

The Tracking Ability of the Divisia Monetary Aggregate Under Risk

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1 Introduction

Over the last decade the area of monetary aggregation has been receiving considerably more and more attention both from economists and from monetary-policy decision makers. This literature of monetary aggregation came about as a result of the Federal Reserve's simple-sum index lacking a strong microeconomic foundation. In Barnett's (1980) seminal paper, Barnett presented the weaknesses of the simple-sum index as a measure of consumer's demand for money, and in the process established a link between Diewert's (1978) "superlative" quantity index and monetary aggregation theory. This integration of monetary aggregation with micro-demand analysis established the superlative-Divisia monetary index, a nonparametric index that provides a second-order approximation of the theoretical monetary aggregate, as the most theoretically correct method of measuring the demand for money.

Both Barnett's original paper and the subsequent work by him and others has been based on the assumption that the representative consumer enters her utility maximization problem with either perfect certainty or risk neutral preferences.¹ However, if the representative consumer is risk averse the current theorems on economic aggregates and statistical indices no longer hold. Currently there is very little known about aggregation and index theory under risk, and even less about the connection between the two. Fortunately, Barnett and Yue (1991) have already proven that a theoretical monetary aggregate does exist under risk aversion.

With the existence of a theoretical monetary aggregate already established the next step is to determine if the link between monetary aggregates and statistical index theory continues to exist under risk averse preferences. One way of tackling this question is to investigate how the approximating ability of the Divisia monetary aggregate for a specific theoretical monetary aggregate is affected as risk aversion increases.

The tracking ability of the Divisia monetary aggregate under risk aversion is critical to both economist studying monetary aggregates and those who utilize aggregates in setting monetary policy. Since Barnett (1980) showed that the theoretical aggregate is burdened by

¹See Barnett, Fisher and Serletis (1992) for an excellent literature review on the area of monetary aggregation.

estimation and model specification, the empirically calculated aggregate will be dependent on the econometrician's method of estimation and on the model he selects. Whereas with the Divisia aggregate every econometrician will obtain the same result regardless, since it is a nonparametric approximation of the theoretical aggregate and is easily calculated from known prices and quantities.

To show how well the Divisia monetary aggregate tracks the 'true' theoretical aggregate we numerically solve for the individual monetary assets from the Euler equations associated with a dynamic optimization problem. We use the calculated rational expectation equilibrium to determine both the Divisia and the theoretical monetary aggregate and compare how well the Divisia tracks the movement of the theoretical aggregate.

Although there are exists a number of numerical methods that solve for the rational expectation equilibrium of such problems, for this paper we have selected the approach advocated by Den Haan and Marcet (1990).² The Den Haan and Marcet approach has been shown to provide good results for a number of complex optimization problems.³ Furthermore, the complexity of the dynamic programming problem found in this paper is an important factor in choosing the Den Haan and Marcet algorithm since this solution method does not require discretizing the state space; a formidable task for any computer when the number of state variables is as plentiful as they are in our model. Hence, in addition to determining if the link between the Divisia and the theoretical monetary aggregate exists under risk aversion, this paper also provides a complex dynamic optimization problem that serves to test the Den Haan and Marcet algorithm's ability to find an approximate rational expectation equilibrium.

2 Modeling Consumer Demand for Monetary Assets

2.1 Preferences

In this section we formulate a discrete time period optimization problem for a maximizing representative consumer in an economy that consists of a quantity aggregate for consumption

²See Taylor and Uhlig (1990) for a list and comparison of the known methodologies to solving dynamic optimization problems.

³See Den Haan and Marcet (1989), Marshall (1992), and Bansal et. al. (1992).

goods and a quantity aggregate for monetary goods. We have chosen to comprise the monetary aggregate with three monetary assets so that we can use the empirical results found by Barnett and Yue (1991) to set the unknown parameters of the model. In addition we have tried to use the same notation that Barnett and Yue (1991) used in an attempt to keep some consistency in the literature.

We assume that the individual consumer maximizes her utility over the finite planning horizon $t, t+1, t+2, \dots, t+T$, where t is the beginning time period and $t+T$ is the terminal time period. The representative consumer's utility function is assumed to have the form

$$V(M(\mathbf{m}_t)^{1-\beta} X_t^\beta) + E_t \left[\sum_{s=t+1}^{t+T-1} \rho^{s-t} V(M(\mathbf{m}_s)^{1-\beta} X_s^\beta) + \rho^T V_T(M(\mathbf{m}_{t+T})^{1-\beta} X_{t+T}^\beta, A_{t+T}) \right] \quad (1)$$

where V is the constant relative risk aversion utility function

$$V(M(\mathbf{m}_s), X_s) = \frac{1}{\sigma} \left[X_s^\beta M_s^{1-\beta} \right]^\sigma \quad (2)$$

with $\sigma \in (-\infty, 0) \cup (0, 1)$, X_s and M_s are respectively the consumption good and monetary good quantity aggregates, and \mathbf{m}_s is a 3×1 quantity vector of monetary assets. If $\sigma \rightarrow 0$ the consumer's utility function is $V(M_s, X_s) = \ln(X_s^\beta M_s^{1-\beta})$. In Eq. (1) the allocation between the two aggregates is determined by β which upon closer observation is the parameter of a subnested Cobb-Douglas utility function, hence, $\beta \in (0, 1)$. The parameter ρ is the subjective discount factor with the restriction $\rho \in (0, 1)$.

The variable A_s represents the quantity of the benchmark asset that the consumer plans on holding in period s . Because the yield on A_s only serves to transfer wealth across time periods and does not yield liquidity nor any other services during the current period, the benchmark asset provides nothing to the consumer and hence only enters her utility function in the final time period of the planning horizon. Furthermore, the benchmark asset's period yield is defined to contain all the premiums of the market for foregoing the services provided by a monetary asset. Hence, the probability of the benchmark asset's yield exceeding the yields of all the other assets is nonzero, otherwise there would be no reason for the agent to hold the benchmark asset.

