Beyond the Risk Neutral Utility Function

by

William A. Barnett and Yi Liu,

Washington University in St. Louis,

January 30, 1995

> 'The economic statistics that the government issues every week should come with a warning sticker: User beware. In the midst of the greatest information explosion in history, the government is pumping out a stream of statistics that are nothing but myths and misinformation.'
>
> Michael J. Mandel, "The Real Truth about the Economy: Are Government Statistics so much Pulp Fiction? Take a Look," *Business Week*, cover story, November 7, 1994, pp. 110-118.

1. Introduction

In the case of perfect certainty, it is well known that the Divisia index exactly tracks any aggregator function. This follows from the fact that the Divisia line integral is directly derivable from the first order conditions for optimizing behavior. For example in the case of consumer behavior, the Divisia index is derived directly from the total differential of the demand function after substitution of the first order conditions for maximizing utility subject to a budget constraint. See, e.g., Barnett (1982a, 1983, 1995a). However, the exact tracking property of the Divisia index also applies to demand for monetary services by firms and supply of produced monetary services by financial intermediaries. See Barnett (1987), Barnett and Hahm (1994), Barnett , Hinich, and Weber (1986), Barnett and Zhou (1994a,b), Drake and Chrystal (1994), Hancock (1987, 1991), and Barnett, Kirova, Pasupathy, and Yue (1995).

Risk aversion is another story. The first order conditions in the case of risk aversion are Euler equations. Since those are not the first order conditions used in deriving the Divisia index under perfect certainty, the tracking ability of the unadjusted Divisia index is compromised.

The degree to which the tracking ability degrades is a function of the degree of risk aversion and of risk. Much earlier work has had as its objective the investigation of the degree of tracking ability compromise produced by risk aversion. See, e.g., Barnett, Hinich, and Yue (1991) and Barnett, Kirova, Pasupathy, and Yue (1995). In those papers, the utility or production functions were estimated by generalized method of moments estimation of the parameters of the Euler equations, and the resulting values of the nested monetary aggregator function were produced. The Divisia index path then was compared with that of the estimated exact rational expectations monetary aggregate. This procedure is in accordance with the one widely advocated as the solution to the Lucas Critique and to what Chrystal and MacDonald (1994, p. 76) recently have called the Barnett Critique. The hope was that the compromise in tracking ability resulting from using the uncorrected Divisia index, despite the existence of risk aversion, would be small, and indeed the tracking error has been found to be small in all such explorations so far.

Nevertheless, we cannot rule out the possibility that monetary assets in the monetary aggregates for some country may one day include very risky assets that may produce a nonnegligible loss in tracking ability of the ordinary Divisia monetary aggregate. Indeed there is at the present time increasing interest in the possibility of inclusion of common stock mutual funds and long term bond mutual funds as components of monetary aggregates in the United States. See, e.g., Barnett and Zhou (1995). However, a corrected Divisia index, derived directly from the Euler equations under risk, has just been discovered by Barnett and Liu (1995). Although there does not yet exist any significant degree of applied experience in the use of that extended Divisia index, its availability needs to be known and understood. Index number theory is no longer necessarily compromised by the existence of risk, since Barnett and Liu's new extended Divisia index tracks exactly correctly, regardless of the degree of risk or of risk aversion. In this paper, we describe the potential use of this little known new discovery.

Barnett and Liu's generalized Divisia index has a direct connection with the capital asset pricing model (CAPM) in finance. In a sense their theory is a generalization of CAPM and of the Divisia index since their theory contains both as nested special cases. In particular, CAPM deals with a two dimensional tradeoff between mean return and risk, while the Divisia index deals with the two dimensional tradeoff between investment return and liquidity. Barnett and Liu's generalized theory includes the three dimensional tradeoff between mean return, risk, and liquidity.

2 The Lucas Critique

According to the Lucas Critique, private sector parameters and the parameters of central bank policy rules are confounded together within the demand and supply solution functions that typically are estimated in macroeconometric models. When modeled dynamically, those demand and supply functions are the feedback rules or contingency plans that comprise the solution functions to dynamic programming or optimal control decisions of consumers and firms. However, the central bank's policy process is among the laws of motion serving as constraints in the private sector's dynamic decision. Hence the feedback rules that solve the private sector's decision depend upon the parameters of those processes as well of the private sector's own taste and technology parameters. Shifts in the parameters of the central bank's policy process will shift the private sector's solution feedback rules.

The source of this confounding is the solution of the first order conditions (Euler equations) of the private sector's decision, since that solution cannot be acquired without augmenting the private sector's Euler equations with the government's policy rules. In particular, the central bank's policy rule, interest rate processes, and other governmentally influenced stochastic processes for variables that are in the private agent's decision but are not under the control of that private decision maker must be augmented to private decision maker's Euler equations, before the solution for the feedback rules (demand and supply functions) can be found. But if the Euler equations of the private sector are estimated directly, the confounding problem is avoided. Hence in macroeconomics in general, there is wide acceptance of the idea that Euler equations should be estimated, rather than the demand and supply functions that are the solution to the augmented system. In addition, generalized method of moments (GMM) estimation has made the estimation of Euler equations practical.

