Financial Crisis and Recovery: Learning-based Liquidity Preference Fluctuations*

Ryo Horii† and Yoshiyasu Ono‡

February 2005

*The authors are grateful to V.V. Chari, Kazuo Mino, Akihisa Shibata, and seminar participants at the University of Tokyo, the Tokyo Institute of Technology, the Delhi School of Economics, and the Summer Workshop on Macroeconomics held at the University of Auckland for their helpful comments and suggestions. This research is a part of the Osaka University 21st Century COE program, the Ministry of Education, Science, Sports and Culture.

†Correspondence: Assistant Professor at Graduate School of Economics, Osaka University, 1-7 Machikaneyama, Toyonaka, Osaka 560-0043, JAPAN; E-mail horii@econ.osaka-u.ac.jp; Telephone +81-6-6850-5254; Fax +81-6-6850-5274

‡Professor at Institute of Social and Economic Research, Osaka University, 6-1, Mihogaoka, Ibaraki-shi, Osaka, 567-0047, JAPAN; E-mail ono@iser.osaka-u.ac.jp

Abstract

This paper examines a mechanism of liquidity-preference fluctuations caused by people's learning behavior. When observing a financial shock, they rationally update their belief so that the subjective probability of encountering it again is higher, immediately raise liquidity preference and reduce consumption. As a period without the shock lasts after that, they gradually decrease the subjective probability, lower liquidity preference and increase consumption. Particularly, when the shock is observed many times in succession, recovery is first slow because people do not easily change their pessimistic view, then gradually accelerates, and eventually slows down as they become fully optimistic.

Keywords: Bayesian Learning, Liquidity Preference, Precautionary Motive, Markov Switching

JEL Classifications: D83, E41, E32

1 Introduction

People hold money for a wide variety of reasons. Those discussed in the literature include the transaction motive, the speculative motive, status preference, and many others. Among those, the precautionary motive is most closely related to people's assessment about economic uncertainty, which evolves over time according to their learning behavior. In particular, their incentive to hold money depends on their estimates about the probability of encountering a shock in which liquidity is needed. By observing whether or not such a shock actually occurs at each point in time, they rationally update the likelihood of encountering the shock in future, causing their liquidity preference to fluctuate. The fluctuation then affects the optimal time paths of money holding and consumption, generating a fluctuation in aggregate demand. This paper develops a theoretical framework that captures this idea and investigates the pattern of decline and recovery in aggregate demand.

We consider a situation where people hold money to prepare for a possible aggregate liquidity shock, such as bank runs or financial system collapses. At normal times, they can settle transactions without money by relying on credit cards or other forms of short-term credits. However, once a financial crisis occurs and obtaining credits becomes difficult, holding money stock yields an explicit benefit. Bank runs can also be regarded as another example of such liquidity shocks. As long as banks normally operate, people need not hold money since they can easily withdraw deposits. Only when a bank run occurs, holding money generates utility. However, since such a crisis cannot exactly be anticipated, people have an incentive to hold money anytime.

Obviously, the expected benefit from holding money is larger when the probability of encountering the liquidity shock is higher. Note that the shock probability is not necessarily constant over time. For example, Japan has experienced a number of bank failures in the 1990s whereas there were very few before, implying some underlying conditions for the economy changed in that period. If people know the possibility of such a structural change but cannot directly observe it, an occurrence of the shock affects not only their current utility but also their belief about the current underlying state of the economy.[1] More precisely, when people observe the shock, they strengthen the belief that they are in a dangerous state, increase their subjective probability for meeting with the shock again, and raise liquidity preference. Conversely, if the shock does not occur for a while, they gradually increase the belief of being in a safer state, reduce the shock probability, and lower liquidity preference.

To highlight this information-driven fluctuation in liquidity preference, we introduce a stochastic version of Sidrauski (1967) model: people receive utility from holding money only when the liquidity shock occurs. There are two unobservable states, between which the economy goes back and forth according to a Markov process. The liquidity shock follows a Poisson process in each state with a different arrival rate. By observing whether or not the shock occurs at each point in time, people rationally update the subjective probability using Bayes' law and revise the time paths of consumption and money holding. Assuming that nominal wage adjustment is sluggish, we show that aggregate demand and employment fluctuate over time as liquidity preference varies.

The fluctuation pattern depends on the frequency of the shock because one occurrence of the shock only partially reveals information about the true state. As long as the liquidity shock occurs sparsely in time, people view the shock as a mere accident. Although consumption decreases temporarily, it quickly recovers as this optimistic view is confirmed by the subsequent observation that no shock occurs thereafter. However, when the shock occurs many times for a short while, people are convinced that they are in the more dangerous state. In this case the recovery process takes a long time even after the shock

[1]If they know the underlying state and it does not change over time, the expected gain of holding money is constant. Thus, they behave as if they had a deterministic utility function of money as originally assumed in Sidrauski (1967). This implies that the fluctuation in the liquidity preference is driven not by the occurrence of the shock *per se* but by information brought by the shock.

ceases. The recovery speed of consumption is initially slow because their pessimistic belief is so strong that it is not easily turned over by the gradually revealed information that no shock occurs.[2] The recovery process gradually accelerates as the belief gets weaker and more sensitive to new information, and eventually slows down again when they are quite sure that they are in the safer state. Those results contrast with the previous studies of informational cycles by Caplin and Leahy (1993) and Zeira (1993), where a shock, once it occurs, completely reveals the true state.[3]

There are a number of attempts to explain cyclical movements of macroeconomic variables by combining unobservable regime changes and Bayesian updating agents. When the signal is noisy, agents slowly change their belief, making the effect of a regime change more persistent than in the case where the state is perfectly observable.[4] Andolfatto and Gomme (2003) and Sill and Wrase (1999), for example, demonstrate that monetary policy has longlasting effects when it periodically switches between low and high monetary growth in an unobservable way. Other papers examine how agents react to unobservable changes in investment opportunities. Chalkley and Lee (1998) show that recovery from a recession is protracted when risk aversion of agents prevents them from acting promptly

---

[2]This mechanism provides a possible explanation of why it took so long a time for consumer confidence in Japan to recover after experiencing a succession of bank failures in the 1990s.

[3]In their models, even when a number of shocks come in a bunch, those except for the first one conveys almost no information because uncertainty vanishes after the first. Thus, the number of shocks that agents observe does not affect their behavior. Boldrin and Levine (2001) also consider a related model, but they rule out the possibility that the shock continually occurs.

[4]The literature of herds also examines the situation in which agents gradually receive noisy signals, and shows that learning from others' decisions sometimes leads to abrupt changes in aggregate variables rather than persistence (See, for example, Chamley and Gale 1994, and Chari and Kehoe 2004). A typical assumption in studies of herd behavior is that each agent has some private information and must make a one-time and irreversible decision, while we consider a representative agent who chooses money holdings at every instant.

on receiving good news. Veldkamp (2005) and Nieuwerburgh and Veldkamp (2005) explain the slow recovery by an endogenous flow of information (see also Potter, 2000). If agents have a pessimistic belief, their activities are low, generating fewer public information, and therefore good news are only slowly revealed in bad times. Those studies are complementary to this paper, where recovery is slow not because information is scarce in recession but people's strong belief dwarfs the significance of new, favorable information.[5]

Another distinctive feature of this paper is that the crash and the slow recovery is produced by fluctuations in consumer demand. We set up a continuous-time model in which infinitely-lived households maximize their expected utility that depends on the paths of consumption and money holding, and characterize the dynamics of the economy as a stationary cycle. We theoretically show that their rational, forward-looking behavior generates a crash and a subsequent slow recovery, whereas previous related studies either solve the model entirely by numerical procedures or consider the case where agents live only for one period so that their decision problems are static.[6]

The remaining of the paper is organized as follows. After modelling the belief-updating behavior of households in the next section, we examine liquidity-preference fluctuations and the optimal consumption behavior in section 3. Section 4 derives the existence, uniqueness and other properties of the stationary equilibrium path and presents the cyclical movements of the belief and consumption. Section 5 summarizes and concludes. A mathematical proof and numerical procedures are provided in the appendix.

[5]In fact, the flow of information brought by no occurrence of the shock is largest when people are convinced of being in the more dangerous state. However, it is also the time when their prior belief is strongest, and hence people only slowly change it.

[6]Although they are not studies on cycles, Driffill and Miller (1993) and Zeira (1999) analytically examine continuous-time models in which agents update their belief based on discrete signals. In their models, however, uncertainty eventually vanishes and the economy reaches a steady state since the unobservable state is time invariant.

2 Liquidity Shock and Bayesian Learning

We use a continuous-time model in which a representative household faces an aggregate liquidity shock that follows an exogenous Poisson process. Liquidity holding generates utility when the shock actually occurs, but does not while the shock does not occur. Since when the shock occurs cannot exactly be anticipated, even during the period without it the household holds liquidity so as to prepare for it.

There are two underlying states with different probabilities of the shock, called states H and L. In state $i \in \{H, L\}$ the shock occurs with probability θ^i per unit time, where $\theta^H > \theta^L > 0$. The household cannot directly observe the current state but knows that the state evolves according to a Markov process: state H changes to state L with Poisson probability p^H per unit time whereas state L changes to state H with probability p^L. We assume that the shock occurs much more frequently in state H than in state L and that the state change is a rare event when compared to the shock in state H. Formally,

Assumption 1 $\theta^H - \theta^L > p^H + p^L$.

By observing whether the shock occurs or not she continuously revises her subjective shock probability in a Bayesian manner. Let $\theta_t \in \{\theta^H, \theta^L\}$ denote the true shock probability at time t, which is unknown to her. Using information available up to time t, she forms a belief that current θ_t is θ^H with probability λ_t^H and θ^L with probability λ_t^L. Obviously,

$$\lambda_t^L + \lambda_t^H = 1 \quad \text{for all } t. \tag{1}$$

In order to find how she updates λ_t^i from t to $t + \Delta t$,[7] we first obtain the subjective probability that the shock does not occur between t and $t + \Delta t$ for given λ_t^i. It is denoted by $\mathrm{Prob}_t\left[S_{(t,t+\Delta t]} = \phi\right]$, where $\mathrm{Prob}_t[\cdot]$ is a probability operator based on information available at t, $S_{(a,b]}$ is the set of dates on which the shock actually occurs

[7] Time interval Δt is taken to be so short that the probability that the liquidity shock and a state change coexist in the interval is negligible.

during $(a, b]$, and ϕ the empty set. Since the underlying state is either H or L at time $t + \Delta t$, this probability is divided into two components, $\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^H\right]$ and $\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^L\right]$.

Each of the two components is further divided into two probabilities. The former is the sum of the probability that 'the state is H at time t and neither the state change nor the shock occurs during the interval' and the probability that 'the present state is L and the state changes to H during the interval.' It is[8]

$$\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^H\right] = \left(1 - (\theta^H + p^H)\Delta t\right) \lambda_t^H + (p^L \Delta t)\lambda_t^L. \tag{2}$$

Similarly, the latter is

$$\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^L\right] = \left(1 - (\theta^L + p^L)\Delta t\right) \lambda_t^L + (p^H \Delta t)\lambda_t^H. \tag{3}$$

Summing up (2) and (3) yields

$$\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi\right] = 1 - \theta_t^e \Delta t, \tag{4}$$

where θ_t^e represents the expected (or subjective) probability of the shock per unit time at time t:

$$\theta_t^e \equiv \theta^H \lambda_t^H + \theta^L \lambda_t^L. \tag{5}$$

Let us consider how the representative household updates her belief if she eventually finds that the shock did not occur during $(t, t + \Delta t]$. In this case the information that $S_{(t,t+\Delta t]} = \phi$ is added to her knowledge. Thus, using Bayes' law we find updated subjective probability $\lambda_{t+\Delta t}^i$ to be

$$\begin{aligned}\lambda_{t+\Delta t}^i &\equiv \text{Prob}_{t+\Delta t}\left[\theta_{t+\Delta t} = \theta^i\right] = \text{Prob}_t\left[\theta_{t+\Delta t} = \theta^i | S_{(t,t+\Delta t]} = \phi\right] \\ &= \frac{\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^i\right]}{\text{Prob}_t\left[S_{(t,t+\Delta t]} = \phi\right]}.\end{aligned}$$

[8]Throughout the paper we ignore the second-order term of Δt and higher because $\Delta t \to 0$.

Since the numerator is given by (2) or (3) and the denominator by (4), $\lambda_{t+\Delta t}^H$ equals[9]

$$\lambda_{t+\Delta t}^H = \frac{\left(1-(\theta^H+p^H)\Delta t\right)\lambda_t^H + (p^L\Delta t)\lambda_t^L}{1-\theta_t^e\Delta t}.$$

From this equation we derive the time derivative of λ_t^H as

$$\dot{\lambda}_t^H = \lim_{\Delta t\to 0}\frac{\lambda_{t+\Delta t}^H - \lambda_t^H}{\Delta t} = (\theta_t^e - \theta^H - p^H)\lambda_t^H + p^L\lambda_t^L. \tag{6}$$

We next consider the case where the shock occurs during $(t, t+\Delta t]$. Since

$$\mathrm{Prob}_t\left[S_{(t,t+\Delta t]} \neq \phi \cap \theta_{t+\Delta t} = \theta^i\right] = \theta^i\lambda_t^i\Delta t \quad \text{for} \quad i \in \{L, H\}, \tag{7}$$

the probability that the shock occurs is

$$\mathrm{Prob}_t\left[S_{(t,t+\Delta t]} \neq \phi\right] = \left(\theta^H\lambda_t^H + \theta^L\lambda_t^L\right)\Delta t = \theta_t^e\Delta t, \tag{8}$$

which is consistent with (4). From Bayes' law dividing (7) by (8) gives the updated subjective probability that $\theta_{t+\Delta t} = \theta^i$ under the condition that the shock occurs during $(t, t+\Delta t]$. It is

$$\lambda_t^i = \lim_{t'\to t-0}\frac{\theta^i\lambda_{t'}^i}{\theta_{t'}^e} \equiv \frac{\theta^i\lambda_{t-0}^i}{\theta_{t-0}^e}, \tag{9}$$

where subscript $t-0$ represents the state just before t.

Finally, we obtain the dynamics of subjective probability θ_t^e. From (1) and (5),

$$\lambda_t^H = \frac{\theta_t^e - \theta^L}{\theta^H - \theta^L}, \quad \lambda_t^L = \frac{\theta^H - \theta_t^e}{\theta^H - \theta^L}. \tag{10}$$

Substituting (6) and (10) into the time derivative of (5) yields the time derivative of θ_t^e in the case where the shock does not occur at time t:

$$\dot{\theta}_t^e = (\theta_t^e - \theta^L - p^L)(\theta_t^e - \theta^H - p^H) - p^Lp^H \equiv g(\theta_t^e) \quad \text{for } t \notin S_{(0,\infty)}. \tag{11}$$

Under Assumption 1, this function has an 'U'-shape as illustrated in figure 1. The figure shows that

$$g(\theta) \lesseqgtr 0 \Longleftrightarrow \theta \gtreqless \theta^* \quad \text{for any } \theta \in \left[\theta^L, \theta^H\right], \text{ where}$$

$$\theta^* \equiv \frac{\theta^L + \theta^H + p^L + p^H - \sqrt{(\theta^H + p^H - \theta^L - p^L)^2 + 4p^Lp^H}}{2} \in (\theta^L, \theta^H). \tag{12}$$

[9] $\lambda_{t+\Delta t}^L$ is analogously obtained. From (1) it equals $1 - \lambda_{t+\Delta t}^H$.

Similarly, by substituting (9) and (10) into (5) we obtain the value of θ_t^e as a function of θ_{t-0}^e in the case where the shock does occur at time t.

$$\theta_t^e = \theta^L + \theta^H - \frac{\theta^L \theta^H}{\theta_{t-0}^e} \equiv h(\theta_{t-0}^e) \quad \text{for } t \in S_{(0,\infty)}. \tag{13}$$

As shown in Figure 2, $h(\theta)$ satisfies

$$h(\theta^H) = \theta^H, \text{ and } \theta^e < h(\theta^e) < \theta^H \text{ for all } \theta^e \in (\theta^L, \theta^H).$$

Equations (13) and (11) respectively describe the dynamics of θ_t^e with and without the shock. They jointly show that θ_t^e fluctuates within interval $\left(\theta^*, \theta^H\right]$. The liquidity shock is a rare event, and therefore causes a discrete change in people's expectation about the present state once it occurs. As function $h(\theta^e)$ is located above the 45-degree line in Figure 2, the more often people observe the shock, the more strongly they believe that they are in state H, and hence θ_t^e becomes closer to θ^H.

Conversely, in the absence of the shock people gradually become more and more optimistic and confident that the economy is in state L. Thus, their subjective probability of the shock gradually declines, converging to θ^*.[10] However, the U-shape of function $g(\theta_t^e)$ implies that the speed of adjusting their belief is slow when θ_t^e is near θ^H. Note that $\theta_t^e \approx \theta^H$ is equivalent to $\lambda_t^H \approx 1$ from (10), which means that the precision of the prior belief is quite high (i.e., people are quite sure that the current state is H). In that case any additional information has only a small impact on the posterior belief.

3 Liquidity Preference and Consumption Behavior

In this section, we investigate how the aforementioned fluctuations in the representative household's belief affects her liquidity preference and consumption behavior. Before doing that, we briefly describe the basic structure and production side of the economy.

[10] θ_t^e never becomes lower than $\theta^* (> \theta^L)$ since people take into account the possibility that state L might have changed to state H even though the shock does not occur.

The economy is inhabited by the representative households with measure one. Each household is infinitely lived and supplies labor to a representative firm. The firm produces $y\ell_t$ units of commodity, where ℓ_t is the level of labor input and y is a constant input-output coefficient.[11] Since there is no investment in our setting and thus consumption c_t equals total commodity demand, total labor demand ℓ_t is

$$\ell_t = c_t/y. \tag{14}$$

In this economy, money affects real variables through the sluggishness of nominal wage adjustment. Instead of explicitly introducing the adjustment cost of nominal wages, we simply assume that there is a reduced-form relationship between labor demand ℓ_t and the rate of nominal wage adjustment $\dot{W}_t/W_t$,

$$\dot{W}_t/W_t = f(\ell_t - 1), \quad f'(\cdot) \geq 0,\ f(0) = 0, \tag{15}$$

i.e., the rate of nominal wage adjustment is an increasing function of labor demand in excess of the 'natural' level, the latter being normalized to unity. Given W_t, the perfect adjustment of commodity price P_t always yields

$$W_t/P_t \equiv w_t = y, \tag{16}$$

which shows that real wage w_t is constant. Thus, from (15), the inflation rate is determined as a function of commodity demand c_t:

$$\pi_t \equiv \dot{P}_t/P_t = f(c_t/y - 1). \tag{17}$$

Note that equation (17) implies that y is the level of consumption (=output) at which the price level become constant.

[11] Labor supplied by each household is potentially differentiated. In that case '$y\ell_t$' should be considered as the amount of output when the representative firm employs the same amount of labor from each household.

Next, we describe the demand side. We introduce a stochastic version of the Sidrauski model that incorporates the random liquidity shock. The representative household gains utility only from consumption c_t when the shock does not occur. However, when it occurs, she gains utility not only from consumption but also from money holding m_t. Her expected utility EU_t is therefore given by

$$EU_t = E_t \left[\int_t^{\infty} u(c_\tau) e^{-\rho(\tau - t)} d\tau + \sum_{\tau \in S_{(t,\infty)}} \beta m_\tau e^{-\rho(\tau - t)} \right], \tag{18}$$

where constant ρ is her subjective discount rate, and function $u(\cdot)$ represents the instantaneous felicity from consumption satisfying twice differentiability and the Inada conditions. Constant β specifies the marginal benefit from money holding, and therefore $\beta m_\tau e^{-\rho(\tau-t)}$ gives the discounted utility from money if she encounters a liquidity shock at date τ. It is summed over $\tau \in S_{(t,\infty)}$, where $S_{(t,\infty)}$ represents the discrete set of dates on which the shock occurs.[12]

The household chooses assets among money and the complete set of contingent claims for future commodities. However, since all households are identical, money is the only asset that they hold after all arbitrage opportunities are exploited.[13] We assume that nominal money supply is constant and that there is no tax-cum-subsidy.[14] Thus, the flow budget

---

[12]This setting is obviously a simplifying approximation. In reality, a financial crisis does not instantly terminate but continues for a few days so that a household receives, say, utility flow $\tilde{\beta} m_{\tau'} e^{-\rho(\tau'-t)}$ throughout $\tau' \in [\tau, \tau + T]$. Since each period of financial crisis is typically short, we approximate the total utility received from money holding during one occurence of the shock by $\beta m_\tau e^{-\rho(\tau-t)}$, where $\beta \equiv \tilde{\beta} T$.

[13]We later consider a complete set of contingent assets to derive arbitrage conditions that define the optimal behavior of agents. However, we obtain the same result even in the absence of some contingent assets since in equilibrium there is no need for homogeneous households to sell and buy them to each other. Also note that the ownership of firms has no value in the present setting since firms use only labor and their profits are always zero. If physical capital exists, however, another state variable is introduced, which might more or less affect the result.

[14]The assumption of constant money supply is not essential. In fact, as shown below, the level of money

equation is

$$\dot{m}_t = w_t \ell_t - \pi_t m_t - c_t. \tag{19}$$

Having the belief mentioned in the previous section, the representative household chooses the time paths of consumption and money holding so as to maximize the expected utility (18) subject to (19).

Objective function (18) depends on only c_t, m_t and the expected pattern of the shock, the last of which is fully described by θ_t^e since θ_t^e is a sufficient statistic for θ_t that governs the current and future probabilities of the shock. Constraint (19) depends on w_t, ℓ_t and π_t, all of which are determined by only c_t on the equilibrium path, as seen from (14), (16) and (17). Therefore, from the perspective of the household that determines c_t, the current status is fully summarized by m_t and θ_t^e. Thus, given the recursive structure of the model, the movement of c_t on the path of stationary dynamics must completely be expressed as a function of m_t and θ_t^e.[15] Furthermore, since objective function (18) and constraint (19) are both linear in m_t, the optimal choice of c_t is independent of the level of m_t.[16] Thus, c_t should be a function of only θ_t^e:

$$c_t = C(\theta_t^e) \quad \text{for all } t. \tag{20}$$

Since θ_t^e fluctuates within interval $(\theta^*, \theta^H]$, as shown in the previous section, we only need to characterize the shape of function $C(\theta_t^e)$ in this interval.[17] To this end we examine

stock does not affect the household behavior in the present setting.

[15]This strategy for finding stationary dynamics is analogous to Lucas (1978) who analyzes the determination of equilibrium price behavior under an exogenous production shock that follows a Markov process.

[16]The linearity of the utility function with respect to m_t is assumed primarily for showing how the fluctuation in liquidity preference affects the consumption path of the utility-maximizing household in the simplest setting. When the marginal utility of holding money is variable, we actually find that the equilibrium dynamics of c_t depends on both θ_t^e and m_t. It substantially complicates the analysis but does not affect our main results, such as the pattern of recovery.

[17]In the following it is assumed that $\theta_t^e \in (\theta^*, \theta^H]$ unless otherwise noticed.

the first-order conditions for the household's optimizing behavior.

Let $1-\mu(\theta_t^e)\Delta t$ denote the price of the claim to a unit of the commodity at $t+\Delta t$ measured in terms of the commodity at t under the condition that the shock does not occur between t and $t+\Delta t$. Note that it is a function of θ_t^e because the value of the claim depends on the probability with which the contingent event occurs.[18] If the shock does not occur during the interval, consumption increases from $C(\theta_t^e)$ to $C(\theta_t^e+g(\theta_t^e)\Delta t)$ since θ_t^e changes by the amount of $g(\theta_t^e)\Delta t$, as shown by (11). Since the probability that the shock does not occur during this interval is $1-\theta_t^e\Delta t$, as given by (4), the first-order condition between the present and future consumption under the condition that the shock does not occur is

$$(1-\mu(\theta_t^e)\Delta t)u'(C(\theta_t^e)) = (1-\theta_t^e\Delta t)u'(C(\theta_t^e+g(\theta_t^e)\Delta t))e^{-\rho\Delta t}.$$

From this equation in which $\Delta t \to 0$ we derive

$$\mu(\theta_t^e) = \rho + \theta_t^e + \frac{C'(\theta_t^e)}{C(\theta_t^e)}\gamma(C(\theta_t^e))g(\theta_t^e) \quad \text{for all } \theta_t^e, \tag{21}$$

where $\gamma(c) \equiv -u''(c)c/u'(c)$ represents the degree of risk aversion.

Analogously, let $\nu(\theta_t^e)\Delta t$ denote the price of the contingent claim to a unit of the commodity at $t+\Delta t$ under the condition that the shock does occur between t and $t+\Delta t$. When the shock occurs, consumption jumps from $C(\theta_t^e)$ to $C(h(\theta_t^e))$, as seen from (13). Since the shock probability is $\theta_t^e\Delta t$, as shown by (8), the first-order condition between the present and the future consumption in this case is

$$\nu(\theta_t^e)\Delta t \cdot u'(C(\theta_t^e)) = \theta_t^e\Delta t \cdot u'(C(h(\theta_t^e)))e^{-\rho\Delta t}.$$

By making $\Delta t \to 0$ in this equation we find

$$\nu(\theta_t^e) = \frac{\theta_t^e u'(C(h(\theta_t^e)))}{u'(C(\theta_t^e))} \quad \text{for all } \theta_t^e. \tag{22}$$

Next, let us consider the arbitrage between these contingent claims and a risk-free asset, such as a riskless bond. Let r_t be the real interest rate of a risk-free asset, then the

[18] In addition, θ_t^e affects consumption and therefore the marginal utility of consumption.

price of a risk-free claim to the future commodity at $t + \Delta t$ is $e^{-r_t \Delta t}$. Since the claim is equivalent to the asset of the synthesis of the claim contingent on the absence of the shock whose price is $1 - \mu(\theta_t^e)\Delta t$ and that conditional on its presence whose price is $\nu(\theta_t^e)\Delta t$, the no-arbitrage condition requires

$$e^{-r_t \Delta t} = 1 - \mu(\theta_t^e)\Delta t + \nu(\theta_t^e)\Delta t.$$

As $\Delta t \to 0$, it reduces to

$$r_t = \mu(\theta_t^e) - \nu(\theta_t^e) \quad \text{for all } t. \tag{23}$$

Substituting (21) and (22) into (23) yields

$$C'(\theta_t^e) = \frac{C(\theta_t^e)}{\gamma(C(\theta_t^e))g(\theta_t^e)} \left[r_t - \rho + \theta_t^e \left(\frac{u'(C(h(\theta_t^e)))}{u'(C(\theta_t^e))} - 1 \right) \right]. \tag{24}$$

Applying (11) and (20) to (24) leads to the dynamics of c_t ($\equiv C(\theta_t^e)$) while the shock does not occur:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma(c_t)} \left[r_t - \rho + \theta_t^e \left(\frac{u'(C(h(\theta_t^e)))}{u'(c_t)} - 1 \right) \right] \quad \text{for } t \notin S_{(0,\infty)}, \tag{25}$$

which is the Keynes-Ramsey rule in the present setting. Note that it is the same as the standard one except for the third term in brackets of the right-hand side. This term represents a jump in the marginal utility caused by the shock. If c_t declines after the shock, causing the marginal utility of consumption to increase, this term is positive and thus the growth rate of c_t is higher than would obtain in the standard Ramsey model. That is, since the representative household anticipates a possible increase in the marginal utility of consumption, she tries to reallocate consumption from the present to the future, raising the growth rate of consumption during the period without the shock.

Having examined the household's intertemporal optimization of consumption, we now turn to the optimal choice between consumption and money holding. By holding a one-unit money between time t and $t + \Delta t$, the household loses $(r_t + \pi_t)\Delta t$ units of consumption, or equivalently $(r_t + \pi_t)u'(c_t)\Delta t$ units in terms of utility, when compared to holding a unit of risk-free asset during this period. At this cost a one-unit increase in real money holding raises the household's utility (18) by β if the shock occurs. Since the subjective probability that the shock occurs between t and $t + \Delta t$ is $\theta_t^e \Delta t$, the increase in the expected utility is $\beta \theta_t^e \Delta t$. The marginal benefit should equal the marginal cost on the optimal path, which yields the first-order condition between money holding and consumption:

$$r_t + \pi_t = \frac{\beta \theta_t^e}{u'(c_t)} \quad \text{for all } t. \tag{26}$$

Equation (26) implies the well-known property—i.e., the marginal rate of substitution between consumption and real money holding equals the nominal rate of interest.

Substituting (17) into (26) and rearranging terms yield

$$r_t = \frac{\beta \theta_t^e}{u'(c_t)} - f\left(\frac{c_t}{y} - 1\right) \equiv R(\theta_t^e, c_t). \tag{27}$$

Intuitively, function $R(\cdot)$ represents the household's preference for liquidity, defined by the expected utility gain minus capital loss from holding money for a unit time. Given the amount of consumption, the preference for liquidity is stronger when the subjective shock probability is higher, i.e.,

$$R_\theta(\theta^e, c) \equiv \partial R(\theta^e, c)/\partial \theta^e = \frac{\beta}{u'(c_t)} > 0 \quad \text{for all } \theta_t^e > 0 \text{ and } c_t > 0. \tag{28}$$

Equation (27) requires that the rate of return from a risk-free asset (riskless bonds), r_t, should be equalized to $R(\cdot)$ for all t. Also, r_t must be consistent with Euler equation (24). From these two equations, the optimizing household behavior is summarized as

$$C'(\theta_t^e) = \frac{C(\theta_t^e)}{\gamma(C(\theta_t^e))g(\theta_t^e)}\left[R(\theta_t^e, C(\theta_t^e)) - \rho + \theta_t^e\left(\frac{u'(C(h(\theta_t^e)))}{u'(C(\theta_t^e))} - 1\right)\right] \quad \text{for all } \theta_t^e \in (\theta^*, \theta^H]. \tag{29}$$

Function $C(\cdot)$ is determined so that it satisfies differential equation (29). To pin down $C(\cdot)$, however, we also need a boundary condition. If $\dot{c}_t/c_t$ remains positive as θ_t^e approaches steady-state value θ^*, then c_t unboundedly explodes. Conversely, if $\dot{c}_t/c_t$ remains negative as $\theta_t^e \to \theta^*$, c_t converges to zero, violating the transversality condition. We rule out such paths by imposing a boundary condition:[19]

$$\lim_{\theta^e \to \theta^*} R(\theta^e, C(\theta^e)) - \rho + \theta^e \left(\frac{u'(C(h(\theta^e)))}{u'(C(\theta^e))} - 1 \right) = 0, \tag{30}$$

under which $\dot{c}_t / c_t$ given by (25) approaches zero as $\theta_t^e \to \theta^*$.

Equations (29) and (30) determine the whole shape of $C(\theta_t^e)$ within interval $\left(\theta^*, \theta^H\right]$. Once it is determined, applying the dynamics of θ_t^e given by (11) and (13) to it provides the dynamic path of c_t.

4 Crisis and Recovery

This section investigates the shape of the consumption path determined by (29) and (30). Note that the right-hand side of differential equation (29) includes $C(h(\theta))$ along with θ and $C(\theta)$, implying that we cannot simply illustrate a phase diagram.[20] Therefore, we theoretically examine basic properties of function $C(\theta)$ and then numerically obtain a typical shape of it.

For the tractability of the analysis we assume the following two properties regarding $u(\cdot)$ and $f(\cdot)$, which are stated in terms of function $R(\cdot)$ defined by (27).

Assumption 2 $\lim_{c \to 0} R(\theta^H, c) < \rho$ *and* $\lim_{c \to \infty} R(\theta^*, c) > \rho$.

Assumption 3 $R_c(\theta^e, c)$ *is continuous and positive for all* $\theta^e \in [\theta^*, \theta^H]$ *and* $c > 0$*, where* $R_c(\theta^e, c) \equiv \partial R(\theta^e, c) / \partial c$.

If the household's preference for liquidity, given by $R(\cdot)$, is stronger than her preference for present consumption ρ, she postpones consumption and holds more money. Therefore,

[19] Throughout this paper we use operator 'lim' to denote the right-hand limit.

[20] This type of equation is called a difference-differential equation or a delay difference equation.

in Assumption 2 the first condition implies that she prefers consumption to liquidity holding when her consumption is quite low even if she expects the highest shock probability θ^H. The second condition implies that she prefers liquidity holding to consumption when her consumption is sufficiently high even if she expects the lowest shock probability θ^*. Assumption 3 extends this relationship between c_t and $R(\cdot)$ to a smooth and monotonic one under a given θ_t^e—i.e., as consumption increases, liquidity preference rises as long as the state of expectation about the shock is unchanged.

Under the two assumptions we find the existence, and a few properties, of function $C(\theta^e)$ that satisfies (29) and (30):

Proposition 1 *Under Assumptions 2 and 3, there exists a unique function $C(\theta^e)$ that satisfies (29) and (30). It is strictly downward sloping for any $\theta^e \in (\theta^*, \theta^H]$ and has positive and finite upper and lower bounds $\overline{c}$ and $\underline{c}$ that are given by*

$$R(\theta^*, \overline{c}) = \rho, \quad R(\theta^H, \underline{c}) = \rho. \tag{31}$$

Proof. See Appendix A.

Given initial belief θ_0^e and the history of the liquidity shock $S_{(0,t]}$, the path of θ_t^e is uniquely determined by (11) and (13). Therefore, the uniqueness of function $C(\cdot)$ implies that of the consumption path: $c_t = C(\theta_t^e)$. The negative relationship between θ_t^e and c_t provides an intuitive figure of the dynamics. Figure 1 shows $\dot{\theta}_t^e < 0$ for any $(\theta^*, \theta^H]$, implying that subjective shock probability θ_t^e gradually declines while the shock does not occur. Thus, preference for liquidity gradually decreases and consumption grows. When the shock occurs, people discretely increase θ_t^e to $h(\theta_t^e)$, as illustrated in Figure 2, causing a negative jump in consumption to occur. If the shock does not occur for a while, people again gradually become optimistic and raise consumption. In this way, consumption persistently fluctuates within finite interval $[\underline{c}, \overline{c}]$.

Note that as the preference for liquidity (i.e., money) changes, not only the demand for consumption but also preferences for all kinds of assets, including riskless bonds, vary over time. However, since there is no stickiness in the price of assets, the rate of return for each asset is adjusted so that the excess demand for any asset other than money is always zero. By contrast, the price of goods cannot change flexibly because of the sluggishness in nominal wage adjustment, and therefore the changes in the liquidity preference have quantitative effects on consumption.

To obtain a typical shape of function $C(\theta^e)$ and the dynamics of consumption more clearly, we numerically analyze the dynamics by assuming

$$u(c_t) = \log c_t \quad \text{and} \quad f(\ell_t - 1) = \alpha \cdot (\ell_t - 1) \quad \text{where } \alpha \text{ is constant.} \tag{32}$$

In this setting, Assumptions 2 and 3 reduce to the following:[21]

$$\beta y \theta^* > \alpha, \quad \rho > \alpha.$$

In the numerical calculation we choose parameter values so that these conditions as well as Assumption 1 are satisfied.[22]

Figure 3 illustrates the shape of $C(\theta_t^e)$ obtained from the numerical analysis. It is in fact downward sloping from $c^* \equiv C(\theta^*)$ to $c^H \equiv C(\theta^H)$. If the shock does not occur while θ_t^e moves from θ^H to θ^* according to equation (11), $C(\theta_t^e)$ moves along the solid curve in Figure 4. Note that consumption first grows slowly, gradually accelerates, and eventually slows down again as it approaches c^*—i.e., it traces an S-shaped trajectory.

The intuition behind is clear from the U-shape of function $g(\theta_t^e)$ as depicted in Figure 1. If people strongly believe that they are in state H and thus θ_t^e is very close to θ^H,

[21] Using a money-in-utility model without uncertainty Ono (1994, pp.86-88; 2001) shows that in the case where $\beta y > \alpha$ and $\rho > \alpha$ there is a unique saddle-stable path. Furthermore, the path accommodates a persistent demand shortage when $\rho < \beta y$, whereas it reaches a full-employment steady state when $\rho > \beta y$. The present condition is the same as his condition except that the former includes shock probability θ^*.

[22] Specifically, $\theta^H = .4$, $\theta^L = .05$, $p^H = .025$, $p^L = .1$, $y = 1$, $\rho = .05$, $\alpha = .025$, and $\beta = .4$. Under these parameter values, we obtain $\theta^* \approx .069$, $c^* \approx 1.3$ and $c^H \approx .60$. Details of the numerical procedure are described in Appendix B.

they do not significantly alter their pessimistic view for a while. In fact, the speed of change in θ_t^e, given by $|g(\theta_t^e)|$, is then small and thus consumption increases very slowly. As the period without the shock lasts, θ_t^e decreases and $|g(\theta_t^e)|$ increases, as Figure 1 shows. People become more and more optimistic and thus θ_t^e declines faster, which accelerates the recovery speed of consumption. As θ_t^e approaches to steady-state value θ^*, people become quite confident that they are in state L, and hence an additional period without the shock provides little information. $|g(\theta_t^e)|$ approaches zero and the growth rate of consumption converges to zero.

Once the liquidity shock occurs, however, their consumption jumps downward since the subjective shock probability jumps upward, raising their liquidity preference. Consumption falls from $C(\theta_t^e)$ to $C(h(\theta_t^e))$, the latter being depicted by the dashed curve in Figure 4 (where vertical arrows express the magnitude of each fall). Thereafter the recovery process 'restarts' from the point that corresponds to the decreased level of consumption (as indicated by horizontal arrows) and consumption again traces the solid curve. If the shock continually occurs for a short period, the subjective probability successively increases and $C(\theta_t^e)$ approaches the lowest level c^H. Thereafter, consumption recovers along the S-shaped trajectory, as mentioned above.

Finally, by simulating the Markov process of the underlying state and the Poisson process for the shock, we numerically obtain an example of the realized time paths of θ_t, θ_t^e and c_t. Figure 5 illustrates them. Consumption in fact traces an S-shaped path, especially after a bunch of liquidity shocks make it close to the lowest value c^H. Since the inflation rate is given by (17) and y is located between c^* and c^H under the present parameter values, in the recovery process serious deflation initially occurs, then its rate reduces, and eventually a boom comes and inflation arises after c exceeds y. If the shock occurs infrequently and hence people do not become too pessimistic, the recovery process starts before deflation occurs and consumption quickly converges to c^*.

It is also worth noting that realized booms and depressions do not exactly match the underlying state of the economy but follow the subjective probability that people have in mind. Even when the economy switches to state H and thus the true probability of the shock jumps up, people do not increase money holding until they actually observe it. Analogously, even if the true shock probability jumps down, they still keep strong liquidity preference and thus the recovery speed is very slow for a while once they become very pessimistic.

5 Conclusions

Liquidity preference depends on people's belief about how frequently they encounter crises in which liquidity is needed. This paper has examined the way they update the belief based on Bayesian inference and its effect on their preference for liquidity holding over consumption, in a circumstance where the economy shifts between two unobservable states with different probabilities of the liquidity shock. Each time they observe the shock, they raise their subjective probability of being in the more dangerous state and increase preference for holding money over consumption. The longer the period without the shock lasts, the larger probability people attach to the safer state and increase preference for consumption over money holding. With incomplete nominal wage adjustment, such movements in liquidity preference fluctuate aggregate demand.

The magnitude and persistence of fluctuations in aggregate demand depend on the *realized* frequency of the shock, which does not necessarily match the underlying state of the economy. As long as the shock occurs sparsely in time, it has only a minor effect on the belief and hence the economic recovery thereafter is fast. However, if people observe the shock many times for a short while, they hold a strong belief of being in the more dangerous state and reduce consumption a lot. Once it occurs, it takes a long time for them to reverse their belief and increase consumption. In this process, the recovery speed is first slow, then gradually accelerates, and eventually declines, tracing an S-shaped curve.

Appendix A: Proof of Proposition 1

Proof of $C(\theta)$ to be downward sloping[23]

Before starting the proof we define $D(\theta)$:

$$D(\theta) \equiv R(\theta, C(\theta)) - \rho + \theta \left(\frac{u'(C(h(\theta)))}{u'(C(\theta))} - 1 \right). \tag{33}$$

Since $D(\theta)$ is the expression in brackets of (29) and (11) shows $g(\theta)$ to be negative for all $\theta \in (\theta^*, \theta^H]$,

$$C'(\theta) \lesseqqgtr 0 \Longleftrightarrow D(\theta) \gtreqqless 0. \tag{34}$$

Using function $D(\theta)$ we first prove

Lemma 1 *Suppose that $C(\theta)$ satisfies (29) and that there exists $\theta^0 \in [\theta^*, \theta^H)$ satisfying $\lim_{\theta \to \theta^0} D(\theta) = 0$. Then, under Assumption 3, $C(\theta)$ is strictly downward sloping for all $\theta \in (\theta^0, \theta^H]$.*

Proof: If Lemma 1 does not hold and hence $C(\cdot)$ is weakly upward sloping somewhere in $(\theta^0, \theta^H]$, either of the following must be the case.

(i) There exists some $\theta^A \in [\theta^0, \theta^H)$ such that $\lim_{\theta \to \theta^A} D(\theta) = 0$ and $C'(\theta) \geq 0$ for all $\theta \in (\theta^A, \theta^H]$.

(ii) There exist some $\theta^A \in [\theta^0, \theta^H)$ and $\theta^B \in (\theta^A, \theta^H)$ such that $\lim_{\theta \to \theta^A} D(\theta) = D(\theta^B) = 0$, $C'(\theta) \geq 0$ for all $\theta \in (\theta^A, \theta^B]$, and $C'(\theta) \leq 0$ for all $\theta \in (\theta^B, \theta^H]$.

Intuitively, if the lemma is false, we can choose interval $(\theta^A, \theta^B]$ in which function $C(\theta)$ is weakly increasing, θ^A is either a local minimum or θ^0, and θ^B is either a local maximum or θ^H. If there are multiple intervals of such, we choose the rightmost one. We shall find neither (i) nor (ii) to be valid.

---

[23] In Appendices A and B we use θ instead of θ_t^e to minimize notation.

We first show that case (i) leads to a contradiction. Since $R_c > 0$ from Assumption 3 and $R_\theta > 0$ from (28), in case (i)

$$\lim_{\theta \to \theta^A} R(C(\theta), \theta) < R(C(\theta^H), \theta^H). \tag{35}$$

Since $h(\theta^A) \in (\theta^A, \theta^H)$, $\lim_{\theta \to \theta^A} C(\theta) \leq C(h(\theta^A))$ whereas $C(\theta^H) = C(h(\theta^H))$ since $h(\theta^H) = \theta^H$ from (13). Since applying these properties and (35) to (33) implies $0 = \lim_{\theta \to \theta^A} D(\theta) < D(\theta^H)$ in case (i), from (34) we find $C'(\theta^H) < 0$, which contradicts (i).

In case (ii) $\lim_{\theta \to \theta^A} C(\theta) \leq C(\theta^B)$. Since $R_c > 0$ from Assumption 3 and $R_\theta > 0$ from (28), this inequality implies

$$\lim_{\theta \to \theta^A} R(C(\theta), \theta) < R(C(\theta^B), \theta^B). \tag{36}$$

Further, $h(\theta^A)$ is located in either $(\theta^A, \theta^B]$ or (θ^B, θ^H). If $h(\theta^A) \in (\theta^A, \theta^B]$, then $C(h(\theta^A)) \geq \lim_{\theta \to \theta^A} C(\theta)$ since we suppose $C'(\theta) \geq 0$ for all $\theta \in (\theta^A, \theta^B]$. Contrastingly, $C(h(\theta^B)) \leq C(\theta^B)$ since $h(\theta^B) \in (\theta^B, \theta^H)$ and $C'(\theta) \leq 0$ for all $\theta \in (\theta^B, \theta^H]$. Using these inequalities, (33) and (36) we find $\lim_{\theta \to \theta^A} D(\theta) < D(\theta^B)$, which contradicts case (ii).

If $h(\theta^A) \in (\theta^B, \theta^H)$, then because $h'(\theta) > 0$ from (13), we find $\theta^A < \theta^B < h(\theta^A) < h(\theta^B) < \theta^H$. In case (ii), this means

$$\lim_{\theta \to \theta^A} C(\theta) \leq C(\theta^B) \geq C(h(\theta^A)) \geq C(h(\theta^B)).$$

Thus,

$$\frac{u'(C(h(\theta^B)))}{u'(C(\theta^B))} - 1 \geq \max\left(0, \lim_{\theta \to \theta^A} \frac{u'(C(h(\theta)))}{u'(C(\theta))} - 1\right).$$

Applying this property and (36) to (33) yields $\lim_{\theta \to \theta^A} D(\theta) < D(\theta^B)$, which contradicts case (ii). Thus, anyway case (ii) results in a contradiction. ■

From (33), boundary condition (30) is equivalent to $\lim_{\theta \to \theta^*} D(\theta) = 0$. By regarding θ^* as θ^0 in Lemma 1, we find $C(\theta)$ that satisfies (29) and (30) to be strictly downward sloping for all $\theta \in (\theta^*, \theta^H]$.

The existence of upper and lower bounds for $C(\theta)$

We first show $\overline{c}$ and $\underline{c}$ to be unique and well defined. Since $\theta^* < \theta^H$ and $R_\theta > 0$ from (28), under Assumption 2

$$\lim_{c\to 0} R(\theta^*, c) \leq \lim_{c\to 0} R(\theta^H, c) < \rho < \lim_{c\to\infty} R(\theta^*, c) \leq \lim_{c\to\infty} R(\theta^H, c).$$

Applying this property and Assumption 3 to the intermediate value theorem implies that there are unique and positive $\underline{c}$ and $\overline{c}$ satisfying (31). Furthermore, since $R(\theta^H, \overline{c}) > R(\theta^*, \overline{c}) = \rho = R(\theta^H, \underline{c})$, Assumption 3 implies $\overline{c} > \underline{c}$.

Next, we prove the following lemma.

Lemma 2 *Suppose that $C(\theta)$ satisfies (29) and that there exists $\theta^0 \in [\theta^*, \theta^H)$ satisfying $\lim_{\theta\to\theta^0} D(\theta) = 0$. Then, under Assumptions 2 and 3, $C(\theta) \in [\underline{c}, \overline{c}]$ for all $\theta \in (\theta^0, \theta^H]$.*

Proof: As shown by Lemma 1, the last term in (33) is positive when $\theta \to \theta^0$ and hence

$$\lim_{\theta\to\theta^0} R(\theta, C(\theta)) < \rho. \tag{37}$$

From the first equation of (31), (28), and Assumption 3, $R(\theta, c) > \rho$ for all $\theta \in [\theta^*, \theta^H]$ and $c > \overline{c}$. Thus, (37) implies $\lim_{\theta\to\theta^0} C(\theta) \leq \overline{c}$.

When $\theta = \theta^H$, the last term in (33) equals zero since $h(\theta^H) = \theta^H$ from (13). Since $C'(\theta^H) \leq 0$ from Lemma 1, (34) implies $D(\theta^H) \geq 0$. Applying these properties to (33) yields

$$R(\theta^H, C(\theta^H)) \geq \rho.$$

Comparing this property with the second equation of (31) and using Assumption 3 yield $C(\theta^H) \geq \underline{c}$. Furthermore, the monotonicity of $C(\theta)$ from Lemma 1 implies $C(\theta) \in [C(\theta^H), \lim_{\theta\to\theta^0} C(\theta)] \subseteq [\underline{c}, \overline{c}]$ for all $\theta \in (\theta^0, \theta^H]$. ■

Under condition (30), θ^* satisfies the requirement for θ^0 in Lemma 2. Thus, $C(\theta) \in [\underline{c}, \overline{c}]$ for all $\theta \in (\theta^*, \theta^H]$.

The uniqueness of $C(\theta)$

Let $\widetilde{C}(\theta, c^H)$ be the solution to differential equation (29) that satisfies the following boundary condition:

$$\widetilde{C}(\theta^H, c^H) = c^H, \tag{38}$$

where $c^H (> 0)$ is an arbitrary constant. We can solve differential equation (29) backward from $\theta = \theta^H$ with boundary condition (38), because $h(\theta)$ is larger than θ and thus $C(\theta)$ and $C(h(\theta))$ are already known when we calculate the gradient of $C(\cdot)$ at θ.[24] Therefore, function $\widetilde{C}(\theta, c^H)$ is uniquely determined within interval $(\theta^*, \theta^H]$.

Using function $\widetilde{C}(\cdot)$, boundary condition (30) can be rewritten as

$$\lim_{\theta \to \theta^*} \widetilde{D}(\theta, c^H) = 0, \text{ where} \tag{39}$$

$$\widetilde{D}(\theta, c^H) \equiv R(\theta, \widetilde{C}(\theta, c^H)) - \rho + \theta \left(\frac{u'(\widetilde{C}(h(\theta), c^H))}{u'(\widetilde{C}(\theta, c^H))} - 1 \right). \tag{40}$$

$\widetilde{D}(\theta, c^H)$ is the expression in brackets of (29) with $C(\theta)$ being replaced by $\widetilde{C}(\theta, c^H)$. Functions $\widetilde{C}(\cdot)$ and $\widetilde{D}(\cdot)$ have the following properties:[25]

Lemma 3 *Under Assumption 3, (a) $\widetilde{C}_c(\theta, c^H) > 0$ and (b) $\widetilde{D}_c(\theta, c^H) > 0$ for all $\theta \in (\theta^*, \theta^H]$ and $c^H > 0$. In addition, (c) there is a constant, $\underline{D_c}$, such that $\widetilde{D}_c(\theta, c^H) > \underline{D_c} > 0$ whenever $\widetilde{C}(\theta, c^H) \in [\underline{c}, \overline{c}]$.*

Proof: By rearranging terms in (29),

$$\widetilde{D}(\theta, c^H) = g(\theta) \frac{\gamma(\widetilde{C}(\theta, c^H)) \widetilde{C}_\theta(\theta, c^H)}{\widetilde{C}(\theta, c^H)} = -g(\theta) M_\theta(\theta, c^H), \tag{41}$$

$$\text{where} \quad M(\theta, c^H) \equiv \ln u' \left(\widetilde{C}(\theta, c^H) \right). \tag{42}$$

Differentiating (41) with respect to c^H yields

$$M_{\theta c}(\theta, c^H) = -\widetilde{D}_c(\theta, c^H)/g(\theta). \tag{43}$$

[24]The numerical analysis follows this way. See Appendix B for it.

[25]$\widetilde{C}_c(\theta, c^H) \equiv \partial \widetilde{C}(\theta, c^H)/\partial c^H$. $\widetilde{D}_c(\theta, c^H)$ and other partial derivatives are defined likewise.

Differentiating (40) with respect to c^H gives

$$\begin{aligned}
&\widetilde{D}_c(\theta, c^H) = \Phi(\theta, c^H) + \Psi(\theta, c^H), \text{ where} \\
&\qquad \Phi(\theta, c^H) \equiv R_c(\theta, \widetilde{C}(\theta, c^H)) \frac{u'(\widetilde{C}(\theta, c^H))}{u''(\widetilde{C}(\theta, c^H))} M_c(\theta, c^H), \\
&\qquad \Psi(\theta, c^H) \equiv \theta \frac{u'(\widetilde{C}(h(\theta), c^H))}{u'(\widetilde{C}(\theta, c^H))} \left(M_c(h(\theta), c^H) - M_c(\theta, c^H) \right).
\end{aligned} \tag{44}$$

Since (38) implies $\widetilde{C}_c(\theta^H, c^H) = 1$, differentiating (42) with respect to c^H yields

$$M_c(\theta^H, c^H) = u''(c^H)/u'(c^H) < 0. \tag{45}$$

From (13), $h(\theta^H) = \theta^H$ and thus $\Psi(\theta^H, c^H) = 0$. Using this property, Assumption 3, (38), (44) and (45) we obtain

$$\widetilde{D}_c(\theta^H, c^H) = R_c(\theta^H, c^H) > 0.$$

Now we extend this property to all $\theta \in (\theta^*, \theta^H]$. To prove this, suppose otherwise. Then, there should be some $\theta^A \in (\theta^*, \theta^H)$ that satisfies $\widetilde{D}_c(\theta, c^H) > 0$ for all $\theta \in (\theta^A, \theta^H]$ and $\widetilde{D}_c(\theta^A, c^H) \leq 0$. This property, combined with (43), (45) and the negativity of $g(\theta)$ from (11), gives

$$\begin{aligned}
M_c(\theta^A, c^H) &= M_c(\theta^H, c^H) + \int_{\theta^A}^{\theta^H} \widetilde{D}_c(\theta, c^H)/g(\theta)\, d\theta \\
&< u''(c^H)/u'(c^H) < 0.
\end{aligned} \tag{46}$$

Assumption 3, (44) and (46) imply $\Phi(\theta^A, c^H) > 0$. Similarly, from (43)

$$M_c(h(\theta^A), c^H) - M_c(\theta^A, c^H) = -\int_{\theta^A}^{h(\theta^A)} \widetilde{D}_c(\theta, c^H)/g(\theta)\, d\theta > 0, \tag{47}$$

which means $\Psi(\theta^A, c^H) > 0$. Substituting these results into (44) yields $\widetilde{D}_c(\theta^A, c^H) > 0$, which contradicts the assumption that $\widetilde{D}_c(\theta^A, c^H) \leq 0$. Thus there is no such θ^A, and therefore property (b) holds.

Property (b) and (43) imply

$$M_{\theta c}(\theta, c^H) > 0 \quad \text{for all } \theta \in (\theta^*, \theta^H] \text{ and } c^H > 0. \tag{48}$$

From (45) and (48),

$$M_c(\theta, c^H) < u''(c^H)/u'(c^H) < 0 \quad \text{for all } \theta \in (\theta^*, \theta^H]. \tag{49}$$

Since (49) is equivalent to $\widetilde{C}_c(\theta, c^H) > 0$ from (42), property (a) holds.

Finally we prove property (c). Define

$$\underline{\Phi}(\theta, c) \equiv R_c(\theta, c)\frac{u'(c)u''(c^H)}{u''(c)u'(c^H)}.$$

From Assumption 3, $\underline{\Phi}(\theta, c)$ is positive and continuous for all $(\theta, c) \in \Theta \equiv [\theta^*, \theta^H] \times [\underline{c}, \overline{c}]$. Since Θ is a compact set, there exists $\underline{D_c} \equiv \min_{(\theta,c)\in\Theta} \underline{\Phi}(\theta, c) > 0$. Combined with (44), (48) and (49), this property implies

$$\widetilde{D}_c(\theta, c^H) > \Phi(\theta, c^H) > R_c(\theta, \widetilde{C}(\theta, c^H))\frac{u'(\widetilde{C}(\theta, c^H))u''(c^H)}{u''(\widetilde{C}(\theta, c^H))u'(c^H)} \geq \underline{D_c}$$

whenever $\widetilde{C}(\theta, c^H) \in [\underline{c}, \overline{c}]$. ■

We now prove the uniqueness of function $C(\theta)$ using Lemmata 2 and 3. Suppose that there are two distinct functions $C_1(\theta)$ and $C_2(\theta)$ both of which satisfy (29) and (30). Let $c_1^H \equiv C_1(\theta^H)$ and $c_2^H \equiv C_2(\theta^H)$. Then $C_1(\theta) = \widetilde{C}(\theta, c_1^H)$ and $C_2(\theta) = \widetilde{C}(\theta, c_2^H)$ for all $\theta \in (\theta^*, \theta^H]$. Note that $c_1^H \neq c_2^H$ because we have assumed that $C_1(\theta)$ and $C_2(\theta)$ are distinct functions. Since (30) is equivalent to (39), both functions satisfy

$$\lim_{\theta\to\theta^*} \widetilde{D}(\theta, c_1^H) = \lim_{\theta\to\theta^*} \widetilde{D}(\theta, c_2^H) = 0. \tag{50}$$

From Lemma 2, $\widetilde{C}(\theta, c_1^H), \widetilde{C}(\theta, c_2^H) \in [\underline{c}, \overline{c}]$ for all $\theta \in (\theta^*, \theta^H]$. Applying it to property (a) of Lemma 3 implies $\widetilde{C}(\theta, c^H) \in [\underline{c}, \overline{c}]$ for all $c^H \in [c_1^H, c_2^H]$ and all $\theta \in (\theta^*, \theta^H]$. Thus, we can use property (c) of Lemma 3 to obtain

$$\lim_{\theta\to\theta^*} |\widetilde{D}(\theta, c_1^H) - \widetilde{D}(\theta, c_2^H)| > |c_1^H - c_2^H|\underline{D_c} > 0, \tag{51}$$

which contradicts (50).

The existence of $C(\theta)$

We obtain another property with respect to $\widetilde{D}(\theta, c^H)$.

Lemma 4 *Under Assumptions 2 and 3, (a) $\widetilde{D}(\theta, \underline{c}) < 0$ and (b) $\widetilde{D}(\theta, \overline{c}) > 0$ for all $\theta \in (\theta^*, \theta^H)$.*

Proof: We first prove property (a). From (28) and (31),

$$R(\theta^H, \underline{c}) = \rho \quad \text{and} \quad R(\theta, \underline{c}) < \rho \ \text{ for all } \theta \in (\theta^*, \theta^H). \tag{52}$$

Since (13) implies $h(\theta^H) = \theta^H$, substituting (38) and (52) into (40) yields

$$\widetilde{D}(\theta^H, \underline{c}) = 0. \tag{53}$$

Since (41) and (53) imply $\widetilde{C}_\theta(\theta^H, \underline{c}) = 0$, and $R_\theta > 0$ from (28), differentiating (40) with respect to θ when $\theta = \theta^H$ gives

$$\widetilde{D}_\theta(\theta^H, \underline{c}) = R_\theta(\theta^H, \underline{c}) > 0. \tag{54}$$

Equations (53) and (54) show that there is a small ε (> 0) such that $\widetilde{D}(\theta, \underline{c}) < 0$ for all $\theta \in (\theta^H - \varepsilon, \theta^H)$.

We now extend the negativity of $\widetilde{D}(\theta, \underline{c})$ to the whole interval of (θ^*, θ^H). To see this, suppose otherwise. Then, there must be some $\theta^A \in (\theta^*, \theta^H)$ such that $\widetilde{D}(\theta, \underline{c}) < 0$ for all $\theta \in (\theta^A, \theta^H)$ and that $\widetilde{D}(\theta^A, \underline{c}) \geq 0$. Since $\widetilde{D} < 0 \Leftrightarrow \widetilde{C}_\theta > 0$ from (41) and the negativity of $g(\theta)$ in (11), we obtain $\widetilde{C}(\theta^A, \underline{c}) < \widetilde{C}(h(\theta^A), \underline{c}) < \widetilde{C}(\theta^H, \underline{c}) = \underline{c}$. With Assumption 3 and (52), these inequalities yield $R(\theta^A, \widetilde{C}(\theta^A, \underline{c})) < R(\theta^A, \underline{c}) < \rho$ and hence from (40) $\widetilde{D}(\theta^A, \underline{c}) < 0$, which is a contradiction.

Next we prove property (b). From Assumptions 2 and 3,

$$R(\theta, \overline{c}) > \rho \ \text{ for all } \theta \in (\theta^*, \theta^H]. \tag{55}$$

It implies $\widetilde{D}(\theta^H, \overline{c}) > 0$ from (40) since $h(\theta^H) = \theta^H$ from (13). To prove that this inequality actually holds for whole $(\theta^*, \theta^H]$, suppose otherwise. Then, there should be some $\theta^A \in$

(θ^*, θ^H) such that $\widetilde{D}(\theta, \overline{c}) > 0$ for all $\theta \in (\theta^A, \theta^H]$ and that $\widetilde{D}(\theta^A, \overline{c}) \leq 0$. From (34), we find $\widetilde{C}(\theta^A, \overline{c}) > \widetilde{C}(h(\theta^A), \overline{c}) > \widetilde{C}(\theta^H, \overline{c}) = \overline{c}$. With Assumption 3 and (55), these inequalities imply $R(\theta^A, \widetilde{C}(\theta^A, \overline{c})) > R(\theta^A, \overline{c}) > \rho$ and therefore from (40) $\widetilde{D}(\theta^A, \overline{c}) > 0$, which is again a contradiction. ■

Applying property (*b*) of Lemma 3 and Lemma 4 to the intermediate value theorem assures that, for any given $\theta^0 \in (\theta^*, \theta^H)$, there uniquely exists $c^H \in [\underline{c}, \overline{c}]$ that satisfies $\widetilde{D}(\theta^0, c^H) = 0$. That is, there is a unique function, $c^H = \zeta(\theta^0)$, satisfying

$$\widetilde{D}(\theta^0, \zeta(\theta^0)) = 0 \;\text{ for all } \theta^0 \in (\theta^*, \theta^H). \tag{56}$$

From Lemmata 1 and 2, function $\widetilde{C}(\theta, \zeta(\theta^0))$ is monotonic and bounded by $[\underline{c}, \overline{c}]$ within interval $\theta \in (\theta^0, \theta^H]$. By taking limit as $\theta^0 \to \theta^*$, we conclude that $C(\theta) = \widetilde{C}(\theta, c^{H*})$ is monotonic and bounded for all $\theta \in (\theta^*, \theta^H]$, where $c^{H*} \equiv \lim_{\theta^0 \to \theta^*} \zeta(\theta^0)$.[26] This implies c_t should neither explode, implode nor oscillate. Thus $\dot{c}_t / c_t \to 0$ as $\theta_t^e \to \theta^*$, which gives the validity of boundary condition (30).

Appendix B: Numerical procedure of finding $C(\cdot)$

Our problem is generally called an IVP (initial value problem), which is usually solved by finite difference methods, such as the Runge-Kutta method and the Euler method.[27] However, we cannot use them since the right-hand side of (29) contains $C(h(\theta))$, which makes impossible to calculate the gradient of $C(\theta)$ before $C(h(\theta))$ is determined. Since $h(\theta)$ is always larger than θ, we cannot solve the differential equation forward from θ^*, where the boundary condition is given, toward θ^H. Instead, we can solve it backward from

[26]Note that $\zeta(\theta)$ is bounded by $\underline{c}$ and $\overline{c}$ from Lemma 2. In addition, by totally differentiating (56) and utilizing Lemma 1, we can prove that $\zeta(\theta)$ is continuous and monotonic for all $\theta \in (\theta^*, \theta^H]$. It means that $\zeta(\theta^0)$ does not oscillate as $\theta^0 \to \theta^*$, hence the existence of $c^{H*} \equiv \lim_{\theta^0 \to \theta^*} \zeta(\theta^0)$ is guaranteed.

[27]See, for example, Judd (1998).

θ^H toward θ^*, during which $C(h(\theta))$ is already known when we calculate the gradient of $C(\theta)$.

This strategy, however, involves another difficulty because the value of the function at the starting point, $c^H \equiv C(\theta^H)$, is not predetermined. Thus, we have to find an appropriate initial value c^H such that boundary condition (30) is eventually met when (29) is solved from it.[28] This method is in fact used when we prove proposition 1 in appendix A—i.e., we show that there is a unique $c^{H*} \in [\underline{c}, \overline{c}]$ that satisfies this property. In the numerical analysis we calculate c^{H*} in the following way:

Step 1. Let $i = 0$, $h_0 = \overline{c}$ and $l_0 = \underline{c}$.

Step 2. Let $c_i^H = (h_i + l_i)/2$. Using the Euler method, solve differential equation (29) (to which (32) is applied) backward starting from boundary value $C(\theta^H) = c_i^H$.

Step 3. If $C(\theta)$ exceeds $\overline{c}$ during the calculation, or if $D(\theta)$ defined by (33) remains positive when θ approaches θ^*, let $h_{i+1} = c_i^H$ and $l_{i+1} = l_i$. Conversely, if $C'(\theta)$ becomes positive during the calculation or if $D(\theta)$ remains negative when θ approaches θ^*, let $h_{i+1} = h_i$ and $l_{i+1} = c_i^H$. Otherwise, c_i^H is the solution.

Step 4. Let $i = i + 1$.

Step 5. Repeat steps 2-4 until h_i and l_i get sufficiently close to each other. Then admit $c^H = (h_i + l_i)/2$ as the solution.

References

Andolfatto, David and Paul Gomme. (2003). "Monetary Policy Regimes and Beliefs." *International Economic Review* **44**(1), 1-30.

[28]This method is usually called 'monkey hunting'.

Boldrin, Michele and David K. Levine. (2001). "Growth Cycles and Market Crashes." *Journal of Economic Theory* **96**, 13-39.

Caplin, Andrew and John Leahy. (1993). "Sectoral Shocks, Learning, and Aggregate Fluctuations." *Review of Economic Studies* **60**, 777-794.

Chalkley, Martin and In Ho Lee. (1997). "Learning and Asymmetric Business Cycles." *Review of Economic Dynamics* **1**, 623-645.

Chamley, Christophe and Douglas Gale. (1994). "Information Revelation and Strategic Delay in a Model of Investment." *Econometrica* **62**, 1065-1085.

Chari, V.V. and Patrick J. Kehoe. (2004). "Financial Crises as Herds: Overturning the Critiques." *Journal of Economic Theory* **119**, 128-150.

Driffill, John and Marcus Miller. (1993). "Learning and Inflation Convergence in the ERM,"*Economic Journal* **103**, 369-378.

Judd, Kenneth L. (1998). *Numerical Methods in Economics.*: The MIT Press.

Lucas, Robert E. Jr. (1978). "Asset Prices in an Exchange Economy." *Econometrica* **46**(6), 1429-1445.

Nieuwerburgh, Stijn Van and Laura L. Veldkamp. (2005). "Learning Asymmetries in Real Business Cycles." forthcoming in *Journal of Monetary Economics.*

Ono, Yoshiyasu. (1994). *Money, Interest, and Stagnation.*: Oxford University Press.

Ono, Yoshiyasu. (2001). " A Reinterpretation of Chapter 17 of Keynes's General Theory: Effective Demand Shortage Under Dynamic Optimization." *International Economic Review* **42**, 207-236.

Potter, Simon M. (2000). "A Nonlinear Model of Business Cycle." *Studies in Nonlinear Dynamics and Econometrics* **4**(2), 85-93.

Sidrauski, Miguel. (1967). "Rational Choice and Pattern of Growth in a Monetary Economy." *American Economic Review* **57**(2), 534-544.

Sill, Keith and Jeff Wrase. (1999). "Exchange Rates, Monetary Policy Regimes, and Beliefs." Federal Reserve Bank of Philadelphia Working Paper, 99-6.

Veldkamp, Laura L. (2005). "Slow Boom, Sudden Crash." forthcoming in *Journal of Economic Theory*.

Zeira, Joseph. (1994). "Informational Cycles." *Review of Economic Studies* **61**, 31-44.

Zeira, Joseph. (1999). "Informational Overshooting, Booms, and Crashes." *Journal of Monetary Economics* **43**, 237-257.

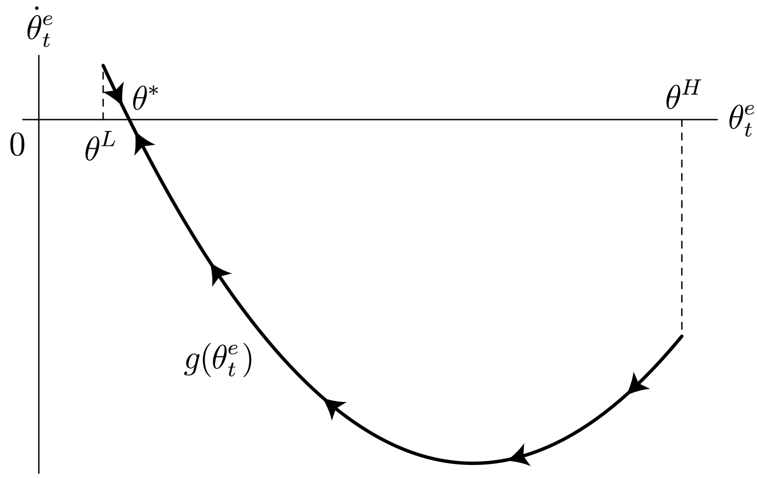

Figure 1: Movement of θ_t^e in absence of the shock

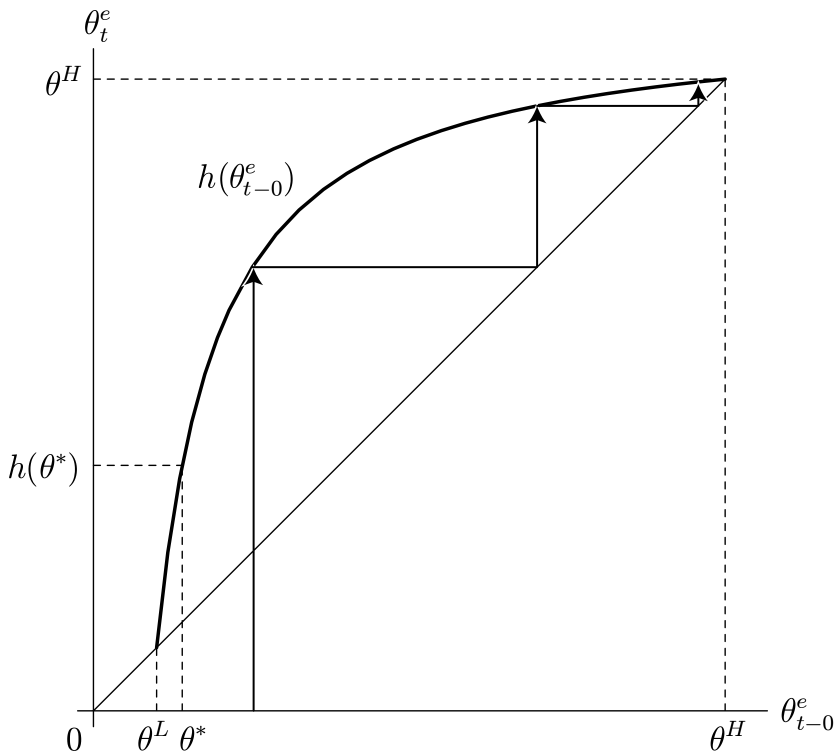

Figure 2: Movement of θ_t^e when the shock occurs

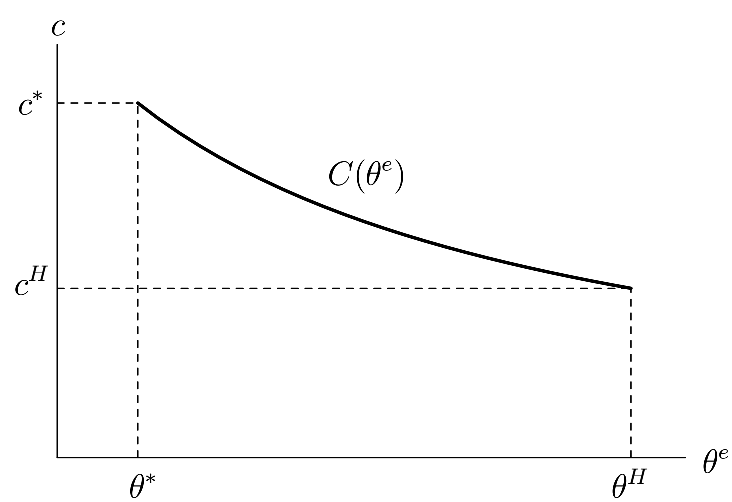

Figure 3: The shape of function $C(\theta_t^e)$

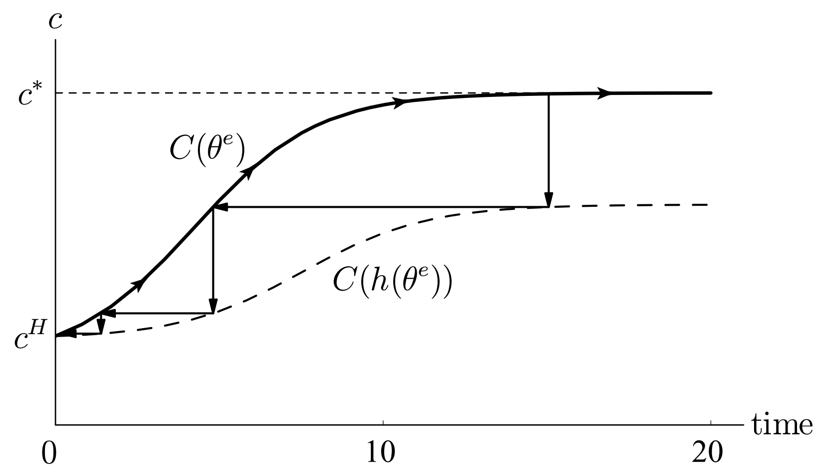

Figure 4: Dynamics of consumption

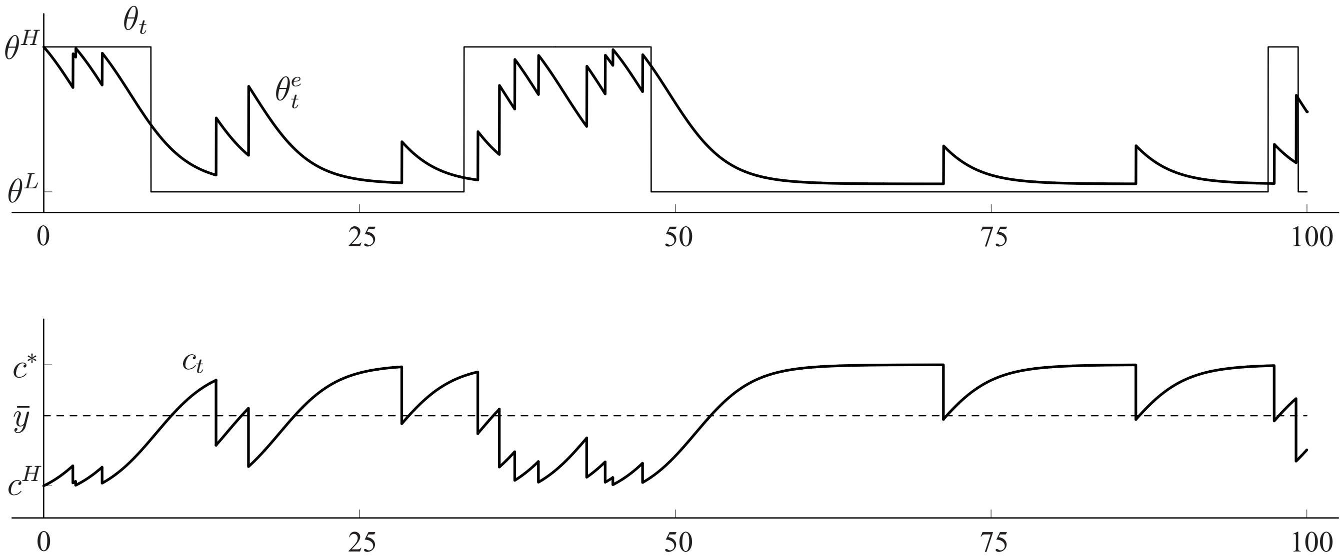

Figure 5: Realized paths of θ_t, θ_t^e and c_t