# Mathematical model of simple business fluctuations

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## Abstract

Presented here the mathematical model with one commodity that describes the acceleration of commodity production as a linear function of commodity's deficit on market. The solution of derived differential equation gives the required fluctuations of the commodity's production.

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Consider model with one commodity, and suppose for simplicity that its consumption is developing with the fixed rate $r_c$**.** For the volume $V_c$ of consumption we can write:

$$V_c \quad = \quad r_c \cdot t + V_c^o , \quad t \in \left[0, +\infty \right)$$

where $V_c^o$ is the volume of consumption at the initial time $t = 0$.

For the volume $V_p$ of production assume that it depends on the volume of consumption as a

$$\frac{d^2V_p}{dt^2} = -\lambda \cdot \left(V_p - V_c\right), \qquad (1)$$

where $\lambda$ is a constant and $\lambda > 0$.

In accordance with (1) acceleration (deceleration) of production at the time *t* is directly proportional to the deficiency (surplus) of commodity on the market.

Analysis and solutions of differential equations in the rest of the article are realized ordinary (e.g. see Piskunov, 1965).

**Case one**

Let $r_c = 0$. Then (1) transforms in

$$\frac{d^2V_p}{dt^2} = -\lambda \cdot \left(V_p - V_c^o\right)$$

or if we use the change of variables $y = V_p - V_c^o$

$$\frac{d^2y}{dt^2} = -\lambda \cdot y \qquad (2)$$

and the solution of (2) is

$$y = C_1 \cdot \cos(\beta t) + C_2 \cdot \sin(\beta t)$$

where $\beta = \sqrt{\lambda}$, $\lambda > 0$. Thus, we obtained the so-called *equation of harmonic oscillations*.

If we reserve the change of variables and replace,

$$A = \sqrt{C_1^2 + C_2^2}\ ,\ \vartheta = arctg\ (C_1 / C_2)$$

we obtain another form of the same equation,

$$V_p\ =\ A \cdot \sin(\beta t + \vartheta_0) + V_c^o\ .$$

The value $A$ is called the amplitude of oscillations, the value $\vartheta_0$ is called the initial phase, and the value $2\pi / \beta$ is called the period of oscillations.

If the volume of production at time $t = 0$ was $V_p^0$, and its rate of change at that time was $r_p^o$ then the values of constants are,

$$A = \sqrt{\left(V_p^o - V_c^o\right)^2 + \left(r_p^o / \beta\right)^2}\ ,$$

$$\vartheta = arctg\left[\left(V_p^o - V_c^o\right) / \left(r_p^o / \beta\right)\right],$$

where $\beta = \sqrt{\lambda}, \quad \lambda > 0.$

Since the amplitude of harmonic oscillations remains constant with elapsing of time we can consider the value $\left(V_p^o - V_c^o\right)$ as a “potential” component of the initial energy of economical system and the value $\left(r_p^o / \beta\right)$ as a “kinetic” component of that energy at the same time $t = 0$.

## Case two

Suppose $r_c \neq 0$. Using the change of variables

$$y = V_p - r_c \cdot t - V_c^o$$

we get the same equation (2) like in case one.

Therefore the solution is

$$V_p = A \cdot \sin(\beta t + \vartheta_0) + r_c \cdot t + V_c^o \ ,$$

and for initial values $V_p^o$ and $r_p^o$ we can find the constants,

$$A = \sqrt{\left(V_p^o - V_c^o\right)^2 + \left(\left(r_p^o - r_c\right) / \beta\right)^2} \ ,$$

$$\vartheta = arctg\left[\left(V_p^o - V_c^o\right) / \left(\left(r_p^o - r_c\right) / \beta\right)\right],$$

where $\beta = \sqrt{\lambda}, \quad \lambda > 0.$

## Case three

It is known from Microeconomics that the change of size of commodity deficiency (surplus) has as a consequence the change of commodity price that draws the change of commodity consumption and hence the change in the size of deficiency (surplus).

To take into account this impact we introduce the force $F_r$ of resistance that is directly proportional to the rate of change of deficiency (surplus), and is oriented to the opposite direction to the change of deficiency (surplus).

