Transitional Dynamics and the Distribution of Assets[*]

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Abstract

We study the evolution of the distribution of assets in a discrete time, deterministic growth model with log-utility, a minimum consumption requirement, Cobb-Douglas technology, and agents differing in initial assets. We prove that the coefficient of variation in assets across agents decreases monotonically in a transition to the steady state from below, if (i) the consumption requirement is zero, or (ii) the consumption requirement is not too big and the initial capital stock is large enough. We also show how a positive consumption requirement or a small elasticity of substitution between capital and labor can generate non-monotonic paths for inequality.

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1 Introduction

This paper studies the dynamics of the distribution of assets in a discrete time, deterministic growth model, with inelastic labor supply and a Cobb-Douglas production function. Preferences are represented by a log-utility function with a minimum consumption requirement, which could be zero as in the standard Neoclassical model of growth, or positive.[1] We consider a decentralized economy with a continuum of agents differing only in their initial level of assets. Our objective is to characterize the evolution of the coefficient of variation in assets across agents over a transition to the steady state from below, that is, starting with an initial aggregate stock of capital below its steady state value.

Our work builds on the investigation by Chatterjee (1994), who studies the dynamics of the distribution of lifetime wealth in a similar economy. He shows that wealth inequality monotonically increases in a transition to the steady state from below if agents face a positive consumption requirement, and that inequality in lifetime wealth remains constant if the consumption requirement is zero. Unlike Chatterjee, we focus on the inequality of assets instead of the inequality in lifetime wealth, and show that in some cases these two measures of inequality evolve in opposite directions. Specifically, we prove that, as long as (i) the minimum consumption requirement is zero, or (ii) the consumption requirement is positive but not too big and the initial capital stock is large enough, the coefficient of variation in assets across agents monotonically decreases in a transition from below. Our result holds in a range of initial conditions for aggregate capital, but it *is not* based on a local approximation of the model around the steady state. As a by-product, we also show that under the same conditions, the discount factor times the interest factor represents a lower bound for the rate of growth of capital.

The intuition is the following. Asset accumulation depends on the balance between the desired rate of growth of consumption and the rate of growth of the lifetime wealth portfolio, which includes assets as well as labor income. With a positive consumption requirement, the desired rate of growth of consumption is lower for poor agents than for rich agents, inducing poor agents to save less. But the rate of growth of lifetime wealth is also different between poor and rich agents because they have different shares of labor

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[1] There is some evidence showing that minimum consumption requirements are important to explain differences in saving rates across income groups at early stages of development. For example, Atkeson and Ogaki (1996) and Rosenzweig and Wolpin (1993) find significant effects of consumption requirements in a panel of Indian villages.

income in their wealth portfolio and factor prices change over time. This is a second mechanism affecting the savings behavior of poor and rich agents differently. Under our assumptions, the labor component of the wealth portfolio grows at a slower rate than average wealth, therefore poor agents need to accumulate assets at a faster rate in order to attain their desired rate of growth of consumption. If the minimum consumption requirement is not too big and the economy is close to the steady state, the second mechanism dominates and the inequality in assets declines.[2]

We also provide numerical simulations to illustrate several interesting cases in which the assumptions used to state our results do not hold. We show that when the minimum consumption requirement is large, or when the elasticity of substitution between capital and labor (in a more general CES production function) is less than one half, the coefficient of variation in assets increases over time. In between – when the minimum consumption requirement is positive but not too large or when the elasticity of substitution between capital and labor lies between one half and one – the dynamics of the distribution of assets is non-monotonic: the inequality first increases and then decreases as the economy approaches its steady state, displaying the so-called Kuznets curve.

Besides its theoretical interest, we think that our exercise is relevant for an additional reason. Empirical studies typically measure and describe inequality in assets (see for instance Díaz-Giménez, Quadrini and Ríos-Rull (1997) for the U.S.), not inequality in lifetime wealth. Therefore, by focusing on asset inequality, we bring theory closer to the available data.

In a related paper, Chatterjee and Ravikumar (1999) do characterize the evolution of the distribution of assets, but in a endogenous growth model with Ak technology. The authors show that, during the transition to a balanced growth path, inequality in assets unambiguously increases with a positive consumption requirement. The result depends on two special features of their framework: (i) there is no labor income, and (ii) by construction, factor prices are constant. In our paper we relax these assumptions and obtain different results for moderate consumption requirements.