Lastly, we assume that the 'true' monetary aggregate, $M(\mathbf{m}_s)$, is defined as the CES

function

$$M(\mathbf{m}_s) = \left(\sum_{i=1}^3 \delta_i m_{is}^\alpha \right)^{1/\alpha} \quad (3)$$

with $\sum_{i=1}^3 \delta_i = 1$, and $\alpha \in (0, 1]$. Even though the theoretical monetary aggregate is an explicit function of the individual monetary assets, we drop the function's arguments to simplify the notation and represent the monetary aggregate as M_s . The monetary aggregate will equal the Cobb-Douglas aggregator function if $\alpha \rightarrow 0$ and the linear aggregator function if α equals one.

2.2 Optimization Problem

The representative consumer's dynamic optimization problem consists of choosing the deterministic point $(\mathbf{m}'_t, X_t, A_t)$ and the stochastic process $(\mathbf{m}'_s, X_s, A_s)$ for $s = t + 1, t + 2, \dots, t + T$ that maximizes Eq. (1) subject to the budget constraints

$$I_s \geq \sum_{i=1}^3 \left[(1 + r_{i,s-1}) p_{s-1}^* m_{i,s-1} - p_s^* m_{i,s} \right] + (1 + R_{s-1}) p_{s-1}^* A_{s-1} - p_s^* A_s - p_s^* X_s \quad (4)$$

for $s = t, t + 1, \dots, t + T$. The budget constraint is comprised of the expected nominal holding period yields of the monetary assets, r_{is} , for $i = 1, 2, 3$, and the expected one-period holding yield on the benchmark asset during period s , R_s . Under the assumption of rational expectation the distribution of each asset's yield is known to the consumer. However, since the payment received on a interest accruing good does not occur until the end of the period the consumer will not know the actual value of either the r_{is} 's nor R_s during period s . On the other hand, the consumption good's price aggregate (or true cost of living index) p_s^* is determined and fully known by the representative agent at the beginning of period s . I_s represents the sum of all other sources of income during period s .

Following Section 2 of Barnett and Yue (1991) the Bellman equation for the above dynamic optimization problem provides the following first-order necessary condition Euler equations⁴

$$X_s^{\sigma\beta-1} M_s^{\sigma(\beta-1)} = E_s \left[\left(\rho \frac{p_s^*}{p_{s+1}^*} (1 + R_s) \right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)} \right] \quad (5)$$

⁴Under more restrictive assumptions Poterba and Rotemberg (1987) also find these same first-order conditions.

$$X_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{1s}^{1-\alpha} = E_s \left[\left(\frac{1}{\delta_1} \frac{\beta\rho}{1-\beta} \frac{p_s^* R_s}{p_{s+1}^*} \right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)} \right] \quad (6)$$

$$X_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{2s}^{1-\alpha} = E_s \left[\left(\frac{1}{\delta_2} \frac{\beta\rho}{1-\beta} \frac{p_s^*(R_s - r_{2s})}{p_{s+1}^*} \right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)} \right] \quad (7)$$

$$X_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{3s}^{1-\alpha} = E_s \left[\left(\frac{1}{\delta_3} \frac{\beta\rho}{1-\beta} \frac{p_s^*(R_s - r_{3s})}{p_{s+1}^*} \right)^{-1} X_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)} \right] \quad (8)$$

In the above dynamic programming problem the consumer faces the exogenous stochastic state vector $\phi_s = (R_{s-1}, r_{2,s-1}, r_{3,s-1}, p_s^*/p_{s-1}^*, I_s/I_{s-1})'$, along with the endogenous state vector \mathbf{m}_{s-1} . Together they define the state vector $\boldsymbol{\sigma}'_s = (\mathbf{m}'_{s-1}, \phi'_s)$, while the representative consumer's control vector for the optimization problem is $\mathbf{z}'_s = (\mathbf{m}'_s, A_s, X_s)$.

We assume that the stochastic process $\{\phi_s\}$ behaves as two independent Markovian processes that have the behavior

$$\phi_{1s} = \mathbf{a}_1 + \mathbf{A}_1 \phi_{1,s-1} + \mathbf{u}_{1s} \quad (9)$$

$$\ln(\phi_{2s}) = \mathbf{a}_2 + \mathbf{A}_2 \ln(\phi_{2,s-1}) + \mathbf{u}_{2s} \quad (10)$$

where $\phi_{1s} = (R_{s-1}, r_{2,s-1}, r_{3,s-1})'$ and $\phi_{2s} = (p_s^*/p_{s-1}^*, I_s/I_{s-1})'$.⁵ \mathbf{u}_{1s} and \mathbf{u}_{2s} are both distributed *i.i.d.* $\mathcal{N}(0, \mathbf{i})$, for $i = 1, 2$, with \mathbf{A}_1 and \mathbf{i}_1 being 3×3 matrices, whereas \mathbf{A}_2 and \mathbf{i}_2 are 2×2 matrices. The processes' intercept terms \mathbf{a}_1 and \mathbf{a}_2 are 3×1 and 2×1 vectors, respectively.

Each Markovian process is a first-order vector autoregressive process (VAR(1)), the only difference between them being the natural logarithmic transformation of ϕ_{2s} . This transformation insures that the process is stationary. Since ϕ_{1s} is only comprised of interest rates there is no reason to transform this process.

To insure that the benchmark asset's yield exceeds r_{2s} and r_{3s} for all s , R_s is generated in the following manner

$$R_s = \begin{cases} \max(r_{2s}, r_{3s}) + \frac{1}{4} \max(r_{2s}, r_{3s}) + u_{1s_1}, & \text{if } \frac{(R_s - \max(r_{2s}, r_{3s}))}{R_s} \leq 0.05 \\ R_s, & \text{otherwise} \end{cases}$$

where u_{1s_1} is the first component of u_{1s} .

⁵A more complex stochastic processes for the exogenous state process can be used if the economist so desires.

