Very Preliminary
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Efficient Bilateral Risk Sharing without Commitment

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Abstract

This paper examines the properties of efficient sustainable allocations in an environment in which two agents want to share risk, have perfect information about each other, but cannot make commitments about future transfers. I describe as *sustainable* any allocation that can be supported as a subgame perfect equilibrium in a game in which individuals make simultaneous transfers. I consider the properties of *efficient* sustainable allocations.

There are three main findings. First, if some first best allocation is sustainable, then any efficient allocation must converge with probability one to a first best allocation. In the long run, the lack of commitment is irrelevant. Second, if no first best allocation is sustainable, then the unconditional probability distribution of an agent's utility converges weakly over time to a nondegenerate distribution. Finally, under any conditions, the conditional contemporaneous covariance of individual income and individual consumption in an efficient allocation is nonnegative; it is zero only if the allocation is first best.

0. Introduction

Commitment is an essential part of risksharing. Yet there are many examples of situations in which individuals share risk even though there is no outside agency forcing them to fulfill the terms of the arrangement. Historically, societies without formal asset markets used large amounts of reciprocal gift-giving to allow individuals to share risk. However, in most of these societies, there was no "gift police" that forced the transfers to take place. Similarly, in families or other groups, an individual who falls on hard times may be the recipient of transfers from other group members. However, this arrangement has no grounding in a contract that has explicit legal backing.

These examples share two common characteristics. First, it seems reasonable to assume that information is symmetric across the agents participating in the arrangement. In a small village, it seems unlikely that an individual could hide any prosperity from others; similarly, within a family, it is hard to falsely proclaim one's poverty. Second, the arrangement is sustainable only because any individual who is supposed to make a transfer believes that the future benefits of the arrangement exceed the current cost of the transfer.

These kinds of considerations make it natural to think about the properties of efficient allocations of risk in a society which faces no asymmetry of information and has no explicit technology of commitment. To address this issue, this paper considers an infinite horizon economy with two agents. Every period, the agents receive a random endowment of a perishable consumption good. There is no aggregate risk and the endowments are independently and identically distributed over time. Moreover, there is no private information problem: each agent knows the current and past realizations of both endowments. Every period, the agents simultaneously transfer a nonnegative amount of consumption to each other. Neither agent is able to commit to promises of future transfers.

I consider subgame perfect equilibria to this infinitely repeated game. I term *sustainable* any allocation of consumption that is supportable by a subgame perfect equilibrium collection of transfer strategies for the two agents (see Chari and Kehoe (1990) for a similar use of the term). I demonstrate that the worst possible sustainable allocation is autarky. Using techniques developed by Abreu (1988), I completely characterize the set of sustainable allocations by using autarky as a way of punishing deviations.

I then consider the properties of *efficient* sustainable allocations: that is, sustainable allocations that are not *ex-ante* Pareto dominated by any other sustainable allocation. I make no attempt to figure out how individuals might actually attain these efficient allocations. Efficient allocations are in general different from *first best* allocations, which in this environment deliver a constant amount of consumption across dates and states to each agent. Indeed, because of the bilateral lack of commitment, it may be impossible to find any first best allocation which is sustainable.

I prove three qualitative results about efficient allocations in this environment with a two-sided lack of commitment. First, if a first best allocation is sustainable, then over time every efficient sustainable allocation converges with probability one to some first best allocation. In the long run,
the inability of agents to commit is irrelevant[1].

Second, if no first best allocation is sustainable, then as time passes, every efficient allocation converges in distribution to the same nondegenerate and symmetric limiting distribution. This proposition about the long run is different from results obtained in environments in the private information literature obtained by Thomas and Worrall (1990), Atkeson and Lucas (1992), and Phelan and Townsend (1991). In their models, utility in efficient allocations converges with probability one to its lowest possible value. On the other hand, in dynamic incomplete market models (for example, Scheinkman and Weiss (1986) and D. Lucas (1992)), and in the private information model of Wang (1992), the equilibrium allocation of resources does eventually settle into a nondegenerate and symmetric stochastic steady state.

Finally, I demonstrate that in any efficient allocation, the conditional covariance of next period's individual income and individual consumption is nonnegative: it is zero only in a first best allocation. At least in a conditional sense, the model replicates the positive correlation between individual income and individual consumption that leads many economists to doubt the existence

[1]Marcet and Marimon (1992) prove in a growth economy with a one-sided lack of commitment that the lack of commitment never affects the steady state. In the latter part of the paper, I argue that their finding is due to the fact that some first best allocation is always sustainable if there is only a one-sided lack of commitment.

of complete risksharing.

The paper is related to two types of other research. The first deals with problems related to lack of commitment. There is a vast literature in international economics discussing how the inability of nations to commit to repay their loans affects international loan markets. Because this literature is dealing with borrowing and lending, it focuses exclusively on situations in which some information is private and there is only a one-sided lack of commitment. Similarly, Phelan (1993) studies the form of the optimal labor contract in an environment in which effort is unobservable and workers can only commit to be with a firm for one period.

The paper is also related to a second line of research which considers the efficient allocation of consumption in dynamic economies with private information. This group of articles includes (among others) Thomas and Worrall (1991), Green and Oh (1991), Phelan and Townsend (1991), Atkeson and Lucas (1992), and Wang (1992). My paper shares a technical feature with the private information literature: I find it convenient to employ the "utility as state variable" approach pioneered by Spear and Srivastava (1987). I also share a common attitude with the authors of those papers: it is instructive to determine the optimal allocations in an economy without necessarily determining how to decentralize those allocations.

The rest of the paper is organized as follows. Section 1 describes the basic environment. Section 2 defines and characterizes sustainable allocations, and Section 3 defines and characterizes efficient sustainable allocations. Section 4 proves that income and consumption are correlated in an efficient sustainable allocation. Section 5 describes the long run behavior of efficient allocations. Section 6 concludes.

1. The Environment

Consider the following environment. There are two infinitely-lived agents. The state of the world in period t is stochastic and is determined by the realization of a discrete iid random variable θ, with support equal to
$\{1,2,3..., S\}$; the probability of θ equalling s is denoted by π_s. There is a single perishable consumption good. The endowment of agent 1 in period t, y_t, is determined by the realization of θ in that period. I assume that the endowment of agent 2 in period t equals $(1-y_t)$. I also require that the joint distribution of the individual endowments be symmetric in the sense that the

outcome (y, 1-y) has the same probability of occurrence as (1-y,y).

An allocation is a stochastic vector process $((c)_{=1})_{=1}$, which is restricted to be measurable with respect to current and past realizations of θ; the realizations of the process must lie in the set $[0,1]^2$. It will be useful to use a Debreu tree to model the uncertainty. In particular, we can think of an allocation as an element of C^2, where:

$$C \equiv [0,1]^S \times [0,1]^{S\wedge 2} \times [0,1]^{S\wedge 3} \times .... = \Pi_{=1} [0,1]^{S\wedge t}$$

We can endow C with the product topology, and then C^2 is a compact space (by Tychonoff's Theorem). A feasible allocation is an element of C^2 such that c + c = 1. Note that the set of feasible allocations is convex and compact.

