Π=P/R	Figure AA4
Experiment B, Π=0	Figure AA5
Description of Experiment C	
Empirical sizes of nominal .01 tests, experiment C	Table AA1
Empirical sizes of nominal .05 tests, experiment C	Table AA2
Empirical sizes of nominal .10 tests, experiment C	Table AA3

II. Proofs:

Lemma A3(b)
Lemma A6
Lemma 4.3 (additional detail)
Proposition 7.2
Proposition 7.1
Theorem 5.1

N.B. The order of the proofs in this additional appendix is not the same as the order of the statements in the paper itself.

III. Illustration of footnote 9 statement about forecast encompassing

IV. Additional references

Description of Experiment C:

The DGP was obtained by solving for the reduced form in Schmidt (1974):

$$y_{1t} = c_1 + \phi_{11}y_{1t-1} + \phi_{12}y_{2t-1} + v_{1t},$$

$$y_{2t} = c_2 + \phi_{21}y_{1t-1} + \phi_{22}y_{2t-1} + v_{2t},$$

$$c_1 = c_2 = 0;\ \phi_{11} = 0.6250,\ \phi_{12} = 0.1250,\ \phi_{21} = 0.6250,\ \phi_{22} = 0.6250;$$

$$v_t \equiv (v_{1t}, v_{2t})' \sim N(0, \Sigma_v);\ \Sigma_v \equiv [\sigma_{ij}];\ \sigma_{11} = 1.9375,\ \sigma_{12} = 2.8125,\ \sigma_{22} = 4.6875.$$

An initial observation on $y_t \equiv (y_{1t}, y_{2t})'$ was drawn from its unconditional normal distribution, and then 200 additional observations were generated using the above. Ordinary least squares was used to estimate the y_{1t} equation, and, for one step ahead predictions, the hypothesis tests performed for experiment A were also performed here.

Table AA1

Size of Nominal .01 Tests, Experiment C

A. Mean Prediction Error

R	P				
	25	50	100	150	175
25	.021	.019	.012	.013	.014
50	.015	.016	.012	.012	
100	.009	.017	.013		

B. Mean Squared Prediction Error

R	P				
	25	50	100	150	175
25	.035	.029	.016	.018	.011
50	.060	.028	.023	.014	
100	.057	.037	.016		

C. First Order Serial Correlation

R	P				
	25	50	100	150	175
25	.008	.012	.018	.016	.015
50	.005	.013	.012	.011	
100	.006	.007	.004		

Table AA2

Size of Nominal .05 Tests, Experiment C

A. Mean Prediction Error

R	P				
	25	50	100	150	175
25	.101	.085	.071	.074	.068
50	.073	.068	.072	.066	
100	.056	.070	.074		

B. Mean Squared Prediction Error

R	P				
	25	50	100	150	175
25	.097	.079	.054	.056	.049
50	.114	.077	.069	.052	
100	.100	.079	.049		

C. First Order Serial Correlation

R	P				
	25	50	100	150	175
25	.045	.064	.072	.067	.077
50	.044	.066	.058	.072	
100	.031	.048	.057		

Table AA3

Size of Nominal .10 Tests, Experiment C

A. Mean Prediction Error

R	P				
	25	50	100	150	175
25	.175	.162	.141	.142	.134
50	.143	.137	.135	.125	
100	.128	.132	.127		

B. Mean Squared Prediction Error

R	P				
	25	50	100	150	175
25	.154	.137	.111	.102	.107
50	.161	.132	.114	.101	
100	.159	.136	.089		

C. First Order Serial Correlation

R	P				
	25	50	100	150	175
25	.098	.127	.130	.130	.146
50	.111	.113	.106	.118	
100	.078	.090	.116		

II. Proofs

Proof of Lemma A3(b): For notational simplicity, set L=0, redefine v_t to be what in the statement of the lemma is $v_t - Ev_t$, and assume that the zero mean process $(v_t, u_t)'$ is fourth order stationary. Let

$$\gamma_j \equiv Ev_t u_{t-j},$$

$$\kappa(i,j,n) = Ev_t v_{t-i} u_{t-j} u_{t-n} - Ev_t v_{t-i} Eu_{t-j} u_{t-n} - \gamma_j \gamma_{n-i} - \gamma_n \gamma_{j-i}.$$

From Andrews (1991), the mixing condition suffices to insure that the fourth cumulant κ satisfies

$$\Sigma\Sigma\Sigma_{i,j,n=-\infty}^{\infty} |\kappa(i,j,n)| < \infty.$$

Let $C_T \equiv P^{-1/2}\Sigma v_t U(t)$. I will show (i) $\lim EC_T = 0$, (ii) $\lim \mathrm{var}(C_T) = 0$, and the desired result will follow.

$$\begin{aligned}
\text{(i)} \quad |EC_T| &= P^{-1/2} \left| [R^{-1}(\gamma_0 + \ldots + \gamma_{R-1}) + \ldots + (R+P-1)^{-1}(\gamma_0 + \ldots + \gamma_{R+P-2})] \right| \\
&= P^{-1/2} \left| \{[(R^{-1} + \ldots + (R+P-1)^{-1}](\gamma_0 + \ldots + \gamma_{R-1}) + \right. \\
&\qquad \left. [(R+1)^{-1} + \ldots + (R+P-1)^{-1}]\gamma_{R+1} + \ldots + (R+P-1)^{-1}\gamma_{R+P-2}\} \right| \\
&\leq P^{-1/2} \{(P/R)(|\gamma_0| + \ldots + |\gamma_{R-1}|) + \\
&\qquad [(P-1)/R]|\gamma_{R+1}| + \ldots + (1/R)|\gamma_{R+P-2}|\} \\
&\leq P^{-1/2}(P/R)(|\gamma_0| + \ldots + |\gamma_{R+P-2}|) \leq P^{-1/2}(P/R)\Sigma_{j=0}^{\infty}|\gamma_j| \to 0
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad & \mathrm{var}(C_T) \\
&= P^{-1}\Sigma_{s=R}^{R+P-1}\Sigma_{i=1}^{s}\Sigma_{t=R}^{R+P-1}\Sigma_{j=1}^{t}\mathrm{cov}(s^{-1}v_s u_i, t^{-1}v_t u_j) \\
&= C_{1T} + C_{2T} + C_{3T},
\end{aligned}$$