Despite the influence of Euler equation estimation and the Lucas Critique in macroeconomics in general, a substantial portion of the literature on monetary economics has continued to base its conclusion on estimates of money demand and money supply functions, which are vulnerable to the Lucas Critique. An exception is Poterba and Rotemberg (1987) who have proposed and applied an approach to Euler equation estimation applicable to consumer decisions in money markets. Surprisingly the Lucas critique seems to have had little influence on monetary policy at any of the world's central banks, since to our knowledge none of those central banks uses a macroeconometric model in which the deep parameters of Euler equations have been estimated. Instead conventional structural parameterizations are used. But it is the deep parameters of the Euler equations and not the parameters of conventional structural equations that are invariant to policy changes, and hence the macroeconomic models used by most central banks for policy simulations remain vulnerable to the Lucas critique. See Barnett, Kirova, Pasupathy, and Yue (1995). In short, it would not be unfair that the Lucas critique, although mathematically correct, is being ignored by the world's central banks.

However, there is another related critique of macroeconomic modeling which indeed is not being ignored by central banks, as is evident from this volume and the conference that produced this volume. The next section relates to that subject.

3. The Barnett Critique

According to the Barnett Critique, as defined by Chrystal and MacDonald (1994, p. 76), an internal inconsistency exists between the microeconomics used to model private sector structure and the aggregator functions used to produce the monetary aggregate data supplied by central banks. For a systematic statement of that critique and its implications for macroeonomics, see Barnett (1994). The result can do considerable damage to inferences about private sector behavior, when central bank monetary aggregate data are used. Chrystal and MacDonald (1994, p. 76) have observed the following regarding "the problems with tests of money in the economy in recent years....Rather than a problem associated with the Lucas Critique, it could instead be a problem stemming from the 'Barnett Critique.'" In fact Barnett Critique issues have been used to cast doubt upon many widely held views in monetary economics, as recently emphasized by Barnett, Fisher, and Serletis (1992), Belongia and Chalfant (1989), Belongia (1995), and Chrystal and MacDonald (1994). Based upon this rapidly growing line of research, Chrystal and MacDonald (1994, p. 108) conclude---in our opinion correctly---that: "Rejections of the role of money based upon flawed money measures are themselves easy to reject."

The Poterba and Rotemberg approach to inference about consumer behavior in the monetary sector circumvents the Barnett Critique by nesting the monetary aggregator function within the consumer's utility function and estimating the aggregator function jointly with the other parameters of the consumer's decision. Hence Poterba and Rotemberg have extended Barnett's (1980,1987) perfect certainty theory to the case of risk. Subsequently, Barnett, Hinich, and Yue (1991), Barnett and Zhou (1994a), and Barnett, Kirova, Pasupathy, and Yue (1995) have investigated the tracking abilities of various nonparametric statistical index numbers, such as the Divisia, to the Poterba and Rotemberg estimated aggregator function under risk and to the analogous weakly separable aggregator functions in the case of demand for financial services by manufacturing firms or the supply of financial services by financial firms, where the aggregator function is estimated from generalized method of moments estimation of the Euler equations. But those papers, after having determined the magnitude of the tracking error produced by risk aversion, did not derive a closed form solution for the risk correction needed to eliminate that error. Only very recently did Barnett and Liu (1995) succeed in deriving that risk correction. This paper explains how in practice that correction can be used to produce an extended Divisia monetary aggregate that is not compromised by risk aversion.

4. Consumers Demand for Monetary Assets

4.1 The Consumer's Decision

This line of research in monetary economics began with Barnett (1980) in the perfect certainty case and Poterba and Rotemberg (1987) in the case of risk. Both papers were produced from models of consumer behavior. In this paper we shall describe newer results, but again we shall present results derived from consumer behavior, despite the fact that analogous results are available for firms.

In this section we formulate a representative consumer's stochastic decision problem over consumer goods and monetary assets. The consumer's decisions are made in discrete time over an infinite planning horizon for the time intervals, t, t+1, ..., s, ..., where t is the current time period and t+T is the terminal planning period. The variables used in defining the consumer's decision are as follows:

$\mathbf{x}_s$ = n dimensional vector of real consumption of goods and services during period s,

$\mathbf{p}_s$ = n dimensional vector of goods and services prices and of durable goods rental prices during period s,

$\mathbf{a}_s$ = k dimensional vector of real balances of monetary assets during period s,

$\boldsymbol{\rho}_s$ = k dimensional vector of nominal holding period yields of monetary assets,

A_s = holdings of the benchmark asset during period s,

R_s = the one-period holding yield on the benchmark asset during period s,

I_s= the sum of all other sources of income during period s,

$p_s^* = p_s^*(\mathbf{p}_s)$ = the true cost of living index.

