Non-neutrality of economic policy: An application of the Tinbergen-Theil's approach to a strategic context[†]

N. Acocella[*] and G. Di Bartolomeo[*]

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Abstract

Issues of policy effectiveness and neutrality are widespread in the economic literature. They have been increasingly raised in specific contexts within the class of LQ (linear-quadratic) policy games in the last 20 years, notably with reference to monetary policy. The more general conditions ensuring non-neutrality in a strategic environment remain however to be inquired. We fill this gap by applying the classical theory of economic policy to a strategic context. This is also useful to highlight some existence conditions for policy game solutions. We restrict ourselves to the common LQ-games in a static perfect information framework, but our simple logic can be extended to other more general situations.

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[*] Department of Public Economics, University of Rome *La Sapienza*.

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1. Introduction

Issues of policy effectiveness were first analyzed in formal terms by Tinbergen (1952, 1956). He also asserted the need for policymakers to have recourse to second-best solutions – by maximizing the value of their preference function subject to the model representing the economy – in the case of a non-perfectly controllable system, an approach later developed by Theil (see Theil, 1964).[1] More general conditions for controllability both in a static and dynamic context were then stated (see Preston and Pagan, 1982; Hughes Hallett and Rees, 1983).

Tinbergen, Theil, and the other founding fathers of economic policy were not concerned with analyzing the effectiveness of specific instruments. However, in the framework of the classical theory of economic policymaking it is not difficult to find the counterpart of the concepts of policy ineffectiveness and neutrality raised in the economic literature with reference to e.g. monetary policy or fiscal policy.[2]

The classical theory of economic policy has been the object of fierce criticism from a number of points of view. The introduction of rational expectations led to an assertion of ineffectiveness of monetary policy more forceful than that stated by Friedman (1968) (see Sargent and Wallace, 1975). In a similar way, with rational expectations fiscal policy was considered to be ineffective on income. A proposition of policy neutrality or "invariance" was then stated. Apart from the critiques advanced with reference to the effectiveness of specific instruments, Lucas (1976) raised

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[1] A good account of the early contributions to the theory of economic policy is in Hughes Hallett (1989).

[2] See Holly and Hughes Hallett (1989).

the more general and forceful argument according to which a Tinbergen-type decision model is inconsistent with the assumption of rational expectations.

In more recent years, following Barro and Gordon (1983),[3] a new approach to the analysis of economic policy has been developed, that of policy games. Within this approach the issues of effectiveness of specific policy instruments have been raised again, mainly with reference to monetary policy, starting from the pioneering article of Gylfason and Lindbeck (1994).[4] Formal conditions leading to monetary policy ineffectiveness – or neutrality – have been explicitly or implicitly investigated in specific setups within the class of LQ-games. Such conditions are apparently very different from those stated in the classical analysis of economic policy: the latter are expressed in terms of matrix ranks, whereas the former have to do with the nature of players' preferences and the strategic interaction.[5]

By applying the classic theory of economic policy to a strategic context, we find general conditions for non-neutrality and highlight some existence conditions for policy games (Nash/Stackelberg) equilibrium. We also check our conditions with references to the effectiveness of monetary policy

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[3] Barro and Gordon (1983) deliver the well-known prediction of monetary neutrality as a result of the private-sector expectations of the monetary policy. The private sector forms rational expectations of the money supply and acts to fully crowd-out the effects of monetary action on the real output by adjusting nominal wages. A socially inefficient inflation bias ensues. Barro-Gordon's analysis can be – and usually is (see, among others, Stokey, 1990, and Sargent, 2002: Chapter 3) – expressed in terms of a Stackelberg game between the central bank and the private sector when the private sector strategies are explicitly modeled.

[4] See, among others, Acocella and Ciccarone (1997), Grűner and Hefeker (1999), Guzzo and Velasco (1999), Lawler (2000 and 2001), Soskice and Iversen (2000), Coricelli *et al.* (2000, 2001), Cukierman and Lippi (2001), Jerger (2002), Acocella and Di Bartolomeo (2004), Lippi (2003).

already found in the literature. The main advantage of our approach is that existence and ineffectiveness can be easily verified *a priori* by applying a simple counting rule, without solving the game. We restrict ourselves to the case of perfect information since it is well known that asymmetric information is itself a source of non-neutrality. However our simple logic can be extended to more complex frameworks. Some intuitions in that direction are provided in the last section.

The rest of the paper is organized as follows. The next section is pedagogical and mainly needed to introduce a clear terminology and definitions.[6] It broadly describes the traditional Tinbergen-Theil's approach (TTA, henceforth) to economic policy in terms of strictly quadratic preferences and a linear model. Section 3 presents our generalization of the TTA to a strategic context, by considering policy games with strictly quadratic preferences, and states the conditions for policy neutrality. This generalization is fully consistent with Lucas criticism since it considers the "private" sector's action as endogenous. Section 4 provides some further generalizations to the case of linear-quadratic preferences. Section 5 checks that the conditions for the effectiveness of monetary policy found in the literature respect those stated in the previous section of the paper. Section 6 concludes and provides some intuitions for further generalizations.

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[5] See Acocella and Di Bartolomeo (2004).

[6] Indeed, throughout the paper, a particular emphasis is placed on definitions and terminology used, since in literature no unanimous consensus on wording exists.

2. The Tinbergen-Theil approach[7]

2.1 The Tinbergen-Theil approach

In this section we consider the optimization problem of a single decision-maker (from now on, without loss of generality, the *Government*). We assume that the Government aims to achieve certain given targets[8] and, if this is not possible, to minimize deviations from them according to a quadratic function. Approaching the problem in this way has the advantage to merge together the fixed and flexible target approach making the former a particular case of the latter (see Preston and Pagan, 1982). It implies that we implicitly assume that, if the fixed target approach fails, the Government sets the instruments according to a flexible approach. Although the flexible approach is not the only alternative to the fixed one, flexible targets seem to be the alternative more in line with our attempt at reformulating the classical theory of economic policy with reference to a strategic context.

The economic relationships between variables are defined by the following general linear algebraic system:

(1) $\mathrm{A}\tilde{y} = \mathrm{B}\tilde{z} + \mathrm{K}$

where A and B are coefficient matrices, $\tilde{y}$ and $\tilde{z}$ are the vectors of *possible* target and instrument variables, K is a vector of linear combination of other variables and/or white noise shocks, which the Government takes as

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[7] Throughout the paper we use the following notation. All vectors are real column vector defined by their dimension; all matrices are real matrices defined by their two dimensions. Considering two vectors, a and b, (a, b) is a column vector; considering matrices A and B with the same number of rows, $[A : B]$ is a matrix formed by merging the two matrices. $A_{i,j}$ is the i,j-element of matrix A, a_i is the i-entry of vector a.

[8] We will later relax the assumption of a given target by considering also the possibility of non-satiation.

given.[9] Indeed, for the sake of exposition, the order of variables is not random as we will later see (see also Appendix A).

First and foremost, in order to consider the Government's problem in the TTA context, we limit our attention to the following linear algebraic sub-system of equation (1):[10]

(2) $Ay = Bu + K$

where $y \in \mathbb{R}^m$ is the vector of the relevant economic variables (Government's target variables), $u \in \mathbb{R}^n$ is the vector of the Government's policy instruments, $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times n}$ are parameter matrices (i.e. the target and instrument coefficient matrices), and $K \in \mathbb{R}^m$ is a vector of constants, i.e. each component is a linear combination of exogenous constants and/or white noise shocks. We assume that $rank\left[A\right] = m$ and $rank\left[B\right] = \min\left\{m, n\right\}$. The full-rank assumptions imply both that all the targets and instrument variables are linearly independent (independence assumption).[11] Intuitively there are m distinct targets and n distinct instruments set by the Government.

The linear reduced-form model can be written in matrix form as:

(3) $y = A^{-1}Bu + A^{-1}K = Cu + \overline{C}$

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[9] For the sake of simplicity, in this paper the stochastic structure is not explicitly considered and the certainty equivalence principle is advocated as we will consider LQ-problems only. Of course, the generalization cannot be advocated for other kinds of uncertainty such as the multiplicative one, incomplete or asymmetric information.

[10] The relationship between the two equation systems is explained in Appendix A. In this section this relationship is not very relevant since there is only one policymaker who acts in a parametric context. Later on it will become significant.

[11] Full-rank assumption is always assured by an appropriate manipulation of the original system (1), i.e. by removing linearly dependent redundant target and instrumental variables.

provided A is non-singular from our rank assumptions.

At this stage it is useful to define Government's policy neutrality by using the reduced form (3). We start by introducing the concept of effectiveness and then we use this concept to define neutrality.

Definition 1 (effectiveness): An instrument is effective with respect to a target variable if changes in the instrument determine changes in the equilibrium value of that target; otherwise it is ineffective.

Definition 2 (exogenous policy neutrality): Economic policy is neutral with respect to a target variable, if all the instruments are ineffective with respect to that target variable.

In the present case each zero in matrix C implies the ineffectiveness of one instrument with respect to one target. Specifically, element $C_{i,j} = 0$ implies that the *j*-instrument is ineffective with respect to *i*-target. Neutrality with respect to *i*-target emerges if the row entries of matrix $C = A^{-1}B \in \mathbb{R}^{m \times n}$ are all zeros. Our rank assumptions rule neutrality.[12]

The aim of the Government is to control the economic system (2), i.e. to determine the values of the target variables. More formally we can define controllability as follows.[13]

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[12] It is trivial that when only one instrument is available effectiveness corresponds to non-neutrality and neutrality corresponds to ineffectiveness. However, in multi-instrument cases this is not in general true.

[13] In this paper we consider only non-differential games (both static and dynamic in the sense of the players' timing). Thus the controllability considered here is the so-called static controllability. In principle, all our results can be easily extended to the differential approach as for the traditional case (e.g. Preston and Pagan 1982: Chapters 1-3), but the differential policy games involve many technical difficulties that are outside the scope of this paper (for a discussion, see, Petit, 1990: Chapters 8-9).

Definition 3 (controllability): A system is controllable if the Government can determine the values of the target variables for any possible vector of desired targets by choosing an appropriate policy (i.e. vector of instruments).