We have now completed the construction of the dynamic optimization problem. The representative agent makes her choice of the deterministic point \mathbf{z}_t and the stochastic points \mathbf{z}_s for $s = t + 1, t + 2, \dots, t + T$ for a given parameter vector describing the economy. If we define the economy's parameter vector as

$$\boldsymbol{\lambda} = (\rho, \sigma, \alpha, \beta, \delta_1, \delta_2, \delta_3, \mathbf{a}'_1, \mathbf{a}'_2, \text{vec}(\mathbf{A}_1)', \text{vec}(\mathbf{A}_2)', \text{vec}(\mathbf{1})', \text{vec}(\mathbf{2})')$$

then for each value of $\boldsymbol{\lambda}$ the above optimization problem defines a nonlinear mapping from the exogenous state process $\{\phi_s\}$ to $\{\mathbf{z}_s\}$. The theoretical monetary aggregate, $\{M_s\}$, and the Divisa monetary aggregate

$$Q_s = Q_{s-1} \prod_{i=1}^3 \left(\frac{m_{is}}{m_{i,s-1}} \right)^{s_{is}^*} \quad (11)$$

where

$$\begin{aligned} s_{is}^* &= \frac{1}{2}(s_{is} + s_{i,s-1}) \\ s_{is} &= \frac{\pi_{is} m_{is}}{\sum_{j=1}^3 \pi_{js} m_{js}}, \\ \pi_{is} &= \frac{p_s^*(R_s - r_{is})}{(1 + R_s)} \end{aligned}$$

can then be calculated from $\{\phi_s\}$ and $\{\mathbf{z}_s\}$.

3 Solving the Optimization Problem

3.1 Parameterized Expectation Approach

Although there has been a large demand for numerical methods that solve a dynamic optimization problem like that found in Section 2 only recently have such algorithms been devised.⁶ Of those currently available to economists the Parameterized Expectation Approach (PEA) of Den Haan and Marcet (1990) has performed well in head-to-head tests with other algorithms. In addition, the PEA approach has been applied to a number of different economic areas, including growth models (den Haan and Marcet (1990)), asset

⁶See Taylor and Uhlig (1990) for excellent source on the available methods and a comparison on their performance record.

markets with heterogenous agents (Ketterer and Marcet (1989), Marcet and Singleton (1990)), and monetary economies (den Haan (1990a, 1990b), Marshall (1992)).

The PEA method simply approximates the expectation operators found in the Euler equations (5)-(8) by parameterizing them with a basis function that spans the set of expectation operators. The basis function is a globally flexible functional form whose arguments are the state variables σ_s .⁷ In this paper we have chosen the first-order polynomial function, however, in future research we plan to utilize the recent advances in wavelet theory to provide a best wavelet basis. One then iterates over the parameters of these flexible functions until a convergence criterion is met.

Intuitively, each iteration of the PEA method can be viewed as a nonlinear least-square learning behavior by the consumer.⁸ Once learning is no longer occurring, the representative agent selects the stochastic solutions $\{z_s\}$ that satisfies the Euler equations, given her learned prediction of the expectation operators. Hence, it can be argued that the numerical solution found with the PEA approach is an equilibrium for a representative agent restricted to the learning associated with a specific globally flexible functional form.

To apply the PEA method to the optimization problem in Section 2 we write the system of Euler equations as

$$\Phi(z_s) = E [\Gamma(z_{s+1}, \phi_{s+1}) | \sigma_s] \quad (12)$$

where $\Phi : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ and $\Gamma : \mathbb{R}^{10} \rightarrow \mathbb{R}^4$. By defining each vector component of Γ 's range as $g_i(z_{s+1}, \phi_{s+1})$, for $i = 1, 2, 3, 4$, we can write the PEA parameterization of each Euler equation's expectation operator as

$$E[g_i(z_{s+1}, \phi_{s+1}) | \sigma_s] = \exp[\text{poly}(\tilde{\sigma}_s, \mathbf{v}_i)] \quad i = 1, 2, 3, 4$$

where $\tilde{\sigma}'_s = (1, \sigma'_s)$, and the \mathbf{v}_i 's are the polynomial's coefficient vector.

An economist using the PEA approach first generates a single realization of the random variables $\{\mathbf{u}_{1s}\}$ and $\{\mathbf{u}_{2s}\}$ from their known distributions. These series along with the two VAR(1) functions, (9) and (10), are then used by the economist to generate the stochastic

⁷See Barnett and Jonas (1982), Barnett and Yue (1988), and Gallant (1982) for the properties and examples of globally flexible functions.

⁸See Marcet and Sargent (1999a, 1989b) for examples of a linear least-square learning model that have a locally stable equilibrium.

exogenous state processes, $\{\phi_{1s}\}$ and $\{\phi_{2s}\}$. To initiate the two Markovian processes we chose to let ϕ_{10} equal $(0.098, 0.039, 0.071)'$, the sample mean of the interest rates from an empirical data source, and ϕ_{20} equal to a vector of ones.⁹

To begin the PEA's iteration process an initial vector is selected by the economist for each of the polynomial's coefficients, which we define as v_i^0 for $i = 1, 2, 3, 4$. Like all numerical algorithms the choice of the v_i^0 's is critical, but even more so with the PEA approach since each iteration does not calculate a directional update. Hence, the PEA algorithm only has good local convergence properties and thus, considerable attention should be made by the economist in selecting v_i^0 .¹⁰

After the v_i^0 's are determined the economist calculates each polynomial $\exp[\text{poly}(\tilde{\sigma}_s^0, v_i^0)]$ and substitutes them into Eq. (12) for $E[\Gamma(z_{s+1}, \phi_{s+1}) | \sigma_s]$. Eq. (12) is a nonlinear system of equations which can be solved explicitly for $\{z_s\}$ for a special limiting case within the parameter space, but for most values of λ Eq. (12) is an implicit system of equations that requires the economist to use a Newton-like algorithm to solve for $\{z_s^1\}$.¹¹

To insure that the Newton-like algorithm converges to the correct solution we rely on homotopy theory to provide initial guesses for the solutions. Later in the paper we will discuss the intricate details of homotopy theory and how it is applied to the current problem of selecting initial solutions for a Newton-like algorithm and the earlier problem of choosing the starting vectors, v_i^0 . However, for now we will only state that homotopy theory suggests first solving the optimization problem for the special parameters case where Eq. (12) can be explicitly solved for z_s . The PEA solution to the optimization problem for that value of λ provides the economist with a reasonably good choice for the starting values of the Newton-like algorithm when the optimization problem has a λ close to the special case.