$$C_{1T} \equiv P^{-1}\Sigma_{s=R}^{R+P-1}\Sigma_{i=1}^{s}\Sigma_{t=R}^{R+P-1}\Sigma_{j=1}^{t}[s^{-1}t^{-1}\gamma_{t-i}\gamma_{s-j}],$$

$$C_{2T} \equiv P^{-1}\Sigma_{s=R}^{R+P-1}\Sigma_{i=1}^{s}\Sigma_{t=R}^{R+P-1}\Sigma_{j=1}^{t}[s^{-1}t^{-1}(Ev_s v_t)(Eu_i u_j)],$$

$$C_{3T} \equiv P^{-1}\Sigma_{s=R}^{R+P-1}\Sigma_{i=1}^{s}\Sigma_{t=R}^{R+P-1}\Sigma_{j=1}^{t}[s^{-1}t^{-1}\kappa(t-s,t-j,t-i)].$$

Note that $|C_{1T}| \leq P^{-1}R^{-2}\Sigma\Sigma\Sigma\Sigma|\gamma_{t-i}\gamma_{s-j}|$. There are at most R+P−1 occurrences of a given pair of values of t−i and s−j ==> $|C_{1T}| \leq P^{-1}[(R+P-1)/R]^2\Sigma\Sigma_{i,j=-\infty}^{\infty}|\gamma_i\gamma_j| \leq P^{-1}[(R+P-1)/R]^2(\Sigma_{i=-\infty}^{\infty}|\gamma_i|)(\Sigma_{j=-\infty}^{\infty}|\gamma_j|) \to 0$. A similar argument shows $C_{2T} \to 0$. In C_{3T}, there are at most R+P−1 occurrences of a given triple of values of t−s, t−j and

$t-i \Rightarrow |C_{3T}| \leq P^{-1}[(R+P-1)/R]^2 \Sigma\Sigma\Sigma_{i,j,n=-\infty}^{\infty} |K(i,j,n)| \to 0.$

Proof of Lemma A6: When $\Pi=0$, the result may be established as in the first part of the proof of Lemma A5. I will sketch the argument when $\Pi \neq 0$. For notational simplicity, in this proof only, let

$$u_{t+1} = f_{t+\tau} - Ef_t,$$

assume q=1, and let

$$\gamma_j = Eu_t h_{t-j} \Rightarrow S_{fh} = \Sigma_{j=-\infty}^{\infty} \gamma_j.$$

The aim, then, is to show that

$$P^{-1}E(u_{R+1}+\ldots+u_{R+P})\Sigma H(t) \equiv B_1 + B_2 \to [1-\Pi^{-1}\ln(1+\Pi)]\Sigma_{j=-\infty}^{\infty}\gamma_j,$$

$$B_1 = P^{-1}a_{R,0}E(u_{R+1}+\ldots+u_{R+P})(h_1+\ldots+h_R),$$

$$B_2 = P^{-1}E(u_{R+1}+\ldots+u_{R+P})(a_{R,1}h_{R+1} + \ldots + a_{R,P-1}h_{R+P-1}),$$

where $a_{R,j}$ is defined as in the proof of Lemma A5. I will show that $B_1 \to 0$, $B_2 \to [1-\Pi^{-1}\ln(1+\Pi)]\Sigma_{j=-\infty}^{\infty}\gamma_j$.

B_1: We have

$$E(u_{R+1}+\ldots+u_{R+P})(h_1+\ldots+h_R) =$$

$$\gamma_1 + \gamma_2 + \ldots + \gamma_R$$

$$+ \gamma_2 + \ldots + \gamma_R + \gamma_{R+1}$$

$$+ \ldots + \gamma_P + \gamma_{P+1} + \ldots + \gamma_{R+P-1}$$

$\equiv B_{11}$ if $P \leq R$, B_{12} if $P>R$,

$$B_{11} = \gamma_1 + 2\gamma_2 + \ldots + P(\gamma_P+\gamma_{P+1}+\ldots+\gamma_R) + (P-1)\gamma_{R+1} + \ldots + \gamma_{R+P-1},$$

$$B_{12} = \gamma_1 + 2\gamma_2 + \ldots + R(\gamma_R+\gamma_{R+1}+\ldots+\gamma_P) + (P-1)\gamma_{P+1} + \ldots + \gamma_{R+P-1}.$$

Consider the case where $P \leq R$ for all T sufficiently large; the case where $P>R$ for all T sufficiently large may be handled similarly. The case where P fluctuates above and below R indefinitely ($\Rightarrow \Pi=1$) follows from the first two cases. We have

$$P^{-1}a_{R,0}B_{11} = a_{R,0}\Sigma_{j=1}^{P}(j/P)\gamma_j + a_{R,0}\Sigma_{j=P+1}^{R}\gamma_j + a_{R,0}\Sigma_{j=1}^{P-1}[(P-j)/P]\gamma_{R+j}.$$

The first two terms go to zero since $a_{R,0}$ is bounded and $\Sigma_{j=0}^{\infty}|\gamma_j| < \infty$. As for the last term:

$$a_{R,0}\left|\Sigma_{j=1}^{P-1}[(P-j)/P]\gamma_{R+j}\right| \le a_{R,0}[(R+P-1)/P]\Sigma_{j=1}^{P-1}[(P-j)/(R+P-1)]\left|\gamma_{R+j}\right|$$
$$\le a_{R,0}[(R+P-1)/P]\Sigma_{j=1}^{P-1}[(R+j)/(R+P-1)]\left|\gamma_{R+j}\right| \to 0$$

since $\Sigma_{j=0}^{\infty}|\gamma_j| < \infty$ and it was shown in the proof of Lemma A5 that $a_{R,0}[(R+P-1)/P]$ is bounded.

