot e^{j\cdot\vartheta_0} \cdot (\alpha + j\cdot\beta) \cdot e^{(\alpha + j\cdot\beta)\cdot t}$$

or

$$\frac{d\overline{V_c}}{dt} = \lambda_2 \cdot \lambda_3 \cdot A \cdot e^{j\cdot\vartheta_0} \cdot e^{(\alpha + j\cdot\beta)\cdot t} + \overline{C_3}$$

and

$$V_c = -\frac{\lambda_2}{\lambda_1} \cdot \frac{\lambda_3^2}{2} \cdot A \cdot e^{\alpha\cdot t} \cdot \sin(\beta \cdot t + \vartheta_0) - \lambda_3 \cdot \sqrt{\frac{\lambda_2}{\lambda_1} - \left(\frac{\lambda_2}{\lambda_1}\right)^2 \cdot \left(\frac{\lambda_3}{2}\right)^2} \cdot A \cdot e^{\alpha\cdot t} \cdot \cos(\beta \cdot t + \vartheta_0) + r_{sys} \cdot t + V_{sys}^0$$

We see that in this case V_p and V_c both asymptotically approach the same linear function $V_{sys} = r_{sys} \cdot t + V_{sys}^0$ with $t \to +\infty$ where P_{sys}^0, r_{sys} and V_{sys}^0 are as in case one.

3. Conclusion

We see that business fluctuations are provoked by the reasons similar to ones existing in some other fields as Mechanics or Electrical Engineering. Acting forces in both economical and physical systems are naturally expressed by the means of *second-order differential equations*. However it rises more questions than gives answers.

4. References

Krouglov, Alexei, 1997a, Mathematical model of simple business fluctuations, EconWPA, Macroeconomics, June 1997, ewp-mac/9706009.

Krouglov, Alexei, 1997b, Mathematical model of interdependency between production and price fluctuations, EconWPA, Macroeconomics, September 1997, ewp-mac/9709002.

Krouglov, Alexei, 1997c, Determination of the lower and upper bounds for savings circulating in national economy and impact of these bounds on the economy's growth or drop, EconWPA, Macroeconomics, June 1997, ewp-mac/9706008.

Piskunov, Nikolai S., 1965, Differential and integral calculus (Groningen P. Noordhoff).