Define Y to be a compact subset of the n+k+2 dimensional nonnegative orthant. The consumer's consumption possibility set, S(s) for $s \in \{t,...,t+T\}$ is:

$$S(s) = \{ (\mathbf{a}_s, \mathbf{x}_s, A_s) \in Y: \sum_{i=1}^{n} p_{is}x_{is} =$$

$$+ \sum_{i=1}^{k} [(1+\rho_{i,s-1})\, p^*_{s-1}\, a_{i,s-1} - p^*_s\, a_{is}] + (1+R_{s-1})p^*_{s-1}\, A_{s-1} - p^*_s\, A_s + I_s\}. \qquad (4.1)$$

Under the assumption of rational expectations, the distribution of random variables is known to the consumer. Since current period interest rates are not paid until the end of the period, they may be contemporaneously unknown to the consumer. The benchmark asset A_s provides no services other than its yield R_s. As a result, the benchmark asset does not enter the consumer's contemporaneous utility function. In a finite planning horizon context, the benchmark asset would enter the intertemporal utility function only in the terminal period, but since we are using an infinite planning horizon model, the benchmark asset never enters the utility function and appears only in constraints as an investment that can be used for saving across periods.

Given the price and interest rate processes, the consumer selects the deterministic point $(\mathbf{a}_t, \mathbf{x}_t, A_t)$ and the stochastic processes $(\mathbf{a}_s, \mathbf{x}_s, A_s)$, s=t+1, ..., to maximize the expected value of utility over the planning horizon, subject to the sequence of choice set constraints.[1] Formally, the consumer's decision problem is the following.

<u>Problem 1</u>: Choose the deterministic point $(\mathbf{a}_t, \mathbf{x}_t, A_t)$ and the stochastic process $(\mathbf{a}_s, \mathbf{x}_s, A_s)$, s = t+1, ..., ∞, to maximize

$$u(\mathbf{a}_t, \mathbf{x}_t) + E_t \left[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u(\mathbf{a}_s, \mathbf{x}_s)\right] \qquad (4.2)$$

subject to $(\mathbf{a}_s, \mathbf{x}_s, A_s) \in S(s)$ for s ³ t, and also subject to

$$\lim_{s\to\infty} E_t \left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0.$$

The latter constraint rules out perpetual borrowing at the benchmark rate of return, R_t. The subjective rate of time discount, ξ, is assumed to be constant.

4.2 Existence of a Monetary Aggregate for the Consumer

In order to assure the existence of a monetary aggregate for the consumer, we partition the vector of monetary asset quantities, $\mathbf{a}_s$, such that $\mathbf{a}_s = (\mathbf{m}_s, \mathbf{h}_s)$. We correspondingly partition the vector of interest rates of those assets, ρ_s, such that $\rho_s=(\mathbf{r}_s,\mathbf{i}_s)$. We then assume that the utility function, u, is blockwise weakly separable in $\mathbf{m}_s$ and in $\mathbf{x}_s$ for some such partition of $\mathbf{a}_s$.[2] Hence there exists a monetary aggregator ("category utility") function, M, and consumer goods aggregator function, X, and a utility function, u*, such that

$$u(\mathbf{a}_s, \mathbf{x}_s) = u^*(M(\mathbf{m}_s), \mathbf{h}_s, X(\mathbf{x}_s)). \tag{4.3}$$

Then it follows that the exact monetary aggregate, measuring the welfare acquired from consuming the services of $\mathbf{m}_s$, is

$$M_s = M(\mathbf{m}_s). \tag{4.4}$$

We define the dimension of $\mathbf{m}_s$ to be k_1, and the dimension of $\mathbf{h}_s$ to be k_2, so that $k=k_1+k_2$.

It is clear that equation (4.4) does define the exact monetary aggregate in the welfare sense, since M_s measures the consumer's subjective evaluation of the services that he receives from holding $\mathbf{m}_s$. However it also can be shown that equation (4.4) defines the exact monetary aggregate in the aggregation theoretic sense. In particular, the stochastic process M_S , s^3t, contains all of the information about $\mathbf{m}_S$ that is needed by the consumer to solve the rest of his decision problem. For the proof, see Barnett and Liu (1995).

The Euler equations which will be of the most use to us below are those for monetary assets. Replacing $X(\mathbf{x}_t)$ by c_t in u, and letting $\rho=\frac{1}{1+\xi}$, the Euler equations for monetary assets become:

$$E_t\left[\frac{\partial u}{\partial m_{it}} - \rho \frac{p_t^*(R_t - r_{it})}{p_{t+1}^*} \frac{\partial u}{\partial c_{t+1}}\right] = 0 \tag{4.5a}$$

for i = 1, . . . , k_1, where $c_t = X(\mathbf{x}_t)$ is the exact quantity aggregate over $\mathbf{x}_t$ and p_t^* is its dual exact price aggregate.[3] Similarly we can acquire the Euler equation for the consumer goods aggregate c_t, rather than for each of its components. The resulting Euler equation for c_t is

$$E_t\left[\frac{\partial u}{\partial c_t} - \rho \frac{p_t^*(1 + R_t)}{p_{t+1}^*} \frac{\partial u}{\partial c_{t+1}}\right] = 0 \tag{4.5b}$$

See Barnett, Kirova, Pasupathy, and Yue (1995) for more details regarding the derivation of the Euler equations.