B_2: Consider the case where $P \le R$ for all T sufficiently large; once again, other cases may be handled similarly. We have

$$E(u_{R+1}+\ldots+u_{R+P})(a_{R,1}h_{R+1} + \ldots + a_{R,P-1}h_{R+P-1}) =$$
$$c_0\Sigma_{j=-P+2}^{P-1}\gamma_j + \Sigma_{j=1}^{P-1}(c_j-c_0)\gamma_j + \Sigma_{j=-P+2}^{0}(c_j-c_0)\gamma_j,$$
$$c_0 = a_{R,1} + \ldots + a_{R,P-1},$$
$$c_1 = a_{R,1} + \ldots + a_{R,P-1},\ c_2 = a_{R,1} + \ldots + a_{R,P-2},\ \ldots,\ c_{P-1} = a_{R,1},$$
$$c_{-1} = a_{R,2} + \ldots + a_{R,P-1},\ c_{-2} = a_{R,3} + \ldots + a_{R,P-1},\ \ldots,\ c_{-P+2} = a_{R,P-1}.$$

By logic such as that in the proof of Lemma A5,

$$P^{-1}c_0 \sim P^{-1}\int_1^{P-1}[\int_y^{P-1}(R+x)^{-1}dx]dy \to [1-\Pi^{-1}\ln(1+\Pi)].$$

where "~" means "is arbitrarily close to for sufficiently large T." The desired result follows if $P^{-1}\Sigma_{j=1}^{P-1}(c_j-c_0)\gamma_j \to 0$, $P^{-1}\Sigma_{j=-P+2}^{0}(c_j-c_0)\gamma_j \to 0$. I will show the first; the second may be shown similarly.

By the definitions of c_j for $j>0$, of $a_{R,j}$, and because $R^{-1} > (R+P-j)^{-1}$ for $0 \le j < P$,

$$|c_j-c_0| \le R^{-1}[1+2+\ldots+(j-1)] = .5R^{-1}j(j-1)$$

Then

$$\left|P^{-1}\Sigma_{j=1}^{P-1}(c_j-c_0)\gamma_j\right| \le .5(RP)^{-1}\Sigma_{j=1}^{P-1}j(j-1)\left|\gamma_j\right| \le .5P^{-1}\Sigma_{j=1}^{P-1}j\left|\gamma_j\right| \to 0.$$

Additional detail on proof of Lemma 4.3: To show $V(\Pi)$ is positive definite: define the following $(\ell+q)x(\ell+q)$ matrices:

$$S = \begin{pmatrix} S_{ff} & S_{fh} \\ S_{fh}' & S_{hh} \end{pmatrix}, \quad A_1 = \begin{pmatrix} I_\ell & 0 \\ 0 & \Pi I_q \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & (\Pi-\Pi^2)S_{hh} \end{pmatrix}.$$

Then $V(\Pi) = A_1SA_1' + A_2$. By assumption, S is positive definite; since $0\le\Pi\le1$, $\Pi^2<\Pi$ so A_2 is positive semidefinite.

Proof of Proposition 7.2: Note: the matrix J, which will be defined more than once below, means different things in different parts of the proof.

Let $Z_t = (1\ Z_{1t}')'$, where Z_{1t} is $(r-1)x1$. It is convenient as well to:

(1)Order the elements of β^* so that the coefficients on the constant terms come first, $\beta^* \equiv (c', \beta_1')'$. (c and β_1 are defined in the Appendix "Formal statement of assumptions" for Proposition 7.2.)

(2)Order the elements in h_t according to $h_t=Z_t\otimes u_t$ (rather than $h_t=u_t\otimes Z_t$). Then $h_t = (u_t', Z_{1t}'\otimes u_t')'$, $H(t) = (U(t)', t^{-1}\Sigma Z_{1s}'\otimes u_s')'$, $B = -(EZ_t'\otimes G_t'WEZ_t\otimes G_t)^{-1}EZ_t'\otimes G_t'W$, and condition (ii) in the formal statement of assumptions becomes

(ii)$W^{-1}=\Sigma_{j=-n}^{n}[(EZ_tZ_t')\otimes(Eu_tu_t')]$.

7.2(a) Here, $F = EG_t$. If (iii) holds, so that $k=\ell r$, then the equation is exactly identified and $B=(EZ_t\otimes G_t)^{-1}$ for any p.d. W. Given that U(t) comprises the first ℓ elements of H(t), it is necessary to show

$$\underset{\ell xk}{EG_t}\underset{kxk}{(EZ_t\otimes G_t)^{-1}} = \underset{\ell x(r-1)\ell}{(I_\ell \quad 0)}$$

$$\Longleftrightarrow EG_t = (I_\ell\ 0)(EZ_t\otimes G_t)$$

which holds since the first element of Z_t is the constant term.