Caselli and Ventura (2000) use a continuous time model and study the dynamics of the distribution of consumption, assets and income in an economy where agents potentially differ in their tastes for a publicly provided con-

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[2] Perhaps surprisingly, this result does not imply that poor agents have larger investment to income ratios (or saving rates) than rich agents. Comment 2 after Theorem 4 elaborates on this issue.

sumption good and in their ability to work, as well as in their initial assets. They provide a variety of examples with monotonic and non-monotonic dynamics for inequality in the distribution of assets.[3] We make similar points in a discrete time model using a very different set of techniques, and extend the results to the case of a positive consumption requirement, which is equivalent to a negative transfer of public goods in their setup.

Finally, Alvarez and Díaz (2003) compute the evolution of the distribution of assets in a framework similar to ours, but using a more general CRRA utility function. They find that inequality decreases as long as the intertemporal elasticity of substitution is high (which includes the log case) and the minimum consumption requirement is low. For other cases, they simulate interesting examples of non-monotonic Kuznets-like paths for inequality, which they calibrate to reproduce the evolution of asset inequality in the U.S. economy. Again, some of their results are similar to ours, but we prove the monotonicity of inequality instead of relying on numerical computations.

The paper is organized as follows. Section 2 describes the model economy and defines a competitive equilibrium. In Section 3 we introduce the planner's problem and summarize what is known about its solution. Our result on the rate of growth of capital is obtained and discussed in Section 4, while in Section 5 we use it to prove our main result on the evolution of the coefficient of variation in assets. In Section 6, we provide numerical simulations to illustrate some interesting cases where the assumptions required for the previous results do not hold. Finally, we conclude.

2 The Model

The economy is inhabited by a continuum of agents indexed by $i \in [0,1]$. Each of these agents behaves so as to maximize the present value of the utility derived from consumption over an infinite horizon: $\sum_{t=0}^{\infty} \beta^t u(c_t^i)$, where $\beta \in (0,1)$ is the common subjective discount factor. We assume that $u(c_t^i) = \log(c_t^i - \bar{c})$, where $\bar{c} \geq 0$. Agents differ only in their initial endowment of assets, denoted a_0^i.[4] In addition, each agent is endowed with a unit of time in the beginning of each period, which they inelastically supply as labor.

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[3] Stiglitz (1969) also states conditions for non monotonic dynamics in a transition towards the steady state using a continuous time Solow model with several *ad hoc* saving functions.

[4] We think of individual portfolios of assets as including mainly capital. Since this is a complete markets economy, our analysis remains unchanged if we include any other asset in zero net supply as long as, in equilibrium, all assets offer the same net rate of return.

In the production side of the economy there is a representative firm. This firm produces the consumption/investment good combining units of capital and labor with the following Cobb-Douglas technology: $Y_t = K_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t$, with $\alpha \in (0,1)$. Here Y_t, K_t and N_t stand for aggregate output, capital and labor in period t, and $\delta \in (0,1)$ is the depreciation rate. The firm chooses capital and labor to maximize profit in each period: $Y_t - K_t R_t - w_t N_t$, where R_t and w_t are the interest rate and wage rate, respectively. Markets for output, capital and labor are competitive.

2.1 Agents' optimal choice

The problem each agent solves can be written formally as follows:

$$\begin{array}{ll} \max_{\{c_t^i, a_{t+1}^i\}} & \sum_{t=0}^{\infty} \beta^t \log(c_t^i - \bar{c}) \\ \text{s. to} & c_t^i + a_{t+1}^i = w_t + R_t a_t^i, \\ & c_t^i \geq \bar{c},\ \forall t \geq 0,\ \text{given } a_0^i. \end{array} \tag{1}$$

The first-order necessary conditions for an interior solution are given by the Euler equation:

$$\frac{c_{t+1}^i - \bar{c}}{c_t^i - \bar{c}} = \beta R_{t+1}, \tag{2}$$

the budget constraint, and the usual transversality condition.[5]