Thus,

$$F_r = \mu \cdot \frac{d\left(V_p - V_c\right)}{dt}, \quad \mu \geq 0.$$

Introduction of that force transforms (1) into

$$\frac{d^2 V_p}{dt^2} = -\lambda \cdot \left(V_p - V_c\right) - \mu \cdot \frac{d\left(V_p - V_c\right)}{dt}. \tag{3}$$

Using the change of variables $y = V_p - V_c$ , and taking into account $V_c = r_c \cdot t + V_c^o$ we get the so-called equation of free oscillations,

$$\frac{d^2 y}{dt^2} + \mu \cdot \frac{dy}{dt} + \lambda \cdot y + 0, \quad \lambda > 0, \mu \geq 0, \tag{4}$$

that has the following roots of its characteristic equation,

$$k_1 = -\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} - \lambda} \,,$$

$$k_2 = -\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} - \lambda} \,.$$

**Subcase One**

If $\frac{\mu^2}{4} > \lambda$ the solution of (4) is

$$y = C_1 \cdot e^{k_1 \cdot t} + C_2 \cdot e^{k_2 \cdot t}$$

and granting the change of variables the solution of (3) is

$$V_p = C_1 \cdot e^{k_1 \cdot t} + C_2 \cdot e^{k_2 \cdot t} + r_c \cdot t + V_c^o \qquad (5)$$

Taking into account that $k_1 < 0, k_2 < 0$ and $k_1, k_2 \in \Re$ the volume $V_p$ of production according to (5) does not have oscillations. It asymptotically approaches to the volume of consumption $V_c = r_c \cdot t + V_c^o$ under the condition $t \to +\infty$.

**Subcase Two**

If $\frac{\mu^2}{4} = \lambda$ the solution of (4) is

$$y = (C_1 + C_2 \cdot t) \cdot e^{-\frac{\mu t}{2}}$$

and the solution of (3) is

$$V_p = (C_1 + C_2 \cdot t) \cdot e^{-\frac{\mu t}{2}} + r_c \cdot t + V_c^o$$

Here the volume $V_p$ of production also approaches to the volume $V_c$ of consumption for $t \to +\infty$ (slowly that in subcase one), and the process of oscillations doesn’t take place.

**Subcase three**

If $\mu = 0$ we arrive to the case two.

**Subcase four**

If $\mu \neq 0$ and $\frac{\mu^2}{4} < \lambda$ then the roots of characteristic equation are complex values

$$k_1 = \alpha + i\beta, \quad k_2 = \alpha - i\beta,$$

where $\alpha = -\frac{\mu}{2} < 0$, $\beta = \sqrt{\lambda - \frac{\mu^2}{4}}$ , and the solution of (4) is

$$y = A \cdot e^{\alpha \cdot t} \cdot \sin(\beta \cdot t + \vartheta_0).$$

Thus the solution of (3) is

$$V_p = A \cdot e^{\alpha \cdot t} \cdot \sin\left(\beta \cdot t + \vartheta_o\right) + r_c \cdot t + V_c^o \ .$$

Since $\alpha < 0$ the value ( $A \cdot e^{\alpha \cdot t}$ ) approaches to zero for $t \to +\infty$, and we have the so-called equation of damped oscillations for the volume $V_p$ of production relative to the volume of consumption $V_c = r_c \cdot t + V_c^o$ .

For the initial values $V_p^o$ and $r_p^o$ the values of constants are

$$A = \sqrt{\left(V_p^o - V_c^o\right)^2 + \left[\left(\left(r_p^o - r_c\right) - \alpha \cdot \left(V_p^o - V_c^o\right)\right) / \beta\right]^2} \quad ,$$

$$\vartheta = arctg\left[\left(V_p^o - V_c^o\right) / \left[\left(\left(r_p^o - r_c\right) - \alpha \cdot \left(V_p^o - V_c^o\right)\right) / \beta\right]\right],$$

where $\alpha = -\frac{\mu}{2}$ , $\beta = \sqrt{\lambda - \frac{\mu^2}{4}}$ .

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## References

Piskunov, Nikolai S., 1965, Differential and integral calculus (Groningen P. Noordhoff).