Controllability can be also defined in weak terms: A system is weakly controllable if, given a vector of desired targets, the Government can determine it by choosing an appropriate policy. Controllability clearly implies weak controllability but not conversely. Moreover, it should be noticed that controllability is an existence condition for the policy (*existence* problem). Thus it assures neither the uniqueness of the policy (*uniqueness* problem) nor how to determine it (*policy design* problem).

Notice that controllability implies policy non-neutrality but the converse is not true.

By defining the vector of the desired targets as $\bar{y} \in \mathbb{R}^m$, existence conditions for a solution are easily found by applying standard mathematical techniques.[14]

Theorem 1 (controllability): the system $y = Cu + \bar{C}$ is weakly controllable for the desired given vector of targets $\bar{y}$ if and only if $rank\left[C : \bar{y}\right] = rank\left[C\right]$; it is controllable for any vector of desired target $y \in \mathbb{R}^m$ if and only if $rank\left[C\right] = m$.

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[14] Formal proofs of the statements of this section are not reported since they summarize some well-known concepts. For an extensive discussion we refer to Preston and Pagan (1982).

Being $rank\left[C\right] \leq \min\left\{m,n\right\}$, $rank\left[C\right] = m$ requires $n \geq m$. Thus controllability embodies the famous Tinbergen's precepts about the number of instruments and targets. Formally Tinbergen Theorem comprises two conditions.

Theorem 2 (Tinbergen Theorem): The Government can achieve any vector of independent targets by an appropriate vector of instruments if and only if the number of independent instruments is equal to, or greater than, the number of targets.

The theorem can be qualified for uniqueness, which requires $rank\left[C\right] = m = n$, i.e. the number of independent targets must be equal to that of the independent instruments. The solution of the policy design is $u = \left(C'C\right)^{-1} C'\left(\bar{y} - A^{-1}K\right)$, which in the case of uniqueness becomes $u = B^{-1}\left(A\bar{y} - K\right)$. Henceforth, we will say that an equation system is TT-controllable by a policymaker if the number of independent instruments equals that of independent targets.

Now assuming that the policy model fails to satisfy the appropriate weak or strong existence criterion (controllability), the first best policy cannot be achieved but still the Government's action can be effective, i.e. give it the possibility to have an influence on its objectives and to "go close to the unreachable targets." Formally, failure to find a solution for the fixed target problem generates a second best policy, in terms of the flexible target approach.

The flexible target approach is based on the minimization of a criterion (loss) function. A useful formalization of the Government's cost for deviations of the relevant variables from their target values is the following quadratic form:

$$(4) \qquad U = \left(y - \bar{y}\right)' Q \left(y - \bar{y}\right)$$

where Q is a symmetric positive semi-definite matrix. In equation (4) we do not explicitly consider instrument costs for a reason that will be clear below. Quadratic functions are used not only for their mathematical tractability, but also for their useful economic properties. In fact, deviations from the target are associated to increasing costs and, therefore, the marginal rate of substitution between any couple of target variables is never constant but depends on the values of the two variables in the point where it is computed.[15] In addition, quadratic forms can be obtained as second-order Taylor approximations of more complex functions (for a recent example, see Woodford, 2003: 392-404).[16]

The flexible target policy is obtained by minimizing equation (4) subject to equation (3). The corresponding first order condition is:

$$(5) \qquad C'QCu = C'Q\left(\bar{y} - A^{-1}K\right)$$

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[15] The relevance of the variability of the marginal rate of substitution as well as that of the instrument costs will be developed more in detail in section 5.

[16] More in detail, "quadratic cost functions are of particular interest in game theory, firstly because they constitute second-order approximations of other types of nonlinear functions, and secondly because they are analytically tractable, admitting in general closed-form equilibrium solutions which provide insights into the properties and features of the equilibrium solution concept into consideration." (Başar and Olsder, 1995: 197). However, Taylor approximations are usually based on a more general specification of quadratic functions, which also includes linear terms. We refer to the latter as LQ-form, but introduce it only later since for the purposes of this section we can use the simpler quadratic form. For a more formal description of LQ-functions and their properties, see Frisch (1969), who is the father of the idea of loss functions representing the players' preferences rather than the aggregation of those of individual agents, and Petit (1990: Chapter 6).

Equation (5) can be rewritten in short terms as $\Phi u = K_{\Phi}$. As in the previous case, existence of a solution is ensured if $rank\left[\Phi : K_{\Phi}\right] = rank\left[\Phi\right]$. Uniqueness requires the non-singularity of Φ. The policy design clearly implies the following policy:

(6) $$u = \left(C'QC\right)^{-1} C'Q\left(\bar{y} - A^{-1}K\right)$$

If $n = m$, the above policy becomes $u = B^{-1}\left(A\bar{y} - K\right)$ – the same expression found in the fixed target case – which implies $y = \bar{y}$. This result directly derives from the omission of explicit instrument costs from equation (4). Hence if we omit explicit instrument costs we can simultaneously consider both the fixed and the flexible target approaches.[17]

Before leaving this section, we must notice that when considering the flexible approach any policy is endogenous. In this context we find it convenient to redefine neutrality as follows.

Definition 4 (endogenous policy neutrality): Economic policy is neutral with respect to a target variable y_i, if its equilibrium value is not affected by any change in policymaker's preferences.

Definition 4 generalizes definition 2 in the same way as the flexible target approach nests the fixed one.

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[17] Notice that, without any loss of generality, each instrument can be decoupled into two variables by considering its possible double nature of instrument and variable of possible interest for the policymaker.

2.2 Controllability and sub (partial) controllability

We have shown that, if the matrix C is square and full rank, the system is controllable. This is also the case of a rectangular matrix C with more columns (instruments) than row (targets) and a rank equal to the number of rows. By contrast, if C is a rectangular matrix with more rows than columns, the system is not controllable and a flexible target approach should be used. In this subsection we focus on this latter case.

If C is a rectangular matrix with $m > n$, the Government cannot achieve the target vector (i.e. his first-best outcome) and will trade off between his targets. Because of the quadratic form of preferences, the Government does not generally find it optimal to reach any given target exactly since the gain of getting closer to it decreases as the deviation from it is reduced. However, there is a *particular* case in which the Government exactly achieves part of his targets and trades off only between the others. In other words, notwithstanding the decreasing marginal rate of substitution between targets implied by the quadratic preferences, there is a case in which the optimal policy of the Government, derived by the flexible target approach, implies that the Government can perfectly control part of the system.

The reason is simple to explain. Imagine two *distinct* problems, one TT-controllable by the Government and another that is not, e.g. a 2 targets by 2 instruments and a 3 targets by 2 instruments independent LQ-problems. Merging the two problems together the Government will face a system of 5 equations (targets) with 4 unknowns (instruments). Although the new system is not controllable in the sense of getting some pre-assigned values for all the 5 targets, by solving the Government's optimization problem with the flexible approach it is clear that the Government will achieve the first two targets (for any possible target vector) and will trade off between the other three as the two problems are independent. Hence the system is not controllable but the Government can always achieve the first two targets. In this case we speak of sub-controllability, i.e. the Government perfectly controls part of the system.[18]

Formally, we first define an M_0-augmented block diagonal matrix as:

$$(7) \qquad \left[M_t, M_{t-1}, \ldots M_1, M_0\right]^{BD} \equiv \begin{bmatrix} M_t & 0 & \cdots & \cdots & 0 \\ 0 & M_{t-1} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & M_1 & 0 \\ 0 & \ldots & \ldots & 0 & M_0 \end{bmatrix}$$

where M_i for $i \in \{1, 2, \ldots, t\}$ are full-rank square matrices.

Second, we define the *BlockDiag* operator, as an operator that by permutations and scalar normalization transforms a matrix in an M_0-augmented block diagonal form:

$$(8) \qquad BlockDiag\left[M\right] = \left[M_t, M_{t-1}, \ldots M_1, M_0\right]^{BD}$$

where M_0 is a square matrix if and only if C is a square matrix. Of course, if $BlockDiag\left[M\right] = M_0$, $M = M_0$.

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[18] It is worth noticing that, in general, the Government can always control part of the system (many target variables as many are controls can always be controlled), but here we are saying that it can perfectly control part of the system as a result of his optimization process.

Without loss of generality, we assume that matrix *C* is already written in a *M*₀-augmented block diagonal form by preliminary appropriate row and/or column permutation:[19]

(9) $C = BlockDiag\left[C\right] = \left[C_t, C_{t-1}, \dots C_1, C_0\right]^{BD}$

Representing the economic system by (9) implies that the Government faces *t* controllable independent systems plus an additional independent system that is not controllable if *C* is not a full-rank square matrix. In other words, the Government can exactly set the values of the first $\sum_{i=1}^{t} rank\left[C_i\right]$ targets independently of the problem of setting the last $n - \sum_{i=1}^{t} rank\left[C_i\right]$. We can define the set of the controllable targets as the set of the first $\sum_{i=1}^{t} rank\left[C_i\right]$ targets by meaning that the Government can always achieve these targets by solving his optimization problem.

By basic linear algebra, the set of controllable targets can be defined in more general terms. By defining $e(i) \in \mathbb{R}^n$ as an eye vector, i.e. a vector of zero entries with the exception of the *i*-th entry, which is equal to one, and $col(M)$ as the column set of matrix *M*,[20] we can define the controllable target set as follows.

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[19] Hence also all the other vectors are adjusted. Invariant normalizations of this kind are rather common (see e.g. Engwerda *et al*., 2002).

[20] Notice that the column set has not a unique representation, but infinite equivalent ones.

Definition 5 (controllable target set). The controllable target set, associated with the Government's problem: max $U = (y-\bar{y})' Q (y-\bar{y})$ subject to $Ay = Bu + K$, is $\Theta = \left\{ y_i \middle| e(i) \in \text{span}\left[col\left(A^{-1}B \right) \right],\ i \in \{1,2,...,n\} \right\}$.