Once $\{z_s^1\}$ is calculated, the economist takes the solution and calculates $g_i(z_{s+1}^1, \phi_{s+1})$ for $i = 1, 2, 3, 4$. She then regresses each of them on a linearized version of $\exp[\text{poly}(\tilde{\sigma}_s^1, v_i)]$. This linearization is performed around v_i^0 , which allows the economist to use ordinary least-squares rather than a nonlinear regression at each step of the iteration.

⁹To reduce the dependence of the exogenous state variables on these initial values we exclude the first five periods from the simulation.

¹⁰Homotopy theory is helpful in insuring that the initial coefficients are within the set which converges to a solution. We will present the essentials of homotopy theory in the next subsection.

¹¹In our actual calculations we use a modified Powell hybrid algorithm based on the MINPACK subroutine *HYBRDI* to solve the implicit system of equations.

Because of the instability involved in the above iteration process for complex optimization problems, Den Haan and Marcet (1990) suggest that the updated coefficients for the polynomials be a convex combination of the estimated and previous iterations coefficients. In other words, the updated coefficients are calculated as

$$\mathbf{v}_i^1 = \eta \mathbf{b}_i^1 + (1 - \eta) \mathbf{v}_i^0 \quad (13)$$

where $\eta \in [0, 1]$ and \mathbf{b}_i^1 is the estimated coefficients vector found in the previous paragraph. The convex combination of the coefficient estimates has a stabilizing affect on the PEA's iterations for models that are normally explosive. By choosing smaller values of η the updating of the polynomial's coefficients is more gradual and hence, more resistant to an explosive path.

The above steps are then repeated by the economist until the solutions $\{\mathbf{z}_s\}$ converge. We use the convergence criterion recommended by Bansal et. al. (1992) which is

$$\max_i \max_s \left| \frac{(z_{is}^k - z_{is}^{k-1})}{(z_{is}^{k-1} + \epsilon)} \right| \leq \xi \quad (14)$$

where ϵ and ξ are small positive number, and k is an index for the number of the iterations performed. Den Haan and Marcet (1990) originally suggested that the estimated coefficients vectors, \mathbf{v}_i^k , be tested for convergence. However, since the optimization problem contained in this paper has eight state variables that are possibly correlated, testing convergence with the coefficients of the polynomials would most likely suffer from the oscillating nature associated with the estimated parameters of a multicollinear model.¹²

3.2 Homotopy Theory

Homotopy theory although very simple is a very powerful tool in finding a solution to a complex fixed point algorithm like the ones found in this paper. The basic idea behind homotopy is to slowly move from a known solution to the solution that the economist desires to find. As a result of this methodology many previously unstable and explosive algorithms are much more well behaved and quickly converge to the correct solution. For

¹²Later we explain and use Marshall's (1992) sum of squared residuals convergence criterion.

a pedagogical example of homotopy theory we apply the theory to solving the nonlinear system Eq. (12).¹³

Suppose that the economist has substituted $\exp[\text{poly}(\tilde{\sigma}_s, \mathbf{v}_i)]$ for $E[\Gamma(\mathbf{z}_{s+1}, \phi_{s+1})|\sigma_s]$, as we instructed in Subsection 3.1, and subtracts the appropriate exponentiated polynomial scalar from both sides so that the system of equations found in Eq. (12) can be written as

$$F(\mathbf{z}, \tilde{\sigma}_s, \boldsymbol{\lambda}) = 0 \quad (15)$$

where $F : \mathbb{R}^4 \times \mathbb{R}^{10} \rightarrow \mathbb{R}^4$. The economist desires to solve Eq. (15) for the parameter value $\boldsymbol{\lambda}^*$, but she does not have a good starting solution for the Newton-like algorithm. Fortunately, the economist happens to know that the solution to Eq. (15) is \mathbf{z}_0 when the parameter vector is $\boldsymbol{\lambda}_0$, i.e. $F(\mathbf{z}_0, \tilde{\sigma}_s, \boldsymbol{\lambda}_0) = 0$.

With this information the economist defines the homotopy function, $H : \mathbb{R}^4 \times [0, 1] \rightarrow \mathbb{R}^4$, where $H(\mathbf{z}_0, 0) = 0$, $H(\mathbf{z}^*, 1) = 0$, and \mathbf{z}^* is the solution to Eq. (15) when the parameter value is $\boldsymbol{\lambda}^*$. This homotopy function provides the economist with a function $\mathbf{z}(\tau)$ that satisfies $H(\mathbf{z}(\tau), \tau) = 0$ for all $\tau \in [0, 1]$. Hence, the economist can gradually move τ from zero to one and $\mathbf{z}(\tau)$ will map out the path to the solution \mathbf{z}^* by providing the economist with the Newton-like algorithm's starting points for the next value of τ .