4.3. The Perfect Certainty Case

In the perfect certainty case, nonparametric index number theory is highly developed and is applicable to monetary aggregation. In the perfect certainty case, Barnett (1978,1980) proved that the nominal user cost of the services of m_{it} is $^1{}_{it}$, where

$$\pi_{it} = p_t^* \frac{R_t - r_{it}}{1 + R_t} \tag{4.6}$$

The corresponding real user cost is $^{1}{}_{it}/p^*$. In the risk neutral case, the user cost formulas are the same as in the perfect certainty case, but with the interest rates replaced by their expected values. See Barnett (1995b). It can be shown that the solution value of the exact monetary aggregate $M(\mathbf{m}_t)$ can be tracked without error in continuous time (see, e.g., Barnett (1983)) by the Divisia index:

$$d \log M_t = \sum_{i=1}^{k_1} s_{it} \; d \log m_{it} \,, \tag{4.7}$$

where the user cost evaluated expenditure shares are $s_{it} = \pi_{it} m_{it} / \Sigma_{j=1}^{k_1} \pi_{jt} m_{jt}$. The flawless tracking ability of the index in the risk neutral case holds regardless of the form of the unknown aggregator function, M. However, under risk aversion the ability of equation (4.7) to track $M(\mathbf{m}_t)$ is potentially compromised.

4.4. An Initial Extension

The fact that the Divisia index tracks exactly under perfect certainty or risk neutrality is well known. However, we show in this section that neither perfect certainty nor risk neutrality is needed for exact tracking of the Divisia index. Only contemporaneous prices and interest rates need be known. Future interest rates and prices need not be known, and risk averse behavior relative to those stochastic processes need not be excluded. The proof is as follows.

Assume that R_t, p_t^*, and $\mathbf{r}_t$ are known at time t, although their future values are stochastic. Then the Euler equations (4.5a) for $\mathbf{m}_t$ are

$$\frac{\partial u}{\partial m_{it}} - \rho p_t^* (R_t - r_{it}) E_t \left[\frac{1}{p_{t+1}^*} \frac{\partial u}{\partial c_{t+1}} \right] = 0 \tag{4.8}$$

for i = 1, . . . , k_1. Similarly the Euler equation (4.5b) for aggregate consumption of goods, c_t, becomes

$$\frac{\partial u}{\partial c_t} - \rho p_t^*(1+R_t)E_t\left[\frac{1}{p_{t+1}^*}\frac{\partial u}{\partial c_{t+1}}\right] = 0 \tag{4.9}$$

Eliminating $E_t\left[\frac{1}{p_{t+1}^*}\frac{\partial u}{\partial c_{t+1}}\right]$ between (4.8) and (4.9), we acquire

$$\frac{\partial u}{\partial m_{it}} = \frac{R_t - r_{it}}{1+R_t}\frac{\partial u}{\partial c_t} \tag{4.10}$$

But by the assumption of weak separability of u in $\mathbf{m}_t$, we have

$$\frac{\partial u}{\partial m_{it}} = \frac{\partial u}{\partial M_t}\frac{\partial M}{\partial m_{it}} \tag{4.11}$$

where $M_t = M(\mathbf{m}_t)$ is the exact monetary aggregate that we seek to track.

Substituting (4.10) into (4.11) and using (4.6), we find that

$$\frac{\partial M}{\partial m_{it}} = \pi_{it}\frac{\partial u / \partial c_t}{\partial u / \partial M_t} \tag{4.12}$$

Now substitute (4.12) into the total differential of M to acquire

$$dM(\mathbf{m_t}) = \frac{\partial u / \partial c_t}{\partial u / \partial M_t}\sum_{i=1}^{k_1}\pi_{it}dm_{it} \tag{4.13}$$

But since M is assumed to be linearly homogeneous, we have Euler's equation for linearly homogeneous functions. Substituting (4.12) into Euler's equation, we have

$$M(\mathbf{m_t}) = \frac{\partial u / \partial c_t}{\partial u / \partial M_t} \sum_{j=1}^{k_1} \pi_{jt} m_{jt} \quad (4.14)$$

Dividing (4.13) by (4.14), we acquire (4.7), which is the Divisia index. Hence the exact tracking property of the Divisia index is not compromised by uncertainty regarding future interest rates and prices or by risk aversion. Nevertheless, this assumption is not trivial, as emphasized by Poterba and Rotemberg (1987), since current period interest rates are not paid until the end of the current period.[4] In fact current period interest rates are not assumed contemporaneously known in our Euler equations (4.5a) and (4.5b).