It is easy to verify that if the number of independent instruments is equal to (or greater than) the number of independent targets, $\forall e(i) \in \text{span}\left[col\left(A^{-1}B \right) \right]$ and, therefore, $y_i \in \Theta$ for $\forall i \in \{1,2,..,n\}$. In other words, the system $Ay = Bu + K$ is controllable.[21]

3. A revised target/instruments approach

3.1 The policy game approach

The well-known Lucas critique highlights the need to model policy decision within a strategic context,[22] where the policymaker interacts with at least another decision maker, usually the private sector. The policy game approach in fact consists of directly modeling the behavior of players by considering separate but not independent optimization problems.

For the sake of simplicity we consider only two players, the public sector (or *Government*) and the private sector (or *Agent*); however, the approach can be clearly generalized to the case of more players. In addition, we mainly discuss the neutrality of the Government's policy (i.e. policy

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[21] An example is provided by Appendix B.

[22] A usual way to model the private sector anticipation is provided by the rational expectation techniques. A more general way to get the same result is to directly model the private sector behavior and, hence, the players' interactions by using the policy game approach. Introducing rational expectations is indeed a particular case in the policy game approach, as we show in Appendix C.

neutrality), but all our results can be applied also to case of the Agent's policy or generalized.

The extension of the TTA to a strategic context needs a number of additional qualifications.

a) We should consider the possibility that the two players share some target variables, while some others are peculiar to each one.
b) A linear algebraic system more general than system (2) must be considered. The latter should be augmented with the relationships concerning the target variables the private sector does not share with the Government.
c) It is useful to decouple the optimization process into two traditional TT-optimization processes, one for each player.
d) Finally, the kind of interactions (equilibrium) between the players must be specified.

More formally, we operate in four steps.

a) We first distinguish three vectors of target variable vectors.

1. The vector of variables of interest for the Government only: $y^u \in \mathbb{R}^g$.
2. The vector of shared target variables, which are of interest for both players: $y^s \in \mathbb{R}^l$. Notice that $y = \left(y^u, y^s\right)$ and $g + l = m$.
3. The vector of variables of interest for the Agent only: $y^w \in \mathbb{R}^v$.

Regarding the instruments:

1. The instrument vector controlled by the Government is, as previously, $u \in \mathbb{R}^n$.

2. The instrument vector controlled by the Agent is $w \in \mathbb{R}^p$.

b) We can then rewrite the system (1) as:[23]

$$(10) \qquad \mathrm{A}\begin{pmatrix} y^u \\ y^s \\ y^w \end{pmatrix} - \begin{bmatrix} D_u^u \\ D_u^s \\ D_u^w \end{bmatrix} u - \begin{bmatrix} D_w^u \\ D_w^s \\ D_w^w \end{bmatrix} w = \begin{pmatrix} \bar{K}^u \\ \bar{K}^s \\ \bar{K}^w \end{pmatrix} \text{ i.e. } \mathrm{A}x - D_u u - D_w w = \mathrm{K},$$

where $x = \left(y^u, y^s, y^w\right)$. Coefficient matrices of appropriate dimensions are indicated by D variables. Notice that the above equation system contains v equations more than sub-system (2). We extract for matrix A two squared sub-matrices $A \in \mathbb{R}^{m \times m}$ and $\tilde{A} \in \mathbb{R}^{m+v \times m+v}$. The former is obtained by eliminating columns and rows from $m+1$ on. The latter is obtained by eliminating the columns and rows from 1 to m. We also assume that $\left[D_u^{u\prime} : D_u^{s\prime}\right]^\prime$, i.e. B, and $\left[D_w^{s\prime} : D_w^{w\prime}\right]^\prime$ are full-rank matrices. As in the parametric decision case, the meaning of the full-rank assumptions is independence of targets and instruments in each single player's problem.

The reduced form of the model is:

$$(11) \qquad x = \mathrm{A}^{-1} D_u u + \mathrm{A}^{-1} D_w w + \mathrm{A}^{-1}\mathrm{K}$$

c) Finally, in order to express the policy game approach in terms similar to the TTA, it is better to decouple equation (11) into two-overlapping equation systems:

$$(12) \qquad y = A^{-1}\begin{bmatrix} D_u^u \\ D_u^s \end{bmatrix} u + A^{-1}\begin{bmatrix} D_w^u \\ D_w^s \end{bmatrix} w + A^{-1}\begin{bmatrix} \bar{K}^u \\ \bar{K}^s \end{bmatrix}$$

i.e. $y = Cu + Ew + F$,

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[23] See Appendix A.

(13) $$z = \tilde{A}^{-1}\begin{bmatrix} D_w^s \\ D_w^w \end{bmatrix} w + \tilde{A}^{-1}\begin{bmatrix} D_u^s \\ D_u^w \end{bmatrix} u + \tilde{A}^{-1}\begin{bmatrix} \bar{K}^s \\ \bar{K}^w \end{bmatrix}$$

i.e. $z = \tilde{C}w + \tilde{E}u + \tilde{F}$.
where $z = \left(y^s, y^w\right)$ is the vector of Agent's target variables.

Equation (12) represents the constraint of the Government in solving the optimization problem in the TTA, i.e. the given Agent's control. Notice that (12) is equivalent to equation (2). Equation (13) is the corresponding constraint for the Agent's problem, i.e. when the Agent solves its problem by considering the Government's instruments as given. Then, taking account of the above constraints, the two players minimize the following quadratic losses:

(14) $$U = \left(y - \bar{y}\right)' Q \left(y - \bar{y}\right)$$

(15) $$W = \left(z - \bar{z}\right)' N \left(z - \bar{z}\right)$$

where $\bar{y} = \left(\bar{y}^u, \bar{y}^s\right)$ and $\bar{z} = \left(\bar{z}^s, \bar{z}^w\right)$ are respectively appropriate vectors of Government's and Agent's targets; *Q* and *N* are symmetric positive semi-definite matrices, which for the sake of simplicity we assume to be both diagonal, i.e., with no cross products.

According to Theorem 2, system (12) ((13)) is TT-controllable by the Government (Agent) if the number of Government's (Agent's) instruments is equal to the number of Government's (Agent's) targets.

d) We consider three common kinds of possible interactions based on different time protocols and, correspondingly, different equilibrium concepts obtained by solving the following optimization problems with respect to *u* and *w*:[24]

$$(16)\quad \mathrm{N} \triangleq \left\{ \min_{u} \text{ eq. (14) s.t. eq. (12)}, \min_{w} \text{ eq. (15) s.t. eq. (13)} \right\}$$

$$(17)\quad \mathrm{C} \triangleq \left\{ \min_{u} \text{ eq. (14) s.t. } \left\{ \min_{w} \text{ eq. (15) s.t. eq. (13)} \right\}, \text{ eq. (12)} \right\}$$

$$(18)\quad \mathrm{D} \triangleq \left\{ \min_{w} \text{ eq. (15) s.t. } \left\{ \min_{u} \text{ eq. (14) s.t. eq. (12)} \right\}, \text{ eq. (13)} \right\}$$

In words, the non-cooperative *Nash equilibrium* (N) corresponds to a situation in which both decision-makers play simultaneously and hence each of them has to form expectations on the opponent's policy. In the case of the *Commitment equilibrium* (C), the Government is the (Stackelberg) game leader, who forms expectations on the opponent's behavior, it can commit himself to a policy rule, i.e. a policy that is not contingent on that of the other player. Finally, in the *Discretionary equilibrium* (D), the Government is a Stackelberg follower and cannot commit its policy whereas the Agent forms expectations on the Government's policy.

In all the above cases the expectations of the players are rational since they are self-fulfilling (see Appendix C). The result directly derives from the rationality assumption of the players and from the Nash equilibrium definition.[25] However, the relationship between a model with rational expectations and a policy game is sometimes a source of confusion.[26]

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[24] In all cases perfect information and players' rationality are assumed.

[25] Notice that formally all the three situations describe a Nash equilibrium, which in the Stackelberg game is also perfect in the sub-games, i.e. it is also a Nash equilibrium of the sub-games.

[26] Rational expectation models are indeed semi-reduced forms of policy games that transform a two-player optimization problem into a one-player optimization problem constrained by some additional condition imposed by the rival's rational expectations.

Summarizing, a decision problem in a strategic context has been reduced to two TT-optimization problems, which can be studied with the tools used in the traditional approach of economic policy.

3.2 Policy neutrality

In the above decoupled representation of the policy game, a straightforward condition for neutrality can be defined as follows. Provided that equilibrium exists, the Government's policy is neutral with respect to the targets shared with the Agent, if the system formed by the last $cw + cs$ equations of (13) is TT-controllable by the Agent.[27]

Although it is intuitive, the above condition nests an apparent contradiction, since the Agent's TT-controllability does not exclude that also the Government can TT-control its sub-system. As we will show, the contradiction is however only apparent. In fact, were this the case, the equilibrium would not exist. The issue of equilibrium existence is indeed crucially related to that of controllability and neutrality, as the following theorem more formally states.

Theorem 3 (Government's policy existence and neutrality). (i) The equilibrium of the game in the target space exists if the intersection of the players' controllable sets is empty or the players share the same target values for the variables therein contained. (ii) The Government's policy is

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Hence they correspond to a policy game of the third kind, the discretionary equilibrium. However, notice that rational expectation games are often solved by using the Nash equilibrium. This occurs because the control variable of the forecaster is the same forecast on a target (not an instrument) variable. In addition, in these, the Nash and Stackelberg equilibria with the forecaster leadership coincide.

[27] The following theorem also generalizes the above claims to the sub-controllability concept introduced in section 2.2.

neutral for all the Government's target variables contained in the Agent's controllable target set.

Indicating formal necessary and sufficient conditions for the existence of Nash equilibrium is not a negligible task.[28] However, our LQ-context simplifies the discussion. Here, the existence problem is related to the solution of a linear-equation system and, therefore, can be reduced to some rank conditions. A proof of the above theorem in the target-variable space follows (See Appendix D for a formal proof in the usual control space).