In our optimization problem we not only apply homotopy theory in solving Eq. (12), we also use the same concept in determining the initial values of the coefficient vectors \mathbf{v}_i , for $i = 1, 2, 3, 4$. In the above discussion on homotopy theory we assumed that the economist knows the solution when the parameter vector is equal to $\boldsymbol{\lambda}_0$. However, for our optimization problem we do not initially know the solution for any value of $\boldsymbol{\lambda}$. To provide an initial solution we instead utilize the PEA approach for a simple case of the optimization problem, i.e. we choose a $\boldsymbol{\lambda}$ for which the PEA converges. We choose the limiting case where $\sigma \rightarrow 0$ and $\alpha \rightarrow 0$ causes the Euler equations of the optimization problem to be

$$X_s = E_s \left[\left(\frac{\rho p_s^*}{p_{s+1}^*} (1 + R_s) \right)^{-1} X_{s+1} \right]^{-1} \quad (16)$$

$$m_{1s} = E_s \left[\left(\frac{1}{\delta_1} \frac{\beta \rho p_s^* R_s}{1 - \beta p_{s+1}^*} \right)^{-1} X_{s+1} \right] \quad (17)$$

¹³See Garcia and Zangwill for a more extensive study on homotopy theory being applied to fixed point problems.

$$m_{2s} = E_s \left[\left(\frac{1}{\delta_2} \frac{\beta \rho}{1 - \beta} \frac{p_s^*(R_s - r_{2s})}{p_{s+1}^*} \right)^{-1} X_{s+1} \right] \quad (18)$$

$$m_{3s} = E_s \left[\left(\frac{1}{\delta_3} \frac{\beta \rho}{1 - \beta} \frac{p_s^*(R_s - r_{3s})}{p_{s+1}^*} \right)^{-1} X_{s+1} \right] \quad (19)$$

and Φ of Eq. (12) to be the identity function.

The reader will notice that under this limiting parameter vector solving for z_s in each iteration of the PEA approach no longer requires a Newton-like numerical procedure. Rather, we are able to explicitly solve for the vector z_s at each iteration by setting the solutions equal to the parameterized value of the expectation operator, $\exp[\text{poly}(\tilde{\sigma}_s, \mathbf{v}_i)]$.

After convergence is reached with the PEA algorithm, homotopy theory suggests that the economist use the calculated coefficient vectors, \mathbf{v}_i , for $i = 1, 2, 3, 4$, as the starting coefficients for the PEA approach and the associated PEA solutions $\{z_s\}$ as the initial guesses for the Newton-like algorithm when the parameter values are either $\sigma \rightarrow 0$ and $\alpha = 0.1$ or $\sigma = -0.1$ and $\alpha \rightarrow 0$. The coefficients found with the PEA approach for the new parameter vector are then used as starting values for the PEA approach when λ 's values are again slightly changed. This iterative process of using the previous optimization problems solutions as the starting values for the next optimization problem continues until the economist reaches the specific combination of parameters she desires.

For the special limiting case of $\sigma \rightarrow 0$ and $\alpha \rightarrow 0$ we set all of the components of \mathbf{v}_i^0 equal to zero except for the coefficients of the constants. These coefficients are equal to the natural logarithm of the expected value of $g_i(\mathbf{1}_{s+1}, \phi_{s+1})$, for $i = 1, 2, 3, 4$.¹⁴ The initial solutions $\{z_s^0\}$ are also all set equal to the expected values of $g_i(\mathbf{1}_{s+1}, \phi_{s+1})$.¹⁵

4 Simulation Results

In this section we present some preliminary finding for the PEA solutions to this paper's dynamic optimization problem. All the calculations were programmed in Fortran 77 and

¹⁴This is the method that is recommend by Bansal et. al. (1992).

¹⁵Another equally valid method for setting the initial values of \mathbf{v}_i 's is to regress $g_{i,s}$ on $\exp[\text{poly}(\tilde{\sigma}_s, \mathbf{v}_i)]$ with $\{z_s\}$ equal to the steady state solution.

compiled and executed on a SPARC 10 workstation.¹⁶ We experimented with different convergence levels and discovered that the strictest convergence criterion for which the PEA approach would consistently converge under 40,000 iterations was $\xi = 0.03$. Convergence of the solutions varied with each model but if the PEA algorithm converged it usually did so within thirty minutes. In addition, by applying homotopy theory once we obtained a PEA solution convergence for similar parameter values was much quicker and took less than a hundred iterations.

However, convergence of the PEA approach does not guarantee that the calculated solutions will be equal to the ‘true’ solutions of the optimization problem. Because the PEA approach is a numerical algorithm it can only provide an approximation of the ‘true’ solution. Hence, to determine if the approximation is close to the ‘true’ solution we employ the den Haan, Marcet test statistic (DHM-stat) [den Haan and Marcet (1992) and Taylor and Uhlig (1990)]. The DHM-stat provides a test of the theoretical martingale property $E[\boldsymbol{\nu}_s \otimes \mathbf{h}_{sij}] = 0$ where $\boldsymbol{\nu}_s$ is the 4×1 residual vector from the Euler equations and $\mathbf{h}_{sij} = \boldsymbol{\sigma}_{s-j}^i$. If Hansen’s (1982) regularity conditions hold and the approximation is an exact solution satisfying the above martingale property then the DHM-stat

$$TBA^{-1}B \tag{20}$$

where $B = 1/T \sum_{s=t}^{t+T} [\boldsymbol{\nu}_s \otimes \mathbf{h}_{sij}]$ and $A = 1/T \sum_{s=t}^{t+T} [\boldsymbol{\nu}_s \otimes \mathbf{h}_{sij}][\boldsymbol{\nu}_s \otimes \mathbf{h}_{sij}]'$, will be distributed in law χ^2 with 20 degrees of freedom. In the following results we have set the DHM-stat’s i and j equal to one.

In each case the order of the approximating polynomial is equal to one. We attempted to increase the order of the polynomial to two, both with and without the cross variable terms, but the PEA approach became explosive after a few iterations. The problem associated with the higher order polynomial may be the result of the the polynomial’s large number of state arguments. If this same problem exists for other flexible functional forms is a topic for future research.

Table 1 shows all the parameter values, except for α and σ , used in the PEA algorithm. To insure that λ accurately reflects the empirical world we set the parameters contained in

¹⁶I am deeply indebted to Bob Hussey for providing and explaining the initial coding from which I was able to change to fit the model found in this paper.

the Euler equations (5)-(8) equal to the generalized method of moment estimates found by Barnett and Yue (1991). The other parameters associated with the two Markovian processes (9)-(10) were set equal to the estimated parameters for the two VAR(1) using monthly data from January 1960 to December 1990. Lastly, in Table 1 the parameter values H_{ijk} for $i = 1, 2$, and $j, k = 1, 2, 3$, represent the Cholesky factorization of the random disturbances covariance matrix, i.e. $H_i H_i' = \Sigma_i$.