5. The Extended Divisia Index

As shown in the above sections, the tracking ability of the ordinary Divisia index is not exact when risk aversion exists relative to the risk of contemporaneous interest rates that are not known with certainty until the end of the current period, even though risk regarding unknown future period interest rates does not compromise the Divisia index's tracking ability. However, Barnett and Liu (1995) recently have derived a generalized Divisia index, which is exact even when risk aversion exists. Their generalized Divisia index reduces to the ordinary Divisia index, under perfect certainty. The form of that new extended Divisia index is identical to that of the ordinary Divisia index, but the user cost prices in the share weights are adjusted by a CAPM risk adjustment depending upon the covariance between the own rates of return of the assets and the consumption stream of consumer goods. The risk of unknown contemporaneous yields has no effect on the consumer, if those yields are uncorrelated with the consumer's consumptions of goods, since it is the consumer goods that enter the consumer's utility function and are the final objects of consumer preferences.

However, risk relative to the current period interest yields on components of existing monetary assets is low and contributes very little to household consumption risk. In short, the covariance between those interest rates and consumption of goods during the same period is very low, and hence we should expect that the CAPM adjustment of the user costs in the Divisia index is so small as to be negligible. As in the risk neutral case, all that is needed to deal with risk is to replace the interest rates in the Divisia index by their expectations.

Nevertheless, there is increasing interest in the possibility of including much riskier assets in monetary aggregates. See, e.g., Barnett and Zhou (1995) regarding the possible relevancy of common stock and bond mutual funds as monetary aggregate components. If this research trend should continue in the U.S. or elsewhere, the CAPM adjustment of the user costs in the Divisia monetary aggregates may become entirely nontrivial. For that reason, we explain below the CAPM adjustment needed to apply Barnett and Liu's (1995) result in practice.

5.1 The User Cost of Money Under Risk Aversion

For notational convenience, we sometimes convert the nominal rates of return, r_{it} and R_t, to real total rates of return, $1+r_{it}^*$ and $1+R_t^*$, such that

$$1+r_{it}^* = \frac{p_t^*(1+r_{it})}{p_{t+1}^*} \text{ and } 1+R_t^* = \frac{p_t^*(1+R_t)}{p_{t+1}^*}, \tag{5.1}$$

where r_{it}^* and R_t^* defined in that manner are called the real rates of excess return. Also let $\rho=\frac{1}{1+\xi}$, where ξ is the subjective rate of time discount defined in (4.2) above. Further to simplify the discussion below, consider the case of aggregation over all of the monetary assets in the utility function, so that all monetary assets are assumed weakly separable within that utility function. Then there exist utility functions V and F and monetary aggregator function M such that $V(\mathbf{m}_s,c_s)=F(M(\mathbf{m}_s), c_s)$, where aggregate consumption of goods is defined by $c_s=X(\mathbf{x}_s)$.

As proven by Barnett and Liu (1995), the risk adjusted user cost of the services of monetary asset i under risk is $\Pi_{it} = \pi_{it} + \psi_{it}$, where

$$\pi_{it} = \frac{E_t R_t - E_t r_{it}}{1 + E_t R_t} \tag{5.2}$$

and

$$\psi_{it} = \rho(1 - \pi_{it}) \frac{\mathrm{Cov}(R_t^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}} - \rho \frac{\mathrm{Cov}(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}} . \tag{5.3}$$

When the covariances in (5.3) are zero, we are back to the risk neutral case in which the user costs are (5.2).

As we have observed, those covariances are indeed very small with the current components of the Federal Reserve's monetary aggregates. But if riskier assets were to be considered as possible components of future monetary aggregates, we would need the ability to compute (5.3). In its current form, equation (5.3) depends upon the form of the utility function V. In CAPM theory, it is well known that dramatic simplifications are possible by assuming quadratic utility or Gaussian processes for random variables. Either assumption produces the same result. Barnett and Liu (1995) have proven that under either of those conventional CAPM assumptions, (5.3) simplifies to

$$\psi_{it} = \frac{1}{1 + R_t^*} H_{t+1} \mathrm{Cov}(r_{it}^*, c_{t+1}) \tag{5.4}$$

where $H_{t+1} = H(M_{t+1}, c_{t+1})$ is the well known Arrow-Pratt measure of absolute risk aversion,

$$H(M_{t+1}, c_{t+1}) = \frac{-E_t[V'']}{E_t[V']}, \tag{5.5}$$

where $V' = \partial V(\mathbf{m}_{t+1}, c_{t+1})/\partial c_{t+1}$ and $V'' = \partial^2 V(\mathbf{m}_{t+1}, c_{t+1})/\partial c_{t+1}^2$. In this definition, risk aversion is measured relative to consumption risk, conditionally upon the level of monetary services produced by M_{t+1}=M($\mathbf{m}_t$). Under risk aversion, H_{t+1} is positive and increases as the degree of absolute risk aversion increases.