Proof. Nash equilibrium is usually computed by the inversion of the coefficient matrix of players' reaction functions, but it can also be derived directly in the target-variable space (dual problem) by simultaneously solving the system of the following first order conditions:[29]

$$(19) \qquad \frac{\partial U}{\partial u} = 2C'Q\left(y-\bar{y}\right) = 0 \in \mathbb{R}^n$$

$$(20) \qquad \frac{\partial W}{\partial w} = 2\tilde{C}'N\left(z-\bar{z}\right) = 0 \in \mathbb{R}^p$$

Since both matrices C and $\tilde{C}$ are full rank by assumption, $rank\left(C\right) = \min\left(n, g+l\right)$ and $rank\left(\tilde{C}\right) = \min\left(p, l+v\right)$. Moreover, we can restrict ourselves to the most relevant cases: $n \le g+l$ and $p \le l+v$.[30]

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[28] After Nash (1951), existence of Nash equilibrium has been studied, among others, by Debreu (1952), Glicksberg (1952), Dasgupta and Maskin (1986). See the textbooks by Rasmusen (1989:124-127), Friedman (1991: 68-77), and Fudemberg and Tirole (1991: 34-35) for general discussions.

[29] Equations (28) and (29) define the system of the "quasi-best reply functions," i.e. the system of the reaction functions in the space of the targets (see e.g. Cubitt, 1992)

[30] The discussion can be easily generalized to the case of a number of instruments greater than that of the target variables.

By assuming that both players can perfectly control their sub-systems, matrices $C'Q$ and $\tilde{C}'N$ are square (of order $n = g + l$ and $p = l + v$, respectively) and thus $y = \overline{y}$ and $z = \overline{z}$ follow. This result is consistent only in the trivial cases of either inexistence of shared targets or $\overline{y}^s = \overline{z}^s$, i.e. a non-conflict solution exists.[31] In other words, if both players TT-control their sub-systems, the system formed by equations (19) and (20) would be over-determined, as $n + p = g + v + 2l$ independent equations would be used to find $g + v + l$ independent unknowns.[32]

Once the first part of the theorem is acquired, the proof of the second part is rather intuitive. If the Agent is able to TT-control its sub-system, it can reach all its targets, including the shared target variables. Hence, Government policy is neutral with respect to the shared target variables, as claimed. If also the Government controlled its sub-system the intersection of the Government and Agent controllable target sets would be not empty, the equilibrium would not exists. Thus the theorem is verified. ■

Proof extension (Stackelberg equilibria). The above inconsistencies can emerge for Stackelberg equilibria as well. In fact, if one player is able to

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[31] It is straightforward to notice that in non-conflict solution neutrality would hold even if the Government achieves its targets. In fact, changes in the Government's preferences either do not affect the equilibrium outcomes (as changes in marginal rates of substitution) or imply non existence (as in the case of changes in target values of shared variables contained in the intersection of the players' controllable sets).

[32] The same inconsistency can also emerge in the case of sub-controllability. In fact, if the same shared target variable could be independently determined by both the Government from a sub-system of equations (19) and by the Agent from a sub-system of equations (20), the system formed by equations (19) and (20) would be consistent if and only if the Government and the Agent also share the same target values for the shared target variables. Otherwise the two sub-systems would be inconsistent with the solution of the system formed by equations (19) and (20) of which they are a part.

control one or many target variables, his optimal policy is independent of the kind of policy-game solution considered. ■

The reason why an issue of existence can emerge in our context may also be usefully discussed in terms of the conditions required by a well-known Nash equilibrium theorem of existence with bounded strategies (i.e. instrument costs). In one-shot games, a Nash equilibrium always exists if i) the space of strategies of each player is convex and compact; ii) the payoff function of each player is a definite, bounded, and continuous function for each strategic combination; iii) the player i's payoff function is concave with respect to each player i's strategy for all possible strategic combinations and for all the players.

In our case condition i) is not met since the players' controls are unbounded.[33] Hence the equilibrium non-existence discussed above is a possible outcome. In policy games, quadratic costs for instrumental variables are often introduced in order to achieve solution existence.[34] In this case instruments become bounded and the Nash equilibrium always exists. Formally, the introduction of quadratic instrument costs in our context would imply that the dimensions of matrices Q and N become $g+l+n$ and $v+l+p$, respectively. Thus, the instruments' number would be less than the targets' and the system would not be TT-controllable by any player.[35]

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[33] See, e.g. Dasgupta and Maskin (1986).

[34] Instrument costs are also often used in numerical optimization problem involving calibration techniques. In such cases, instrument costs are introduced to smooth the Government's policy in order to get result in line with the empirical observations.

[35] The same argument applies to sub-controllability. In fact, by introducing instrument costs all the sub-systems that were sub-controllable would also become non-controllable since all

The above results can be intuitively summarized and generalized to *Z*-players as follows.[36]

Theorem 4 (equilibrium existence and neutrality generalized). (i) The equilibrium of a policy game between *Z* player exists (in the target space) if the intersection of their controllable target sets is empty or if the variables contained in the set are associated with the same quantitative target for all the players which control them. (ii) Player *z*'s policy is neutral for all its targets contained in the intersection set of the other players' controllable target sets.

4. A generalization to LQ-losses[37]

In the previous section we have assumed quadratic losses; here we generalize our results to LQ-preferences. In policy games, LQ-losses are often used, especially for computational purposes.[38] LQ-expressions for players' losses are:

(21) $U = \left(y - \overline{y}\right)' Q \left(y - \overline{y}\right) + y'R$

(22) $W = \left(z - \overline{z}\right)' N \left(z - \overline{z}\right) + z'H$

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the sub-systems that were square (controllable) in the M_0-augmented block diagonal form would no longer be square including now instrument costs.

[36] Proof is intuitive and directly follows from Theorem 3.

[37] Notice that until now we have considered a linear model with strictly quadratic preferences and we have referred to it as the LQ-case. In this section, instead, we consider a linear model with LQ-preferences. Throughout this section, with no loss of generality (extensions of results are trivial), we will not formally discuss the sub-controllability, i.e. we assume $BlockDiag\left[\tilde{C}\right] = \tilde{C}_0$.

[38] Indeed, LQ-functions generalize quadratic ones. However, if some arguments enter the loss function only linearly, computations are easier.

which generalize equations (14) and (15). We assume that for each player at least one target variable enters its loss function in a quadratic form – otherwise the problem becomes trivial. In addition, for the sake of simplicity, the players are assumed not to be interested in cross products between their target variables.[39]

The extension of conditions for policy existence and neutrality to LQ-losses is simple and can be done along the lines of our previous analysis; however, it nests some further technical complexities. In particular, stronger conditions for the equilibrium existence and a weaker form of neutrality are involved.

We will deal with existence first. However, before passing to this issue, we must clarify the implications of LQ preferences for optimal policies. A generic entry of the LQ-loss for the Government is $Q_{i,i}\left(y_i-\overline{y}_i\right)^2+R_i y_i$, which collapses to a quadratic term for $R_i=0$. Hence, the *optimum optimorum* for the target variable y_i is $\overline{y}_i-\frac{1}{2}\frac{R_i}{Q_{i,i}}$, instead of $\overline{y}_i$ (as in the quadratic case), and it does not exist as a finite value if $Q_{i,i}=0$ (in such a case $y_i=\pm\infty$ is optimal for the player according to the sign of R_i). Hence, if a decision-maker is able to TT-control the system, it will optimally set its instrument vector at the value associated with its *optimum optimorum* instead of having zero deviations from the target vector.

Now we can deal with the issue of existence. Because of the linear terms in the loss functions, a specific problem arises, leading to more stringent

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[39] Formally, with reference to equation (29), $Q_{i,j}=0$ for $i\neq j$; if $Q_{i,j}=0$, $R\neq 0$ and vice

conditions for the equilibrium existence.[40] If a player can TT-control a system, it sets its instruments in order to achieve values of its target variables equal to its *optima optimorum*, for example $\bar{z}_i - \frac{1}{2}\frac{H_i}{N_{i,i}}$ in the case of the Agent. But, if $N_{i,i} = 0$ for some *i*, the *optima optimorum* for those target variables no longer exist and the Agent's problem cannot be solved (for finite values of instrumental variables). Hence, if a target variable that enters *U* (*W*) only linearly is in Government's (Agent's) controllable target set, the equilibrium does not exist. Generalizing the first part of Theorem 4 to the case of LQ preferences, the existence problem can be summarized as in the theorem below, which implies stronger conditions for the equilibrium existence.

Theorem 5 (equilibrium existence extended). An equilibrium in the target space of a policy game between the *Z* players exists if i) The first part of Theorem 4 holds *or* ii) any player's controllable target set does not contain any target variable that enters its loss linearly only.

Proof. Formally, by deriving first order conditions from equations (21) and (22), we can write them as quasi-reaction functions as follows:

$$(23) \qquad \frac{\partial U}{\partial u} = 2C'Q\left(y - \bar{y}\right) + C'R = 0 \in \mathbb{R}^n$$

$$(24) \qquad \frac{\partial W}{\partial w} = 2\tilde{C}'N\left(z - \bar{z}\right) + \tilde{C}'H = 0 \in \mathbb{R}^p$$

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versa; however, $Q_{i,j} \neq 0$ at least for one target. Similar assumptions hold for equation (22).

[40] As, in LQ preferences, target variables that enter only linearly imply unbounded payoff functions.

Equations (23) and (24) are obtained from two separate optimization problems and represent the optimal values of the target variables that assure the minimization of each player's loss, given the policy of the other one. Thus, by definition of Nash equilibrium, both condition (23) and (24) have to be mutually verified to describe the Nash equilibrium of the policy game.[41] Formally, Equations (23) and (24) define $n+p$ conditions for $u+w+2s$ unknowns, by mapping the vector of target variables into that of the target desired values.[42] Now assume that the Government's system is TT-controllable. If $Q_{i,i}=0$, the *i*-th column of matrix $C'Q$ is a zero vector. Hence, matrix $C'Q$ cannot be inverted and system (23) is over-determined as from Theorem 5. In words, the Government can control the system, but one of its targets takes an infinite value and a finite solution does not then exist.[43] ■

The existence of target variables that enter the players' loss functions only linearly implies also additional complications from the point of view of policy neutrality. The second part of Theorems 3 and 4 becomes.