To estimate the parameters of Eq. (10) we used the Consumer Price Index for p_s^* and nominal GNP for I_s . The interest rate data for Eq. (9)'s regression was supplied to us by the St. Louis Federal Reserve Bank. Each investment's rate of return was adjusted to a common one-month maturity with Farr and Johnson's (1985) yield curve adjustment to eliminate liquidity premiums associated with longer maturity investments. In addition, the investment's interest rates were adjusted to an annualized one-month yield on a bond interest basis (365 day) as opposed to a bank basis (360 day). For the regression of Eq. (9) the benchmark assets rate of return was measured by

$$R_s = \max[r_{BAA}, (r_{is}, i = 1, 2, 3)]$$

where r_{BAA} is the rate on Moody's BAA corporate bond, r_{2s} was measured by the average of the following investment returns

- RMMDAC: Rate paid on Money Market Deposit Accounts at commercial banks
- RMMDAT: Rate paid on Money Market Deposit Accounts at thrift institutions
- RSDCB: Rate on savings deposits less RMMDAC
- RSDSL: Rate on savings deposits at FDIC-Insured savings banks
- RSNOWC: Rate paid on Super NOWs at commercial banks
- RSNOWT: Rate paid on Super NOWs at thrift institutions
- RONRP: Rate paid on overnight dealer financing in the repurchase market
- RONED: Rate paid on overnight eurodollars from London
- RMMMF: Average yield on Money Market Mutual Funds

and r_{3s} was the average of the returns

- RLTDCEB: Rate on large time deposits at commercial banks
- RSTDTH: Rate paid on small time deposits and retail repurchase agreements at thrifts.

In the above definition for interest rates we have implicitly assumed that m_s contains three aggregated elements. In other words, there exists a vector of investment vehicles a_s that are weakly separable in the blocks described above.¹⁷

The size of our simulation is equal to 100, i.e. $T = 100$. We chose a small simulation size because of the explosive nature of the PEA approach for large simulations if the solutions to the Euler equations (5)-(8) are not stationary processes.¹⁸ This volatile behavior of the PEA approach results from the size of the set containing the endogenous state variables becoming bigger with larger T . The increased size of the endogenous state variable set presents the PEA approach with the difficulty of trying to determine which observations in the state space are relevant and which ones are not. Because the PEA approach weights each observation equally as more and more irrelevant exogenous state observations occur the PEA method becomes explosive trying to fit the irrelevant observations with the relevant states. To overcome this problem of endogenous oversampling for large simulations the economist can produce a number of small simulations.¹⁹

4.1 Limiting Case

Table 2 shows the summary statistics for the limiting case. The Den Haan-Marcet statistic is also found in Table 2 with its p -value in parenthesis. In the limiting case the DHM-stat statistically accepts the hypothesis that the PEA algorithm has provided correct solutions.

In Fig. 1 the growth rates of the theoretical and Divisia monetary aggregate for the limiting case are plotted. From these plots we found that the Divisia monetary aggregate's

¹⁷Barnett and Yue (1990) used this same sub-nested blocking of a_s to obtain the GMM estimates for this model.

¹⁸Marcet and Marshall (1992) provide a set of conditions and an alternative PEA approach that overcomes the problems associated with unstable solutions. They advocate taking a large number of samples of random disturbances to generate the exogenous state variables and use the sample average of the polynomial's coefficient estimates, v_i 's, as the coefficients for approximating the expectation operators.

¹⁹See Marshall (1992) and Marcet and Marimon (1991) for examples of this approach.

growth rate identically tracks the theoretical's growth rate up to two decimal places. This result favorably compares to the three decimal accuracy of the Divisia aggregate when the representative agent has risk neutral preferences [Barnett (1980)]. Furthermore, our discovery supports Barnett and Yue's (1990) empirical findings that the Divisia aggregate tracked reasonable well a GMM estimated theoretical monetary aggregate. Hence, by increasing the measure of relative risk aversion from zero to one the Divisia monetary aggregate loses its second-order explanatory powers, but it continues to provide information about the theoretical monetary aggregate up to the second decimal place.

Using the solutions from the limiting case we applied homotopy theory to obtain the solutions for the cases where $\sigma \rightarrow 0$ but α is allowed to move from zero to 0.6.²⁰ By keeping σ the same and allowing α to change we are able to keep the level of relative risk aversion equal to one, but alter the theoretical monetary aggregate from a Cobb-Douglas function to a CES aggregation function. Furthermore, this change in parameter values moves us closer to the empirically estimated value of α , which Barnett and Yue (1991) found to be equal to 0.8426. Unfortunately, we were unable to obtain convergence with the PEA approach for values of α larger than 0.7.

The summary statistics for these cases are found in Table 3 and 4. The reader will note that in each of these tables the DHM-stat rejects the hypothesis that the approximate solutions are equal to the true rational expectation equilibrium. These results suggest that the PEA approach loses its ability to find approximate solutions that are close to the real Euler equation solutions when the Euler equations can not be written in reduced form. This shortcoming could be the result of either the PEA approach or the inability of the Newton-like algorithm we used to solve the system of equations.

Even with the rejection of the approximate solutions, in Fig. 2 through 7 the Divisia monetary aggregate's growth rate continues to closely track that of the theoretical aggregate. Although the tracking by the Divisia monetary aggregate's growth rate is no longer identical to the second decimal place of the true aggregate's growth, the closeness found in these figures suggests that the Divisia monetary aggregate continues to produce reliable

²⁰We applied homotopy theory for the cases $\alpha = 0.1, 0.2, 0.3, 0.4$ by using the PEA solutions from $\sigma \rightarrow 0$ and $\alpha \rightarrow 0$ as the initial starting values for the Newton-like solutions of the system of equations. For the other two cases $\alpha = 0.5, 0.6$ we respectively used the polynomial coefficients from the PEA solutions for $\alpha = 0.45, 0.5$ as the starting coefficients for the PEA approach.

nonparametric approximations of the theoretical aggregate.