In what follows we characterize paths for consumption and assets of an agent i given his initial stock of assets and a sequence of prices $\{R_t, w_t\}$. We proceed as in Chatterjee (1994) and define the *lifetime wealth* of agent i in period t as

$$\omega_t^i = R_t \left[a_t^i + \sum_{j=0}^{\infty} \frac{w_{t+j}}{\prod_{s=0}^{j} R_{t+s}} \right]. \tag{3}$$

Repeated substitutions of Equation (2) in the budget constraint and the use of the previous definition provides the following expression for an agent's consumption in period t:

$$c_t^i = (1-\beta)\omega_t^i + \bar{B}_t, \tag{4}$$

where $\bar{B}_t = \bar{c} \sum_{j=0}^{\infty} \frac{\beta R_{t+1+j} - 1}{\prod_{s=0}^{j} R_{t+1+s}}$.

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[5] Notice that we write the Euler equation with equality. Therefore, we are implicitly assuming that there is a borrowing limit sufficiently generous such that agents never find optimal to exhaust it. See Hernández (1991) for a study of aggregate dynamics with binding borrowing constraints.

It follows from the previous expression that consumption is linear in lifetime wealth. In other words, Engel curves are linear. Inserting the expression for consumption into the budget constraint in period t, and using the definition of ω_t^i, we get:

$$a_{t+1}^i = \beta R_t a_t^i + D_t, \forall t \geq 0, \tag{5}$$

with the sequence D_t defined as:

$$D_t = w_t - (1-\beta) R_t \sum_{j=0}^{\infty} \frac{w_{t+j}}{\prod_{s=0}^{j} R_{t+s}} - \bar{B}_t.$$

Notice that D_t is independent of i. The equation in expression (5) will be useful to characterize the evolution of the distribution of assets in a transition to the steady state.

2.2 Firm's problem

We write the representative firm's problem as follows:

$$\begin{array}{ll} \max_{\{Y_t, K_t, N_t\}} & Y_t - R_t K_t - w_t N_t \\ \text{s. to} & Y_t = K_t^{\alpha} N_t^{1-\alpha} + (1-\delta) K_t. \end{array} \tag{6}$$

The first order conditions for optimality equate wages to the marginal product of labor and the interest rate to the marginal product of capital: $w_t = (1-\alpha) K_t^{\alpha} N_t^{-\alpha}$, and $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} + (1-\delta)$.

2.3 Competitive equilibrium

We are interested in the competitive evolution of the distribution of assets. The following definition introduces a notion of competitive equilibrium for the economy.

Definition 1: A competitive equilibrium for this economy is a list of sequences for individual consumptions and assets $\{c_t^i, a_t^i\}$ *and a sequence of prices* $\{R_t, w_t\}$ *such that: 1)* $\{c_t^i, a_t^i\}$ *solve the problem in (1) for each agent i taking* $\{R_t, w_t\}$ *as given; 2) prices are competitive:* $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} + (1-\delta)$ *and* $w_t = (1-\alpha) K_t^{\alpha} N_t^{-\alpha}$*; 3) markets clear:* $K_t = \int_0^1 a_t^i di$*, and* $N_t = 1$.

Notice that Definition 1 implies market clearing for the output good.[6] With this definition in place, to study the evolution of asset's holdings we only need to characterize the evolution of equilibrium prices. As shown in Chatterjee (1994), linear Engel curves provide an aggregation result so that equilibrium prices depend only on the aggregate stock of capital, not on its distribution across agents. This observation is useful because it implies that equilibrium prices can be recovered from the optimal allocation of a planner's problem, where the initial endowment of capital corresponds to the average initial capital of the market economy. We study this planner's problem in the next section.