Theorem 6 (Government's extended policy neutrality). Provided that either the Nash or Commitment equilibrium of the policy game between the Government and the Agent exist, the Government's policy is neutral for all

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[41] We will later extend our discussion to the Stackelberg cases (i.e. Commitment and Discretion). However, see also the proof extension of Theorem 3.

[42] It is worth noticing that condition (23) represents the Government's dual problem of that described by equation (5), i.e. Government's reaction function. Clearly if the Government can solve equation (5) for any vector of desired target variables (TT-controllability), it can also control the system (23). If the reaction function system, i.e. equation (5), is over-determined, the quasi-reaction function system is under-determined; and vice versa. The same is true for the Agent's problem.

the shared Government's target variables, if the number of instruments of the Agent is equal to the number of its quadratic target variables.

Notice that Theorem 6 generalizes the second part of Theorem 3 only for Nash and Commitment, whereas neutrality under Discretion still depends on the second part of Theorem 3.

Proof. By assuming that the Agent's system is TT-controllable, i.e. $p = l + v$, the Government policy is neutral, if an equilibrium exists. However, if $N_{i,i} = 0$, the *i*-th column of matrix $\tilde{C}'N$ is a zero vector. Hence, system (24) is under-determined, i.e. the corresponding reaction function system that maps instruments into desired targets is over-determined (the difference between equations and independent unknowns is one) and the equilibrium does not exist since in such a case it is optimal for the Agent to set the p–1 instruments to equalize the square targets to its *optima optimorum* and to set the remaining one equal at an infinite value to achieve an infinite value of the *i*-th target variable. But if the number of the Agent's instruments is equal to m–1, system (24) is exactly determined, as the number of independent equations is equal to the number of independent unknowns. More generally, if the number of independent instruments of the Agent is equal to the number of independent square-target variables, the Agent can control equation (24), and either Government's policy is neutral or the Nash equilibrium does not exists. ■

Proof extension (Stackelberg equilibria). The above discussion can be also extended to the case of Commitment since in such a case equation (24) must

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[43] A similar situation arises when the system is TT-controllable by the Agent and some targets appear only linearly in the Agent's preference function.

be verified in equilibrium. By contrast, the conditions stressed in this section cannot be applied to Discretion. In such a case, in equilibrium the quasi-reaction function of the Agent does not hold and neutrality does not emerge since the equilibrium is determined by the tangency of the Agent's preferences to equation (23). ■

It is finally worth noticing that even if Government's neutrality holds, the Nash and Commitment equilibria may be not a first best for the Agent when some targets enter its preference only linearly.[44] Thus the Agent could raise its utility if it is able to change the equilibrium finite value of linear targets. This is possible in the Discretion case, where the Agent may be able to use its first-move advantage to achieve a loss lower than those associated with Nash and Commitment equilibria taking account of the Government preferences.[45]

5. A closer look at the literature on monetary policy neutrality

5.1 Barro and Gordon (1983)

The most celebrated policy game is probably Barro and Gordon (1983).[46] A quite simple version of this model can be represented as follows:

$$(25) \qquad G = -\frac{a_1}{2}\left(n - n_g\right)^2 - \frac{a_2}{2}\left(p - p_g\right)^2$$

$$(26) \qquad P = -\frac{1}{2}\left(n - n_p\right)$$

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[44] Differently from the strictly quadratic case, here the first best can never be reached since it implies an infinity value for linear target variables.

[45] See Acocella *et al.* (2003) and Acocella and Di Bartolomeo (2004) for some examples.

[46] See also Stokey (1990), Cubitt (1992), Sargent (2002).

(27) $n = \left(p - p^e\right) + n_p$

where G is the Government's loss, depending on employment (n) and price (p) deviations from a target, P is a similar function for the private sector, and equation (27) describes the baseline structure of the economy. The government controls inflation whereas the private sector controls inflation expectations. Alternatively, but without different implications, one can assume that the private sector loss is defined in terms of expectation deviations, i.e. $-\frac{1}{2}\left(p - p^e\right)$.

The trivial well-known Nash solution of the model is:

(28) $n = n_p$

(29) $p = p_g + \frac{a_2}{a_1}\left(n_g - n_p\right)$,

By assuming a_1 and a_2 finite and different from zero, in equation (25) the Government has two independent target variables and one instrument. Hence, system (27) is not TT-controllable for the Government and its controllable target set, Θ_G, is empty. By contrast, the private sector has one target variable and one control variable. Thus, equation (27) is TT-controllable by the private sector, i.e. its controllable target set is $\Theta_P = \{n\}$. Being $\Theta_G \cap \Theta_P = \varnothing$ and $\Theta_P = \{n\}$, a solution exists and government's policy is neutral with respect to employment.

If $a_1 = 0$, we have $\Theta_G = \{p\}$, $\Theta_P = \{n\}$, and $\Theta_G \cap \Theta_P = \varnothing$. Hence we a solution exists, but the Government is neutral with respect to employment and the private sector is neutral with respect to price. This result can be easily checked from equations (28)-(29). By contrast, for $a_2 = 0$ system (27) is TT-controllable for both policymakers, who also share the same target variable. Thus, $\Theta_G \cap \Theta_P = \{n\}$ is non-empty and two possibilities arise. Either the two policymakers share the same target value (i.e. $n_g = n_p$), and a trivial solution exists, or the solution does not exist since the equilibrium price goes to infinity.

It is easy to verify that all these claims hold,[47] if we consider a more complex economy where the government directly sets the quantity of money, *m*, instead of the price level according to (demand side):

(30) $\quad n = m - p$

and the private sector sets the nominal wage, *w*, rather than price expectations, which are finally determined by (supply side):

(31) $\quad n = p - w$

5.2 Gylfason and Lindbeck (1994) and the union's inflation aversion

Gylfason and Lindbeck (1994)[48] make a step further by considering the private sector distortion as endogenous. These authors consider the economic structure (30)-(31) and assume that a monopoly union directly controls the nominal wage, *w*, to maximize:

(32) $$P = -\frac{b_1}{2}\left(w - p - \omega_p\right)^2 - \frac{b_2}{2}\left(n - n_p\right)^2 - \frac{b_3}{2}\left(p - p_p\right)^2$$

where ω_p is the union's desired real wage.

The Nash equilibrium of the above game turns out to be:

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[47] See Gylfason and Lindbeck (1994) or the next sub-section for $b_3 = 0$.

[48] See also, among others, Acocella and Ciccarone (1997) or Cubitt (1997).

$$(33) \qquad n=\frac{\left(a_1 p_g+a_2 n_g\right)\left(b_1+b_2\right)+a_2\left(b_1 \omega_p+b_3 p_p\right)}{a_1\left(b_1+b_2\right)+a_2 b_3}$$

$$(34) \qquad p=\frac{\left(a_1 p_g+a_2 n_g\right) b_3+a_1\left(b_2 n_p-b_1 \omega_p-b_3 p_p\right)}{a_1\left(b_1+b_2\right)+a_2 b_3}$$

The problem of the government is clearly the same already discussed. For a_1 and a_2 finite and different from zero, system (30)-(31) is not TT-controllable by the government and the set Θ_G is empty.

The union's apparently faces three-targets and one-instrument problem. However, equation (31) clearly highlights that the real wage and employment are not independent target variables. Thus, by using equation (31), it is convenient to rewrite equation (32) in a different form:

$$(35) \qquad P=\frac{\tilde{b}_1}{2} n^2+\tilde{b}_2 n-\frac{b_3}{2}\left(p-p_p\right)^2+K$$

where $\tilde{b}_1=-\left(b_1+b_2\right)$, $\tilde{b}_2=-\left(b_1 \omega_p-b_2 n_p\right)$, and $K=-\left(b_1 \omega_p^2+b_2 n_p^2\right)$. Now preferences of the union are expressed in terms of independent target variables.

With references to equation (35), first consider the case of indifference to inflation (i.e. $b_3 = 0$). In such a case we have that the union preference is linear quadratic and the union has one instrument for one target variable. The union can thus control the system and get its *optima optimorum* as the number of its instruments equals the number of its (quadratic) target variables. According to Theorem 6 this implies that – if the equilibrium exists – the Government is neutral with respect to employment. The result can be easily verified for $b_3 = 0$ from equations (33) and (34). Notice that if $\tilde{b}_1 = -(b_1 + b_2) = 0$, Nash equilibrium does not exist as stated by Theorem 5. An inflation-averse union would have two independent target variables (one linear quadratic and another quadratic only) and one instrument. Hence, Theorem 5 would not apply. A solution exists (see Theorem 5) and no policy is neutral.[49]

5.3 The wage-wedge in a small-open economy

A different way used to obtain non-neutrality of monetary policy is that of inserting inflation indirectly into the union's preference as e.g. in Acocella and Di Bartolomeo (2004). Together with the supply equation (31), consider now the following simple logarithmic demand instead of equation (30):[50]

$$(36) \qquad n = m - p - (\mu - 1)(p - e - p^*)$$

where *e* and p^* are the exogenously given nominal exchange rate and the foreign price level respectively, $(\mu - 1)$ is the real exchange rate elasticity of output.[51] For the sake of brevity, without loss of generality, we assume *e* and p^* equal to zero.

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[49] By considering a similar setup, a further application can be made by using Cubitt (1997) and Acocella and Ciccarone (1997). The former allows us to verify that our conditions hold in the cooperative solution, whereas the latter allows us to verify the existence condition provided by Theorem 6 since Acocella and Ciccarone (1997) use union's preferences that are linear in the real wage and quadratic in employment. In a more complex setup, Acocella and Ciccarone (1997) use the public deficit instead of inflation as a shared objective to obtain non-neutrality. See also Detken and Gärtner (1994) doe a different approach.

[50] Here we use a simplified version of Acocella and Di Bartolomeo (2002) to which we refer for more details.

[51] By assuming $h = 1$ and $\mu = 0$, we have the traditional closed economy setup already analyzed.