Now let condition (i) or (ii) hold. For g_{1t} defined in the formal statement of assumptions, let G_{1t} be the $\ell x(k-\ell)$ matrix of derivatives of g_{1t} with respect to the $k-\ell$ elements of β_1. Since c, the vector of coefficients on the constant term, is unrestricted,

$$G_t = \underset{\ell xk}{(I_\ell} \quad \underset{\ell x(k-\ell)}{G_{1t}})$$

==>

$$\text{(AA-1)} \qquad \underset{kx\ell r}{Z_t'\otimes G_t'} = \begin{pmatrix} Z_t'\otimes I_\ell \\ \\ Z_t'\otimes G_{1t}' \end{pmatrix}$$

For n defined in the formal statement of assumptions in the Appendix, let

$$J = \Sigma_{j=-n}^{n} E u_t u_{t-j}',$$

so that J is the $(\ell x \ell)$ upper left hand block in W^{-1}. Below, I will show that

$$\text{(AA-2)} \qquad (EZ_t'\otimes I_\ell)W = \underset{\ell x(r-1)\ell}{(J^{-1} \quad 0)},$$

$$\text{(AA-3)} \qquad EG_t(EZ_t'\otimes G_t'WEZ_t\otimes G_t)^{-1} = \underset{\ell x(k-\ell)}{(J \quad 0)}.$$

From (AA-1), (AA-2) and (AA-3), it follows that

$$\text{(AA-4)} \qquad EG_t(EZ_t'\otimes G_t'WEZ_t\otimes G_t)^{-1}EZ_t'\otimes G_t'W = \underset{\ell x(k-\ell)}{(J \quad 0)}EZ_t'\otimes G_t'W$$

$$= J(EZ_t'\otimes I_\ell)W$$

$$= \underset{\ell x(r-1)\ell}{(I_\ell \quad 0)},$$

and the desired result follows.

Proof of (AA-2): Let $Z_t' = (1\ z_{2t}\ \ldots\ z_{rt})$. Postmultiply both sides of (AA-2) by W^{-1}. Given the $\ell x(r-1)\ell$ matrix of zeroes on the right hand side of (AA-1), what results is: J^{-1} times the product of the first ℓ rows of W^{-1}. By assumption (i) or (ii), this product is

$$J^{-1}(\Sigma_{j=-n}^{n} E u_t u_{t-j}' \quad \Sigma_{j=-n}^{n} E z_{2t} E u_t u_{t-j}' \quad \ldots \quad \Sigma_{j=-n}^{n} E z_{rt} E u_t u_{t-j}')$$

which indeed reduces to $EZ_t' \otimes I_\ell$ given that $J^{-1} \equiv (\Sigma_{j=-n}^{n} E u_t u_{t-j}')^{-1}$.

Proof of (AA-3): Postmultiply both sides of (AA-3) by $EZ_t' \otimes G_t' W E Z_t \otimes G_t$. The right hand side becomes

$$
\begin{aligned}
&(J \quad 0_{\ell x (k-\ell)}) E Z_t' \otimes G_t' W E Z_t \otimes G_t \\
&= J (E Z_t' \otimes I_\ell) W E Z_t \otimes G_t \\
&= J (J^{-1} \quad 0_{\ell x (r-1)\ell}) E Z_t \otimes G_t \\
&= (I \quad 0_{\ell x (r-1)\ell}) E Z_t \otimes G_t \\
&= E G_t,
\end{aligned}
$$

where the second equality uses AA-2, the last the fact that the first element of Z_t is the constant term.

7.2(b.i) Write the structural equations as

$$y_t = X_t' \beta^* + u_t,$$

where X_t' is $\ell x k$. (To understand the notation, it may be helpful to note that since the constant terms appear first in β^*, the first ℓ columns of X_t' contain the $(\ell x \ell)$ identity matrix.) Let X_{it}' be the $1 x k$ i'th row of X_t'. Since $v_t = C^{-1} u_t$ it is necessary to show that $-FB = (C^{-1} \quad 0)$.

To show this: Since f_t is a column vector, $f_t = vec(f_t') \equiv vec[y_t' - Z_t'(C^{-1}D)'] = vec(y_t) - vec[Z_t'(C^{-1}D)'] = vec(y_t) - (I_\ell \otimes Z_t') vec[(C^{-1}D)'] \Rightarrow F = -(I_\ell \otimes E Z_t') \partial\{vec[(C^{-1}D)']\}/\partial\beta$. Goldberger et al. (1961) shows that $\partial\{vec[(C^{-1}D)']\}/\partial\beta = (C^{-1} \otimes I_r) Q$, where

$$
\text{(AA-5a)} \qquad Q = \begin{pmatrix} ((EZ_tZ_t')^{-1} EZ_t X_{1t}') \\ (\ldots) \\ ((EZ_tZ_t')^{-1} EZ_t X_{\ell t}') \end{pmatrix}.
$$

(See Hausman (1983,p418).) This means that $-F = (I_\ell \otimes E Z_t')(C^{-1} \otimes I_r) Q = (C^{-1} \otimes E Z_t') Q$. Given that the first element of Z_t is a constant term, it is straightforward to show that $EZ_t'(EZ_tZ_t')^{-1} EZ_t X_{it}' = E X_{it}'$. Let $[c^{ij}] = C^{-1}$. Routine manipulations then

yield

$$(AA\text{-}5b)\quad (C^{-1}\otimes EZ_t')QB = \begin{matrix}(c^{11}EZ_t' & \dots & c^{1\ell}EZ_t')((EZ_tZ_t')^{-1}EZ_tX_{1t}')\\ (\dots & & \dots)(\dots)\\ (c^{\ell 1}EZ_t' & \dots & c^{\ell\ell}EZ_t')((EZ_tZ_t')^{-1}EZ_tX_{\ell t}')\end{matrix}$$

$$= \begin{matrix}(c^{11}EX_{1t}'+\dots+c^{1\ell}EX_{1t}')\\ (\dots)\\ (c^{\ell 1}EX_{1t}'+\dots+c^{\ell\ell}EX_{\ell t})\end{matrix} B = C^{-1}EX_t'B = (C^{-1}\ 0),$$

the final equality following from (AA-4).