In period t, the two agents have identical preferences over C described by the utility function:

$$E_t \Sigma_{=0} \beta^s u(c_{t+s}), 0 < \beta < 1.$$

The cardinal utility function u is increasing, strictly concave, and continuously differentiable over [0,1]; without loss of generality, I assume that u(0) = 0 and u(1) = 1. With these assumptions, the utility function is continuous over C in the product topology. (When I refer to the utility derived by agent j from a given allocation, I mean his period zero utility, which is evaluated before any uncertainty has been resolved.)

It is natural in this environment to use the following definition of first-best.

Definition 1.1: A *first best* feasible allocation is an element (c,1-c) of C^2 such that the allocation c is a time and state invariant constant.

Because the aggregate resources are constant over dates and states, a first best allocation prescribes perfect insurance for all individuals. (Of course, the split of the pie is left undetermined.)

2. Sustainable Allocations

The two agents interact in the following way. At the beginning of period t, the realization of θ_t becomes known to both of them. At that point, each of the two individuals simultaneously transfers a nonnegative amount of his current income to the other individual (note that this transfer may be zero).

Why should an individual ever want to make a positive transfer to a person about whom he cares nothing? The structure of the environment is such that any rich agent is sure to be poor again sometime in the future. At that later date, he will want to receive some kind of transfer from a person who may now be poor. Rich individuals make transfers today so that they will receive transfers in the future.

The following definitions from dynamic game theory provide a way of formalizing this intuition. A period t history in this dynamic game is a sequence of realizations for θ and nonnegative transfers made by the agents:

$$(\theta_1, \tau_1, \theta_2, \tau_2, ..., \tau_{t-1}, \theta_t)$$

A strategy for an agent specifies his action after each possible history; thus, in period t, agent j's strategy is a mapping from possible histories into transfer amounts.

A subgame perfect equilibrium specifies a strategy for each agent such that a player's action (that is, choice of transfer amounts) at a given history is optimal given the other players' strategies. A subgame perfect equilibrium path is the allocation of consumption (feasible by construction) that results from the implementation of these strategies. (This last definition is somewhat nonstandard, because it is in terms of consumption not transfers.)

The following definition uses subgame perfection to describe the types of allocations that can survive in this world.

Definition 2.1: A feasible allocation is *sustainable* if it is a subgame perfect equilibrium path.

It is easy to characterize the set of sustainable allocations using the techniques of Abreu (1988). We first identify the sustainable allocation that provides the least utility to both of the agents.

Lemma 2.1: The autarkic allocation, c = y for all j and t, is sustainable and provides less utility to both agents than any other sustainable allocation.

Proof: In Appendix.

In what follows, I will use V_{aut} to denote the utility derived by any agent in autarky; note that the symmetry of the joint distribution of the aggregate endowments guarantees that V_{aut} is the same for all individuals.

Since autarky provides less utility than any other sustainable allocation, it is the worst possible punishment that can be provided in this environment for an agent who deviates from a proposed allocation. This intuition allows us to deduce Proposition 2.1.

Proposition 2.1: A feasible allocation (c, (1-c)) in C^2 is sustainable if and only if it satisfies the following two conditions:

(1) $u(c_t) + E_t \Sigma_{=1} \beta^s u(c_{t+s}) \geq u(y_t) + \beta V_{aut}$ for all dates and states.

(2) $u(1-c_t) + E_t \Sigma_{=1} \beta^s u(c_{t+s}) \geq u(1-y_t) + \beta V_{aut}$ for all dates and states.

Proof: Suppose an allocation satisfies these two conditions. Write down a nonnegative transfer scheme that generates the allocation (this is possible because it is feasible). Then specify the following strategy for each individual. If in the past, all individuals have acted so as to follow the transfer scheme, then follow it yourself today. If they have not, then make no transfers. This collection of strategies is clearly a subgame perfect equilibrium because the proposed allocation satisfies (1).

Now suppose that the allocation (c,1-c) is sustainable. Consider agent 1 at a particular date and state. We know that it must be at least as good for him to consume c_t as to make no transfers; hence:

$$u(c_t) + E_t \Sigma \beta^s u(c_{t+s}) \geq u(c') + \beta V'$$

where c_t' is his current consumption and V' is his expected utility from future consumption if he makes no transfers today. Since $c_t' \geq y_t$, and $V' \geq V_{aut}$, (1) follows. (2) can be established in a similar fashion. QED

This proposition is not especially surprising given the work of Abreu (1988). However, it demonstrates that the set of sustainable allocations (which I will henceforth label Γ) can be fully characterized using some simple constraints. In particular, note that (1) and (2) imply that Γ is compact in the product topology and that Γ is convex.

Are there any sustainable allocations which are nonautarkic? The following result provides a simple sufficient condition for the existence of such allocations. Let y_{min} and y_{max} be the lower and upper bounds on the support of y.

Proposition 2.2: Suppose the probability of agent 1's receiving endowment y_{max} is given by $\pi < 1/2$. Then a nonautarkic sustainable allocation exists if:

$$(1-\beta\pi)u'(y_{max}) - \beta\pi u'(y_{min}) < 0 \qquad \text{(C)}$$

Proof: Let c^* maximize the function $(1-\beta\pi)u(c) + (\beta\pi)u(1-c)$, over the range $[y_{min}, y_{max}]$. We know from the condition (C) that $0.5 < c^* < y_{max}$. Define (y) as follows:

$$\begin{aligned} (y) &= y \text{ for y not in } \{y_{min}, y_{max}\} \\ &= c^* \text{ for } y = y_{max} \\ &= (1-c^*) \text{ for } y = y_{min} \end{aligned}$$

and consider the allocation which gives agent 1 (y_t) in period t. This allocation is nonautarkic. Also, when $y_t = y_{max}$:

$$u(c^*) + \beta\{\pi u(c^*) + \pi u(1-c^*)\} > u(y_{max}) + \beta\{\pi u(y_{max}) + \pi u(y_{min})\}$$

Hence, this allocation is sustainable.

The sufficient condition (C) is automatically satisfied for power utility functions if $y_{min} = 0$.

Throughout the rest of the paper, I maintain the following assumption.

Assumption 2.1: The specification of the joint distribution of the individual endowments, β and the utility function u are such that there exists some nonautarkic sustainable allocation.

In the environment described in this paper, individuals make transfers to others in order to receive transfers in the future. It is natural to ask when this "tit-for-tat" process is sufficient to support a first best allocation. The following three corollaries help to answer this question. The first result characterizes the set of sustainable first best allocations.

Corollary 2.1: Some first best allocation is sustainable if and only if:

$$u(1/2)/(1-\beta) \geq u(y_{max}) + \beta V_{aut}$$

where y_{max} is the upper bound of the support of the distribution of an individual's endowment in period t.