4.2 Higher Risk Case

In this subsection we present the PEA solutions for our dynamic optimization problem when the value of σ becomes smaller, i.e. the level of relative risk aversion increases. However, unlike the previous solutions in this subsection we use the convergence criterion advocated by Marshall (1992) to determine the stopping point for the PEA algorithm.²¹ Marshall argues that the PEA algorithm converges to an approximate equilibrium if the coefficient vectors, \mathbf{v}_i for $i = 1, 2, 3, 4$, are the optimal prediction vectors. The \mathbf{v}_i 's are considered optimal coefficient vectors if they minimize the sum of squared residuals associated with the PEA approaches' least square regressions. Hence, the convergence criterion used in this subsection is

$$\max_i \left| \frac{SSR(\mathbf{v}_i^k) - SSR(\mathbf{v}_i^{k-1})}{SSR(\mathbf{v}_i^{k-1})} \right| \leq \xi \quad (21)$$

where $SSR(\mathbf{v}_i^k) = \sum_{s=t+1}^{t+T} \nu_{isk}^2$, and ν_{isk} is the s observation of the residuals associated with the estimated coefficient vector from the k th iteration. As before all the results were calculated using $\xi = 0.03$.

The summary statistics for the PEA solutions are found in Table 5 and 6.²² Once again the DHM-stat with p -values approximately equal to zero soundly rejects the hypothesis that the approximated solutions are equal to the true rational expectation solutions. Furthermore, the Divisia and the theoretical monetary aggregate's growth rates plotted in Fig. 8 to 15 reveal either the poor performance of the Divisia monetary aggregate in tracking the true aggregate, or the inability of the PEA approach to come close to approximating the true solutions. From the encouraging results found when $\sigma \rightarrow 0$ and $\alpha \rightarrow 0$ we suspect that it is the later.

The reader will note that as σ becomes negative the Euler equations (5)-(8) include both the current and future period's theoretical monetary aggregate. This, along with the

²¹We initially used the convergence criterion found in Eq. (14) but the PEA algorithm failed to converge within 40,000 iterations.

²²In each case we used homotopy theory to provide the initial values for the polynomial's coefficients. For those models with $\sigma = -0.1, -0.2, -0.3$ we used the parameter values found for the solution to the limiting case as initial values, while for $\sigma = -0.4, -0.5, -0.6, -0.7$ we used the PEA's solution parameters from the model with $\sigma = -0.3$ and $\alpha \rightarrow 0.0$.

complex nature of the implicit system of equations, seems to create a serious problem for the PEA approach. We can only hypothesize that this is the case, but in the future we desire to determine if this hypothesis is correct by finding the approximate rational expectation solutions to our dynamic optimization problem with one of the other Euler equation techniques supplied by Coleman (1990, 1991), Baxter (1991), or Baxter, Crucini, and Rouwenhorst (1990).

5 Conclusion

In conclusion we have provided the PEA solutions to a dynamic optimization problem where a representative agent allocates her earnings between a consumption aggregate good and a monetary aggregate comprised of three monetary assets. It is important to find the approximate rational expectation equilibrium to this optimization problem in order to determine how strong the link between index and aggregation theory is under risk. The stronger this link the more reliable the Divisia monetary aggregate is as a nonparametric approximation to the ‘true’ theoretical monetary aggregate. In addition, by incorporating the PEA approach in this task we have also provided a dynamic programming problem that proves to be a formidable test to the PEA’s ability to produce reliable solutions. We found both success and failure in these objectives.

The results found in this paper show that the PEA method performed poorly in those cases where the parameter values caused the Euler equations to become an implicit set of equations. This occurred whenever the dynamic programming problem’s α or σ did not equal the limiting case values of $\alpha \rightarrow 0$ and $\sigma \rightarrow 0$. Hence, when either the level of relative risk aversion increased above one, or if the theoretical monetary aggregate was a CES function rather than the limiting case’s Cobb-Douglas, the PEA approach failed to provide approximate solutions that were close to the ‘true’ rational expectation equilibrium.

The poor tracking performance of the Divisia monetary aggregates also occurred in these same cases. Although the Divisia’s tracking ability was considerably worse for those cases where the level of relative risk aversion was higher, the Divisia aggregate did provide helpful information about the growth of the theoretical monetary aggregate when a CES function was used as the theoretical aggregate. However, knowing how poorly the PEA

approach performed in providing correct solution for these cases we reserve judgement on how good or bad the Divisia aggregate tracks the theoretical aggregate when relative risk increases above one or the theoretical monetary aggregate is a CES function. A better numerical method for calculating the rational expectation solutions needs to be developed before any conclusion on this matter can be made.

The one case where both the PEA approach and the Divisia aggregate performed at their best was the limiting case where $\sigma \rightarrow 0$ and $\alpha \rightarrow 0$. Under this special limiting case the PEA approach obtained solutions that were statistically close to the optimization problem's true solutions as measured by the Den Haan, Marcet statistic. The success of the PEA approach also provided the Divisia aggregate with those solutions that caused it to identically track the growth of the theoretical aggregate up to its second decimal place. Hence, our findings for the special limiting case validates Barnett and Yue's (1990) empirical results that the Divisia aggregate closely tracks the estimated theoretical aggregate, but it also provides the area of economic aggregation with the incentive to find a Diewert-like theorem that links aggregation theory with index theory when risk aversion is present.