7.2(b.ii) Define

$$N \equiv \begin{matrix}(1 & 0)\\ (c & \Phi)\end{matrix}$$

$$J' = \underset{\ell x(\ell+1)}{(0\ I_\ell)}\quad \ell x1$$

$$\Rightarrow\quad J'Z_{t+1} = y_t.$$

where the "0" in the definition of N is a 1xℓ vector of zeroes. Note that $Z_{t+1}=NZ_t+(0,u_t')'$ ==> the population prediction of $y_{t+\tau}$ is $J'N^\tau Z_{t+1}$ (= $(I+\Phi+\dots+\Phi^{\tau-1})c+\Phi^\tau y_t$). Thus, $vec(f_{t+\tau}) = vec(f_{t+\tau}') = vec(y_{t+\tau}') - vec(Z_{t+1}'N^{\tau\prime}J) = vec(y_{t+\tau}) - (J'\otimes Z_{t+1}')vec(N^{\tau\prime})$ ==> $F = -(J'\otimes EZ_t')[\partial vec(N^{\tau\prime})/\partial\beta]$. Note that $vec(N'J) = (J'\otimes I_{\ell+1})vec(N')$, and, since $J'N = (C^{-1}D)'$, $[\partial vec(N'J)/\partial\beta] = (C^{-1}\otimes I_{\ell+1})Q$, where Q is defined in (AA-5a) ==> $F = -(J'\otimes EZ_t')[\partial vec(N^{\tau\prime})/\partial vec(N')](J\otimes I_{\ell+1})(C^{-1}\otimes I_{\ell+1})Q$. Finally, Schmidt (1973) shows that $\partial vec(N^{\tau\prime})/\partial vec(N') = \Sigma_{j=0}^{\tau}(N^j\otimes N^{\tau-1-j\prime})$ ==>

$$FB\Sigma H(t) = -\Sigma_{j=0}^{\tau}[J'N^jJC^{-1}\otimes EZ_t'N^{\tau-1-j\prime}']QB\Sigma H(t).$$

Note that $Z_{t+1} = NZ_t+(0,u_t')'$ ==> $EZ_t=NEZ_t$ ==> $EZ_t'N^{\tau-1-j\prime} = EZ_t'$. The right hand side of the above, then, is

$$-\Sigma_{j=0}^{\tau}(J'N^jJC^{-1}\otimes EZ_t')QB\Sigma H(t)$$

$$= -\Sigma_{j=0}^{\tau}(J'N^jJ\otimes 1)(C^{-1}\otimes EZ_t')QB\Sigma H(t)$$

$$= -\Sigma_{j=0}^{\tau}(J'N^jJ)[C^{-1}\Sigma U(t)]$$

$$\equiv -\Sigma_{j=0}^{\tau}(J'N^jJ)[C^{-1}(\Sigma_t t^{-1}\Sigma_s u_s)]$$

$$= -(\Sigma_{j=0}^{\tau}\Phi^j)[C^{-1}(\Sigma_t t^{-1}\Sigma_s u_s)]$$

$$= - \Sigma_t t^{-1}\Sigma_s(\Sigma_0^{\tau-1}\Phi^j C^{-1}u_s).$$

where the second equality uses AA-5b, the next to last the definitions of J and N. Let $V(t) \equiv t^{-1}\Sigma_{s=1}^{t}v_s$. Note that $v_s = C^{-1}u_s+\Phi C^{-1}u_{s-1}+...+\Phi^{\tau-1}C^{-1}u_{s-\tau+1}$. It is easily shown that $P^{-1/2}\Sigma_t t^{-1}\Sigma_s(\Sigma_0^{\tau-1}\Phi^j C^{-1}u_s) = P^{-1/2}\Sigma V(t) + o_p(1)$, and the desired result follows.

Proof of Proposition 7.1: In light of the discussion of Theorem 6.1, this result would be immediate if were known that for in-sample inference about autocorrelation under the conditions of the Proposition, the middle term in the definition of $\tilde{\Omega}$ in Theorem 6.1 is exactly -2 times the third and final term. Unfortunately, while that result is no doubt implicit in papers such as Pagan and Hall (1983) and Cumby and Huizanga (1992), it does not appear to be explicit, and this proof does nothing but tediously grind through the requisite algebra.

Since $B=V_\beta(EX_t\otimes Z_t')W$, the condition given in the sketch of the proof in the Appendix may be rewritten

$$Ev_{1t-1}Z_t'MV_\beta(EX_t\otimes Z_t')W(Eu_tv_{1t}\otimes EZ_tv_{1t-1}) = Ev_{1t-1}Z_t'MV_\beta M'EZ_tv_{1t-1}$$

sufficient for which is

$$(EX_t\otimes Z_t')W(Eu_tv_{1t}\otimes EZ_tv_{1t-1}) = M'EZ_tv_{1t-1} \qquad \text{(AA-6)}$$

Write the $(k_j x 1)$ vector of r.h.s. variables in the j'th equation as x_{jt}, $k = \Sigma_{j=1}^{\ell}k_j$ ==> $EX_t\otimes Z_t'$ is block diagonal with j'th block $Ex_{jt}Z_t'$. Let $C^{-1} = [c^{ij}]$. Define the $(\ell x 1)$ vector e_1 as $e_1 = (1\ 0\ ...\ 0)'$ ==> $v_{1t} = v_t'e_1 = u_t'C^{-1}'e_1$ ==> the l.h.s. of (AA-6) is

$$(EX_t\otimes Z_t')[(Eu_tu_t')^{-1}\otimes(EZ_tZ_t')^{-1}](Eu_tu_t'C^{-1}'e_1\otimes EZ_tv_{1t-1})$$
$$= (EX_t\otimes Z_t')\{C^{-1}'e_1\otimes[(EZ_tZ_t')^{-1}EZ_tv_{1t-1}]\}$$

$$= \begin{pmatrix} (c^{11}Ex_{1t}Z_t'(EZ_tZ_t')^{-1}EZ_tv_{1t-1} \\ \ldots \\ (c^{1\ell}Ex_{\ell t}Z_t'(EZ_tZ_t')^{-1}EZ_tv_{1t-1} \end{pmatrix}$$

$$= M'EZ_tv_{1t-1},$$

the final equality following from Goldberger et al. (1961).

Proof of Theorem 5.1: (a) Consider $\hat{\Gamma}_{ff}(0) \equiv P^{-1}\Sigma[f_{t+\tau}(\hat{b}_t)-\bar{f}]^2$; other autocovariances may be handled similarly. Expand $f_{t+\tau}(\hat{b}_t)$ around $f_{t+\tau}(b^*)$:

$$f_{t+\tau}(\hat{b}_t)-\bar{f} = \phi_{t+\tau} + r_{t+\tau},$$

$$\phi_{t+\tau} \equiv f_{t+\tau}(b^*)-Ef_t, \quad r_{t+\tau} \equiv (Ef_t-\bar{f}) + [f_{t+\tau,b}(\hat{b}_t-b^*)] + w_{t+\tau}. \tag{AA-7}$$

==>

$$\hat{\Gamma}_{ff}(0) = P^{-1}\Sigma\phi_{t+\tau}^2 + 2P^{-1}\Sigma[\phi_{t+\tau}r_{t+\tau}] + P^{-1}\Sigma r_{t+\tau}^2 \tag{AA-8}$$

(Recall that I am assuming that ℓ=1 for notational simplicity.) The first term on the right hand side of (AA-8) converges in probability to $\Gamma_{ff}(0)$ by White (1984, Corollary 3.48). The second term on the r.h.s. of (AA-8) is 2 times

$$(Ef_t-\bar{f})P^{-1}\Sigma\phi_{t+\tau} + P^{-1}\Sigma[\phi_{t+\tau}f_{t+\tau,b}(\hat{b}_t-b^*)] + P^{-1}\Sigma(\phi_{t+\tau}w_{t+\tau}) \tag{AA-9}$$

The first term on the r.h.s. of (AA-9) is $o_p(1)$ by Theorem 4.2 and White (1984, Corollary 3.48). The absolute value of the second term on the r.h.s. of (AA-9) is bounded above by

$$(L+1)\ \sup_t|\hat{b}_t-b^*|\ (P^{-1}\Sigma\phi_{t+\tau}^2)^{1/2}\ (P^{-1}\Sigma|f_{t+\tau,b}|^2)^{1/2},$$

which is $o_p(1)$ because the first term is $o_p(1)$ by Lemma A2(c), the second and third are $O_p(1)$ by Markov's inequality. Finally, the third term on the r.h.s. of (AA-8) is

$$P^{-1}(Ef_t-\bar{f})^2 + 2(Ef_t-\bar{f})P^{-1}\Sigma[f_{t+\tau,b}(\hat{b}_t-b^*)] + 2(Ef_t-\bar{f})P^{-1}\Sigma w_{t+\tau}$$
$$+ \{P^{-1}\Sigma[f_{t+\tau,b}(\hat{b}_t-b^*)]\}^2 + 2P^{-1}\Sigma[f_{t+\tau,b}(\hat{b}_t-b^*)]w_{t+\tau} + P^{-1}\Sigma w_{t+\tau}^2.$$

That each of the six terms converges in probability to zero follows from Theorem

4.2, Lemma A2(c), and Assumption 2(b). For example, since $w_{t+\tau}$ = $.5(\hat{b}_t-b^*)'(\partial^2 f_{t+\tau}(\tilde{b}_t)/\partial b\partial b')(\hat{b}_t-b^*)$ for some $\tilde{b}_t$ between $\hat{b}_t$ and b^*

$$|P^{-1}\Sigma[f_{t+\tau,b}(\hat{b}_t-b^*)]w_{t+\tau}| \leq$$
$$(L+1)^3 \sup_t|\hat{b}_t-b^*|^3 \ (P^{-1}\Sigma|f_{t+\tau,b}|^2)^{1/2} \ (P^{-1}\Sigma|(\partial^2 f_{t+\tau}(\tilde{b}_t)/\partial b\partial b')|^2)^{1/2},$$

the first term of which is $o_p(1)$ by Theorem 4.2, the next two of which are $O_p(1)$ by Assumption 2(b) and Markov's inequality.

5.1(b) The result follows if $\hat{S}_{hh} \xrightarrow{p} S_{hh}$, $\hat{S}_{ff} \xrightarrow{p} S_{ff}$ and $\hat{S}_{fh} \xrightarrow{p} S_{fh}$. That $\hat{S}_{hh} \xrightarrow{p} S_{hh}$ follows from Andrews (1991). I will sketch the proof that $\hat{S}_{ff} \xrightarrow{p} S_{ff}$; that $\hat{S}_{fh} \xrightarrow{p} S_{fh}$ follows from similar logic.