In this economy, the policymaker sets the nominal money supply and the union sets the nominal wage. The Government loss is derived from equation (25):

$$(37) \qquad G = -\frac{\beta}{2} p^2 - \frac{1}{2}\left(n - n_g\right)^2$$

where $\beta = a_1 / a_2$ and $p_g = 0$. The union loss is a generalization of equation (26):[52]

$$(38) \qquad P = \alpha_1 \left(w - cpi\right) - \frac{\alpha_2}{2}\left(w - cpi - \omega_p\right)^2 - \frac{1}{2}\left(n - n_p\right)^2 - \frac{\vartheta}{2} p^2$$

where $cpi = \left(1 - h\right) p + hp^*$ is the consumer price index and h is the weight of foreign goods in the consumption basket of wage-earners. Here the relevant real wage for firms could differ from the real wage relevant for the union. The relevant real wage for firms, $w - p$, is expressed in terms of producer prices while the one relevant for the union, $w - cpi$, is in terms of the consumer price index.

The Nash equilibrium level of output and the Nash equilibrium price level are:

$$(39) \qquad n = \frac{\beta\mu n_P - \beta\left(\mu + h\right)\left(\alpha_2\omega_P + \alpha_1\right) + \left[h\left(\mu + h\right)\alpha_2 + \vartheta\right] n_G}{\left(\beta + h\right)\left(\mu + h\right)\alpha_2 + \left(\beta\mu + \vartheta\right)}$$

$$(40) \qquad p = \frac{\left(\mu + h\right)\left(\alpha_1 + \alpha_2\omega_P\right) - \mu n_P + \left[\left(\mu + h\right)\alpha_2 + \mu\right] n_G}{\left(\beta + h\right)\left(\mu + h\right)\alpha_2 + \left(\beta\mu + \vartheta\right)}$$

By assuming that the union is the game leader, and solving the game by backward induction, Stackelberg outcomes are:

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[52] Notice that both equations (37) and (38) are normalized by the employment weight without loss of generality.

$$(41)\qquad n=\frac{\beta^2 n_P-\beta\left(\beta+h\right)\left(\alpha_2\omega_P+\alpha_1\right)+\left[h\left(\beta+h\right)\alpha_2+\vartheta\right]n_G}{\left(\beta+h\right)^2\alpha_2+\left(\beta^2+\vartheta\right)}$$

$$(42)\qquad p=\frac{\left(\beta+h\right)\left(\alpha_1+\alpha_2\omega_P\right)-\beta y_P+\left[\left(\beta+h\right)\alpha_2+\beta\right]y_G}{\left(\beta+h\right)^2\alpha_2+\left(\beta^2+\vartheta\right)}$$

Let us analyze the above result in the TTA terms. First of all, by using equation (31) and the *cpi* definition, in order to obtain a relation between only independent target variables, we rewrite equation (38) as:

$$(43)\qquad P=-\frac{1}{2}\begin{pmatrix} n \\ p \end{pmatrix}'\begin{bmatrix} 1+\alpha_2 & -\alpha_2 h \\ -\alpha_2 h & \vartheta+\alpha_2 h^2 \end{bmatrix}\begin{pmatrix} n \\ p \end{pmatrix}+\begin{pmatrix} n \\ p \end{pmatrix}'\begin{pmatrix} n_p-\alpha_1-\alpha_2\omega_p \\ \alpha_2 h\omega_p+\alpha_1 h \end{pmatrix}+$$
$$-\frac{1}{2}\left(\alpha_2\omega_p^2+n_p^2\right)$$

Notice that if either α_2 or ϑ are different from zero the union has two independent target variables and one instrument; thus $\Theta_P=\varnothing$. If α_1 is finite and both α_2 and ϑ are equal to zero, the union cares for inflation linearly and for employment in a quadratic manner. If α_1, α_2 and ϑ are all zero, the union takes account of employment only; thus $\Theta_P=\{n\}$.

By looking at the government's problem, the system is TT-controllable in two cases: a) for $\beta=0$ (i.e. $\Theta_G=\{n\}$); and b) for $\beta=+\infty$ (i.e. $\Theta_G=\{p\}$) – otherwise $\Theta_G=\varnothing$.

Putting all together, from equations (39)-(42), it is easy to verify the following properties.

1. If α_1, α_2, ϑ, β are zero, then $\Theta_P \cap \Theta_G = \{n\}$ is non empty and a solution does not exists.[53]

2. If α_1, α_2, ϑ are zero and $\beta \neq 0$, then $\Theta_P \cap \Theta_G = \varnothing$ and $\Theta_P = \{n\}$. Thus monetary policy is neutral with respect to employment, i.e. $n = n_P$.

3. If α_1, α_2, ϑ are zero and $\beta = +\infty$, then $\Theta_P \cap \Theta_G = \varnothing$, $\Theta_P = \{n\}$, and $\Theta_G = \{p\}$. Thus monetary policy is neutral with respect to employment and wage policy is neutral with respect to the price level, i.e. $n = n_P$ and $p = 0$.

In addition, if linear-quadratic preferences are also considered (i.e. $\alpha_1 \neq 0$), by referring to Theorem 5 and 6, we can verify the following properties.

4. If α_2, ϑ are zero and $\beta \neq 0$, then in the Nash equilibrium monetary policy is neutral with respect to employment, but it not always true that $n = n_P$ (that holds only for $\alpha_1 = 0$ since $n = n_P - (1 + h/\mu)\alpha_1$). By contrast, in the Stackelberg equilibrium it is not neutral, i.e. $n = n_P - (1 + h/\beta)\alpha_1$. Moreover, although neutrality does not hold, it can be verified the loss of the union under the discretion is lower than the loss in the Nash regime. This highlights the different kind of neutrality arising in the LQ-preference case, i.e. neutrality does not imply the achievement of the best possible outcome.

5.4 Monopolistic competition and wage setters

More recent contributions in the policy game literature stress a new channel of monetary non-neutrality. Among other innovative results directly related

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[53] The proof is trivial from denominator inspection.

to the non-neutrality of monetary policy, Soskice and Iversen (1998, 2000), Coricelli *et al*. (2002) and Cukierman and Lippi (2002) show that if there are a multiplicity of unions and product markets are monopolistically competitive, a Barro-Gordon framework delivers policy non-neutrality, even if unions are not directly averse to inflation.[54]

We can describe a model of the above kind in a simple way.[55] In the economy *n* unions and a central bank are active. The central bank seeks to maximize the following quadratic objective function:

$$(44) \qquad B = -\frac{\beta}{2} p^2 - \frac{1}{2} u^2 .$$

where *p* is the price level and *u* is the unemployment rate. Each union seeks to maximize a linear-quadratic preference function with the membership's log real wage, $w_i - p$, and unemployment rate, u_i, as arguments:

$$(45) \qquad U_i = b_1 \left(w_i - p \right) - \frac{1}{2} u_i^2 \qquad\qquad i \in \{1, 2, ...n\} ,$$

The economy consists of three equations:[56]

$$(46) \qquad u_i = \frac{\eta}{\alpha + \eta(1-\alpha)} \left(w_i - p \right) - \frac{1}{\alpha + \eta(1-\alpha)} \left(m - p \right)$$

$$(47) \qquad p = \alpha w + (1-\alpha) m$$

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[54] See Cukierman (2004) for a survey.

[55] We simplify Jerger (2002) and Acocella *et al*. (2004) to which we refer for more details. For the sake of brevity, we present the model in its essential aspects only. All intermediate computations are not described.

[56] Equation (46) refers to the (micro) disaggregate equilibrium conditions whereas Equation (46) and (48) to the (macro) aggregate ones. More in detail, equation (46) is the union's employment function stemming from a traditional labor demand derived by real profit maximization assuming a Blanchard and Kiyotaki's (1987) firm's demand. Equation (46) and (48) are the price level and unemployment rate.

(48) $$u = -\frac{1}{1-\alpha}\left(w-p\right)$$

where w_i is the wage set by the *i* union; $\eta > 1$ is the degree of monopolistic competition and $\alpha \in \left(0,1\right)$ is the labor coefficient of the productions function, $w = \sigma w_i + \left(1-\sigma\right) w_j$ is the average wage, the general level of prices is defined according to the Dixit-Stigliz's tradition as $p = \int_0^1 p_{ij} dj$. After manipulations, the model reduced form turns out to be:

(49) $$\begin{pmatrix} p \\ u \\ u_i \end{pmatrix} = \begin{pmatrix} 1-\alpha & \sigma\alpha & \left(1-\sigma\right)\alpha \\ -1 & \sigma & \left(1-\sigma\right) \\ -1 & \frac{\eta-\alpha\sigma(\eta-1)}{\alpha+\eta(1-\alpha)} & -\frac{\alpha(1-\sigma)(\eta-1)}{\alpha+\eta(1-\alpha)} \end{pmatrix} \begin{pmatrix} m \\ w_i \\ w_{-i} \end{pmatrix}$$

where w_{-i} is the average wage of the unions different of union *i*. Notice that the three target variables are independent.

By solving the model the Nash equilibrium is:

(50) $$p = \frac{\left(1-\alpha\sigma\right)\left(\eta-\alpha\left(\eta-1\right)\right)\left(\alpha-\phi+\alpha\phi\right)}{\left(\eta-\alpha\sigma\left(\eta-1\right)\right)\left(1+\phi\right)} b_1 \geq 0$$

(51) $$u = \frac{\left(1-\alpha\sigma\right)\left(\eta-\alpha\left(\eta-1\right)\right)}{\eta-\alpha\sigma\left(\eta-1\right)} b_1 > 0 .$$

where $\phi = \frac{\alpha\left(1-\alpha\right)\beta-1}{\left(1-\alpha\right)^2\beta+1}$. In order to evaluate possible non-neutrality, notice that ϕ is the only parameter containing central bank's preference.

By solving the model the Stackelberg equilibrium (unions' leaders) is:

(52) $$p=\frac{\left(1-\left(\alpha-\phi+\alpha\phi\right)\sigma\right)\left(\alpha-\phi+\alpha\phi\right)\left(\alpha+\left(1-\alpha\right)\eta\right)}{\left\{\eta\left(1-\sigma\left(\alpha-\phi+\alpha\phi\right)\right)+\alpha\sigma\left(1+\phi\right)\right\}\left(1+\phi\right)}b_1\geq 0$$

(53) $$u=\frac{\left(1-\left(\alpha-\phi+\alpha\phi\right)\sigma\right)\left(\alpha+\left(1-\alpha\right)\eta\right)}{\eta\left(1-\left(\alpha-\phi+\alpha\phi\right)\sigma\right)+\alpha\sigma\left(1+\phi\right)}b_1>0\,.$$

Let us analyze the above result in the TTA terms. First of all, one can notice that the central bank problem is the same as that analyzed in the above subsection. The sub-system formed by the first two rows of equation (49) is TT-controllable by the central bank only in two cases: $\beta=0$ (i.e. $\Theta_G=\left\{n\right\}$) and $\beta=+\infty$ (i.e. $\Theta_G=\left\{p\right\}$); otherwise $\Theta_G=\varnothing$.

Regarding a representative union, for convenience, after manipulations, we can rewrite equation as

(54) $$U_i=-\frac{1}{2}u_i^{\,2}+\frac{\gamma\left[\alpha+\eta\left(1-\alpha\right)\right]}{\eta}u_i-\frac{\gamma}{\eta}u$$

Each union has one instrument and two target variables. Thus if γ is finite and different from zero, $\Theta_P=\varnothing$. However, not surprisingly, even if the system (49) is not TT-controllable by the representative union,[57] by Theorem 6, we can claim that the model implies neutrality in the Nash equilibrium, because the linear-quadratic nature of equation (54), and non-neutrality in the Stackelberg one (discretion) as equations (51) and (53) confirm. As a result, if β and b_1 are different from zero neutrality does not emerge unless $\frac{\alpha+\eta(1-\alpha)}{\eta}=0$. In fact, for $\eta\rightarrow+\infty$ (perfect competition),

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[57] I.e. union *i*'s equilibrium unemployment is not zero.

equation (54) becomes $U_i = -\frac{1}{2} u_i^2$ and standard results arise. In such a case $u_i = u$ and $\Theta_P = \{u_i\}$. Other checks similar to those used above can be easily made by varying parameter values.

6. Concluding remarks

This paper generalizes the classic theory of economic policy to the more recent strategic approach of policy games. We have shown how a revised version of the Tinbergen-Theil's traditional theory can deal with policy neutrality problems, taking fully account of the Lucas critique. We have also shown how it can be profitably used to deal with equilibrium existence conditions in policy games.

In a game theoretical perspective, controllability and neutrality are dual concepts. Controllability for one player implies neutrality for the other one. Of course, controllability needs to be reinterpreted to take account of the strategic nature of context. Once this has been done, neutrality merely becomes an instrument/target accounting problem in the traditional Tinbergen-Theil's vein.

We have shown that controllability by a player of a subset of variables always implies neutrality of all the others' policies for the same subset (if an equilibrium exists), but the reverse does not hold. In particular, by generalizing our investigation to the case of constant marginal rates of substitution between targets, we have shown how neutrality can emerge if the counting rule of the number of instruments and targets is violated. However, the neutrality emerging in such a case has a different nature since it does not imply the realization of the player's *optimum optimoru*m (which does not exist because of the non-satiation). Hence, it leaves an open room for different arrangements as cooperation or policy leadership, which could be associated with lower loss levels.

It is finally worth summarizing and briefly discussing the main assumptions implicitly or explicitly used.

1. Complete, symmetric and perfect information contexts.
2. LQ-preference functions with linear economic relationships.
3. Static policy games.

All the above assumptions were also used in the seminal Tinbergen and Theil approach and almost all of them have been relaxed by sequels without deeply affect their simple logic. Our results can be similarly generalized, but some additional complications arise in some cases.

As in Tinbergen and Theil, a straightforward generalization is possible with the perfect information assumption, which can be relaxed by introducing linear uncertainty and discussing our results in expected terms. In general, a similar extension is not possible for model uncertainty or incomplete and asymmetric information since the certainty equivalence no longer holds. Indeed, we are not very interested in similar cases where – as is well known – neutrality generally does not hold.[58] However, our results may still apply as particular cases.[59]

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[58] See e.g. Brainard (1967), Rogoff (1983) and Vickers (1986) as preeminent examples. It is also worth to notice that complete information rules out Bayesian equilibria, which are however generally characterized by non neutrality. See Holly and Hughes Hallett (1998) for a complete discussion on model uncertanty.

[59] More precisely, if uncertainty directly or indirectly only affects the equation blocks or the players' preferences that are not relevant for the neutrality conditions.

We have considered the LQ-case to follow Preston and Pagan (1982), who generalize the fixed and flexible approaches. Our extension to LQ preferences highlights the complexities that can arise when non-satiation is introduced. Further extensions to non-linear contexts and more general preferences exhibit similar problems and should probably be analyzed case by case. We have considered the LQ-case also for reasons of tractability and because it is rather common in the policy games we are interested in.

The static assumption, rather common in policy games,[60] is mainly justified for its tractability. Although, in principle, the simple logic of our discussion can be extended to dynamic contexts,[61] as for the Tinbergen and Theils's seminal contribution, formal generalizations face many computational and practical problems. For instance, general existence conditions for differential or difference games are difficult to derive, especially in the infinite horizon case.[62] Moreover, many different kinds of equilibrium must be considered since time implies various information assumptions, as e.g. feedback or open-loop structures.[63] Our results have however an immediate practical extension to the steady state existence and long-run neutrality in Nash differential or difference games or cob-web dynamics and some learning adjustment process[64] since, in all these cases, our propositions can be directly applied to steady state relationships.

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[60] A notable exception, among others, is Başar *et al.* (1988).

[61] See Acocella and Di Bartolomeo (2005).

[62] See Engwerda (1998) for an example.

[63] See Petit (1990) or Başar and Olsder (1995) for a complete discussion.

[64] See among others Evans and Honkapohja (2001), Sargent (2002: Chapter 3), and Di Bartolomeo and Pauwels (2005) as examples for cob-web dynamics or learning adjustment process where our results holds.

Appendix A – The linear model

Throughout the paper we consider linear relationships among economic variables. In different parts of the paper we use different relations and often express them – in particular in sections 3 and 4 – in terms of partitioned matrices. This Appendix aims to clarify the relationship between the different representations of the economic system used.

Consider the general linear-equation system (1) in Section 2. By using partitioned matrices it can be rewritten as:

$$\text{(A1)} \qquad \mathrm{A}\begin{pmatrix} y_u \\ y_s \\ y_w \end{pmatrix} = \begin{bmatrix} D_u \\ D_w \end{bmatrix}\begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} \bar{K}^u \\ \bar{K}^s \\ \bar{K}^w \end{pmatrix}$$

where:

1. $y^u \in \mathbb{R}^{cu}$ is the vector of variables contained in the Government's loss function only; the vector $y^w \in \mathbb{R}^{cw}$ is the vector of variables contained in the Agent's loss function only; $y^s \in \mathbb{R}^{cs}$ is the vector of variables contained in both loss functions.
2. $u \in \mathbb{R}^n$ is the vector of Government's instruments; the vector $w \in \mathbb{R}^p$ is the vector of Agent's instruments.
3. D_u and D_w are appropriate coefficient matrices.

After manipulations, system (A1) can be rewritten as:

$$\text{(A2)} \qquad \mathrm{A}\begin{pmatrix} y_u \\ y_s \\ y_w \end{pmatrix} - \begin{bmatrix} D_u^u \\ D_u^s \\ D_u^w \end{bmatrix} u - \begin{bmatrix} D_w^u \\ D_w^s \\ D_w^w \end{bmatrix} w = \begin{pmatrix} \bar{K}^u \\ \bar{K}^s \\ \bar{K}^w \end{pmatrix} \text{ i.e. } \mathrm{A}x - D_u u - D_w w = \bar{K},$$

where: D_u^i and D_w^i for $i \in \{u, s, w\}$ are appropriate coefficient matrices.

From (A2) we extract the first $m = g + l$ equations and by simple manipulations we get the following linear-equation system:

$$\text{(A3)} \quad A\begin{pmatrix} y_u \\ y_s \end{pmatrix} = \begin{bmatrix} D_u^u \\ D_u^s \end{bmatrix} u + \begin{pmatrix} \bar{K}^u \\ \bar{K}^s \end{pmatrix} + \begin{bmatrix} D_w^u \\ D_w^s \end{bmatrix} w$$

i.e. equation (12) in Section 3. Matrix $A \in \mathbb{R}^{m \times m}$ is obtained by eliminating from A columns and rows from $m+1$ on. The system (13) can be derived from equation (A2) in a similar way by extracting the last $l+v$ equations. Moreover, notice that, by defining $B = \left[D_u^{u\prime} : D_u^{s\prime} \right]'$; $K = \left(\bar{K}^u, \bar{K}^s \right) + \tilde{A}w$, we can rewrite (A3) as equation (2) in the Section 2.

Appendix B – A Numerical example of sub-controllability

Example 1. Consider the following numerical values for the general model described by equation (2):

$$\text{(B1)} \quad \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \\ 3 & 4 & 0 \end{bmatrix} \begin{pmatrix} y_1 - \bar{y}_1 \\ y_2 - \bar{y}_2 \\ y_3 - \bar{y}_3 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

where it is easy to check that full-rank conditions are satisfied, i.e. $rank(A) = 3$ and $rank(B) = 2$. The structural model (B1) can be rewritten in a reduced form as:

$$\text{(B2)} \quad \begin{pmatrix} y_1 - \bar{y}_1 \\ y_2 - \bar{y}_2 \\ y_3 - \bar{y}_3 \end{pmatrix} = -\begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \\ 3 & 4 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ \frac{1}{3} \end{pmatrix} = -\begin{bmatrix} 0 & -3 \\ 0 & \frac{5}{2} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ \frac{1}{3} \end{pmatrix}$$

where $rank(C) = 2$. Moreover, $BlockDiag[C]$ is never equal to C; in fact by swapping the first with the third-target variable, system (B2) can be rewritten in the following M_0-augmented block diagonal form:

$$\text{(B3)}\quad \begin{pmatrix} y_3 - \bar{y}_3 \\ y_2 - \bar{y}_2 \\ y_1 - \bar{y}_1 \end{pmatrix} = -\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{5}{2} \\ 0 & -3 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 1 \\ -1 \end{pmatrix}$$

According to our discussion, from system (B3), the Government can control the third target variable, i.e. y_3. The same is evident by computing the column set of C directly from the more general system (B2), e.g. $col(C) = \left\{ (0,0,1)', \left(-6, \frac{15}{2}, 0\right)' \right\}$, and verifying if the eye vectors can be derived as linear combinations of it. For example, in order to check if the first-target variable, y_1, is controllable, we must verify that $(1,0,0) \in span[col(C)]$, which is clearly untrue. By contrast, in the case of the third target variable, y_3, $(0,0,1) \in span[col(C)]$ is verified.

Appendix C – Rational expectations and game theory

This appendix aims to clarify the relationship between the rational expectation hypothesis and the policy games, in particular, between the former and the different concepts used to solve games. We provide a simple informal discussion to stress the main relationship and describe the terminology that is used in the paper. For a more formal description see Petit (1990: Chapters 8-10).

Assuming that an operator is interested to predicting the future (or unknown)[65] value of a certain variable z or a vector of variables, we refer to the operator's forecast (or expectation) as $\mathcal{E}(z)$. The rational expectation hypothesis implies that $\mathcal{E}(\;)$ is said rational if:

(C1) $\mathcal{E}(z) = E[z|I]$

where $E[\;|I]$ is the statistical expectation conditional on informational available at the time the forecast is made. In other words, the rational expectation hypothesis requires that the prediction made by the forecaster be consistent with the prediction generated by the model, conditional on information available at the time the forecast is done. Definition (C1) implies:

(C2) $\mathcal{E}(z) = z + \varepsilon$

where is ε a purely random shock with zero mean. In a deterministic model $\mathcal{E}(z) = z$, i.e. perfect foresight.

In models with explicit rational expectation variables as, e.g., that in Barro-Gordon (1983), the effects and the meaning of the rational expectations is rather clear. Models with explicit rational expectations consider the action of the players constrained by the rational expectations: The traditional player's problem is augmented with the additional constraint that expectations are rational and the structural model describing the economy

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[65] The variable can be also set at the same time the operator has to make his choices (and thus he needs to know the variable value), but if the operator cannot observe it, he should make a forecast. This situation is rare in the traditional expectation models. By contrast, it is rather usual in policy game, for example in Nash equilibrium, each player in setting the

depends also on the private sector forecast of economic variables (and thus of policy variables). By contrast, in policy games the strategies of the private sector are "simultaneously modeled" to the players' behavior by implicitly considering that the private sector forms rational expectations in forming its strategies.[66] In policy games the relationship between the players' action and the rational expectation hypothesis is more obscure. However, as we will show, the policy game approach is richer and more general and rich since it easily permits to take account of different information structures extending the rational expectation case.

By considering the policy game approach, the rational expectations imply the operator perfectly anticipates the actions of the other players given the information the operator has (see, e.g., Mas-Colell *et al*. (1995: 439) for a clear example). Hence the natural way to model rational expectations is the Nash sub-game perfect equilibrium (e.g. the Stackelberg equilibrium of two-player static games).

In a game between two players (U and W), which have two policy instruments (respectively, u and w), the time protocol associated with the Stackelberg equilibrium implies that the game leader (e.g. W) moves first without knowing the value of the instrument set by his rival (i.e. player U). Moreover, by definition, it is assumed that both players know the rules of the game and the other player's preference. In such a situation the leader of

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control variable has to make an expectation on the simultaneously-determined rival's choice since he cannot observe it.

[66] Notice that here when we use the expression "simultaneously modeled" we do not mean that the private sector and the Government play simultaneously but only that their actions are modeled simultaneously, i.e. by directly considering their strategies instead of the Government strategies constrained by the private sector expectations on public sector action.

the game in setting *u* should form an expectation on the value *w* that the follower will set.

Consider the game described by the following figure in the space of strategies (i.e. reaction functions).

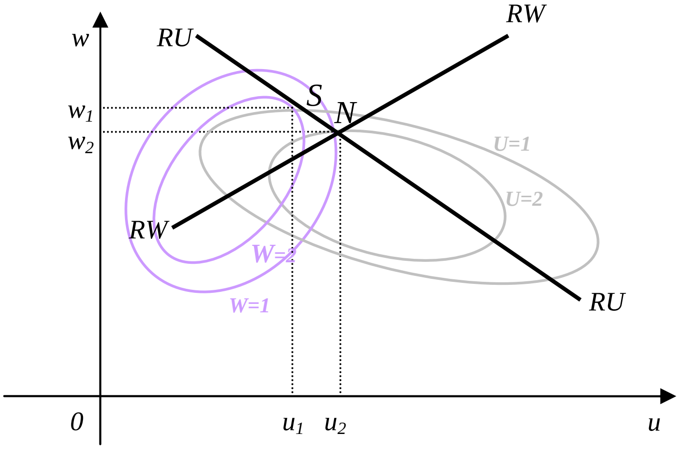

Figure 1

The reaction function of the leader (follower) W (U) is *RW* (*RU*). Gray (pink) lines are the follower (leader)'s indifference curves, who sets *w* (*u*). Point S clearly represents the Stackelberg (w_S, u_S) equilibrium of the game. As the leader (W) in setting her policy cannot observe the policy of the follower she should form an expectation on it. She conjectures $\mathcal{E}(u)=u_1$ and, therefore, plays $w_1 = f(\mathcal{E}(u)) = f(u_1)$. Notice that nothing assures that $\mathcal{E}(.)$ is rational. We just assume that for some reason $\mathcal{E}(u)=u_1$ is the leader's conjecture about the action of the follower. The follower (U) observes w_1 and plays u_1, which is optimal for him. Hence the Stackelberg equilibrium is represented by point S, where $w_S = w_1$ and $u_S = u_1$. In equilibrium leader conjecture on the rival's behavior is realized.[67] Hence Stackelberg equilibrium is supported by self-confirming beliefs ($\mathcal{E}(u) = u_1 = u_S$), in such a sense it can be seen as the outcome of the rational expectation hypothesis (see equation (B2)). In other words, operator $\mathcal{E}(\ )$ is a rational expectation.

Now consider the Nash equilibrium N. The time protocol associated with it implies that each player moves without knowing the value of the instrument set by his rival. Moreover, as in the previous case, we assume that both players know the rules of the game and the other player's preference. In such a situation both players in setting their instruments should form expectations on the other player's policy.

As in the Stackelberg case, when W sets *w*, she cannot observe *u* and has to make a conjecture (expectation), i.e. $\mathcal{E}(u) = u_1$. Simultaneously, U makes a conjecture that W will play w_2, i.e. $\mathcal{E}(w) = w_2$. In equilibrium, $w_N = w_2$ and $u_N = u_2$, but this means that expectations are fulfilled: $\mathcal{E}(u) = u_N$ and $\mathcal{E}(w) = w_N$.

Thus Nash equilibrium can be interpreted as the result of a situation in which both players form rational (conjectures) expectations on the other player's policy since in equilibrium each player conjecture on the rival's behavior is realized. In other word Nash equilibrium is supported by self-

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[67] Notice that U does not form expectations since she observes the value of *w*.

confirming beliefs and can be seen as the outcome of the rational expectation assumption.

Appendix D – Nash equilibrium existence and controllability

Recall that, if the systems (19) and (20) are TT-controllable, C and $\tilde{C}$ are square matrices. By solving problem (16), we derive the following first order conditions:

(D1) $C'QCu+C'QEw-C'Q\left(\bar{y}+F\right)=0$

(D2) $\tilde{C}'N\tilde{C}w+\tilde{C}'N\tilde{E}u-\tilde{C}'N\left(\bar{z}+\tilde{F}\right)=0$

i.e. $\begin{bmatrix} C'QC & C'QE \\ \tilde{C}'N\tilde{E} & \tilde{C}'N\tilde{C} \end{bmatrix}\begin{pmatrix} u \\ w \end{pmatrix}=\begin{bmatrix} C'Q\bar{y}+C'QF \\ \tilde{C}'N\bar{z}+\tilde{C}'N\tilde{F} \end{bmatrix}$. A necessary and sufficient condition for the existence of a solution is that the inverse of $\Delta=\begin{bmatrix} C'QC & C'QE \\ \tilde{C}'N\tilde{E} & \tilde{C}'N\tilde{C} \end{bmatrix}$ exists. The inverse existence requires that $C'QC$, $\tilde{C}'N\tilde{C}$, $\left(I-\tilde{C}^{-1}\tilde{E}C^{-1}E\right)$, and $\left(I-C^{-1}E\tilde{C}^{-1}\tilde{E}\right)$ are non singular.

It is easy to verify that if the players share all the target variables the inverse of matrix Δ does not exist, since, in such a case, $C=\tilde{E}$ and $E=\tilde{C}$. More in detail, $\tilde{C}^{-1}\tilde{E}C^{-1}E=\left(D_u^s\right)^{-1}D_w^s\left(D_w^s\right)^{-1}D_u^s=I$. Moreover, notice that for $C=\tilde{E}$ and $E=\tilde{C}$, the first order condition can be rewritten as $0=-E^{-1}CC^{-1}\left(\bar{y}-F\right)+E^{-1}\left(\bar{z}-F\right)$, which is clearly satisfied only for $\bar{y}=\bar{z}$. In other words, even if a solution in the space of controls does not exists, for $\overline{y} = \overline{z}$ a trivial solution in the space of target exists such that $x = \overline{y} = \overline{z}$.

More in general, matrix Δ can be rewritten as the product of two square partitioned matrices as $\Delta = \begin{bmatrix} C'Q & \varnothing \\ \tilde{C}'N & \varnothing' \end{bmatrix} \begin{bmatrix} C & E \\ \tilde{E} & \tilde{C} \end{bmatrix} = \Gamma_1 \Delta_1$, where $\varnothing \in \mathbb{R}^{g+l \times l+v}$ is a zero rectangular matrix. Clearly, $\det(\Gamma_1) = 0$. Hence Δ is singular. Notice that if the two systems are not TT-controllable by decision-makers Γ_1 and Δ_1 are rectangular.

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