To apply this formula to any country, the first step would be to compute the covariances $\mathrm{Cov}(r_{it}^*, c_{t+1})$ from data on aggregate consumption of goods and on the excess rates of return r_{it}^* on the component assets. If there is a component asset i having sufficiently risky rate of return so that the covariance $\mathrm{Cov}(r_{it}^*, c_{t+1})$ is not negligible, it becomes worthwhile to compute the adjustment (5.4). Otherwise the ordinary Divisia index with the risk neutral user costs (5.2) are adequate. However, if the covariances $\mathrm{Cov}(r_{it}^*, c_{t+1})$ are found to be nonnegligible for at least one asset i, it becomes necessary to acquire an estimate of the Arrow-Pratt measure of absolute risk aversion H_{t+1}. For most countries, many papers have been published containing estimates of that degree of risk aversion and it makes sense to use an existing estimate. At that point, the computed values of (5.4) and (5.2) can be substituted into $\Pi_{it} = \pi_{it} + \psi_{it}$, which then is the correct user cost to substitute into the Divisia index formula, (4.7). Clearly the resulting generalized user cost reduces to the usual one only if ψ_{it}=0, so that $\Pi_{it}=\pi_{it}$.

An alternative, although mathematically equivalent form, exists for that adjusted user cost formula. Define $Z_t=H_{t+1}c_t$, where Z_t is a modified (time shifted) Arrow-Pratt relative risk aversion measure. Barnett and Liu (1995) have proven that

$$\Pi_{it} = \frac{E_t R_t^* - \left(E_t r_{it}^* - \phi_{it}\right)}{1 + E_t R_t^*}. \tag{5.6}$$

where

$$\phi_{it}=Z_t \mathrm{Cov}\left(r_{it}^{*},\frac{c_{t+1}}{c_t}\right) \tag{5.7}$$

As is evident from equation (5.6), the function ϕ_{it} is a clear risk adjustment to the unadjusted expected excess rate of return $E_t r_{it}^{*}$.

There is no such risk adjustment to the benchmark rate of return, since in CAPM theory, it is conventional to treat the benchmark rate as a risk free rate. So in applications of (5.6), the benchmark rate must be computed from rates that already have been risk adjusted. We advocate the use of an envelope rate of the form $E_t R_t^{*}=\max\{E_t r_{it}^{*}-\phi_{it}: \ i=1,k\}$, where the $k=k_1+k_2$ risk-adjusted rates on the $\mathbf{a}_t = (\mathbf{m}_t, \mathbf{h}_t)$ assets within the envelope should include those of all of the k_1 components assets, $\mathbf{m}_t$, in the Divisia index (4.7) along with the k_2 rates on the unused assets $\mathbf{h}_t$. Assuming data are available on enough assets $\mathbf{h}_t$, there should rarely if ever be a case in which any user cost is zero, and so long as all of the k_1 assets in $\mathbf{m}_t$ are included within the envelope, a negative value for a user cost is impossible.

Even if a zero user cost is encountered occasionally, the resulting zero weight in the Divisia index applies temporarily only at the margin for the corresponding component asset's growth rate and does not mean that the level of the asset has no weight in the level of the aggregate. Also observe that the asset that is on the upper surface of the envelope is not likely to be the same asset throughout a relevant time period. The upper envelope is a proxy for the expected risk-adjusted benchmark asset, and not a direct measurement on that asset. In principle the benchmark asset is one that has so little liquidity that its expected rate of return contains no liquidity premium. Such a pure investment asset cannot have a market of sufficient quality to produce regular data on its rate of return. In theory, the benchmark asst often is viewed as being the rate of return on human capital in a world without slavery. The adjusted rate of return on such an asset should be higher than that of the envelope rate, which always tracks the expected adjusted excess rate of return on an actual market asset, so we should not be concerned that no market asset can produce the benchmark rate. We should be more concerned with the fact that any market asset can ever equal that rate of return. The best way to raise the envelope towards the unmeasurable benchmark rate is to broaden the scope of the assets included in $\mathbf{h}_t$ and thereby included in the envelope but not in the aggregate.

The adjustment formula (5.7) has a very revealing and useful form. Observe that the covariance now is between a rate of return and a measure of the growth rate of consumption, rather than with the level of consumption, as in the earlier formula, (5.4). Also observe from (5.6) and (5.7) that a positive covariance results in a subtraction of a risk premium from the unadjusted expected rate of return. The reason is that positive correlation between an asset's rate of return and the consumption growth rate represents an increase in consumption risk from holding the asset. So the asset contributes positively to the consumer's risk, and hence a positive risk premium must be subtracted out of the expected rate of return to get the risk-adjusted rate of return. Also observe that the size of that risk premium adjustment depends not only upon the size of the covariance, but also upon the degree of risk aversion.

With very risky assets, especially those having substantial principal risk, such as common stock and bond mutual funds, we should expect that the covariance will be positive, since such assets are likely to contribute positively to household consumption risk. But many of the currently existing assets within monetary aggregates contribute only very slightly to contemporaneous household consumption risk. With such assets, after aggregating over all of the consumers in the country, it would not be surprising to find a small negative covariance between the asset's rate of return and the representative consumer's consumption growth. In theory, such assets can be viewed as "diversifying" household risk, and hence holding such assets tends to decrease consumption risk. In such cases, the negative covariance results in an addition to the unadjusted rate of return of that asset. Since the benchmark rate is an envelope rate, a risk adjustment increasing a component rate of return cannot result in a negative user cost, since the rates of return within the envelope have already been risk adjusted, whether positively or negatively.

While small negative covariances for low risk assets are likely to be common, the word small should be understand here to mean "very small." In short, we do not expect such positive risk adjustment additions to yield ever to be more than tiny for any asset, since substantial household risk diversification by holding low risk, liquid monetary assets seems very unlikely.

In the finance literature the well known consumption based beta of CCAPM theory is defined by

$$\beta_{ic} = \frac{Cov(r_{it}, c_{t+1})}{Var(c_{t+1})}.$$

The subscript c in β_{ic} designates "consumption based" beta, and the lack of a time subscript in the notation β_{ic} results from the assumption of stationarity of the interest rate and consumption bivariate process in most of that literature. Clearly the risk adjustment needed to get the Divisia monetary aggregates to track exactly under risk aversion can be interpreted in terms of that beta.

6. Conclusions

We conclude that the Barnett critique can be circumvented by using Divisia monetary aggregates rather than simple sum aggregates. Under perfect certaintly the ordinary Divisia index tracks exactly correctly under perfect certainty. Under risk neutrality, the exact tracking ability of the Divisia index still holds, so long as the rates of return in the user cost formulas are replaced be their expectations. Under risk aversion, the expected rates of return must be risk adjusted in accordance with the formula derived by Barnett and Liu (1995). As explained in this paper, that risk adjustment is easily computed and used, but is likely to be negligible for most if not all of the assets contained within current monetary aggregates. But if the trend continues towards absorbing increasingly risky assets into monetary aggregates, soon perhaps even including assets having substantial principal risk, such as stock and bond mutual funds, the risk adjustment method described in this paper may become necessary.

References

Arrow, K. J. and F. Hahn (1971), *General Competitive Analysis* . San Francisco, Holden-Day.

Barnett, William A. (1978), "The User Cost of Money," *Economics Letters*, vol. 1, pp. 145-149.

Barnett. William A. (1980), " Economic Monetary Aggregates: An Application of Index Number and Aggregation Theory," *Journal of Econometrics*, 14, 11-48.

Barnett, William A. (1982a), "Divisia Indices," in Samuel Kotz and Norman Johnson (eds.), *Encyclopedia of Statistical Sciences*, vol. 2, pp. 412-415.

Barnett, William A. (1982b), "The Optimal Level of Monetary Aggregation," *Journal of Money, Credit, and Banking*, Nov., part 2, vol. 14, no. 4, pp. 687-710.

Barnett, William A. (1983), "Understanding the New Divisia Monetary Aggregates," *Review of Public Data Use*, vol. 11, December pp. 349-355.

Barnett, William A. (1987), "The Microeconomic Theory of Monetary Aggregation," in Barnett, William A. and Kenneth J. Singleton eds., *New Approaches to Monetary Economics, Cambridge University Press.*

Barnett, William A. (1994), "Perspective on the Current State of Macroeconomic Theory," *International Journal of Systems Science*, vol. 25, no. 5, pp. 839-848.

Barnett, William A. (1995a), "The Divisia Monetary Aggregates," in David Glasner (ed.), *Encyclopedia of Business Cycles, Panics, Crises and Depressions*, Garland Publishing Company, forthcoming

Barnett, William A. (1995b), "Exact Aggregation under Risk," in Barnett, William A., Moulin, Herve, Salles, Maurice, and Norman Schofield (Eds.), *Social Choice, Welfare, and Ethics*, Cambridge University Press, forthcoming.

Barnett, William A. and Seungmook Choi (1989), "A Monte Carlo Study of Tests of Blockwise Weak Separability," *Journal of Busniess and Economic Statistics,* July 1989, vol. 7, no. 3, pp. 363-377.

Barnett, William A., D. Fisher and S. Serletis (1992), " Consumer Theory and the Demand for Money," *Journal of Economic Literature,* 92, 2086-119.

Barnett, William A. and Jeong Ho Hahm (1994), "Financial-Firm Production of Monetary Services: A Generalized Symmetric Barnett Variable-Profit-Function Approach," *Journal of Business & Economic Statistics,* vol. 12, January .

Barnett, William A., Hinich, Melvin, and Warren Weber (1986), "The Regulatory Wedge between the Demand-Side and Supply-Side Aggregation-Theroetic Monetary Aggregates," *Journal of Econometrics*, 33, pp. 165-185.

Barnett, William A., Hinich, Melvin, and Piyu Yue (1991), "Monitoring Monetary Aggregates under Risk Aversion," in Michael T. Belongia (ed.), *Monetary Policy on the 75th Anniverary of the Federal Reserve System,* Proceedings of the Fourteenth Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis, Kluwer, pp. 189-222.

Barnett, William, Kirova, Milka, Pasupathy, Meenakshi, and Piyu Yue (1995), "Estimating Policy-Invariant Deep Parameters in the Financial Sector, when Risk and Growth Matter," *Journal of Money, Credit, and Banking*, proceedings volume of Cleveland Federal Reserve Bank Conference on Liquidity, Monetary Policy, and Financial Intermediation, forthcoming.

Barnett, William and Yi Liu (1995), "The CAPM-Extended Divisia Monetary Aggregate with Exact Tracking under Risk," Working Paper #194, Washington University in St. Louis, St. Louis, Missouri.

Barnett, William A. and Ge Zhou (1994a), "Financial Firm's Production and Supply-Side Monetary Aggregation Under Dynamic Uncertainty," *St. Louis Federal Reserve Bank's Monthly Review*, pp. 133-165.

Barnett, William A. and Ge Zhou (1994b), "Response to Brainard's Commentary," *St. Louis Federal Reserve Bank's Monthly Review*, pp. 169-174.

Barnett, William A. and Ge Zhou (1995), "Mutual Funds and Monetary Aggregates: Commentary," with Ge Zhou, Federal Reserve Bank of St. Louis *Review*, November/December 1994, special edition containing proceedings of the *Symposium on Mututal Funds and Monetary Aggregates*, vol. 76, No. 6, pp. 53-62.

Belongia, Michael T. (1995), "Measurement Matters: Recent Results from Monetary Economics Re-examined," *Journal of Political Economy*, forthcoming.

Belongia, Michael and James Chalfant (1989), "The Changing Empirical Definition of Money: Some Estimates from a Model of the Demand for Money Substitutes," *Journal of Political Economy* , 97, April, pp. 387-398.

Chrystal, K. Alec and Ronald MacDonald (1994), "Empirical Evidence on the Recent Behavior and Usefulness of Simple-Sum and Weighted Measures of the Money Stock," Federal Reserve Bank of St. Louis *Review*, March/April, pp. 73-109.

Diewert, W.E. (1980), "Recent Developments in the Economic Theory of Index Numbers: Capital and the Theory of Productivity Measurement," *American Economic Review*, May.

Drake, Leigh and Alec Chrystal (1994), "Company-Sector Money Demand: New Evidence on the Existence of a Stable Long-run Relationship for the United Kingdom," *Journal of Money, Credit and Banking*, August 1994, Part 1, pp. 412-438.

Feenstra, Robert C. (1986), "Functional Equivalence Between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics*, March, 271-291.

Hancock, Diana (1987), "Aggregation of Monetary Goods: A Production Model," In Barnett, William A. and Kenneth J. Singleton eds., *New Approaches to Monetary Economics*, Cambridge University Press.

Hancock, Diana (1991), *A Theory of Production for the Financial Firm*, Boston: Kluwer Academic Publishers.

Phlips, Louis and Frans Spinnewyn (1982), "Rationality versus Myopia in Dynamic Demand Systems," in R. L. Basmann and G. F. Rhodes (eds.), *Advances in Econometrics* , JAI Press, pp. 3-33.

Poterba, James M. and Julio J. Rotemberg (1987), "Money in the Utility Function: An Empirical Implementation," in William A. Barnett and Kenneth J. Singleton (eds.), *New Approaches to Monetary Economics*, Cambridge, Cambridge University Press, pp. 219-240.

Rotemberg, Julio J., Driscoll, John C., and James M. Poterba (1994), "Money, Output, and Prices: Evidence from a New Monetary Aggregate," *Journal of Business and Economic Statistics*, forthcoming.

Sargent, Thomas J. (1987), "*Dynamic Macroeconomic Theory*," Harvard University Press. Cambridge, Massachusetts and London, England.

Sidrauski, Miguel (1967), "Rational Choice and Patterns of Growth in a Monetary Economy," *American Economic Review*, 57, 2, May, pp. 534-544.

Swofford, James L. and Whitney, Gerald A. (1987), "Nonparametric Test of Utility Maximization and Weak Separability for Consumption, Leisure, and Money, *Review of Economics and Statistics,* August, 69(3), pp. 458-64.

FOOTNOTES

[1]As is well known in general equilibrium theory, a derived utility function containing money exists, so long as money has positive value in equilibrium. See, e.g., Feenstra (1986), Arrow and Hahn (1971), Sidrauski (1967), and Phlips and Spinnewyn (1982). We assume that money has positive value in equilibrium and use the resulting derived utility function. The inverse mapping from the derived utility function back to the explicit motive for holding money is not unique, and hence the derived utility function cannot be used to reveal the explicit motive. But we have no reason to seek to determine that explicit motive, and the nonuniqueness of the inverse mapping proves that putting money into the utility function produces a generalization over any model based upon an explicit motive, such as a cash in advance constraint.

[2]A long literature exists on testing that weak separability assumption. See, e.g., Barnett (1982b), Barnett and Choi (1989), Belongia and Chalfant (1989), and Swofford and Whitney (1987).

[3]Assuming that X is linearly homogeneous, the exact price aggregator function is the unit cost function.

[4]Also see Rotemberg, Driscoll, and Poterba (1994).