Let $k_j \equiv k(j/M)$, suppressing for simplicity dependence of k_j on M and thus P. I will show that $\Sigma_{j=0}^{P-1-L}k_j\hat{\Gamma}_{ff}(j) \xrightarrow{p} \Sigma_{j=0}^{\infty}\Gamma_{ff}(j)$, from which it will also follow that $\Sigma_{j=1}^{P-1-L}k_{-j}\hat{\Gamma}_{ff}(-j) \xrightarrow{p} \Sigma_{j=1}^{\infty}\Gamma_{ff}(-j)$ and hence $\hat{S}_{ff} \xrightarrow{p} S_{ff}$. Notation: in this proof only,

$\Sigma_j \equiv \Sigma_{j=0}^{P-1-L}$.

Define $\phi_{t+\tau}$ and $r_{t+\tau}$ as in (AA-7). An expansion such as in (AA-8) then yields

$$\Sigma_j k_j\hat{\Gamma}_{ff}(j) = \Sigma_j k_j\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}\phi_{t+\tau-j}]\} + \Sigma_j k_j\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}r_{t+\tau-j}]\}$$
$$+ \Sigma_j k_j\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau-j}r_{t+\tau}]\} + \Sigma_j k_j\{P^{-1}\Sigma_{t=R+j+L}^{T}[r_{t+\tau}r_{t+\tau-j}]\}$$

It follows from Andrews (1991) that $\Sigma_j k_j\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}\phi_{t+\tau-j}]\} \xrightarrow{p} \Sigma_{j=0}^{\infty}\Gamma_{ff}(j)$, so it suffices to show that the other three double summations converge in probability to zero. I will show this for the second double summation; the argument for the third and fourth double summations are similar.

Note that

$$|P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}f_{t+\tau-j,b}]| \leq (P^{-1}\Sigma_{t=R+j+L}^{T}\phi_{t+\tau}^2)^{1/2}(P^{-1}\Sigma_{t=R+j+L}^{T}|f_{t+\tau-j,b}|^2)^{1/2}$$
$$\leq (P^{-1}\Sigma_{t=R}^{T}\phi_{t+\tau}^2)^{1/2}(P^{-1}\Sigma_{t=R}^{T}|f_{t+\tau,b}|^2)^{1/2},$$

and, similarly,

$$|P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}w_{t+\tau-j}]| =$$

$$.5\ \left|P^{-1}\Sigma_{t=R+j+L}^{T}\{\phi_{t+\tau}(\hat{b}_{t-j}-b^{*})'[\partial^{2}f_{t+\tau-j}(\tilde{b}_{t-j})/\partial b\partial b'](\hat{b}_{t-j}-b^{*})\}\right| \leq$$

$$.5\ \sup_{t}\left|\hat{b}_{t}-b^{*}\right|^{2}(L+1)^{2}(P^{-1}\Sigma_{t=R}^{T}\phi_{t+\tau}^{2})^{1/2}(P^{-1}\Sigma_{t=R}^{T}m_{t+\tau}^{2})^{1/2},$$

where m_t is defined in assumption 2. Let $\underline{a}$ be defined as in the statement of the Theorem. Then

$$\left|\Sigma_{j}k_{j}\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}r_{t+\tau-j}]\}\right| = \left|(Ef_{t}-\bar{f})\Sigma_{j}k_{j}[P^{-1}\Sigma_{t=R+j+L}^{T}\phi_{t+\tau}]\right.$$

$$\left. + \Sigma_{j}k_{j}\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}f_{t+\tau-j,b}(\hat{b}_{t}-b^{*})]\} + \left|\Sigma_{j}k_{j}\{P^{-1}\Sigma_{t=R+j+L}^{T}[\phi_{t+\tau}w_{t+\tau-j}]\}\right|\right.$$

$$\leq (M/P^{a})(M^{-1}\Sigma_{j}|k_{j}|)\{\ \left|P^{a}(Ef_{t}-\bar{f})\right|P^{-1}\Sigma_{t=R}^{T}|\phi_{t+\tau}| +$$

$$P^{a}\sup_{t}\left|\hat{b}_{t}-b^{*}\right|(L+1)(P^{-1}\Sigma_{t=R}^{T}\phi_{t+\tau}^{2})^{1/2}(P^{-1}\Sigma_{t=R}^{T}|f_{t+\tau,b}|^{2})^{1/2} +$$

$$.5\ P^{a}\sup_{t}\left|\hat{b}_{t}-b^{*}\right|^{2}(L+1)^{2}(P^{-1}\Sigma_{t=R}^{T}\phi_{t+\tau}^{2})^{1/2}(P^{-1}\Sigma_{t=R}^{T}m_{t+\tau}^{2})^{1/2}\ \ \}.$$

By Theorem 4.1 and Lemma A2(c), $|P^{a}(Ef_{t}-\bar{f})| \xrightarrow{p} 0$ and $P^{a}\sup_{t}|\hat{b}_{t}-b^{*}| \xrightarrow{p} 0$; by Markov's inequality each of the summations inside the braces is $O_p(1)$. By assumption, $M/P^{a} = O(1)$ and $M^{-1}\Sigma_{j}|k_{j}| \to \int_{0}^{\infty}|k(x)|dx < \infty$, so the desired result follows.

III. Illustration of footnote 9 statement about forecast encompassing

This documents the statement made in footnote 9, using the model (7.2). In the notation of (7.2), suppose that u_{1t} is a martingale difference with respect to past u_{1t} and u_{2t} and current and past X_{1t} and X_{2t}; that $EX_{2t}u_{2t}=0$; and that both models are estimated by OLS. Chong and Hendry (1986) recommend testing H_0: $Ef_t=0$ where $f_t=u_{1t}(X_{2t}'\beta_2)$. Then

$$F = (-\beta_2'EX_{2t}X_{1t}' \quad 0),$$

$$B = \begin{pmatrix} ((EX_{1t}X_{1t}')^{-1} & 0 \\ 0 & (EX_{2t}X_{2t}')^{-1} \end{pmatrix},$$

$$S_{ff} = E[u_{1t}^2(X_{2t}'\beta_2)^2],$$

$$S_{hh} = \begin{pmatrix} EX_{1t}X_{1t}'u_{1t}^2 & EX_{1t}X_{2t}'u_{1t}u_{2t} \\ EX_{2t}X_{1t}'u_{1t}u_{2t} & EX_{2t}X_{2t}'u_{2t}^2 \end{pmatrix},$$

$$S_{fh}' = \begin{pmatrix} EX_{1t}X_{2t}'u_{1t}^2\beta_2 \\ EX_{2t}X_{2t}'u_{1t}u_{2t}\beta_2 \end{pmatrix}.$$

Let $\hat{u}_{1t}=y_t-X_{1t}'\hat{\beta}_{1t-1}$, $\bar{f} = P^{-1}\sum_{t=R}^{R+P-1}\hat{u}_{1t+1}(X_{2t+1}'\hat{\beta}_{2t})$. From Theorem 4.1, the asymptotic variance of $P^{1/2}(\bar{f}-Ef_t) = P^{1/2}\bar{f}$ is

$$\Omega = E[u_{1t}^2(X_{2t}'\beta_2)^2] - 2\Pi\beta_2'(EX_{2t}X_{1t}')(EX_{1t}X_{1t}')^{-1}(EX_{1t}X_{2t}'u_{1t}^2)\beta_2 + 2\Pi\beta_2'(EX_{2t}X_{1t}')(EX_{1t}X_{1t}')^{-1}(EX_{1t}X_{1t}'u_{1t}^2)(EX_{1t}X_{1t}')^{-1}(EX_{1t}X_{2t}')\beta_2 \tag{AA-10}$$

Suppose first that $E(X_{1t}X_{2t}'u_{1t}^2) = EX_{1t}X_{2t}'Eu_{1t}^2$, $E(X_{1t}X_{1t}'u_{1t}^2) = EX_{1t}X_{1t}'Eu_{1t}^2$, sufficient for which is that u_{1t} is <u>independent</u> of X_{1t} and X_{2t}. It may be seen that Ω reduces to $E[u_{1t}^2(X_{2t}'\beta_2)^2]$ $(=S_{ff})$: sampling error in estimation of β^* is irrelevant asymptotically. Chong and Hendry's (1986) argument that such sampling error will have little affect on a test statistic that ignores such sampling error when (in my notation) R>>P is therefore to be expected as well even when R<<P as long as R and P are both large. As discussed in section 6, this is a case in which $(FBS_{fh}'+S_{fh}B'F') = -2FV_\beta F'$. So from Theorem 6.1 (or Davidson and

MacKinnon (1981,p787)), even with u_{it} iid the comparable in-sample test has asymptotic variance $\tilde{\Omega} = E[u_{1t}^2(X_{2t}'\beta_2)^2] - \beta_2'EX_{2t}X_{1t}'Eu_{1t}^2(EX_{1t}X_{1t}')^{-1}EX_{1t}X_{2t}'\beta_2$, and for this in-sample test it is necessary to account for sampling error in estimation of β^* (except in special cases such as when X_{1t} and X_{2t} are orthogonal).

But (AA-10) indicates that if u_{1t} is heteroskedastic conditional on X_{1t} and/or X_{2t}, so that $E(X_{1t}X_{2t}'u_{1t}^2) \neq EX_{1t}X_{2t}'Eu_{1t}^2$, $E(X_{1t}X_{1t}'u_{1t}^2) \neq EX_{1t}X_{1t}'Eu_{1t}^2$, sampling error in estimation of β^* is relevant asymptotically in general. Suppose that one tests the null by regressing $\hat{u}_{1t+1}$ on $X_{2t+1}'\hat{\beta}_{2t}$ and using a t-statistic computed from a heteroskedasticity consistent covariance matrix (Fair and Shiller (1990)). Note that this will not, in general, properly account for sampling error in estimation of β^*. Use of such a t-statistic rather than the usual OLS t-statistic will substitute an estimate of $E[u_{1t}^2(X_{2t}'\beta_2)^2]$ for an estimate of $Eu_{1t}^2E(X_{2t}'\beta_2)^2$. This is a step in the right direction. But the proper quantity is neither of these but an estimate of Ω.

On the other hand, suppose one tests whether the covariance between a given model's prediction error and its own prediction is zero (e.g., Pagan and Schwert (1990))--H_0: $Ef_t=0$ for $f_t \equiv u_t(X_t'\beta^*)$ in the model (7.1), for example. Then sampling error in estimation of β^* is in general irrelevant asymptotically, even with a conditionally heteroskedastic disturbance. (The proof follows from Lemmas 4.1 and A.7. Theorem 4.1 is not applicable since the relevant S is not positive definite.)

IV. Additional references

Hausman, Jerry A., 1983, "Specification and Estimation of Simultaneous Equations Models," 391-448 in Z. Griliches and M.D. Intriligator (eds.), Handbook of Econometrics Vol I., Amsterdam: North Holland.

White, Halbert, 1984, Asymptotic Theory for Econometricians, New York: Academic Press.