# Multiple Critiques of Woodford's Model of a Cashless Economy

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## ABSTRACT

**Woodford's (2003) model of a cashless economy is the basis for his book *Interest and Prices*. Since Woodford assumes complete markets, this paper explicitly includes state-contingent securities with either temporary money or a cash-in-advance constraint to analyze Woodford's logic. This analysis finds four logical problems with Woodford's analysis: (1) Woodford's assumption that his solution is bounded is inappropriate. (2) Any finite version of Woodford's model is incomplete. (3) Woodford's central bank does not control nominal interest rates. (4) Woodford argument that interest rates determine prices and that prices affect the interest rate is circular and hence invalid.**

## Multiple Critiques of Woodford's Model of a Cashless Economy

Woodford's (2003) book *Interest and Pri*ces is considered by some to be the cornerstone of the New Keynesian Economics. Woodford bases his book on his Chapter 2 model of a cashless economy. No monetary frictions exist and the price level is flexible in that model. This paper reports four logical flaws in Woodford's analysis of this model.

I cannot help but be reminded of the children's story, "The Emperor Who Had No Clothes." This story is about a tailor who supposedly makes the Emperor an outfit with such "fine" thread that people can see through it. In fact no one can even see the thread. When the Emperor is parading his "outfit" in front of his subjects, a child speaks up, "but he has no clothes on." I am like that child, reporting that Woodford's model is naked once we remove the cloak of infinity that clouds our ability to see his model for what it really is. You the readers are like the rest of the Emperor's subjects. The question remains whether you will react to what I report.

The first flaw is Woodford's assumption that the solution to his model must be bounded. It is true that a precedent for making that assumption does exist in the economic literature and in particular the literature related to price determinacy. However, this paper reports that that precedent is invalid and unsound at least in its general application. Just because a major economist says one should make that assumption does not mean we should. Making such an assumption when mathematically solving for a solution can only be justified by a mathematical proof that it is valid. This paper reports on a literature review that has traced the bounded assumption to Sargent (1979). The literature review concludes that Sargent never proved that the bounded assumption applies in all cases. In fact, personal correspondence with Sargent himself confirms the lack of such a universal proof. Furthermore, this paper cites other work where it has been shown that Sargent's precedent can lead to incorrect solutions. Personal correspondence with other economists confirms that the bounded assumption does not universally apply.

Sargent (1979) did show that his precedent was valid in one case by assuming a finite horizon and then looking at the resulting transversality condition as the horizon goes to infinity. When we look at a finite-version of Woodford's model, however, we find that his model is incomplete. That a finite version of his model is incomplete is the second flaw of Woodford's analysis.

Woodford assumes that markets are complete, but he does not explicitly model the securities that make markets complete. In this paper, we do explicitly include state-contingent securities to complete the markets. By doing so, we demonstrate a third flaw in Woodford's analysis – Woodford's central bank cannot affect the market nominal interest rate.

As part of his argument that prices are determined in his model, Woodford assumes a feedback policy rule whereby the current price level affects the interest rate the central bank pegs. The fourth flaw in Woodford's analysis is that Woodford's feedback rule interjects circular logic in his analysis. While he argues that the nominal interest rate determines the price level; Woodford also assumes that the price level affects the nominal interest rate. Because this again is clouded by the issue of the infinite horizon, this paper looks at a model of an "almost cashless" economy with a finite horizon. In the limit as the horizon approaches infinity, this "almost cashless" economy approaches Woodford's model. In the "almost cashless" economy, the current nominal interest rate does determine the current price level, which means that the current price level cannot affect the current nominal interest rate as assumed in Woodford's policy feedback function.

We limit this paper to Woodford's Chapter 2 model with no monetary frictions and flexible prices. While Woodford later does try to extend his model to include monetary frictions and to include somewhat rigid prices, we do not try to evaluate such extensions since his underlying model is flawed.

Section II discusses the essence of Woodford's model of a cashless economy and his argument that under certain policy feedback rules his model determines prices. Section III discusses Sargent's precedent for solving expectational difference equations, shows that that precedent does not universally apply, and argues that it does not apply to Woodford's model. Furthermore, this section shows that a finite version of Woodford's model is incomplete.

The paper then proceeds to complete Woodford's model. While Woodford assumes complete markets without including the securities that complete the markets, we choose to avoid confusion by explicitly including state-contingent securities that complete the markets. As a first step, section IV reviews an Arrow-Debreu economy and an important property of that economy. Section V then adds Eagle and Domian's (2003a, 2003b) temporary money to this Arrow-Debreu economy to show that Woodford's central bank is impotent, i.e., it is unable to affect the market nominal interest rate.

Next, we modify the model of section V so that temporary money only exists at time T, the final period. The result is a model of a finite, "almost cashless" economy that in the limit would correspond to Woodford's model. In this "almost cashless" economy, the current interest rate determines the current price level, which makes Woodford's policy feedback rule impossible. Also, section V uses the "almost cashless" model to show that an uncountably infinite number of price-level paths exist in Woodford's model.

Section VII then summaries and reflects upon this paper's findings.

## II. The Essence Of Woodford's Model of a Cashless Economy

In his Chapter 2 model of a cashless economy, Woodford assumes a representative-consumer, pure exchange economy with flexible prices. While he does not explicitly include the securities necessary to complete markets, Woodford does assume markets are complete. Woodford then assumes the central bank sets the nominal interest rate by offering to borrow at the rate $i^M$. He then obtains an equation which we will call the Fisher-Euler equation, which is:[1]

$$\frac{1}{1+i_t} = \beta \frac{E_t[\xi_{j,t+1} U'_{j,t+1}(c_{j,t+1}) / P_{t+1}]}{\xi_{jt} U'_{jt}(c_{jt}) / P_t} \tag{1}$$

where $i_t$ is the nominal interest rate, $P_t$ is the price level at time t, and $\beta$ is the time preference discount factor which is assumed to be the same for all consumers. To avoid confusion caused by the representative consumer assumption, I have added j subscripts to consumption and to the utility shocks. I also have added subscripts to the utility function and to consumption and to the utility shocks. Consumer j's utility function $U'_{jt}(c_{jt})$ is assumed to be well behaved and reflects that each consumer is strictly risk averse. This basic function is the same for all states of nature, with any utility shocks being reflected in the variables $\xi_{jt}$.[2]

Woodford (p. 17) states that his equation (1.21), which is essentially the same as equation (1) above "takes the form of a 'Fisher' equation for the nominal interest rate, where the

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[1] Because of my personal preference, equation (1) is actually the reciprocal of Woodford's (1.21).

[2] Woodford's utility function is $U(c,\xi)$. However, define $\tilde{U}(c) \equiv E[U(c,\xi)]$ where the expectations operator is over all possible values of $\xi$ keeping c constant. Then define $\tilde{\xi} \equiv \frac{U(c,\xi)}{\tilde{U}(c)}$. It therefore follows that $U(c,\xi) = \tilde{\xi}\tilde{U}(c)$, which shows that this paper's symbolization and Woodford's symbolization are equivalent.

intertemporal marginal rate of substitution of the representative household plays the role of the interest rate."

While this paper points out several flaws in Woodford's analysis, it is important that we also give Woodford credit for correctly focusing on the Fisher-Euler equation (1) as the equation through which interest rates determine prices if the model is complete. Woodford (p. 73) states, "I then have a system of two equations at each date, (1.15) and (1.21) to determine the two endogenous variables $P_t$ and $i_t$…" Since Woodford's (1.15) just states that the nominal interest rate is pegged by the central bank, Woodford is saying that his Fisher-Euler equation with the nominal interest rate given determines prices under certain policy rules. In section IV, we will confirm Woodford's contention that the Fisher-Euler equation is how interest rates determine prices in a complete model.[3] However, in the next section we will learn that Woodford's model is not complete and as a result, prices are not determined in his model.

Because Woodford assumes a representative consumer, he replaces the consumption in the Fisher-Euler equation with real aggregate supply. He then loglinearizes the equation to get:

$$\hat{i}_t = \hat{r}_t + E_t[\hat{P}_{t+1}] - \hat{P}_t + E_t[\pi^*_{t+1}] \tag{2}$$

where $\hat{i}_t$ is the loglinearized nominal interest rate, $\hat{r}_t$ is the loglinearized real return, $\pi^*_{t+1} = \ln\left( P^*_{t+1} / P^*_t \right)$, and $\hat{P}_t = \ln\left( P_t / P^*_t \right)$ where $P^*_t$ is the targeted price level.

Woodford in his equation (1.34) then assumes an interest-rate pegging policy with feedback from prices:

$$\hat{i}_t = \phi_P \hat{P}_t + v_t \tag{3}$$

[3] Also, Eagle (2005c) uses the Fisher-Euler equation (1) as the basis for this analysis of the inflation dynamics of pegging interest rates.

where $v_t$ is random. He then finds that a unique bounded solution exists for $\hat{P}_t$. From this, Woodford concludes that prices are determined in his model.

The first problem with Woodford's analysis is his assumption that his solution must be bounded. Woodford never states why he assumes his solution is bounded. This absence of the justification of such an important assumption is academically inappropriate. References should be provided for something this important. This academic deficiency is even more compounded by Woodford's refusal in personal correspondence with me to explain why he makes this assumption despite my requests for him to provide me this information.

At this time, I can only presume that Woodford bases his analysis on Sargent's (1979) precedent for solving expectational difference equations. The next section addresses this precedent.

## III. Solving Expectational Difference Equations

Woodford (2003) assumes that the solution to his expectational difference equation must be bounded apparently because of Sargent's precedent to do so. Sargent (1979) says that to solve expectational difference equations, one should solve convergent equations backwards and divergent equations forward. He goes on to say that when solving expectational difference equations forward, one should assume that solution is bounded.

Eagle and Murff (2005) have reviewed the literature concerning Sargent's precedent. They found that Sargent never proved his precedent works in all cases, although for one case he did start with a model with a finite horizon and showed that as that horizon approached infinity the resulting transversality condition implied the solution was bounded. Eagle and Murff also show several examples where Sargent's precedent leads to incorrect answers. Eagle and Murff recommend that instead of Sargent's precedent, one solve expectational difference equations forward by assuming a finite horizon and taking the limit of the resulting terminal condition as the horizon approaches infinity.

Personal correspondence between the author and several economists confirms that Sargent did not prove his precedent applied in all cases. Sargent (2004) wrote that he has been out of touch in the literature related to his precedent and states that, "Everything that I know about the issue is in Gabel and Roberts and in Whiteman,…"[4] Levin (2004) wrote, "You may not be aware that there is a large literature in economics that attacks the issue of the boundedness of solutions to infinite horizon problems generally with the use of examples and models from economics... . In the case of single person optimization problems, this falls under the category of transversality conditions. Infinite horizon existence results generally follow from carefully relating a finite horizon truncated economy to the infinite horizon economy. In some cases, boundedness follows naturally from economic assumptions. In other cases (most obviously models of growth) they do not."

The comments by Sargent and Levine indicate that since the assumption of boundedness does not universally apply, the appropriate approach is as Eagle and Murff (2005) suggest, which is to assume a finite horizon, determine the terminal conditions, and then take the limit as the horizon goes to infinity. Thus, to determine whether the assumption of boundedness applies to Woodford's model, we need to look at a finite version of his model.

Let T be the final period of such a finite version of the model. Then there would be T+1 periods in the model – periods t=0,1,2,…T. Therefore, there would be T+1 price levels that would have to be determined by the model. However, the Fisher-Euler equation (1) applies only for periods t=0,1,2,…,T-1. The reason that the Fisher-Euler equation does not apply to period T is because no interest rate exists at time T. Because there is no period T+1, no borrowing or lending at time T can take place. As a result, Woodford's model has T+1 unknown price levels and only T Fisher-Euler equations. Therefore, Woodford's model is incomplete.

This actually understates the incompleteness of Woodford's model as the price levels are random variables, so for each time period the number of price level values that need to be determined equals the number of states of nature at that time. If *n* is the number of branches per node in a "decision" tree, Eagle and Murff (2005) argue that the ratio of the number of unknown price levels to the number of equations in Woodford's model will approach n as the horizon approaches infinity.

It is true that Woodford's optimization problem does include a transversality condition (See p. 72, equations (1.23) and (1.24)). However, neither of Woodford's conditions lead to a conclusion that the price level must be bounded. In fact, unbounded price levels rather than bounded ones would help ensure that one of Woodford's conditions holds (See his equation (1.24)). Also, Woodford's argument does include his policy feedback rule. A question exists as to whether that policy feedback rule completes his model. I argue that the model remains incomplete. First, Woodford's policy feedback rule is contradictory in that equation (1) is how the current interest rate determines current prices when future prices are determined, and therefore current prices cannot affect the current interest rate. This contradiction is made clearer in section VI of this paper. Second, since it also involves the interest rate, Woodford's policy feedback rule only applies for periods 0,1,2,…T-1. Hence, including this feedback rule doubles the number of equations. If there are *n* nodes per branch in a decision" tree where n>2, then the resulting number of equations will still be less than needed to complete Woodford's model.

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[4] Sargent refers to Gabel & Roberts (1987) and Whiteman (1983)

Section VI does complete the finite version of Woodford's model by adding Eagle and Domian's (2003a, 2003b) temporary money to the final period of the model, which then determines the price level for that final period. We then use this model to show that Woodford's feedback rule is contradictory and that an uncountable infinity of price sequences exist that in the limit are consistent with Woodford's model.

Because the model of section VI explicitly includes state-contingent securities to complete the markets, the next section reviews an Arrow-Debreu pure exchange economy.

## IV. A Quick Review of An Arrow-Debreu Pure Exchange Economy

This section reviews a standard Arrow-Debreu pure exchange economy without storage consisting of one nonstorable consumption good. It also discusses the important Consumption-Aggregate-Supply Invariance Property of that economy.

Assume each consumer j's time-separable utility function is:

$$\xi_{j0} U_{j0}(c_{j0}) + \sum_{t=1}^{T} \beta^{t} \sum_{s=1}^{S_t} \pi_{st} \xi_{jst} U_{jt}(c_{jst}) \tag{4}$$

where $c_{j0}$ is j's consumption at time 0, $c_{jst}$ is j's consumption in state s at time t, $\beta$ is the time preference discount factor, and $\pi_{st}$ is the probability of state s occurring at time t. The T represents the last period of the economy and $S_t$ is the number of possible states at time t. The utility functions $U_{jt}(c_{jt})$ are continuous, twice differentiable, strictly concave (i.e., strickly risk averse), and strictly increasing. To rule out corner solutions, assume $\lim_{c\to 0} U'_{j0}(|c|) = \lim_{c\to 0} U'_{jt}(|c|) = +\infty$. The time frame for the *s* subscript is determined by the *t* subscript to the right of the s subscript. For example, the $s$ in $c_{js1}$ refers to one of the possible states that can occur at time 1.

At time 0, consumers can buy or sell state-contingent securities. These state-contingent securities are prepaid securities where the buyer pays the seller the price of the security at time 0. Let $x_{jst}$ represent individual j's demand at time 0 for the state-contingent security that delivers one consumption good at time t iff state s occurs at time t. Define $\Omega_{st}$ so that the price of this security equals $P_0 \pi_{st} \Omega_{st}$. With it so defined, $\Omega_{st}$ represents the real pricing kernel.

Each consumer j chooses $x_{jst}$ for all s and t to maximize (4) subject to:

$$P_0 c_{j0} + P_0 \sum_{t=1}^{T} \sum_{s=1}^{S_t} \pi_{st} \Omega_{st} x_{jst} = P_0 y_{j0} \tag{5}$$

$$c_{jst} = y_{jst} + x_{jst} \tag{6}$$

where (6) applies for all $s$=1,2,…,$S_t$ for all $t$=1,2,…,T.

The market clearing conditions are that $\sum_{j=1}^{n} c_{j0} = Y_0$, $\sum_{j=1}^{n} c_{jst} = Y_{st}$, and $\sum_{j=1}^{n} x_{jst} = 0$ for all states $s$ at time t and for $t$=1,2,…$T$, where the aggregate supply of the consumption good is represented by $Y_0$ at time 0 and $Y_{st}$ in state s at time t respectively. Consumer j's optimization problem is satisfied when $\frac{\xi_{j0} U'_{j0}(c_{j0})}{P_0} = \frac{\beta^t \pi_{st} \xi_{jst} U'_{jt}(c_{jst})}{P_0 \pi_{st} \Omega_{st}}$ for all s=1,2,…,$S_t$ and for all t=1,2,…,T, which implies that

$$\Omega_{st} = \frac{\beta^t \xi_{jst} U'_{jt}(c_{jst})}{\xi_{j0} U'_{j0}(c_{j0})} \tag{7}$$

The left side of (7) is the real pricing kernel and the right side is the intertemporal marginal rate of substitution. Some literature mistakenly defines the pricing kernel as the intertemporal marginal rate of substitution (See, for example, Campbell, Lo, and MacKinlay, 1997, p. 294). The equality between the real pricing kernel and the intertemporal marginal rate of substitution shown in (7) is an equilibrium condition, not a definition.

Since this is a standard one-good, Arrow-Debreu pure-exchange economy with well behaved utility functions, a unique competitive equilibrium exists and that competitive equilibrium is Pareto efficient. Also, the following property holds:

**Consumption-Aggregate-Supply Invariance Property**: Let 1 and 2 represent any two different states of nature. If $Y_{1t}$=$Y_{2t}$, and $\xi_{j1t} = \xi_{j2t}$ for all j, then $c_{j1t} = c_{j2t}$ for all j.

> Proof by contradiction. Assume there is some consumption allocation in a competitive equilibrium where for some t and some states 1 and 2 at time t, $\xi_{j1t} = \xi_{j2t}$ for all j, , $Y_{1t}$=$Y_{2t}$, and there are two individuals j and k such that $c_{j1t} < c_{j2t}$ and $c_{k1t} > c_{k2t}$. Since this is an Arrow-Debreu competitive equilibrium, the consumption allocation must be Pareto efficient. Define $\tilde{c}_{j1t} \equiv \frac{1}{2}\left(c_{j1t} + c_{j2t}\right)$ and $\tilde{c}_{k1t} \equiv \frac{1}{2}\left(c_{k1t} + c_{k2t}\right)$. Define a new consumption allocation where for all consumers, for all states of nature, and for all time periods, the new consumption equals the old consumption except that j's consumption in states 1 and 2 are both $\tilde{c}_{j1t}$ and k's consumption in states and 2 are both $\tilde{c}_{k1t}$. The new consumption allocation is obviously feasible since the original allocation was feasible. Because both j and k are strictly risk averse, they are both better off with this new consumption allocation. However, that contradicts the statement that the original consumption allocation is Pareto efficient. We, therefore, conclude that $c_{j1t} = c_{j2t}$ for all j. Q.E.D.

The Consumption-Aggregate-Supply Invariance Property has important implications to economic theory. In particular, Eagle and Domian (2003a, 2003b) use the property to show that their quasi-real bonds dominate nominal bonds in the sense that no one will choose to issue or hold nominal bonds in their model. The reason is that nominal bonds expose both the issuer and the holder to the risk that the real value of the nominal bond payments will change as a result of nominal aggregate demand changing even though real aggregate supply does not change. Since the Consumption-Aggregate-Supply Invariance Property states that, in the absence of utility shocks, Pareto-efficient consumption does not change when real aggregate supply remains the same; exposing oneself to this nominal risk would be not be Pareto efficient. This raises the question as to how the central bank could affect the economy by controlling the interest rate on nominal bonds. The next section helps answer this question. In doing so, it finds that Woodford's central bank is impotent, i.e., it is unable to affect the market level of the nominal interest rate.

## V. Central Bank Impotence

Woodford argues that his central bank can control the nominal interest rate even though there are no monetary frictions in his economy. He does so with an arbitrage argument that is typically used in the field of Finance. Let $i_t$ be the market interest rate on one-period nominal bonds at time t and let $i_t^M$ be the interest rate paid on central-bank-issued, one-period nominal bonds issued at time t. Woodford argues that if $i_t^M > i_t$, then consumers will borrow at $i_t$ and lend at $i_t^M$ driving the market nominal interest $i_t$ up to $i_t^M$. Even though this is a typical Finance arbitrage argument, it is invalid. This section develops a model to show that Woodford's central bank is impotent in its ability to affect the market nominal interest rate. For any finite amount of money the central bank borrows, no matter how large, the "market" nominal interest rate will be unaffected; in other words the market nominal interest rate will not be bid up.

The model of this section adds Eagle and Domian's (2003a, 2003b) temporary money to the Arrow-Debreu economy of the previous sections. It also adds government nominal subsidies and nominal taxes, since Woodford's model includes such subsidies and taxes.

When consumers wake up each period, they wake up to endowments of both the consumption and another good, called type M good, that does not enter their utility functions.

Neither the type M good nor the consumption good can be stored from one period to the next. While the individual consumers receive the endowments of the type M good, the consumption good is collectively received. We assume that a law of the economy is that the consumption good will be distributed based on the one's relative holdings of the type M good. From now on in this paper, we refer to the type M good as "temporary money."

Let $N_{st}$ be the aggregate amount of temporary money in state *s* at time *t* and let $\eta_{jst}$ be the amount of temporary money endowed to consumer j in state *s* at time *t*. Then if consumer j keeps his/her endowed amount of temporary money, he/she will be allocated $\frac{\eta_{jst}}{N_{st}} Y_{st}$ units of the consumption good. This means that in essence the price of the consumption good in terms of the temporary money is $\frac{N_{st}}{Y_{st}}$, which implies that $N_{st} = P_{st} Y_{st}$. Given the level of real aggregate supply, the aggregate amount of temporary money then determines the price level in this economy by $P_{st} = \frac{N_{st}}{Y_{st}}$.

Let $Z_{j0}$ and $Z_{jst}$ be the nominal government subsidies paid to consumer j at time 0 and in state s at time t respectively. Negative values represent nominal lump-sum taxes paid by consumer j. Also, let $B_{j0}$ and $B_{jst}$ represent j's demands for non-central-bank-issued nominal bonds. These bonds can either be issued by the government or issued privately. Negative values for these demands represent bonds that j chooses to issue.

In this model we assume that the central bank lends a finite amount of nominal bonds. The amount the central bank lends can be arbitrarily large, but it must be finite. If $i_t^M > i_t$, then each consumer will want to buy an infinite amount of the central-bank-issued nominal bonds. Therefore, when the central bank issues only a finite amount of these bonds, those bonds will have to be rationed. Let $M_{j0}$ and $M_{jst}$ respectively represent j's rationed amount of these bonds for time-0 bonds and for time-t bonds issued when state s occurs at time t. If $i_t^M = i_t$ then these symbols represent j's demand for these bonds.

Below is the consumer's optimization problem of the previous section modified to incorporate the temporary money, the government subsidies/taxes, central-bank-issued bonds and privately-issued nominal bonds.

Each consumer j maximizes the utility function (4) subject to the following constraints:

$$P_0 c_{j0} + M_{j0} + B_{j0} + P_0 \sum_{t=1}^{T} \sum_{s=1}^{S_t} \pi_{st} \Omega_{st} x_{jst} = P_0 \frac{\eta_{j0}}{N_0} Y_0 + Z_{j0} \tag{8}$$

$$P_{st} c_{jst} + M_{jst} + B_{jst} = P_{st} \frac{\eta_{jst}}{N_{st}} Y_{st} + P_{st} x_{jst} + Z_{jst} + B_{js,t-1}(1 + i_{s,t-1}) + M_{js,t-1}(1 + i_{s,t-1}^M) \tag{9}$$

where (9) holds for s=1..$S_t$ and for t=1..T. These constraints are the same as constraints (6) and (7) except we include temporary money, government subsidies and taxes, and nominal bonds whether issued by the central bank or by others. Also, equation (9) includes the price level in state *s* at time *t*, which is determined from the temporary money.

We need to be careful not to let the government subsidies and taxes confuse the analysis. The government subsidies or taxes could end up redistributing wealth. Such redistribution will then affect the consumption allocation. Since this paper's analysis requires that we compare the resulting consumption allocation to the consumption under the original Arrow-Debreu model, we need to assume that government subsidies and taxes not be wealth redistributing. Similarly, we need make sure the rationing of the central-bank-issued nominal bonds does not redistribute wealth. In a competitive economy with complete markets, in order that the subsidies, taxes, and rationing not be wealth redistributing, consumers must be able to afford the same consumption allocation after the subsidies, taxes, and rationing as in the original Arrow-Debreu model.

Another way of saying this is that if wealth is not redistributed in this economy, then the original Arrow-Debreu consumption allocation must also satisfy the budget constraints when subsidies, taxes, and rationing are included.

An issue Woodford does not completely address in his analysis is the cash-flow constraint faced by the government. Below are the cash flow constraints facing the government:

$$\sum_{j=1}^{m} B_{j0} + \sum_{j=1}^{m} M_{j0} = \sum_{j=1}^{m} Z_{j0} \tag{10}$$

$$\sum_{j=1}^{m} B_{jst} + \sum_{j=1}^{m} M_{jst} = \sum_{j=1}^{m} Z_{jst} + (1+i_{s,t-1})\sum_{j=1}^{m} B_{js,t-1} + (1+i_{s,t-1}^{M})\sum_{j=1}^{m} M_{js,t-1} \tag{11}$$

Note that the sum of the quantity demands for privately issued bonds should equal zero. Thus the sum of the quantity demands for all non-central-bank-issued (NCBI) bonds should equal the amount of government-issued bonds. These funds plus the funds from the central-bank-issued (CBI) bonds will equal the amount of subsidies the government pays at time 0. For future periods, the sum of the government-issued bonds plus central-bank-issued bonds will equal the subsidies the government makes plus the principal and interest payments the government or central bank must pay on the bonds they issued the previous period.[5]

**The Central-Bank-Impotence and Nominal-Hedging Proposition**: If government subsidies, taxes and central-bank-bond rationing does not redistribute wealth, then the following conditions hold in any competitive equilibrium:

(i) Consumers have the same consumption allocation as in the original Arrow-Debreu model.

[5] In addition to assuming a government that spends no funds, Woodford later in Chapter 2 analyzes a government that does spend funds. In order to avoid the issue of whether government spending is Pareto efficient, this paper only considers a government that spends no money on government consumption.

(ii) For temporary money given in each period, the market-determined nominal interest rates are the same as if no government subsidies or taxes and no central bank bonds exist.

(iii) Each individual's demand to hold or issue nominal bonds exactly hedges that individual against nominal government subsidies or taxes. Therefore, each individual j's demands for nominal bonds are given by (Equation (13) holds for all states s at time t and for t=1,2,…,T.:

$$B_{j0} = Z_{j0} - M_{j0} \tag{12}$$

$$B_{ist} = Z_{jst} - M_{jst} + B_{js,t-1}(1+i_{s,t-1}) + M_{js,t-1}(1+i^{m}_{s,t-1}) \tag{13}$$

Proof: Since the subsidies, taxes and rationing do not redistribute wealth, each consumer can afford the same consumption allocation as in the original Arrow-Debreu economy. Since that consumption allocation is Pareto-efficient, they will not change. . This proves (i). As long as temporary money and real aggregate supply are the same for each state, the price levels for each state will also be the same. Given these price levels and given that the consumption levels will not change, the Fisher-Euler equation (1) implies that the nominal interest rates will be the same with subsidies, taxes, and rationing as without. This proves (ii). Nominal bonds expose both the bond issuer and the bond holder to nominal risk. Therefore, risk-averse investors would not choose to hold any nominal bonds in a competitive equilibrium where no other nominal contracts existed. However, the nominal government subsidies, taxes, and central-bank-bond rationing do expose the consumers to nominal risk. As a result, the consumers will choose to issue or hold nominal bonds as a hedge against these other nominal contracts. Specifically, each consumer would issue or hold exactly the amount of nominal bonds necessary to hedge against the nominal subsidies, taxes, and central-bank-bond rationing. The exact amount that the nominal (NCBI) bonds each consumer j would need to hold in order to get this perfect hedge are given in equations (12) and (13) because if we subtract (12) and (13) from (8) and (9) respectively, we will get the constraints that would exist if no subsidies, taxes, and central-bank-bond rationing existed. By (ii), those no-subsidy, no-tax, no-central-bank-rationing constraints must hold even though subsidies, taxes, and central-bank-bond rationing do exist. In fact, the only way that those no-subsidy, no-tax, no-central-bank-rationing constraints can be met while (8) and (9) are also met is for conditions (12) and (13) to hold. This proves (iii). Q.E.D.

The hedging statement of the above proposition is relatively easy to understand. The only reason that risk-averse consumers would choose to hold nominal bonds is to hedge against other nominal contracts, which in this model are nominal subsidies and taxes. We can use this hedging statement to more fully understand the central-bank impotence part of the proposition. To do so, suppress the s subscripts for notational simplicity and define $\tilde{Z}_t \equiv \sum_{j=1}^{m} Z_{jt}$ , $\tilde{M}_t \equiv \sum_{j=1}^{T} M_{jt}$ , and $\tilde{B}_t^d \equiv \sum_{j=1}^{m} B_{jt}$ , which respectively represent the aggregate subsidies (taxes), aggregate central-bank-issued nominal bonds, and aggregate demand for NCBI nominal bonds (NCBI stands for "non-central-bank-issued").

Next, define:

$$\tilde{\tilde{M}}_t \equiv \tilde{M}_t - (1 + i_{t-1}^M)\tilde{M}_{t-1} \tag{14}$$

which then represents the net new central-bank issued nominal debt at time t (net of principal and interest paid on previously issued such debt). Note that $\tilde{\tilde{M}}_0 = \tilde{M}_0$ since at time t=0 there is no previously held such debt. Also, note that $\tilde{\tilde{M}}_T \equiv -(1 + i_{T-1}^M)\tilde{M}_{T-1}$ since at time T, no new debt whatsoever can be issued but the central bank still must pay back the principal and interest on its previously issued debt.

Next define:

$$\tilde{\tilde{B}}_t^d \equiv \tilde{B}_t^d - (1 + i_{t-1})\tilde{B}_{t-1}^d \tag{15}$$

which then presents the net new NCBI nominal debt at time t (net of principal and interest paid on previously issued such debt). Again, $\tilde{\tilde{B}}_0^d = \tilde{B}_0^d$ and $\tilde{\tilde{B}}_T^0 = -(1 - i_{T-1})\tilde{B}_{T-1}^0$

Finally, define $\tilde{\tilde{B}}_t^s$ to be the net new government-issued (but not central-bank-issued) nominal debt. Again $\tilde{\tilde{B}}_t^s$ is net of principal and interest the government pays in period t on previously issued government debt. Given the definitions of $\tilde{\tilde{B}}_t^s$ and $\tilde{\tilde{M}}_t$, the government cash flow constraints (10) and (11) can be rewritten as just one constraint:

$$\tilde{\tilde{B}}_t^s + \tilde{\tilde{M}}_t = \tilde{Z}_t \tag{16}$$

This states that the net new government nominal debt plus the net new central-bank debt must equal the total amount of subsidies paid by the government. (Even though negative subsidies represents taxes, I will from now on just refer to subsidies to try to make the reading easier.)

We will consider two scenarios when there is an increase in $\tilde{M}_t$ for some t between 0 and T-1, but no change in $\tilde{M}_\tau$ occurs for any $\tau \neq t$. Under the first scenario the subsidies remain unchanged even though $\tilde{M}_t$ changes. Under the second scenario, the subsidies change to accommodate any resulting change in the net new central-bank-issued debt that results from $\tilde{M}_t$ changing.

Let us begin with Scenario I where for some t between 0 and T-1, $\tilde{M}_t$ increases by one "buck," but all subsidies remain the same as before the change and no change in $\tilde{M}_\tau$ occurs for any $\tau \neq t$. The first effect is that $\tilde{\tilde{M}}_t$ increases by one buck. The second effect is that $\tilde{\tilde{B}}_t^d$ decreases by one buck since consumers only use nominal bonds to hedge against nominal subsidies. Since subsidies under Scenario I remain unchanged, consumers' demands for all nominal bonds, whether central-bank-issued or otherwise, will not change. Hence when $\tilde{\tilde{M}}_t$ increases by one buck, consumers will reduce their demand for NCBI nominal bonds by an equal amount so that $\tilde{\tilde{B}}_t^d$ will decrease by one buck.

The third effect under Scenario 1 is that $\tilde{\tilde{B}}_t^s$ will decrease by one buck. This is clear from (16). When $\tilde{Z}_t$ remains the same, an increase in $\tilde{\tilde{M}}_t$ will be accompanied by an equal decrease in $\tilde{\tilde{B}}_t^s$.

In summary for Scenario 1, $\tilde{\tilde{B}}_t^d$ and $\tilde{\tilde{B}}_t^s$ will each decrease by one buck. In other words, both the demand curve and supply curve for NCBI nominal bonds will shift to the left by one buck. When both the demand and supply shift to the left equally, the "equilibrium" interest rate is unaffected. Here, "equilibrium" refers to equilibrium in the NCBI nominal bond market.

Now let us consider the effect on the net new nominal debt demand and supply at time t+1. Since $\tilde{M}_t$ increases by one buck and $\tilde{M}_{t+1}$ does not change, (14) implies that $\tilde{\tilde{M}}_{t+1}$ will decrease by $(1+i_t^M)$. However, since no subsidy has changed, the aggregate demand by consumers for nominal bonds from whatever source is unchanged. Therefore, this decrease in $\tilde{\tilde{M}}_{t+1}$ will be accompanied by an increase in $\tilde{\tilde{B}}_{t+1}^d$ by $(1+i_t^M)$. However, (16) implies that $\tilde{\tilde{B}}_{t+1}^s$ will also decrease by $(1+i_t^M)$. These equal increases in $\tilde{\tilde{B}}_{t+1}^d$ and $\tilde{\tilde{B}}_{t+1}^s$ mean both the demand and supply curves for net new NCBI nominal bonds at time t+1 shift to the right by equal amounts, resulting in no change in the "equilibrium" nominal interest rate. (Again, "equilibrium" is defined as equilibrium in the NCBI nominal bond market.)

For τ=t+2,…,T; $\tilde{\tilde{M}}_{\tau+1}$ does not change. We therefore conclude that under Scenario 1, no change in the equilibrium nominal interest rate will occur for a one buck, one-time increase in central-bank-issued and rationed nominal bonds. This argument would reach the same conclusion for any finite such increase in the central-bank-issued and rationed nominal bonds.

Now consider Scenario 2 where any change in $\tilde{\tilde{M}}_t$ is accompanied by an equal increase in the subsidies $\tilde{Z}_t$. However, because $\tilde{Z}_t$ increases exactly with any increase in $\tilde{\tilde{M}}_t$, the consumers will have more nominal subsidies to hedge against and therefore will demand an equal increase in the net new nominal debt they hold, whether central-bank-issued or otherwise. Since that increase in net new nominal debt they demand will equal the increase in the net new central-bank-issued debt they are rationed, their demand for net new NCBI nominal debt will be unaffected. In other words, $\tilde{\tilde{B}}_t^d$ will not change. Also, by (16) $\tilde{\tilde{B}}_t^s$ will not change when $\tilde{\tilde{M}}_t$ and $\tilde{Z}_t$ increase by equal amounts. If neither the demand nor supply for net new NCBI debt changes, then there will be no change in the "equilibrium" nominal interest rate.

We, therefore, conclude that Woodford's central bank is impotent. For any finite increase in central-bank-issued and rationed bonds, neither the demand for nor the supply of NCBI bonds shifts under scenario 2, resulting in no change in the equilibrium nominal interest rate. On the other hand, in scenario 1, the demand and supply curves for NCBI nominal bonds shift in the same direction by the same amount, again resulting in no change to the equilibrium nominal interest rate.

## VI. An "Almost Cashless" Economy

In this section, we modify the model of the previous section, by assuming that temporary money only exists at time T. To bypass the problem of central bank impotence discussed in the previous section, we assume that the central bank is given the authority to legally dictate the nominal interest rate paid in the market. The resulting economy is one which is "almost cashless." It is cashless for periods t=0,1,…,T-1. Only in period T is there any monetary friction, and that is the monetary friction of temporary money. We show in this economy that when the central bank dictates or pegs the nominal interest rate, that nominal interest rate for time t determines the price level for time t. This then makes it impossible for the price level at time t to affect the nominal interest rate at time t. We show that Woodford's model of a cashless economy can be looked at as a limit of this section's "almost cashless" economy. In doing so we learn that there are an uncountably infinite number of price paths consistent with Woodford's model.

At time T, the consumers collectively receive the endowment of the consumption good, which they allocate among themselves based on each consumer's relative endowment of temporary money. Since temporary money only exists in period T, we make the usual assumption that consumers individually receive endowments of the consumption good for periods prior to period T.

The amount of temporary money and real aggregate supply at time T will determine the price level at time T:

$$P_{sT} = \frac{N_{sT}}{Y_{sT}} \qquad (17)$$

Now we move backwards using Bellman's principle. If we solve the Fisher-Euler equation (1) for $P_{st}$, we get:

$$P_{st} = \frac{\xi_{jst} U'(c_{jst})}{\beta(1+i_{st}) E_{st}[\xi_{sj,t+1} U'(c_{js,t+1}) / P_{s,t+1}]}$$

which can be rewritten as:

$$P_{st} = \frac{\left( \dfrac{\xi_{jst} U'(c_{jst})}{\beta E_{st}[\xi_{sj,t+1} U'(c_{js,t+1}) / P_{s,t+1}]} \right)}{(1+i_{st})} \qquad (18)$$

Because markets are complete, we know that the consumption allocation will be the same as for the original Arrow-Debreu economy. Assume that $P_{s,t+1}$ is given for all s at time t+1 (which it would if t+1=T). Then if the central bank pegged the interest rate, then all random variables on the right side of (18) would be given which means that the price level at time t would be determined. By backward recursion, we conclude that prices are determined.

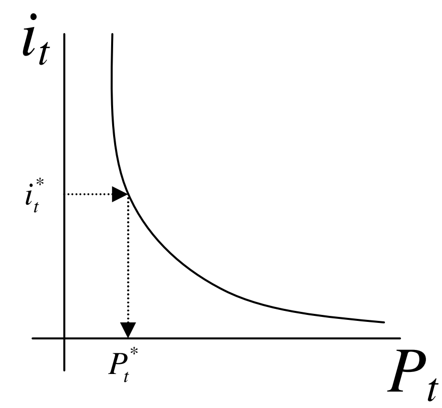

Figure 1: Price Determination in "Almost cashless" Economy

Figure 1 plots relationship (18). Under the complete markets assumption, every consumer will know his/her own consumption allocation for each state and for all time t. Also, the future price level in state s at time t+1 is known. We, therefore, conclude that the numerator of (18) is known at time t. Therefore, if the central bank dictates or pegs the nominal interest rate at $i_t^*$, then (18) implies the price level that will result. If the central bank were to increase the nominal interest rate, a lower price level would result; whereas a lower nominal interest rate would result in a higher price level.[6]

Since the current nominal interest rate determines the current price level, it cannot be the case that the central bank uses the current price level to determine the nominal interest rate it will dictate or peg.

A relatively simple example can make this clear. Assume the economy lasts for two periods: period 0 and period 1. Assume consumers have identical logarithmic utility functions without utility shocks and have perfect foresight concerning period 1. Then, consumer j will maximize:

[6] This relationship is more fully explored in Eagle (2005c).

$$U(c_{j0}, c_{j1}) = \ln(c_{j0}) + \beta \ln(c_{j1}) \tag{19}$$

subject to

$$P_0 c_{j0} + B_0 + P_0 \Omega_t x_{jt} = P_0 y_{j0} \tag{20}$$

$$P_1 c_{j1} = P_1\left(\frac{\eta_{j1}}{N_1} Y_1 + x_{j1}\right) + (1 + i_0) B_0 \tag{21}$$

In this simple example, there is only one state-contingent security, since there is only one future time and one state for that time because of the assumption of certainty.

Assume that $N_1$=100 and $Y_1$=100, which imply by (17) $P_1$=1.00. Also, assume that $\beta = \frac{1}{1.05}$ and $Y_0$=100. Then (18) becomes

$$P_0 = \frac{\left(\frac{1.05}{c_{j0} / c_{j1}]}\right)}{(1 + i_0)} \tag{22}$$

In the above we also used that $P_1$=1.00, that there are no utility shocks, and that the utility function is (19). Since markets are complete, the logarithmic utility functions imply that $\frac{c_{j0}}{Y_0} = \frac{c_{j1}}{Y_1}$ (See Eagle and Domian, 2003b). If we also assume $Y_0$=$Y_1$, then $c_{j0} = c_{j1}$, which implies that (22) can be rewritten as:

$$P_0 = \frac{1.05}{(1 + i_0)} \tag{23}$$

Therefore, if the central bank dictates or pegs $i_0$=5%, then $P_0$ will equal 1.00. On the other hand, if the central bank dictates or pegs $i_0$=2%, then $P_0$=1.05/1.02≈1.03.

We now ask, "How does Woodford's feedback rule fit into this model?" Remember, that Woodford's feedback rule is where the central bank uses the current price level to determine the nominal interest rate it pegs. However, at the time when the central bank makes its decision as to what nominal interest rate to peg, the current price level is not yet determined; it will be determined only after the central bank pegs the nominal interest rate.

Suppose Woodford's feedback rule is:

$$1+i_t^* = \left(1+\bar{i}_t\right)\left(1+\varphi(P_t - P_t^*)\right) \tag{24}$$

where $i_t^*$ is the nominal interest rate that the central bank should peg according to Woodford's feedback rule, $\bar{i}_t$ is the nominal interest rate that would be consistent with the targeted price level $P_t^*$. If we assume that $P_0^* = 1.00$, then (23) implies that $\bar{i}_t$ is 5%. If we then take (24) for t=0 and φ>0 and combine it with (23) and $P_0^* = 1.00$, we get $1 = P_0 + \varphi P_0^2 - \varphi P_2$, whose only positive solution is $P_0$=1.00. While Woodford's "feedback" equation (24) can be combined with (23) to find a particular $P_0$ and $i_0$, it would be a mistake to call equation (24) a "feedback" equation. The solution to (23) and (24) is just a consistency solution. It is merely saying the only observation that would be consistent with Woodford's "feedback" function and (23) would be when the price level in fact turns out to be 1.00 and the nominal interest rate equals 5%. As stated before, in this model the current nominal interest rate determines the current price level. Thus, at the time the central bank dictates or pegs the current nominal interest rate, the current price level is unknown.

Since this section's model is "almost cashless," one might wonder what would happen if we took the limit of the model as $T$ goes to infinity. We can think about Woodford's Chapter 2 model of a cashless model as such a limit.

Note that $P_T$ is determined by $N_T$, which we are assuming is just an endowment from nature, not something under the control of the central bank. The value of $N_T$ could be any positive real number. Since $P_T=N_T/Y_T$, $P_T$ could also be any positive real number. Since the number of positive real numbers is uncountably infinite, the number of possible values of $P_T$ will be uncountably infinite. Since $P_t$ for $t$=0,...,$T$-1 depends on the expected value of $P_T$, there will be an uncountably infinite number of price sequences over time in the "almost cashless" economy unless we anchor $N_T$. However, if we take the limit of the "almost cashless" economy, then in that limit, we never get to period $T$ because in the limit there is no terminal period. Hence, in the limit there is nothing to specify $N_T$; hence there will be an uncountably infinite number of price sequences consistent with that limit. In other words, in the limit prices are not determined.

For example, again assume perfect foresight and logarithmic utility functions without utility shocks. Then each consumer j maximizes:

$$U(c_{j0}, c_{j1}, \ldots, c_{jT}) = \sum_{t=0}^{T} \beta^t \ln(c_{jt}) \tag{25}$$

s.t.:

$$P_0 c_{j0} + B_0 + P_0 \sum_{t=1}^{T} \Omega_t x_t = P_0 y_{j0} \tag{26}$$

$$P_t c_{jt} + B_t = P_t \left(y_{jt} + x_{jt}\right) + \left(1 + i_{t-1}\right) B_{t-1} \text{ for t=1,2,…,T-1} \tag{27}$$

$$P_T c_{jT} = P_T \left( \frac{\eta_{jT}}{N_T} Y_T + x_{jT} \right) + \left(1 + i_{T-1}\right) B_{T-1} \tag{28}$$

Assume also that the central bank dictates the nominal interest rate so that $(1+i_t)\beta=1$ for all t and assume that real aggregate supply is constant over time. Under these assumptions (18) becomes $P_t = P_T$. Since $P_T=N_T/Y_T$, it follows that $P_t=N_T/Y_T$. Now think about the set of all such "almost cashless" finite economies where $N_T$ is a particular positive real number. For each possible value of $N_T$, there is a unique price sequence where all $P_t=N_T/Y_T$. Since $N_T$ is a particular positive real number, there can be an uncountably infinite number of possible values for $N_T$ and therefore an uncountably infinite number of possible price sequences. Since the "almost cashless" economy with any of the uncountable infinite number of possible price sequences will converge to Woodford's model as the horizon T goes to infinity, we conclude that there will be an uncountable infinite number of price sequences that are consistent with Woodford's cashless model. To me that means that prices are indeterminate in Woodford's model.

## VI. Summary

This paper has shown that Woodford makes four logical fallacies in his analysis of his Chapter 2 model of a cashless economy. First, he erroneously assumes that the solution to his expectational difference equation must be bounded. While his making that assumption does have precedent in the economic literature, that literature never proves that one can make that assumption in all cases and in some cases it does lead to erroneous solutions.

The appropriate way to solve the expectational difference equations is to assume a finite horizon, determine the terminal condition, and then take the limit of that terminal condition as the horizon goes to infinity. Doing so with Woodford's model, shows that any finite version of his model is incomplete, which is Woodford's second fallacy.

This paper looks at the economics behind Woodford's claim that the central bank in his model can control the nominal interest rate paid on non-central-bank-issued bonds. We conclude that Woodford's claim is vacuous, that Woodford's central bank is impotent. In particular, we find that any finite issuance of central-bank-issued bonds will not be able to affect the "equilibrium" nominal interest rate where "equilibrium" refers to equilibrium in the non-central-bank-issued bond market.

This paper also looks at an "almost cashless," finite economy that in the limit approaches Woodford's model (with the exception that we had to assume that the central bank dictates the nominal interest rate to avoid the central-bank-impotence problem). We find that the current interest rate determines the current price level, which makes Woodford's feedback rule impossible. At the time the central bank dictates or pegs the nominal interest rate, the current price level has yet to be determined; the current price level at the point is unknown.

While this paper finds four fallacies with Woodford's analysis, this does not necessarily say that we should reject all of Woodford's policy recommendation. For example, Eagle (2005b) has modified Woodford's feedback rule to be

$$1 + i_t^* = \left(1 + \bar{i}_t\right)\left(1 + \varphi(E_{t-1}[P_t] - P_t^*)\right) \tag{23}$$

While the central bank at time t does not know $P_t$, the public's expectations at time $E_{t-1}[P_t]$ may be known. The feedback rule (23) does enhance the credibility of a central bank according to Eagle's analysis.

However, I would recommend against any attempt to totally make an economy cashless as Woodford has argued is possible. Only if future prices are determined, can pegging the interest rate affect current prices today through the Fisher-Euler equation (1). In Woodford's totally cashless economy, future prices are not determined and therefore the Fisher-Euler equation (1) does not determine prices.

Many mathematical techniques used in economics came from applications in rocket science including Ito calculus, the Wiener process, and much of control theory. Suppose the transfer of knowledge in the future goes from economics to rocket science. In particular, let us consider the absurd implications that rocket scientists would encounter if they were to adopt Woodford's approach to solving expectational difference equations.

Suppose a rocket is located in space at $x_0$. The rocket has one thrust to send it off at a constant speed. While the control center has targeted a path for the rocket, it has no mechanism to control the rocket's direction. Define $x_t$ and $x_t^*$ to respectively represent the actual and targeted location vectors of the rocket at time t.

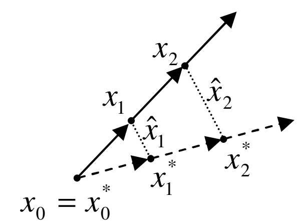

Figure 1. Actual vs. targeted time paths

Define $\hat{x}_t \equiv \left| x_t - x_t^* \right|$, which is the distance between $x_t$ and $x_t^*$. For the moment, suppose the rocket scientists were to follow Woodford's lead of assuming the solution of $\hat{x}_t$ is bounded. If the rocket's actual trajectory differs from the targeted trajectory, then $\lim_{t\to\infty} \left| x_t - x_t^* \right| = \infty$ (See Figure 1). Therefore, the only bounded solution of $\hat{x}_t$ is where the rocket is forever on target. If the rocket scientists were to apply Woodford's type of analysis to this rocket example, they would then conclude that they could perfectly determine the rocket's trajectory even though they had no steering mechanism to cause that to happen. Such a conclusion would be absurd.

Just as there is no steering mechanism in the rocket example, Woodford's model has no monetary frictions and hence no mechanism by which monetary policy affects nominal aggregate demand and hence prices. It is true that Woodford has the Fisher-Euler equation (1), but that equation is more like a law of motion, not a steering mechanism.

Is Woodford's analysis rocket science? I do not think it is. I myself hope never to be on any rocket (or an economy) that has been designed on the basis of Woodford's type of analysis.

## References


Campbell, Lo, and MacKinlay, (1997), *The Econometrics of Financial Markets* (Princeton University Press – Princeton).

Eagle, David (2005a). "Completing Markets in a One-Good, Pure Exchange Economy Without State-Contingent Securities," Finance 0501009, Economics Working Paper Archive at WUSTL.

Eagle, David (2005b). "Price Indeterminacy Reinvented: Pegging Interest Rates While Targeting Prices, Inflation, or Nominal Income," Macroeconomics 0501028, Economics Working Paper Archive at WUSTL.

Eagle, David (2005c). "The Inflation Dynamics of Pegging Interest Rates," Macroeconomics 0502029, Economics Working Paper Archive at WUSTL.

Eagle, David & Elizabeth Murff (2005). "Logical Pitfalls of Assuming Bounded Solutions to Expectational Difference Equations," GE, Growth, Math methods 0501002, Economics Working Paper Archive at WUSTL.

Eagle, David & Dale Domian (2003a). "Sounding the Alarm on Inflation Indexing and Strict Inflation Targeting," Macroeconomics 0312010, Economics Working Paper Archive at WUSTL.

Eagle, David & Dale Domian, (2003b). "Quasi-Real Indexing -- The Pareto-Efficient Solution to Inflation Indexing," Macroeconomics 0312012, Economics Working Paper Archive at WUSTL.

Gabel, Robert A and Richard A. Roberts (1987), *Signals and Linear Systems*, (Wiley – New York)

Levine, David (2004). Personal Correspondence with David Eagle.

Lucas, Robert E. (1976). "Econometric Policy Evaluation: A Critique," *Carnegie-Rochester Conference Series on Public Policy* 1, 19-46.

Sargent, Thomas J. (1979). *Macroeconomic Theory – Economic Theory, Econometrics, and Mathematical Economics* (Academic Press – New York).

Sargent, Thomas J. (1986). *Dynamic Macroeconomic Theory* (Harvard University Press – Cambridge, Massachusetts).

Sargent, Thomas J. (2004). Personal Correspondence with David Eagle

Sargent, Thomas J. and Neil Wallace (1975). "'Rational' Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule," *Journal of Political Economy* 83:241-254.

Whiteman, Charles (1983), *Linear Rational Expectations Models – A User's Guide*, (University of Minnesota Press – Minneapolis)

Woodford, Michael (2003). *Interest and Prices – A Foundation of a Theory of Monetary Policy* (Princeton University Press – Princeton, New Jersey).