Proof: Suppose c^* is an asymmetric first best allocation that is sustainable. Then there exists some individual j such that $c^j < 1/2$ and:

$$u(1/2)/(1-\beta) \geq u(y_{max}) + \beta V_{aut}$$

But then:

$$u(1/2)/(1-\beta) \geq u(y_{max}) + \beta V_{aut}$$

and so the symmetric first best allocation must also be sustainable. QED

Corollary 2.1 demonstrates that a first best allocation is only sustainable if and only if the symmetric first best allocation is sustainable. It also shows that it is not always possible to support a first best allocation[2]. This latter implication is a consequence of the two-sided lack of

[2]For example, suppose $u(x) = x^{0.5}$, $\beta = 0.5$, and there are two equally likely states: $y_1 = 1$ and $y_2 = 0$. Then:

commitment. To see why, suppose that agent 2 can commit to an allocation but agent 1 cannot. In this setting, it is possible to support any first best allocation in which agent 1 receives a sufficiently large fraction of the aggregate pie. The bilateral lack of commitment makes it more difficult to achieve a Pareto optimal sharing of risk because both agents have to be happy with the split in every date and state.

It is simple to use the condition in Corollary 2.1 to perform comparative statics. If the distribution of agent one's endowment, y, is made more risky (in the sense of second order stochastic dominance) without altering its support, then V_{aut} declines: it is easier to support first best allocations. If β increases, then it is easier to support first best allocations. Finally, if u becomes more risk averse, then it is easier to support first best allocations.

The next corollary proves a folk theorem.

Corollary 2.2: Given a first best allocation c in which all agents receive more utility than in autarky, there exists β^* such that c is sustainable for all $\beta \geq \beta^*$.

Proof: Let c_{min} be the smallest level of consumption mandated by the allocation c. Since both agents receive more utility from c than in autarky, we know that:

$$u(c_{min}) > Eu(y^j)$$

We know that the allocation c is sustainable if and only if:

$$u(c_{min}) \geq (1-\beta)u(y_{max}) + \beta Eu(y^j)$$

For β close to zero, the left-hand side is guaranteed to be smaller than the right hand side. For β sufficiently close to one, the right-hand side is sufficiently close to $Eu(y^j)$ to make it less than $u(c_{min})$. QED

On the other hand, it is never possible to sustain all allocations on the contract curve.

Corollary 2.3: For all β, there exists some nonsustainable first best allocation in which all agents

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$$u(0.5)/(1-\beta) < u(1) + \beta[0.5u(1)+0.5u(0)]/(1-\beta)$$

The symmetric first best allocation is not supportable and so no first best allocation is supportable.

receive utility greater than or equal to V_{aut}.

Proof: Fix β. Let c^* satisfy the equation:

$$u(c^*)/(1-\beta) = V_{aut}$$

Then:

$$u(c^*)/(1-\beta) < u(y_{max}) + \beta V_{aut}$$

Hence, a first best allocation in which one agent receives c^* and the other agent gets $(1-c^*)$ is not sustainable.

These corollaries make clear that not all first best allocations are sustainable. Indeed, the two-sided lack of commitment implies that in some environments, no first best allocations will be sustainable (because there exist environments in which the condition in Corollary 2.1 is not satisfied). These results lead to a natural question: what allocations are efficient among the class of all sustainable allocations?

In order to answer this question, it is useful to better understand the properties of the set of sustainable utility vectors. In particular, define U to be the set of all vectors (u^1, u^2) in $[0,1]^2$ such that there exists some sustainable allocation which provides utility u^j to agent j.

Proposition 2.3: The set U of sustainable utility vectors is compact.

Proof: In Appendix.

As we will see, the compactness of U makes it easy to prove the existence of an efficient allocation.

3. Efficient Allocations

I define an efficient sustainable allocation as follows.

Definition 3.1: An allocation $(c^*, 1-c^*)$ in Γ is *efficient* if there exists no other element in Γ that provides both individuals with at least as much utility and one of them with more.

I make no attempt to explain what societal mechanisms might lead agents to play strategies that would lead to efficient allocations (although I think it is simple enough to think of ones). Instead, I concentrate on the characteristics of efficient allocations.

The following proposition uses the compactness of U to guarantee the existence of efficient allocations.

Proposition 3.1: Suppose (u^1, u^2) is an element of U. Then, there exists an efficient allocation in Γ which provides utility vector (u^1, u) to the two agents, where u is defined as follows:

$$u = \max_{u^{1\prime}, u^{2\prime}} u^{2\prime}$$

$$\text{s.t. } (u^{1\prime}, u^{2\prime}) \in U \qquad \text{(P)}$$

$$\text{s.t. } u^{1\prime} \geq u^1 .$$

Proof: In Appendix.

The proposition shows that the last constraint is always binding because the solution to (P) involves setting $u^{1\prime} = u^1$. This differs from some private information settings in which the reservation utility constraint need not be binding in an efficient allocation.

Define V_{max} to be the maximal level of utility available to agent 1 from an allocation in Γ; V_{max} clearly exists (since U is compact) and is the same for the two agents (because of the symmetry of the environment). The convexity of Γ and the continuity of the utility function implies that we can use the intermediate value theorem to conclude that for all u^1 in $[V_{aut}, V_{max}]$, there exists an element (u^1, u^2) in U. Define the function $V:[V_{aut}, V_{max}] \rightarrow [V_{aut}, V_{max}]$ by:

$$V(u_0) = \underset{c}{\text{Max}}\ E_0 \Sigma_{=1} \beta^{t-1} u(1-c_t)$$

$$\text{s.t. } (c, (1-c)) \in \Gamma$$

$$\text{s.t. } E_0 \Sigma_{=1} \beta^{t-1} u(c_t) = u_0$$

From Proposition 3.1, we know that for all u_0 in $[V_{aut}, V_{max}]$, there exists an efficient sustainable allocation in which agent 1 receives ex-ante utility equal to u_0 and agent 2 receives ex-ante utility $V(u_0)$. The function V is clearly decreasing and concave; the Theorem of the Maximum implies that it is continuous. Monotonicity guarantees that V is differentiable almost everywhere; I will further assume that there are no kinks in V and so it is differentiable everywhere.

The definition of V is clear conceptually but not very practical if we want to investigate the properties of efficient allocations. The following characterization of V is more helpful in this regard.

Proposition 3.2: V must satisfy the functional equation:

$$\text{(FE)} \quad V(u_0) = \underset{(c_s,\, u_s)}{\text{Max}} \quad \Sigma_{=1} \pi_s[u(1-c_s) + \beta V(u_s)]$$

$$\text{s.t. } \Sigma_{=1} \pi_s[u(c_s) + \beta u_s] = u_0 \qquad \text{(P1)}$$

$$\text{s.t. } u(c_s) + \beta u_s \geq u(y_s) + \beta V_{aut} \text{ for all s.} \qquad \text{(P2)}$$

$$\text{s.t. } u(1-c_s) + \beta V(u_s) \geq u(1-y_s) + \beta V_{aut} \text{ for all s.} \qquad \text{(P3)}$$

$$\text{s.t. } c_s \,\varepsilon\, [0,1]$$

$$\text{s.t. } u_s \,\varepsilon\, [V_{aut}, V_{max}]$$

Proof: Suppose V does not satisfy (FE). Then there exists (c_s, u_s) that satisfy (P1-P3) such that:

$$V(u_0) < \Sigma_{=1} \pi_s[u(1-c_s) + \beta V(u_s)]$$

For each $(u_s, V(u_s))$, there exists a sustainable allocation c^s that gives the two agents these levels of utility in period zero. Define a new allocation c' in C by:

$$c'(s, \theta_2, ..., \theta_t) = c^s(\theta_2, ..., \theta_t)$$

$$c'(s) = c_s$$

Agent 2 receives (1-c'). This allocation is sustainable because (c_s, u_s) satisfies (P2-P3) and it provides agent 1 with utility equal to u_0 because (c_s, u_s) satisfies (P1). It provides agent 2 with utility equal to $\Sigma_{=1}\ \pi_s[u(1-c_s) + \beta V(u_s)] > V(u_0)$; but this contradicts the definition of V as the maximal amount of utility available to agent 2 in a sustainable allocation that gives agent 1 utility of at least u_0. QED

Proposition 3.2 is similar to ones derived in the dynamic private information literature (see for example Spear and Srivastava (1987)). It shows that we can think about the construction of efficient sustainable allocations using the following mechanical metaphor. A social planner enters period t having promised agent 1 a certain amount of *ex-ante* utility, u_0. Taking this promise as given in the form of (P1), the planner seeks to maximize the amount of *ex-ante* utility agent 2 receives. The planner determines how much consumption to give to or take from agent 1 and how much future utility to promise agent 1, contingent on each state of the world. He must take into account the *sustainability constraints* (P2-P3) that capture his inability to force the agents to give up consumption beyond threatening them with future autarky.

The sustainability constraints (P2-P3) may appear similar to incentive compatibility constraints. Their economic content is of course quite different: a sustainability constraint says that an agent has to be promised enough utility today that he will not quit the risksharing arrangement forever while an incentive compatibility constraint (in the presence of unobservable but exogenous income shocks) says that truth-telling must be a Nash equilibrium. Both do serve to constrain the social planner in his choices.

There is also an important technical distinction. In an incentive compatibility constraint, the choice variables occur on both sides of the inequality: there is no guarantee that the constraint set is convex. In a sustainability constraint, the right hand side of the inequality is exogenous, as in a standard consumer choice problem. As a result, the constraint set is convex, which makes the problem more tractable.

Unlike the Bellman equation in more standard settings, the functional equation (FE) is not helpful in solving for V. (For one thing, the functional equation features multiple solutions: the function $V(u_0) = V_{aut}$ is always a solution to (FE).) However, as we shall see in the next section, the functional equation does aid us in characterizing the behavior of consumption in an efficient allocation.

4. The Efficiency of Correlated Individual Income and Consumption

This section begins a general discussion of the properties of efficient allocations. In it, I show that individual consumption and income are positively correlated in any efficient allocation which is not first best.

Consider an efficient allocation which provides agent 1 with utility equal to u_0. The maximization problem in (FE) makes clear that we can divide the possible states of the world in period one into three groups.

S_1: states in which constraint (P2) binds.
S_2: states in which constraint (P3) binds.
S_3: states in which neither constraint binds.

(By "binds", I mean that the multiplier on the constraint is positive.) It is easy to see that the intersection of S_1 and S_2 must be empty. Suppose s lies in S_1. Then, $c_s \leq y_s$ because $u_s \geq V_{aut}$. If s lies in S_2, then $c_s \geq y_s$. It follows that if s lies in S_1 and S_2, then $c_s = y_s$ and $u_s = V_{aut}$ and $V(u_s) = V_{aut}$. But this means that $V(V_{aut}) = V_{aut}$, which is impossible as long as there exists some sustainable allocation that is nonautarkic.

The first order conditions with respect to u_s in the maximization problem in (FE) take the following form:

$$\beta\pi_s V'(u_s) + \lambda\beta\pi_s + \mu_s + \nu_s V'(u_s) = 0$$

where λ is the multiplier on (P1), μ_s is the multiplier on (P2), and ν_s is the multiplier on (P3). The envelope theorem tells us that $\lambda = V'(u_0)$. Hence, if s lies in S_1, then $\mu_s > 0$ and $\nu_s = 0$, so $V'(u_s) < V'(u_0)$, which means $u_s > u_0$. Similarly, if s lies in S_2, then $V'(u_s) > V'(u_0)$, and $u_s < u_0$. In words, this analysis tells us that it is efficient to induce an agent with a binding sustainability constraint to provide consumption today by promising him more utility in the future.

If s lies in S_3, on the other hand, then μ_s and ν_s are both zero, and so $V'(u_s) = V'(u_0)$; in other words, $u_s = u_0$. The first order conditions with respect to c_s tell us that:

$$u'(c_s)/u'(1-c_s) = -\lambda = -V'(u_s) = V'(u_0).$$

Hence, c_s is the same for all states s that lie in S_2. Note that c_s is determined by u_0; I will use the notation $c(u_0)$ to refer to the level of consumption in states that lie in S_3. Note that $c(u_0)$ is an increasing function.

The dependence of c_s on u_0 does not carry over to states that lie in S_1 or S_2. Suppose for example that s lies in S_1. Then, we know that agent 1 must receive utility equal to $u(y_s) + \beta V_{aut}$ in state s. An efficient choice of c_s and u_s thus solves the following maximization problem:

$$\text{MAX}_1 \qquad \text{Max } u(1-c_s) + \beta V(u_s)$$
$$\text{s.t. } u(c_s) + \beta u_s = u(y_s) + \beta V_{aut}$$

Similarly, if s lies in S_2, then an efficient choice of c_s and u_s solves the problem:

$$\text{MAX}_2 \qquad \text{Max } u(c_s) + \beta u_s$$
$$\text{s.t. } u(1-c_s) + \beta V(u_s) = u(1-y_s) + \beta V_{aut}.$$

Thus, if s lies in S_1 or S_2, then the level of utility and consumption received by agent 1 in those states is not affected by u_0. (Of course, whether s does lie in S_1 or not is affected by how big u_0 is.)

To understand how individual consumption and income are correlated in an efficient allocation, we need three lemmas.

Lemma 4.1: If s lies in S_1 and r lies in S_3, then $c_s > c_r$. Similarly, if s lies in S_2 and r lies in S_3, then $c_s < c_r$.

Proof: Suppose s is in S_1. Then we can use the first order conditions from MAX_1 to conclude that:

$$\begin{aligned} -u'(1-c_s)/u'(c_s) &= V'(u_s) \\ &< V'(u_0) \\ &= -u'(1-c_r)/u'(c_r) \end{aligned}$$

which implies that $c_s > c_r$. The proof for states that lie in S_2 is similar. QED

Thus, agents consume more in states in which the sustainability constraint binds than in states in which it does not bind. Intuitively, when the sustainability constraint binds, the threat of autarky is not sufficient to convince rich agents to follow full insurance.

The second lemma demonstrates that the sustainability constraint binds an agent only when he is rich.

Lemma 4.2: If s lies in S_1, and $y_s \leq y_r$, then r lies in S_1 also.

Proof: Suppose not. Then r either lies in S_2 or S_3. In either case, $c_r < c_s$ and $u_r < u_s$. Hence:

$$u(c_r) + \beta u_r < u(c_s) + \beta u_s = u(y_s) + \beta V_{aut} \leq u(y_r) + \beta V_{aut}$$

which violates the supposition that r does not lie in S_2. QED

The two lemmas give us a lot of information about the structure of efficient allocations. Given u_0, there are two cutoff points in the support of the endowment y_1. For high income levels, the sustainability constraint binds; the agent receives a large amount of utility and consumption. For middle income levels, neither sustainability constraint binds. The split of consumption across agents is the same in all of these states. Finally, for low income levels, the other agent's sustainability constraint binds.

To complete the picture, we need to know how consumption and income are correlated within S_1 and S_2.

Lemma 4.3: If s and r lie in S_1, and $y_s > y_r$, then $u_s > u_r$ and $c_s > c_r$.

Proof: Suppose $u_s \leq u_r$. Then:

$$\begin{aligned} -u'(1-c_s)/u'(c_s) &= V'(u_s) \\ &\geq V'(u_r) \\ &= -u'(1-c_r)/u'(c_r) \end{aligned}$$

It follows that $c_s \leq c_r$. But $u(c_s) + \beta u_s = u(y_s) + \beta V_{aut} > u(y_r) + \beta V_{aut} = u(c_r) + \beta u_r$, so we have a contradiction. A similar contradiction can be derived by assuming that $c_s \leq c_r$.

QED

Thus, within the group of states in which the sustainability constraint binds, income and consumption are positively correlated.

Graph 1 is a picture of the interaction between agent 1's consumption and agent 1's income described by Lemmas 4.1-4.3. When agent 1's income is low, then agent 2's sustainability constraint is binding; Lemma 4.3 tells us that consumption and income are positively correlated in that region. When agent 1's income is about average, then neither sustainability constraint is binding: in that region, his consumption is flat. Finally, when agent 1's income is high, his sustainability constraint binds and his consumption is positively correlated with income.

The size of the regions S_1, S_2 and S_3 is determined by u_0: one or two of the regions may be empty depending on the value of u_0. In particular, if S_3 contains all S states, then consumption and income are uncorrelated. Note that this implies that the allocation is first best: $u_s = u_0$ for all s, so the solution to the maximization problem in (FE) is the same in every state next period.

The above analysis can be summarized in this proposition.

Proposition 4.1: In an efficient allocation, the conditional covariance between individual income and individual consumption is nonnegative. If the conditional covariance is zero at any point in time, then the efficient allocation is date and state invariant (that is, first best) in all ensuing states.

5. The Long Run Behavior of Efficient Allocations

In this section, I examine the long run dynamics of efficient allocations. There are two separate cases of interest. Define c_{FB} to be the solution to the equation:

$$u(c_{FB})/(1-\beta) = u(y_{max}) + \beta V_{aut}$$

In the first type of environment, $c_{FB} \leq 1/2$ and so a first best allocation is sustainable. In the second type of environment, $c_{FB} > 1/2$: no first best allocations are sustainable.

In both settings, I will focus on the long run behavior of the *ex-ante* utility received by agent 1:

$$u_t = E_t \Sigma_{=0} \beta^s u(c_{t+s+1})$$

I show that when a sustainable first best allocation exists, then u_t converges with probability one to some element of the set $I \equiv [u(c_{FB})/(1-\beta), u(1-c_{FB})/(1-\beta)]$. The limiting point of u_t depends on the initial level of utility promised to agent 1, u_0. If u_0 lies in I, then $u_t = u_0$ for all dates and states. If $u_0 < u(c_{FB})/(1-\beta)$, then u_t converges monotonically to $u(c_{FB})/(1-\beta)$. On the other hand, if $u_0 > u(1-c_{FB})/(1-\beta)$, then u_t converges monotonically to $u(1-c_{FB})/(1-\beta)$. It is not possible to say as much analytically when no first best allocation exists. However, it is possible to show that the unconditional distribution of u_t converges weakly to the same limiting distribution independent of the initial value u_0. In contrast to some results in the private information literature[3], I show that the limiting distribution has a support which is not concentrated on $\{V_{aut}\}$ or $\{V_{max}\}$.

The following lemma is useful in proving these convergence results.

Lemma 5.1: The S policy functions $u_s(u_0)$ are increasing in u_0.
Proof: In Appendix.

The lemma says that future utility is a "normal" good in its response to changes in u_0: it is optimal for the social planner to provide more future utility to agent 1 when the latter is promised more current utility.

A. When Sustainable First Best Allocations Exist

Suppose $c_{FB} \leq 1/2$, and define $u_{FB} = u(c_{FB})/(1-\beta)$ and $u_{FB}' \equiv u(1-c_{FB})/(1-\beta)$. Suppose u_0 lies in the set $I = [u_{FB}, u_{FB}']$. Then the allocation $(c_0, (1-c_0))$ where:

$$u(c_0)/(1-\beta) = u_0$$

is first best and sustainable; it is therefore an efficient sustainable allocation that provides utility u_0 to agent 1. In these kinds of allocations, u_t is constant over all dates and states at its initial level u_0.

It is more interesting to think about efficient allocations in which u_0 does not lie in I.

[3]Wang (1992) obtains a similar nondegeneracy result in a model with two risk averse agents who cannot observe each other's effort levels.

Lemma 5.1: Suppose $u_0 > u_{FB}' = u(1-c_{FB})/(1-\beta)$. Then agent 1's sustainability constraint does not bind in period 1.

Proof: If it does bind, then it must bind when $y_s = y_{max}$ (Lemma 4.2). On the other hand, Lemma 4.3 demonstrates that $u(c_s) + \beta u_s$ is highest when $y_s = y_{max}$. Hence:

$$u(c_s) + \beta u_s \leq u(y_{max}) + \beta V_{aut} = u(c_{FB})/(1-\beta) \text{ for all s.}$$

But this implies that $u_0 = \Sigma_{=1}\, \pi_s[u(c_s) + \beta u_s] \leq u(c_{FB})/(1-\beta)$, which contradicts the fact that $u_0 > u(1-c_{FB})/(1-\beta)$.

We can use this lemma to prove that if $u_0 > u(1-c_{FB})/(1-\beta)$, then u_t converges monotonically to $u(1-c_{FB})/(1-\beta)$.

Proposition 5.1: Suppose there exists a sustainable first best allocation, and u_0 is greater than the largest amount of utility available to agent 1 in a sustainable first best allocation (that is, $u(1-c_{FB})/(1-\beta)$). Then $u(1-c_{FB})/(1-\beta) \leq u_{t+1} \leq u_t$ for all t, with the last inequality strict with some positive probability conditional on time t.

Proof: I will prove the proposition by induction. We know from Lemma 5.1 that agent 1's constraint does not bind in period one. This immediately implies that $u_1 \leq u_0$. As well, agent 2's constraint must bind in at least one state in period one (because the allocation is not first best). It follows that $u_1 < u_0$ in some state of the world.

It remains to prove that $u_1 \geq u(1-c_{FB})/(1-\beta)$ in all states. Suppose not. Then there exists some state s in which:

$$u_{1s} < u(1-c_{FB})/(1-\beta)$$

We know that since $u_{1s} < u_0$, agent 2's sustainability constraint binds in state s. It follows that:

$$\begin{aligned} u(1-c_{1s}) + \beta V(u_{1s}) &= u(1-y_{1s}) + \beta V_{aut} \\ &\leq u(y_{max}) + \beta V_{aut} \\ &= u(c_{FB})/(1-\beta) \end{aligned}$$

Because V is decreasing, $V(u_{1s}) > V(u_{FB}') = u_{FB} = u(c_{FB})/(1-\beta)$. Hence:

$$u(1-c_{1s}) < u(c_{FB})$$

and so $c_{1s} > (1-c_{FB})$.

But we obtain c_{1s} by solving MAX_2:

$$\text{Max } u(c_{1s}) + \beta u_{1s}$$

$$\text{s.t. } u(1-c_{1s}) + \beta V(u_{1s}) = u(1-y_{1s}) + \beta V_{aut}$$

The first order conditions for this problem are:

$$-u'(1-c_s)/u'(c_s) = V'(u_{1s})$$

$$> V'(u_{FB}')$$

$$= -u'(1-c_{FB})/u'(c_{FB})$$

which implies that $c_{1s} < c_{FB}$ which is a contradiction. We conclude that $u_{1s} \geq u_{FB}'$ for all s. This same argument can be replicated for any period t, so the proposition follows. QED

The proposition implies a more interesting corollary.

Corollary 5.1: If $u_0 \geq u(1-c_{FB})/(1-\beta)$, then with probability one, u_t converges monotonically to $u(1-c_{FB})/(1-\beta)$. Similarly, if $u_0 \leq u(c_{FB})/(1-\beta)$, then with probability one, u_t converges monotonically to $u(c_{FB})/(1-\beta)$.

Proof: It is immediate from the above proposition that u_t converges with probability one to some random variable greater than or equal to u_{FB}'. Consider a state s such that $y_s = y_{min}$. Define the sequence $\{v_n\}_{=1}$ recursively by setting $v_0 = u_0$ and $v_n = u_s(v_{n-1})$. From Lemma 5.1, we can conclude that for all t, v_t is the smallest realization of u_t. From Proposition 5.1, the sequence v_n is decreasing. Hence, it converges to some v^*. Because u_s is continuous (Theorem of the Maximum), $u_s(v^*) = v^*$. But Proposition 5.1 tells us that if $v^* < u_{FB}'$, then $u_s(v^*) < v^*$, so v^* must equal u_{FB}'.

.

Thus, if a sustainable first best allocation exists, then all efficient allocations converge over time to a first best allocation. In the long run, the lack of commitment is irrelevant (see Marcet and Marimon (1992) for a similar result).

B. When No Sustainable First Best Allocations Exist

It is clear from the form of the functional equation (FE) that in an efficient allocation, u_t must follow a stationary Markov process with realizations in the compact set $[V_{aut}, V_{max}]$. In particular, given an efficient allocation:

$$\Pr(u_{t+1} = u | u_t, u_{t-1}, ..., u_0) = \Pr(u_{t+1} = u | u_t) \equiv \phi(u,u_t)$$

The above analysis shows that this Markov process is nonergodic when multiple sustainable first best allocations exist. In this subsection, I demonstrate that the Markov process is ergodic if no sustainable first best allocations exist.

To do so, I first prove in the following two lemmas that the stationary Markov process ϕ satisfies the Feller property and is monotone (see Stokey and Lucas with Prescott (1991) - henceforth termed SLP - for definitions).

Lemma 5.2: The Markov process ϕ satisfies the Feller property.

Proof: Let $u_s(u_0)$ be the policy functions in the maximization problem in (FE) and let f be some continuous function. To show that ϕ satisfies the Feller property, we need to demonstrate:

$$\Sigma_{=1} f(u_s(u_0))\pi_s$$

is continuous in u_0. But this follows from the continuity of u_s, which is a consequence of the Theorem of the Maximum.

Lemma 5.3: The Markov process ϕ is monotone.

Proof: ϕ is monotone as long as $\Sigma_{=1} f(u_s(u_0))\pi_s$ is increasing in u_0 for any increasing f. This follows from Lemma 5.1.

These two lemmas are true regardless of whether a first best allocation is sustainable; together, they guarantee the existence of some limiting distribution for u_t. However, they are not sufficient to preclude the possibility that utility in different efficient allocations (that is, different initial conditions for u_0) converges to different limiting distributions. The following proposition demonstrates that if there are no sustainable first best allocations, then u_t has the same limiting distribution in all efficient allocations.

Lemma 5.3: Suppose there is no sustainable first best allocation. There exists u^* and t such $\Pr(u_t > u^* | u_0 = V_{aut})$ is positive and $\Pr(u_t < u^* | u_0 = V_{max})$ is positive.

Proof: V is continuous and maps $[V_{aut},V_{max}]$ into itself; hence, V has a fixed point u^* such that $V(u^*) = u^*$. Thus, there is an efficient sustainable allocation that provides both agents with the same level of utility. (This fixed point is unique because V is decreasing.) Suppose S_1 is empty when $u_0 = u^*$. The symmetry of the problem then says that S_2 should be empty also. But this is impossible because there is no sustainable first best allocation. Thus, in the efficient allocation in which both agents receive utility u^*, S_1 and S_2 are both nonempty in period one.

I want to prove that starting from any $u_0 < u^*$, there is some t such that $u_t > u^*$ with positive probability. Start with an arbitrary $v_0 < u^*$, and let s be some state in which $y_s = y_{max}$. From the proof of Lemma 5.3, we know that S_1 is nonempty, so that agent 1's constraint binds in state s. Define the sequence $\{v_n\}_{=1}$ recursively by the formula $v_n = u_s(v_{n-1})$ and $v_0 = u_0$. The set S_1 must be nonempty for any initial level of utility less than u^* so $v_n > v_{n-1}$ for any $v_{n-1} \leq u^*$.

Suppose there does not exist any n such that $v_n > u^*$. Then, v_n is a strictly increasing sequence that is bounded from above by u^*; v_n converges to some limit v^* that is less than or equal to u^*. Since u_s is continuous (from the Theorem of the Maximum), this limit must satisfy $u_s(v^*) = v^*$. But this is impossible because $u_s(u) > u$ for any $u \leq u^*$.

Thus, it is possible to start with any $u_0 < u^*$ and find n such that the probability that u_n exceeds u^* is positive. Similarly, I can prove that starting at any $u_0 > u^*$, there is some t such that $u_t < u^*$ with positive probability. QED

Lemma 5.3 establishes that the Markov process ϕ satisfies the mixing condition 12.1 described in SLP.

The following proposition is a simple consequence of Lemmas 5.1-5.3.

Proposition 5.2: The distribution of u_t conditional on u_0 converges weakly to a nondegenerate limiting distribution that is independent of u_0. .

Proof: Lemmas 5.1-5.3 show that ϕ satisfies the assumptions of Theorem 12.12 of SLP; hence, we know that there exists some unique limiting distribution. The symmetry of the environment tells us that the limiting distribution must be symmetric in the sense that Pr(A) = Pr(A'), where A is a subset of U^2, and (u_1, u_2) lies in A' iff (u_2, u_1) lies in A. It follows that the limiting distribution can be degenerate if and only if u_t converges in distribution to u^*, the fixed point of V. But we know that if $u_0 = u^*$, then $Pr(u_1 > u^*) > 0$, because S_1 is nonempty.

QED

Thus, the limiting distribution of u_t is the same in all efficient allocations. In contrast to some results in the private information literature (e.g., Atkeson and Lucas (1992)), the limiting distribution is not degenerate. Since c_{t+1} is a continuous increasing function of u_t, the same can be said of consumption. Notice that consumption c_{t+1} is a function of u_t and y_{t+1}; hence, individual consumption may depend upon lagged individual income as well as current individual income.

C. Discussion of Long Run Behavior

Why does the existence of a sustainable first best allocation make such a difference in the long run behavior of utility? Graphs 2-4 illustrate the answer to this question[4]. In Graph 2, I have drawn typical sample paths of agent 1's utility in an efficient allocation for various initial values. The barriers represent the utility available to agent 1 in the best (barrier B) and worst (barrier W) sustainable first best allocations. Now think about what happens if we decrease β. Then the barriers in Graph 2 will move closer together. Eventually, they will overlap entirely, as they do in Graph 3; this overlap takes place when β is sufficiently small that $u(1/2)/(1-\beta) = u(y_{max}) + \beta V_{aut}$.

In the above discussion, I interpreted barrier W as representing the lowest amount of utility available to agent 1 in a sustainable first best allocation. There is however an alternative interpretation of barrier W: we can also think of it as being the lowest amount of initial utility such that agent 1's sustainability constraint never binds in the following period. Similarly, barrier B is the highest amount of initial utility for agent 1 such that agent 2's sustainability constraint never binds in the following period.

Suppose we keep decreasing β, so that $u(1/2) < (1-\beta)u(y_{max}) + \beta V_{aut}$. The barriers will keep moving in the same direction as before, so now the barrier W will be above the barrier B.

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[4]I emphasize that these graphs are not the products of actual model solutions but are merely my depiction of what would happen in such solutions along "typical" sample paths.

The barriers still bear the same interpretation as mentioned in the above paragraph, so when individual 1's utility level lies between B and W, the two individuals' sustainability constraints both bind in the next period with positive probability (although not simultaneously!). Graph 4 shows what this means for the time path of utility: once it enters the region between B and W, it bounces around randomly. (It goes up if agent 1's sustainability constraint binds, down if agent 2's sustainability constraint binds, and stays the same if neither binds.) Eventually, utility settles into some stationary distribution. The barriers B and W serve to bound the limiting distribution described in Proposition 5.2.

Graph 5 helps us understand the effect of two-sided as opposed to one-sided lack of commitment in this economy. For example, suppose only agent 1 faces a sustainability constraint. Then barrier B disappears because the social planner can promise agent 2 any amount of utility, no matter how low, without affecting the existence of a first best allocation. Graph 5 uses this insight to depict the time path of utility in an economy with one-sided lack of commitment. If the initial level of utility promised to agent 1 lies above W, then utility remains constant at that level. If the initial level of utility lies below W, then utility rises to eventually converge to W.

Now suppose β decreases. Then the barrier W rises. However, there is still no barrier B because, as indicated earlier, in any environment with one-sided lack of commitment, there is a sustainable first best allocation. Hence, the long run behavior of utility is always qualitatively the same as described in Proposition 5.1.

Thus, in environments with one-sided lack of commitment, the lack of commitment always ceases to matter in the long run because there always exists some sustainable first best allocation. However, if there is a two-sided lack of commitment, and no first best allocation is sustainable, then the lack of commitment affects long run allocations and induces a nondegenerate stochastic steady state distribution for individual utility.

6. Conclusions

The purpose of this paper was to examine the properties of efficient sustainable allocations in an environment in which two agents want to share risk but are unable to make commitments about future transfers. There are three main findings. First, if the discount factor is high enough (or individuals are sufficiently risk averse or individual income is sufficiently risky) that some first best allocation is sustainable, then any efficient allocation must converge with probability one to a first best allocation. In the long run, the lack of commitment is irrelevant. Second, if no first best allocation is sustainable, then there exists a unique limiting distribution of utility and consumption. Finally, under any conditions, the conditional covariance of individual income and individual consumption in an efficient allocation is nonnegative; it is zero only if the allocation is first best.

We see that at least in a qualitative sense, a model with two-sided lack of commitment and no sustainable first best allocations is capable of generating implications that are consistent with (extremely?) casual characterizations of United States data. For example, in stochastic steady state, there is idiosyncratic risk remaining in individual consumption. Individual consumption is conditionally positively correlated with current individual income and is unconditionally correlated with many lags of income. The degree of inequality, at least as measured by the cross-sectional distribution of utility or consumption, is stable over time.

Interestingly, these qualitative features are shared by the equilibrium allocations in an incomplete markets environment (Scheinkman and Weiss (1986)) and by the efficient allocations in an economy with two risk averse agents who cannot observe each other's level of effort (Wang (1992)). For this reason, it would be useful to better understand the *quantitative* implications of a two-sided inability to commit. To do so, we need to be able to construct numerical approximations to efficient allocations. This can be done using the techniques of Abreu, Pearce and Stacchetti (1986) to generate the set U of sustainable utility vectors, using the information in this set to construct a numerical approximation to the value function V, and then employing the functional equation (FE) to solve for the efficient allocations. (See Wang (1992) for a particularly lucid discussion.) Presumably, any empirically relevant quantitative work would have to incorporate persistence in the individual endowments; it should not be difficult to augment the theoretical analysis in this paper to allow for that possibility. It would then be instructive to compare the quantitative implications of this full information, no-commitment environment to the implications of an imperfect information, full commitment world and to the implications of a dynamic incomplete markets model.

My use of subgame perfection as an equilibrium concept and the closed nature of the economic system combine to produce a harsh threat point - namely, autarky - for the individuals in this economy. Using some version of renegotiation-proofness as an equilibrium concept or introducing an outside option will make the threat point less dire. In turn, fewer allocations will be sustainable. However, as long as the threat point remains independent of past choices of the individual, and as long as there exists some sustainable nonautarkic allocation, the qualitative results derived in the paper about the properties of efficient allocations will remain unchanged[5].

It would be interesting to examine the properties of efficient risksharing in multiple (that is, more than two) agent environments without commitment. It is trivial to generalize the theoretical analysis through Section 3 to include this case. However, it is difficult (at least for me) to see if the main Propositions 4.1, 5.1 and 5.2 carry over to this richer environment. In particular, it remains unclear whether the stochastic steady state is nondegenerate in an environment with a large number of agents if there are no sustainable first best allocations.

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[5]Here's an example of when these qualifications might break down. Suppose consumption is storable, so individuals can self-insure. Then the threat point will depend upon past choices of the agent and I am not sure if the results presented in this paper will still be true.

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Appendix

Proof of Lemma 2.1: It is clear that autarky is sustainable. Now consider the set of possible period zero utility levels that agent 1 derives from sustainable allocations. This set is bounded from below (by 0). Hence, it has an infimum; call this value M.

Consider an arbitrary sustainable allocation (c,1-c), and let V denote the utility derived by agent 1 from this allocation. From the point of view of period zero, there are S possible outcomes associated with the allocation in period 1:

$$V_0 = \Sigma_{=1} \pi_s(u(c_{1s}) + \beta V_{1s})$$

Agent 1 must be better off choosing c than opting not to make any transfers; hence:

$$V_0 \geq \Sigma_{=1} \pi_s(u(c_{1s}') + \beta V_s')$$

where c_s' and V_s' are his state s consumption and continuation utilities respectively if he fails to make any transfers. But $c_s' \geq y_s$ and $V_s' \geq M$; this implies that:

$$V_0 \geq \Sigma_{=1} \pi_s(u(y_s) + \beta M)$$

Since V is any sustainable utility level, it follows that:

$$M \geq (\Sigma_{=1} \pi_s(u(y_s))/(1-\beta).$$

Since autarky is sustainable, the weak inequality is actually an equality. QED

Proof of Proposition 2.2: Since U is a subset of $[0,1]^2$, it is clearly bounded; to prove it compact, we need only show that it is closed. Let $(u_n)_{=1}$ be a sequence of utility vectors in U that converges to $u*$. For each n, let c_n be a sustainable allocation that generates u_n. Since Γ is compact, there exists a subsequence $(c_nk)_{=1}$ that converges to a limit $c*$ in Γ. Since $(u_n)_{=1}$ converges to $u*$, $(u_nk)_{=1}$ also converges to $u*$. It follows from the continuity of the utility functions over C that the utility derived from $c*$ is given by $u*$, and so $u*$ lies in U. QED

Proof of Proposition 3.1: The compactness of U immediately implies that there exists a solution to the maximization program. Now suppose (u, u) solves this maximization problem. Then I claim $u = u^1$. Suppose not, and $u > u^1$. Then $u > V_{aut}$. There exists an allocation c that provides utility u. Because $u > V_{aut}$, it follows that in some state of the world in period one, the sustainability constraint does not bind agent 1:

$$u(c_1) + \beta E_t \Sigma_{=1} \beta^s u(c_{t+s}) > u(y) + \beta V_{aut}$$

If I transfer a small amount of consumption in this state and date from agent 1 to agent 2, I can increases agent 2's utility without violating the constraint that $u^{1\prime} \geq u^1$. Further, I can make the transfer small enough that the new allocation of consumption is sustainable (because the sustainability requirement is not binding in this state of the world). Hence, the solution to the maximization problem must lie on the constraint.

Thus, there exists an element (u^1, u) in U that solves the maximization problem in (P'). From the definition of U, we know that there is an allocation in Γ that provides this vector of utilities to the agents in the economy. This allocation is clearly efficient because (u^1, u) solves the maximization problem (P). QED

Proof of Lemma 5.1: Suppose u_0' is larger than u_0. As before, define $c(u_0)$ to be the solution to the equation:

$$u'(1-c(u_0))/u'(c(u_0)) = -V'(u_0)$$

If s lies in S_3, then $c_s = c(u_0)$ and $u_s = u_0$. Hence, if s lies in S_3 for both u_0 and u_0', then $u_s(u_0) < u_s(u_0')$. If s lies in S_1 or S_2 for both u_0 and u_0', then $u_s(u_0) = u_s(u_0')$. Thus, if the increase in u_0 does not alter when the sustainability constraints bind, then the lemma is clearly true.

Of course, some increases in u_0 will change whether a given state lies in S_1, S_2 or S_3. The following characterization tells us how:

s lies in S_1 if and only if $u(c(u_0)) + \beta u_0 < u(y_s) + \beta V_{aut}$
s lies in S_2 if and only if $u(1-c(u_0)) + \beta V(u_0) < u(1-y_s) + \beta V_{aut}$

Why is this characterization true? If s lies in S_1, then $u(c_s) + \beta u_s = u(y_s) + \beta V_{aut}$; the inequality then follows from Lemma 4.3. On the other hand, suppose $u(c(u_0)) + \beta u_0 < u(y_s) + \beta V_{aut}$; then, s can't be in S_3 because the sustainability constraint is violated and Lemma 4.1 further implies that it can't be in S_2 either.

The characterization tells us that if u_0 increases enough, then s might shift from S_1 to S_3, from S_3 to S_2, or (in a combined move) from S_1 to S_2. Consider what happens if s shifts from S_1 to S_3. Then, u_s changes from the solution to MAX_1 to instead equal u_0'. Suppose $u_s > u_0'$. Then, it can be shown as in Lemma 4.1 that $c_s > c(u_0')$. But this means $u(y_s) + \beta V_{aut} = u(c_s) + \beta u_s > u(c(u_0')) + \beta u_0'$, which violates the fact that s has shifted into S_3. We conclude that if s shifts from S_1 to S_3, $u_s \leq u_0'$. A similar argument implies that if s shifts from S_3 to S_2, then the u_s that solves MAX_2 is at least as large than u_0. Shifts from S_1 to S_3 are merely combinations of moves from S_1 to S_3 and S_3 to S_2.

Thus, $u_s(u_0)$ is nondecreasing. QED

Very Preliminary
Comments Welcome

Efficient Bilateral Risk Sharing without Commitment

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Abstract

This paper examines the properties of efficient sustainable allocations in an environment in which two agents want to share risk, have perfect information about each other, but cannot make commitments about future transfers. I describe as *sustainable* any allocation that can be supported as a subgame perfect equilibrium in a game in which individuals make simultaneous transfers. I consider the properties of *efficient* sustainable allocations.

There are three main findings. First, if some first best allocation is sustainable, then any efficient allocation must converge with probability one to a first best allocation. In the long run, the lack of commitment is irrelevant. Second, if no first best allocation is sustainable, then the unconditional probability distribution of an agent's utility converges weakly over time to a nondegenerate distribution. Finally, under any conditions, the conditional contemporaneous covariance of individual income and individual consumption in an efficient allocation is nonnegative; it is zero only if the allocation is first best.

Graph 1

Efficient Consumption versus Individual Income

This graph depicts the relationship between period (t+1) consumption and income of agent 1, conditional on information known at time t. S_1 is the set of states in which agent 1's sustainability constraint binds, S_2 is the set of states in which agent 2's sustainability constraint binds, and S_3 is the set of states in which neither agent's sustainability constraint binds.

Graph 2

The Efficient Time Path of Utility

Case 1: Multiple Sustainable First Best Allocations

This graph depicts typical sample paths of agent 1's utility in an efficient allocation. If the initial level of utility lies below W (= $u(c_{FB})/(1-\beta)$), then utility converges over time to W. If the initial level of utility lies above B (= $u(1-c_{FB})/(1-\beta)$), then utility converges over time to B. If the initial level of utility lies in between B and W, then utility stays constant over time.

Graph 3

The Efficient Time Path of Utility

Case 2: Unique Sustainable First Best Allocation

This graph depicts the typical sample path of utility when there is a unique sustainable first best allocation. In that case, utility always converges to $u(1/2)/(1-\beta)$.

Graph 4

The Efficient Time Path of Utility

Case 3: No Sustainable First Best Allocation

This graph depicts a typical sample path of agent 1's utility when there is no sustainable first best allocation. In stochastic steady state, utility is randomly drawn from a distribution with support [B, W].

Graph 5

The Efficient Time Path of Utility with One-Sided Commitment

This graph depicts two typical sample paths of agent 1's utility in an efficient allocation if agent 2 is able to commit. If the initial level of utility is below W, then utility converges over time to W. If the initial level of utility is above W, then utility is constant over time. Note that lowering or raising W will not affect the qualitative nature of this picture.