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ρ	0.89750
β	0.95350
δ_1	0.46560
δ_2	0.33710
δ_3	0.19730
a_{11}	0.00252
a_{12}	0.00040
a_{13}	0.00178
a_{21}	0.00149
a_{22}	0.00414
A_{111}	0.90260
A_{121}	-0.00031
A_{131}	-0.01017
A_{112}	0.00755
A_{122}	0.99056
A_{132}	0.01083
A_{113}	0.09682
A_{123}	0.00181
A_{133}	0.98477
A_{211}	0.65470
A_{221}	-0.32513
A_{212}	-0.02609
A_{222}	-0.15840
H_{111}	0.00817
H_{112}	0.00181
H_{122}	0.00192
H_{113}	0.00451
H_{123}	0.00148
H_{133}	0.00352
H_{211}	0.00246
H_{212}	2.0D-05
H_{222}	0.00624

Table 1: Parameter Values

$\sigma \rightarrow 0 \alpha \rightarrow 0$		
	μ	σ^2
X	24655.300	1.4D+07
m_1	6954.800	1.8D+06
m_2	10662.800	5.1D+06
m_3	17586.900	3.9D+07
M	9688.920	4.0D+06
\dot{Q}	77.120	3.2D+02
\dot{M}	-0.005	7.9D-03
\dot{Q}	-0.006	8.4D-03
DMS	0.0018 (1.0)	

Table 2: Limiting Case

Case		μ	σ^2
$\sigma \rightarrow 0.0 \alpha = 0.1$	X	2.4D+07	
	m_1	6.5D+06	
	m_2	1.0D+07	
	m_3	1.8D+07	
	M	9.6D+06	0.4D+13
	Q	74.6529	0.3D+03
	\dot{M}	-0.004	0.9D-02
	\dot{Q}	-0.005	0.1D-01
	DMS	98.99 (0.0)	
$\sigma \rightarrow 0.0 \alpha = 0.2$	X	6.7D+12	
	m_1	1.7D+12	
	m_2	2.9D+12	
	m_3	5.5D+12	
	M	*****	3.2D+23
	Q	69.78200	3.2D+02
	\dot{M}	-0.00371	1.1D-02
	\dot{Q}	-0.00486	1.3D-02
	DMS	98.99 (0.0)	
$\sigma \rightarrow 0.0 \alpha = 0.3$	X	1.2D+05	
	m_1	3.0D+04	
	m_2	5.4D+04	
	m_3	1.1D+05	
	M	5.0D+04	1.2D+08
	Q	65.30593	3.3D+02
	\dot{M}	-0.00369	0.00124
	\dot{Q}	-0.00478	0.00146
	DMS	98.99 (0.0)	

Table 3:

Case		μ	σ^2
$\sigma \rightarrow 0.0 \alpha = 0.4$	X	1.2D+05	
	m_1	3.0D+04	
	m_2	5.5D+04	
	m_3	1.3D+05	
	M	5.0D+04	1.4D+08
	Q	60.65087	3.4D+02
	\dot{M}	-0.00350	0.00141
	\dot{Q}	-0.00439	0.00174
	DMS	98.99 (0.0)	
$\sigma \rightarrow 0.0 \alpha = 0.5$	X	3.1D+05	
	m_1	5.9D+04	
	m_2	1.4D+05	
	m_3	3.9D+05	
	M	1.3D+05	9.2D+08
	Q	55.79256	3.5D+02
	\dot{M}	-0.00267	0.00172
	\dot{Q}	-0.00313	0.00224
	DMS	98.99 (0.0)	
$\sigma \rightarrow 0.0 \alpha = 0.6$	X	1.5D+07	
	m_1	2.3D+06	
	m_2	6.7D+06	
	m_3	2.4D+07	
	M	6.8D+06	2.6D+12
	Q	51.51806	3.6D+02
	\dot{M}	-0.00048	0.00231
	\dot{Q}	-0.00055	0.00304
	DMS	98.99 (0.0)	

Table 4:

Case		μ	σ^2
$\sigma = -0.1 \alpha \rightarrow 0.0$	X	11486.50	
	m_1	3289.34	
	m_2	5211.88	
	m_3	7938.32	
	M	67.58	30.61
	Q	87.76	378.42
	\dot{M}	-0.00244	0.00064
	\dot{Q}	-0.00472	0.00851
	DMS	98.99 (0.0)	
$\sigma = -0.2 \alpha \rightarrow 0.0$	X	7973.33	
	m_1	2314.42	
	m_2	3742.51	
	m_3	5637.62	
	M	58.70	22.89
	Q	89.22	392.83
	\dot{M}	-0.00232	0.00069
	\dot{Q}	-0.00403	0.00938
	DMS	98.99 (0.0)	
$\sigma = -0.3 \alpha \rightarrow 0.0$	X	11230.4	
	m_1	3179.4	
	m_2	4875.3	
	m_3	7877.2	
	M	66.22	0.00291
	Q	68.48	0.00028
	\dot{M}	-0.00253	0.00114
	\dot{Q}	-0.00167	0.01704
	DMS	98.99 (0.0)	

Table 5:

Case		μ	σ^2
$\sigma = -0.4 \alpha \rightarrow 0.0$	X	6147.51	
	m_1	1791.93	
	m_2	2855.18	
	m_3	4586.94	
	M	52.81606	1.7D+01
	Q	78.64297	3.1D+02
	\dot{M}	-0.00240	0.00062
	\dot{Q}	-0.00462	0.00892
	DMS	98.97 (0.0)	
$\sigma = -0.5 \alpha \rightarrow 0.0$	X	3500.93	
	m_1	1040.49	
	m_2	1706.88	
	m_3	2784.05	
	M	42.68298	1.1D+01
	Q	83.70820	3.4D+02
	\dot{M}	-0.00227	0.00058
	\dot{Q}	-0.00448	0.00859
	DMS	98.97 (0.0)	
$\sigma = -0.6 \alpha \rightarrow 0.0$	X	2138.06	
	m_1	644.63	
	m_2	1081.45	
	m_3	1780.79	
	M	35.44056	0.7D+01
	Q	87.30851	3.7D+02
	\dot{M}	-0.00217	0.00059
	\dot{Q}	-0.00414	0.00894
	DMS	98.97 (0.0)	
$\sigma = -0.7 \alpha \rightarrow 0.0$	X	1412.10	
	m_1	424.82	
	m_2	710.43	
	m_3	1201.31	
	M	30.10261	0.5D+01
	Q	86.53571	3.5D+03
	\dot{M}	-0.00209	0.00060
	\dot{Q}	-0.00392	0.00913
	DMS	98.99 (0.0)	

Table 6: