Preliminary
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On the Concavity of the Consumption Function

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Abstract

Zeldes (1989), Carroll (1992; 1993), and others have shown that optimal consumption behavior for consumers facing income uncertainty can be remarkably different from the certainty-equivalent case. Carroll (1992; 1993) observes that many of the differences can be attributed to the concavity of the consumption function under uncertainty, but he does not describe the conditions under which the consumption function will be concave. We show that if labor income is stochastic, the consumption function will be concave for many commonly used utility functions, and if both labor income and capital income are stochastic, the consumption function is concave for an even broader group of utility functions.

Keywords: consumption function, policy rule, uncertainty, concavity, stochastic income, certainty equivalent, precautionary saving, marginal propensity to consume.

JEL Classification Codes: C6, D91, E21

1 Introduction

One of the more surprising discoveries in macroeconomic theory over the past ten years has been how much the solution to the optimal intertemporal consumption problem can change when income uncertainty is introduced into the problem. In an important early paper, Zeldes (1989), using numerical methods, found that introducing labor income uncertainty made the consumption function concave, with the marginal propensity to consume everywhere higher than in the certainty case. Kimball (1990a; 1990b) gives the analytical explanation for the increase in the slope of the consumption function, but until now there has been no analytical explanation for the concavity of the consumption function that income uncertainty seems to induce. This gap in our understanding is important because, as Carroll (1992; 1993) argues, much of the distinctive behavior of consumption and saving under uncertainty stems from the concavity of the optimal consumption rule.

This paper fills that gap, providing an analytical demonstration of the concavity of the consumption function under uncertainty for a key class of utility functions. We show that the consumption rule is concave whenever the intertemporally separable period utility function is drawn from the class of functions which exhibit Hyperbolic Absolute Risk Aversion, i.e. those functions which satisfy the condition $u'''u'/u''^2 = k \geq 0$. Most commonly used utility functions are of the HARA class: Quadratic utility corresponds to $k = 0$, Constant Absolute Risk Aversion (CARA) corresponds to $k = 1$, and Constant Relative Risk Aversion (CRRA) utility functions satisfy $k > 1$.[1]

We further show that if $k \geq 1$ the consumption function will be *strictly* concave except under very special circumstances. The exceptions to strict concavity include two

[1]This class can also accomodate CRRA utility functions with a shifted origin, which are sometimes used to model a necessity level of consumption.

well-known cases: CARA utility if all of the risk is to labor income (no rate of return risk), and CRRA utility if all of the risk is rate-of-return risk (no labor income risk). These special cases have been widely used because of their analytical convenience (they yield a linear consumption function), but the analytical results in this paper bolster the argument (made forcefully by Kimball (1990a), Carroll (1993), Deaton (1992) and others) that it is in most cases unwise to rely on these analytically convenient formulations because the behavior they imply is qualitatively quite different from behavior in the general case.

2 Proofs

We assume that the consumer is maximizing the time-additive present discounted value of utility from consumption $u(c)$. Denoting the (possibly stochastic) gross interest rate and time preference factors as $\tilde{R}_t \in (0, \infty)$ and $\tilde{\beta}_t \in (0, \infty)$, respectively, and labelling consumption c_t, stochastic labor income $\tilde{y}_t$, and gross wealth (inclusive of labor income) w_t, the consumer's problem can be written as:

$$\begin{aligned} V_t(w_t) &= \max_{c_t} \; u(c_t) + E_t \sum_{s=t+1}^{T} \left(\prod_{j=t+1}^{s} \tilde{\beta}_j \right) u(c_s) && (1) \\ & \quad s.t. \;\; w_{t+1} = \tilde{R}_{t+1}(w_t - c_t) + \tilde{y}_{t+1}. && (2) \end{aligned}$$

As usual, the recursive nature of the problem allows us to rewrite this equation as:

$$V_t(w_t) = \max_{c_t} \; u(c_t) + E_t \tilde{\beta}_{t+1} V_{t+1}(\tilde{R}_{t+1}(w_t - c_t) + \tilde{y}_{t+1}), \qquad (3)$$

or, for convenience defining $\phi_t(s_t) = E_t \tilde{\beta}_{t+1} V_{t+1}(\tilde{R}_{t+1} s_t + \tilde{y}_{t+1})$, where $s_t = w_t - c_t$ is the portion of period t resources saved, we have:

$$V_t(w_t) \quad = \quad \max_{c_t} \; u(c_t) + \phi_t(w_t - c_t). \tag{4}$$

We assume that all stochastic variables in the problem are i.i.d., which implies that there is only one state variable in the problem, wealth. Optimal consumption or saving behavior in a given period can therefore be expressed as a function simply of the level of wealth in that period. Call the policy rule which indicates the optimal value of consumption in period t $c_t^*(w_t)$.

We are now in a position to state the main theorem of this paper. Defining a "permissible income process" as any income process which permits the agent to ensure that consumption remains within the domain over which $u(c)$ is defined, the theorem is:

Theorem 1 *For utility functions in the HARA class, for any permissible income process, the optimal consumption rule is concave, i.e.* $c_t^{*''}(w_t) \leq 0$.

In order to prove this theorem we will first prove three lemmas.

Lemma 1 *If* $\frac{V_{t+1}^{'''} V_{t+1}^{'}}{\left[V_{t+1}^{''}\right]^2} \geq k$ *then* $\frac{\phi_t^{'''} \phi_t^{'}}{\left[\phi_t^{''}\right]^2} \geq k$.

The expression $\frac{\phi_t^{'''} \phi_t^{'}}{\left[\phi_t^{''}\right]^2}$ will be $\geq k$ if and only if the expression $\phi_t^{'''} \phi_t^{'} - k \left[\phi_t^{''}\right]^2$ is nonnegative. But this expression is merely the determinant of the matrix:

$$\Phi_t = \begin{bmatrix} \phi_t^{'} & \sqrt{k} \; \left[\phi_t^{''}\right] \\ \sqrt{k} \; \left[\phi_t^{''}\right] & \phi_t^{'''} \end{bmatrix}. \tag{5}$$

So we know that we will have $\frac{\phi_t^{'''} \phi_t^{'}}{\left[\phi_t^{''}\right]^2} \geq k$ iff the matrix Φ_t is positive semidefinite. But we can rewrite Φ_t as:

$$\Phi_t = E_t\ \tilde{\beta}_{t+1} \begin{bmatrix} \tilde{R}_{t+1}\ V'_{t+1} & \sqrt{k}\ \tilde{R}^2_{t+1}\ \left[V''_{t+1}\right] \\ \sqrt{k}\ \tilde{R}^2_{t+1}\ \left[V''_{t+1}\right] & \tilde{R}^3_{t+1}\ V'''_{t+1} \end{bmatrix}. \tag{6}$$

Notice that, for all possible realizations of $\tilde{R}$, $\tilde{\beta}$, and $\tilde{y}$, the matrix whose expectation we are taking is positive semidefinite because $\tilde{R}^4_{t+1} V'_{t+1} V'''_{t+1} - k\tilde{R}^4_{t+1} \left[V''_{t+1}\right]^2 \in (0, \infty)$ from our initial assumptions that $V'_{t+1}V'''_{t+1}/\left[V''_{t+1}\right]^2 \geq k$, that $\tilde{R}_{t+1} > 0$, and that the period utility function is increasing and concave (the value function inherits these characteristics of the utility function). The expectation operator acts as a weighted sum across possible states, and because the sum of positive semidefinite matrices is positive semidefinite, Φ_t must be positive semidefinite, and the Lemma is proved.

The second lemma we need to prove states that, if the period t expected value function ϕ and the utility function both have HARA parameters $\geq k$, the period t value function V also has HARA parameter $\geq k$. This generalizes Neave's (1971) proof that the value function inherits decreasing absolute risk aversion from the utility function.

Lemma 2 *If* $\frac{\phi'''_t \phi'_t}{\left[\phi''_t\right]^2} \geq k$ *and* $\frac{u'''u'}{u''^2} \geq k$ *then* $\frac{V'''_t V'_t}{\left[V''_t\right]^2} \geq k$.[2]

We begin with a graphical illustration of the proposition for three possible values of the HARA parameter: $k = 0$, corresponding to quadratic utility; $k = 1$, corresponding to CARA utility; and $k > 1$, which applies for CRRA utility functions. Denoting the marginal utility of consumption at the optimal consumption level as $z_t = u'(c^*_t(w_t))$, we wish to define the functions $f_t(z_t)$, $g_t(z_t)$, and $h_t(z_t)$ as the inverses of u', ϕ', and V':

$$\begin{aligned} f_t(z_t) &= u'^{-1}(z_t) &= c_t, \\ g_t(z_t) &= \phi_t'^{-1}(z_t) &= s_t, \\ h_t(z_t) &= V_t'^{-1}(z_t) &= w_t. \end{aligned}$$

[2]We prove the lemma for $\frac{u'''u'}{u''^2} \geq k$ for generality; we will actually use the lemma only for HARA utility functions for which $\frac{u'''u'}{u''^2} = k$.

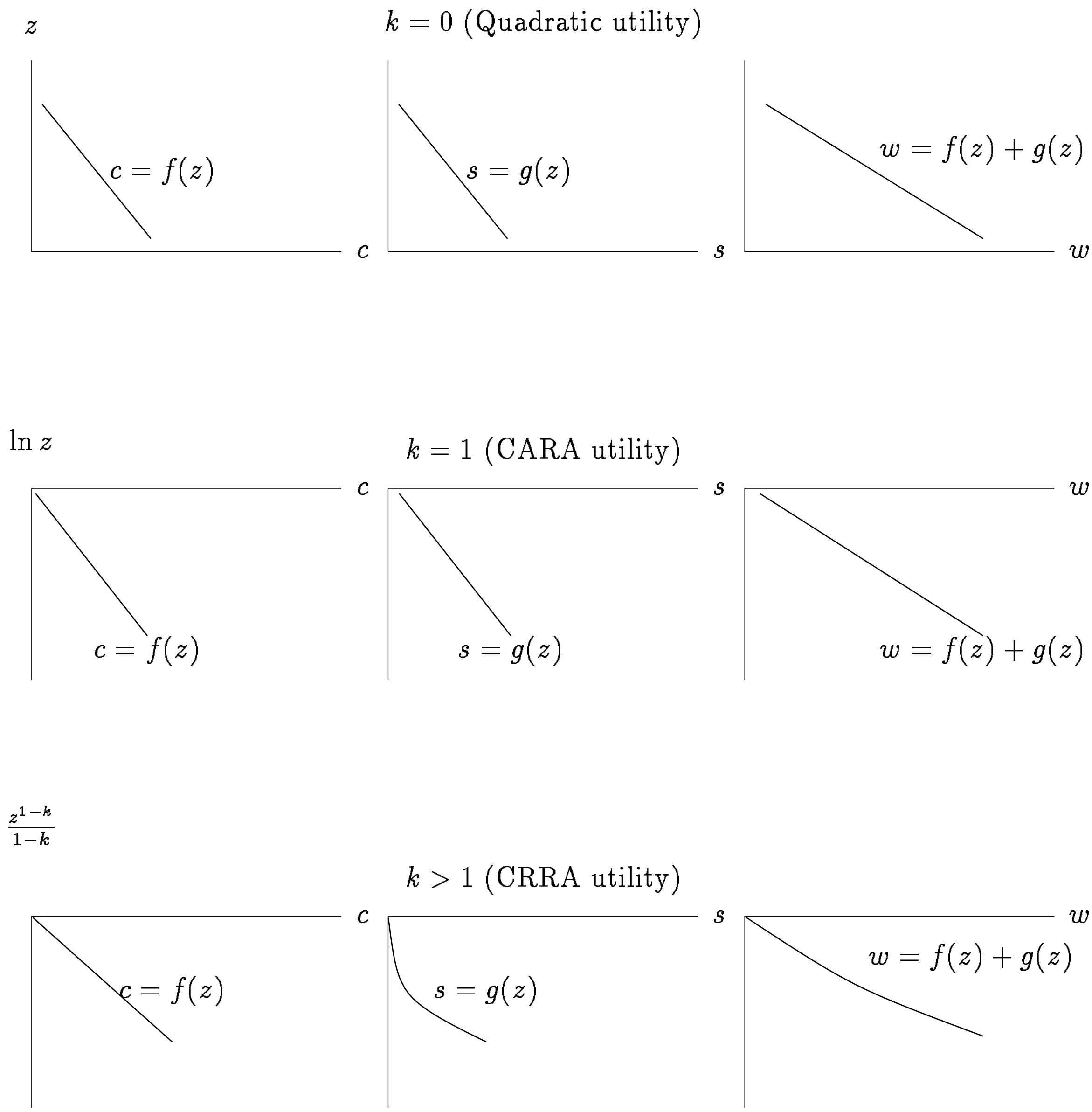

Figure 1: Illustration for Lemma 2

where the second column of equalities will hold true at the optimum. Our task is to prove that the rightmost figures, for $h_t(z_t)$, are convex. The diagrams illustrate that, because the two functions that are being added to produce $h_t(z_t)$ are convex, so, too, is $h_t(z_t)$.

The formal proof of the lemma proceeds as follows. Dropping the time subscripts from f, g and h for convenience, we have:

$$f' = \frac{1}{u''}, \tag{7}$$

$$f'' = -\frac{u'''}{u''^2} f' = -\frac{u'''}{u''^3}, \tag{8}$$

and so

$$-\frac{zf''}{f'} = \frac{u'''\,u'}{u''^2} \geq k. \tag{9}$$

Similarly,

$$-\frac{zg''}{g'} = \frac{\phi_t^{'''}\,\phi_t^{'}}{\left[\phi_t^{''}\right]^2} \geq k \tag{10}$$

and

$$-\frac{zh''}{h'} = \frac{V_t^{'''}\,V_t^{'}}{\left[V_t^{''}\right]^2}, \tag{11}$$

but since $h = f + g$, $h' = f' + g'$ and $h'' = f'' + g''$, so

$$-\frac{zh''}{h'} = -z\frac{f'' + g''}{f' + g'} \tag{12}$$

$$= \underbrace{\frac{f'}{f' + g'}}_{>0} \underbrace{\left(\frac{-zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{g'}{f' + g'}}_{>0} \underbrace{\left(\frac{-zg''}{g'}\right)}_{\geq k}. \tag{13}$$

Thus, $V_t^{'''} V_t^{'} / \left[V_t^{''}\right]^2$ is a weighted average of two quantities both of which are greater than or equal to k, and therefore

$$\frac{V_t^{'''} V_t^{'}}{\left[V_t^{''}\right]^2} \geq k, \tag{14}$$

and the lemma is proven.

The third lemma picks up where Lemma 2 left off. It states that if the value function has HARA parameter $\geq k$ (as Lemma 2 proved it will) and the utility function has HARA parameter $= k$ (as assumed), then the consumption function is concave.

Lemma 3 *If* $\frac{V_t^{'''} V_t^{'}}{\left[V_t^{''}\right]^2} \geq k$ *and* $\frac{u^{'''}u^{'}}{u^{''2}} = k$*, then the optimal consumption policy rule is concave,* $c_t^{*''}(w_t) \leq 0$.

We begin by defining a function which yields the amount of saving corresponding to any optimally chosen level of consumption, $\theta_t^*(c_t) = w_t^*(c_t) - c_t$ where $w_t^*(c_t)$ is the inverse of the optimal consumption rule $c_t^*(w_t)$. Note that an alternative definition of $\theta_t^*(c_t)$ is $\theta_t^*(c_t) = g(f^{-1}(c_t))$. Because $w_t^*(c_t) = c_t + \theta_t^*(c_t)$ it is immediate that if $\theta_t^*(c_t)$ is convex, $w_t^*(c_t)$ is convex. But if $w_t^*(c_t)$ is convex, its inverse, $c_t^*(w_t)$, must be concave, because both are increasing functions. Thus, if we can prove that $\theta_t^*(c_t)$ is convex we will have proven that the consumption rule must be concave.

The proof that $\theta_t^*(c_t)$ is convex follows closely proofs in Pratt (1964) and Kimball (1990b). It proceeds by directly calculating the derivatives of $\theta_t^*(c_t)$:

$$\theta_t^*(c_t) = g(f^{-1}(c_t)), \tag{15}$$

$$\theta_t^{*'}(c_t) = \frac{g'(f^{-1}(c_t))}{f'(f^{-1}(c_t))}, \tag{16}$$

$$\theta_t^{*''}(c_t) = \frac{\left[g''(f^{-1}(c_t))f'(f^{-1}(c_t)) - g'(f^{-1}(c_t))f''(f^{-1}(c_t))\right]/f'(f^{-1}(c_t))}{[f'(f^{-1}(c_t))]^2} \tag{17}$$

$$= \frac{g'(f^{-1}(c_t))}{[f'(f^{-1}(c_t))]^2}\left[\frac{g''(f^{-1}(c_t))}{g'(f^{-1}(c_t))} - \frac{f''(f^{-1}(c_t))}{f'(f^{-1}(c_t))}\right] \tag{18}$$

which, from Lemma 2, (19)

$$= \frac{g'(z_t)}{[f'(z_t)]^2}\underbrace{\underbrace{\left(\frac{1}{z_t}\right)}_{>0}\left[\underbrace{\frac{-z_t f''(z_t)}{f'(z_t)}}_{=k} - \underbrace{\frac{-z_t g''(z_t)}{g'(z_t)}}_{\geq k}\right]}_{\leq 0}. \tag{20}$$

So $sign\left(\theta_t^{*''}(c_t)\right) = -sign\left(g'(z_t)\right))$. But

$$g'(z_t) = \frac{1}{\phi_t^{''}(c_t)} \tag{21}$$

and

$$\phi_t^{''}(c_t) = E_t\tilde{\beta}_{t+1}\left(\tilde{R}_{t+1}^2\right)V_{t+1}^{''}(w_{t+1}), \tag{22}$$

so $sign(\phi_t^{''}(c_t)) = sign(V_{t+1}^{''}(w_{t+1}))$. But

$$V_{t+1}^{''}(w_{t+1}) = u'(c_{t+1}^*[w_{t+1}]) \tag{23}$$

$$V_{t+1}^{''}(w_{t+1}) = \underbrace{\underbrace{u''(c_{t+1}^*[w_{t+1}])}_{<0}\underbrace{c_{t+1}^{*'}[w_{t+1}]}_{>0}}_{<0} \tag{24}$$

so $sign(V_{t+1}^{''}(w_{t+1})) < 0$, implying $sign(g'(z_t)) \leq 0$, and thus $sign\left(\theta_t^{*''}(c_t)\right) \geq 0$. As noted above, if $\theta_t^*(c_t)$ is convex, $w_t^*(c_t)$ is convex, implying that $c_t^*(w_t)$ is concave, and the lemma is proved.

We are now in position to prove the main theorem of the paper.

Proof of Theorem 1. First, note that in the last period of life, $V_T(w_t) = u(w_t)$ so we have that $V_T^{'''}V_T^{'}/\left[V_T^{''}\right]^2 = u'''u'/u''^2 = k$. Therefore, by Lemma 1 we know that $\phi'''_{T-1}\phi'_{T-1}/\left[\phi''_{T-1}\right]^2 \geq k$, and by Lemma 2 we know that $V'''_{T-1}V'_{T-1}/\left[V''_{T-1}\right]^2 \geq k$. Continued iteration using Lemma 1 and Lemma 2 demonstrates that for any $t < T$, $V'''_tV'_t/\left[V''_t\right]^2 \geq k$. Then use Lemma 3 to show that the consumption rule is concave in all time periods t, and the theorem is proven.

The final question we wish to address is under what conditions the consumption rule is *strictly* concave. We will consider separately the cases $k > 1$ and $k = 1$.

Corollary 1 *For utility functions in the HARA class, for any permissible income process, the optimal consumption rule is strictly concave, i.e.* $c^{*''}_t(w_t) < 0$, *whenever* $k > 1$ *and future labor income is to any degree uncertain, and is not perfectly correlated with the future interest rate.*

Proving this corollary requires us to revisit Lemma 1. We will call the new version Lemma 4.

Lemma 4 *If* $\frac{V'''_{t+1}V'_{t+1}}{\left[V''_{t+1}\right]^2} \geq k$ *and labor income in period t+1,* $\tilde{y}_{t+1}$, *is uncertain and imperfectly correlated with* $\tilde{R}_{t+1}$, *then* $\frac{\phi'''_t\phi'_t}{\left[\phi''_t\right]^2} > k$.

Recall our matrix Φ

$$\Phi_t = E_t\ \tilde{\beta}_{t+1} \left[\begin{array}{cc} \tilde{R}_{t+1}\ V'_{t+1} & \sqrt{k}\ \tilde{R}^2_{t+1}\ \left[V''_{t+1}\right] \\ \sqrt{k}\ \tilde{R}^2_{t+1}\ \left[V''_{t+1}\right] & \tilde{R}^3_{t+1}\ V'''_{t+1} \end{array} \right]. \tag{25}$$

In proving lemma 1 we showed that for any possible realizations of $\tilde{R}$, $\tilde{\beta}$, and $\tilde{y}$, the Φ matrix is positive semidefinite. In order for the expectation (the weighted sum across states) to be positive *definite*, we need only that the matrices being added not be scalar multiples of each other. The conditions under which the matrices are scalar multiples are quite restrictive. To see this, consider the last period of life in which $V_T(\tilde{w}_T) =$

$u(\tilde{w}_T) = u(\tilde{R}_T s_{T-1} + \tilde{y}_T)$. For $k > 1$, HARA utility functions can be written in the form $u(c) = \frac{(c-\bar{c})^{1-\gamma}}{1-\gamma}$ where $\bar{c}$ is the necessity level of consumption and γ is the coefficient of relative risk aversion. If there is labor income uncertainty but no interest rate uncertainty, then in order for the Φ matrices to be scalar multiples under different realizations of $\tilde{y}$ it must be the case that

$$\frac{d}{dy}\left(\log[R_T u'(R_T s_{T-1} + y)]\right) = \frac{d}{dy}\left(\log[R_T^2 u''(R_T s_{T-1} + y)]\right). \tag{26}$$

A few lines of algebra show that this requirement reduces to the equation $\gamma = \gamma + 1$, which is of course impossible. In the more general case with both interest rate and labor income uncertainty, it is straightforward to show that the matrices will be scalar multiples only if the interest rate and labor income are perfectly correlated, with $\frac{dR}{dy} = \frac{R}{y-\bar{c}}$.[3]

The final lemma is used to establish the concavity of the consumption rule whenever there is any labor income uncertainty in any future period, even if the uncertainty is not in the successive period (as in Lemma 4).

Lemma 5 *If* $\frac{V'''_{t+1} V'_{t+1}}{\left[V''_{t+1}\right]^2} > k$, *then* $\frac{\phi'''_t \phi'_t}{\left[\phi''_t\right]^2} > k$.

Consider again the definition of the matrix Φ_t in equation 25. For each possible realization of $\tilde{R}$, $\tilde{\beta}$, and $\tilde{y}$, the matrix whose expectation we are taking is positive definite because $\frac{V'''_{t+1} V'_{t+1}}{\left[V''_{t+1}\right]^2} > k$ and V' and V''' are both positive. As the sum of positive definite matrices, Φ_t itself must be positive definite, implying that $\phi'''_t \phi'_t - k\left[\phi''_t\right]^2$ is positive

[3]In the most general case, with uncertain labor income, interest rates, and a necessity level of consumption, the matrices are scalar multiples only if $R(1 - \frac{d\bar{c}}{dy}) = \frac{dR}{dy}(y - \bar{c})$ or if $\tilde{y} = \tilde{\bar{c}}$, i.e. if the pattern of stochastic labor income exactly matches the stochastic necessity level. However, the simplest way to deal with a stochastic necessity level (as one might get, for instance, in a simple model with inelastically necessary medical expenses) is simply to subtract the necessity level from income. For that matter, the simplest way to model a nonstochastic necessity level is also by subtracting it from income.

from the definition of Φ_t in equation 5. Of course, if $\phi_t^{'''}\phi_t^{'} - k\left[\phi_t^{''}\right]^2 > 0$ then we have $\frac{\phi_t^{'''}\phi_t^{'}}{\left[\phi_t^{''}\right]^2} > k$, and the lemma is proven.

With lemmas 4 and 5 in hand, the proof of Corollary 1 is trivial.

Proof of Corollary 1. Imagine there is some labor income uncertainty that is imperfectly corellated with interest rate uncertainty in period t+1. Then Lemma 4 establishes that the consumption rule in period t is strictly concave, and recursion using Lemma 5 establishes strict concavity of the consumption rule in all periods before period t.

The second corollary addresses the case where utility is of the Constant Absolute Risk Aversion form.

Corollary 2 *For utility functions in the HARA class with $k = 1$ (i.e. the CARA utility function), for any permissible income process, the optimal consumption rule is strictly concave, i.e. $c_t^{*''}(w_t) < 0$, whenever the future interest rate is to any degree uncertain.*

Again we illustrate the point using the utility function in the last period of life. Write the CARA utility function as $u(c) = \frac{e^{-\alpha c}}{-\alpha}$. For interest rate uncertainty to leave the Φ matrix unchanged, equation 26 must again hold, which, after a bit of algebra, implies that $1/R = 2/R$, which is again impossible.

The proof of Corollary 2 is the same as the proof of Corollary 1.

The final Corollary just combines Corollaries 1 and 2.

Corollary 3 *For utility functions in the HARA class with $k \geq 1$, for any permissible income process, the optimal consumption rule is strictly concave, i.e. $c_t^{*''}(w_t) < 0$, whenever future labor income and the future interest rate are both uncertain, and are not perfectly correlated with each other.*

The proof is obvious.[4]

3 Conclusion

Many economists from at least Keynes on have had the intuition that the consumption function is concave, with a marginal propensity to consume lower for rich consumers than for poor consumers. One of the most striking features of Zeldes's (1989) numerical solutions for the consumption rule under uncertainty is the introduction of concavity to consumption rules that were linear in the absence of uncertainty. Carroll (1992; 1993) shows how important the curvature of the consumption function can be for reasoning about the behavior of consumption under uncertainty. This paper provides a solid analytical basis for concavity of the consumption rule under uncertainty.

[4]The proofs can actually be extended to the range where $0 < k < 1$, except for the complications induced by the existence of bliss or inflection points at some upper level of consumption. It is also worth noting in this context that the reason consumption is unaffected by uncertainty when utility is quadratic is because $\sqrt{k} = 0$, making the off-diagonal terms of Φ zero, while $u'''(c) = 0$, making the bottom right term zero.

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