3 The Planner's Problem

The social planner solves the following problem

$$\begin{array}{ll} \max_{\{c_t,k_{t+1}\}} & \sum_{t=0}^{\infty}\beta^t \log(c_t-\bar{c}) \\ \text{s. to} & c_t+k_{t+1}=k_t^{\alpha}+(1-\delta)k_t \\ & c_t\geq\bar{c},\ k_{t+1}\geq 0,\ \forall t\geq 0,\ \text{given } k_0. \end{array} \tag{7}$$

To study the properties of a solution to the previous problem it is convenient to define $\tilde{c}_t \equiv c_t-\bar{c}$ and rewrite the problem as:

$$\begin{array}{ll} \max_{\{\tilde{c}_t,k_{t+1}\}} & \sum_{t=0}^{\infty}\beta^t \log(\tilde{c}_t) \\ \text{s. to} & \tilde{c}_t+\bar{c}+k_{t+1}=k_t^{\alpha}+(1-\delta)k_t \\ & \tilde{c}_t\geq 0,\ k_{t+1}\geq 0,\ \forall t\geq 0,\ \text{given } k_0. \end{array} \tag{8}$$

A solution to the previous problem, if it exists, satisfies the following first order conditions

$$\frac{\tilde{c}_{t+1}}{\tilde{c}_t}=\beta[\alpha k_{t+1}^{\alpha-1}+(1-\delta)], \tag{9}$$

$$\tilde{c}_t=k_t^{\alpha}+(1-\delta)k_t-k_{t+1}-\bar{c}, \tag{10}$$

and the transversality condition $\lim_{t\to\infty}\beta^t\,(k_{t+1}/\tilde{c}_t)=0$. Inspection of (10) suggests that a solution may fail to exist if $\bar{c}$ is an arbitrarily large number and/or if the initial capital is too small. Thus, before we proceed we need to introduce additional assumptions. To this end, let

$$k^*=\left(\frac{\alpha\beta}{1-(1-\delta)\beta}\right)^{\frac{1}{1-\alpha}}. \tag{11}$$

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[6] If one-period bonds or other assets in zero net supply are explicitly included in the model, then the corresponding market-clearing conditions for those assets have to be added in the previous definition.

We define $\bar{c}_{max} \equiv (k^*)^\alpha - \delta k^*$. If $\bar{c} > 0$, then we also define $\hat{k}$ as the smallest solution of $\bar{c} - k^\alpha + \delta k = 0$. The definition of $\bar{c}_{max}$ ensures that the previous equation has two solutions when $0 < \bar{c} < \bar{c}_{max}$. Finally, we define

$$k_{min} = \begin{cases} 0 & \text{, if } \bar{c} = 0 \\ \hat{k} & \text{, if } \bar{c} > 0 \end{cases}$$

Throughout the analysis we will assume that the following assumption is satisfied:

A1: $0 \leq \bar{c} < \bar{c}_{max}$ and $k_{min} < k_0 \leq k^*$.

Assumption A1 will be discussed in detail after we introduce Theorems 1 and 2. Theorem 1 summarizes some well-known properties of the solution to the problem in (8). The proof (included in the Appendix) is a straightforward extension of existing results. The non-standard case of $\bar{c} > 0$ involves some technical difficulties, which are solved as in Theorem 1 from Hugget (1991).

Theorem 1. *Consider the problem in (8) and assume A1. Then,*

i) there exists a solution $\{\tilde{c}_t, k_t\}$*;*

ii) the solution $\{k_t\}$ *can be represented by a continuous, non decreasing and strictly concave decision rule* g *which delivers* $k_{t+1} = g(k_t)$*;*

iii) the decision rule g *is such that* $k^* = g(k^*)$*, where* k^* *is as given in (11);*

iv) for all $k_0 < k^*$*, the solution* $\{k_t\}$ *converges monotonically to* k^**,* $k_{t+1} > k_t$ *and* $k_{t+1}/k_t > k_{t+2}/k_{t+1}$*,* $\forall t$*. Moreover,* $\{\tilde{c}_t\}$ *also converges monotonically to* $\tilde{c}^* = (k^*)^\alpha - \delta k^* - \bar{c}$*.*

Proof: see Appendix. ■

Theorem 2 is a version of the Welfare Theorems and states the relation between the solution to the planner's problem and the competitive equilibrium. We omit the proof, which follows standard arguments.

Theorem 2. *Let* $\{\tilde{c}_t, k_t\}$ *be the solution to the planner's problem (8), and* $\{c_t^i, a_t^i; R_t, w_t\}$ *be competitive equilibrium allocations and prices satisfying Definition 1. Then, the following equivalences hold:* $\tilde{c}_t = \int_0^1 c_t^i di - \bar{c}$*,* $k_t = \int_0^1 a_t^i di$*,* $R_t = \alpha(k_t)^{\alpha-1} + (1-\delta)$*, and* $w_t = (1-\alpha)k_t^\alpha$*.*

Discussion of A1

1. Assumption A1 is necessary to ensure that a stationary solution exists to the problem in (8). Since k^* is independent of $\bar{c}$, we can use the feasibility constraint to ask what is the largest $\bar{c}$ that would deliver $\tilde{c}_t = 0$ if $k_t = k^* = k_{t+1}$. The answer is $\bar{c}_{max}$. Thus, if $\bar{c} < \bar{c}_{max}$ then $\tilde{c}_t > 0$ when $k_t = k^*$.

However, $\tilde{c}_t$ can be strictly positive for all $k_t > 0$ only when $k_0 > k_{min}$. This is discussed next.

2. k_{min} is the smallest amount of capital that could be sustained without violating $\tilde{c}_0 \geq 0$. It is straightforward to check that if $0 < \bar{c} < \bar{c}_{max}$, then $0 < k_{min} < k^*$. Thus, A1 guarantees that it is always possible to at least keep the stock of capital constant and still have $\tilde{c}_t > 0$ in all periods.

3. Without imposing A1, the problem in (8) may have some other interesting solutions. To see this, notice that if $\bar{c}$ is unrestricted, then equation $\bar{c} - k^\alpha + \delta k = 0$ may have no solution, or up to two, when $\bar{c} > 0$. Assume for a moment that $\bar{c} > 0$ and sufficiently large so that $\bar{c} - k^\alpha + \delta k = 0$ has only one solution. It follows by construction that $\alpha(\hat{k})^{\alpha-1} + (1-\delta) < \alpha(k^*)^{\alpha-1} + (1-\delta)$, thus, $\hat{k} > k^*$. Choosing $k_0 = \hat{k}$, then $\tilde{c}_t = 0$ and $k_t = \hat{k}$ in all periods, i.e., there exists a degenerate steady state. If we reduce slightly the value of $\bar{c}$, then the equation would admit two solutions, but still $\hat{k} > k^*$. Choosing $k_0 > \hat{k}$ in this case will produce $\tilde{c}_t > 0$ and $k_t > \hat{k}$ in all periods, and a steady state fails to exist.

We call the solution in Theorem 1 *(iv)* a *transition from below.* The evolution of asset holdings over this transition is the main focus of the paper. In the following section we develop a number of results that will help us characterize the evolution of the coefficient of variation in assets across agents (our measure of inequality) in such a transition. We then present the main result in Section 5.

4 A Lower Bound for the Rate of Growth of Capital

Before we continue with the analysis, we define some auxiliary variables and functions, and we introduce assumption A2 for future reference. We define the sequence $\{z_t\}$ as

$$z_t \equiv k_t/\tilde{c}_{t-1}.$$

We also define the function $\phi : [k_{\min}, k^*] \to$R by

$$\phi(k) \equiv \frac{\alpha\beta k^\alpha + \beta(1-\delta)k}{(1-\alpha\beta)k^\alpha + (1-\beta)(1-\delta)k - \bar{c}},$$

and the function $\varphi : [k_{\min}, k^*] \to$R by

$$\varphi(k) \equiv \frac{(1-\delta)k^\alpha(1-\alpha)^2}{\alpha^2 k^{\alpha-1} + (1-\delta)}.$$

It is straightforward to show that φ is increasing in k. Let $\bar{c}^o = \varphi(k^*) > 0$. Also, if $\bar{c} > 0$ we denote by k^o the solution to $(1-\alpha\beta)k^\alpha + (1-\beta)(1-\delta)k - \bar{c} = 0$, and by $\hat{k}^o$ the solution to $\varphi(k) = \bar{c}$.

A2: Either $\bar{c} = 0$, or both $0 < \bar{c} < \min\{\bar{c}^o, \bar{c}_{max}\}$ and $k_0 > \max\left\{k^o, \hat{k}^o\right\}$.

Discussion of A2

1. If $\bar{c} = 0$ then the function ϕ is well defined and positive for all $k > 0$. However, if $\bar{c} > 0$ then the denominator of ϕ equals zero for $k_t = k^o$. To ensure $\phi > 0$ along the transition from below, we impose $k_0 > k^o$. Notice that $k^o < k^*$, as required in a transition from below. To see that this is the case, we use the definition of $\bar{c}_{max}$ and the fact that $0 < \bar{c} < \bar{c}_{max}$, to obtain $(1-\alpha\beta)(k^o)^\alpha + (1-\beta)(1-\delta)k^o < (k^*)^\alpha - \delta k^*$. Assuming that $k^o \geq k^*$, it follows from the previous inequality that $(k^*)^{1-\alpha} < \alpha\beta/(1-\beta(1-\delta))$, which is a contradiction. Similarly, $k^o > k_{\min}$, since by definition $(1-\alpha\beta)(k^o)^\alpha + (1-\beta)(1-\delta)k^o = (k_{\min})^\alpha - \delta k_{\min}$, so $k^o \leq k_{\min}$ implies that $(k_{\min})^{1-\alpha} \geq \alpha\beta/(1-\beta(1-\delta)) = (k^*)^{1-\alpha}$, another contradiction. Finally, $\bar{c} < \bar{c}^o$ implies $\hat{k}^o < k^*$, as required, since φ is strictly increasing.

2. The assumption that $\bar{c} < \bar{c}^o$ and $k_0 > \hat{k}^o$ implies that $\varphi(k) > \bar{c}$ in a transition from below, since $\varphi(\hat{k}^o) = \bar{c}$ and φ is strictly increasing. This result will be used in the proof of Lemma 1. We should point out that, without further assumptions, $\bar{c}^o$ could be larger or smaller than $\bar{c}_{max}$. To see this, assume for a moment that $\bar{c}^o > \bar{c}_{max}$ for all possible α, β and δ. Using the definition of k^* in Equation (11), we obtain $(2-\delta)\beta[\alpha + \beta(1-\delta)(1-\alpha)] > 1$. The previous inequality is violated for any α and δ in $(0,1)$ if β is selected arbitrarily small. Therefore there are configurations for α, β and δ for which $\bar{c}^o < \bar{c}_{max}$. Finally, $\hat{k}^o$ could also be larger or smaller than k^o.

Under assumptions A1 and A2, we prove some useful properties of ϕ and $\{z_t\}$, which we collect as Lemma 1.

Lemma 1. *Assume A1 and A2. In any transition from below,* $\phi(k_{t+1}) > \phi(k_t)$, $\forall t$, *with* $\lim_{j\to\infty} \phi(k_{t+j}) = k^*/\tilde{c}^*$. *Furthermore,* $\forall t$,

$$z_{t+1} = \left[\frac{1}{\phi\left(k_t\right)} + 1\right] z_t - 1 \tag{12}$$

Proof: see Appendix. ■

Having established the monotonicity of ϕ, we next show that the sequence $\{z_t\}$ is monotonic and strictly increasing. This is done in Lemmas 2 and 3.

Lemma 2. *Assume A1 and A2. In any transition from below* $z_{t+1} \leq z_t$ *implies* $z_{t+2} < z_{t+1}$.

Proof: We proceed by contradiction, so suppose there exists a period $t \geq 1$ for which $z_{t+1} \leq z_t$ and $z_{t+2} \geq z_{t+1}$. Using Equation (12) we obtain $z_t \geq \left[\frac{1}{\phi(k_t)} + 1\right] z_t - 1$, and reordering:

$$z_t \leq \phi\left(k_t\right), \tag{13}$$

where we have used the fact that $\phi(k_t) > 0$. Proceeding in the same way with $z_{t+2} \geq z_{t+1}$, we obtain $z_{t+1} \geq \phi\left(k_{t+1}\right)$. Therefore, since $z_{t+1} \leq z_t$, (13) implies $\phi\left(k_t\right) \geq \phi\left(k_{t+1}\right)$, a contradiction with Lemma 1. ■

Corollary 1. *Assume A1 and A2. In any transition from below,* $z_{t+1} \leq z_t$ *implies* $z_{t+j+1} < z_{t+j}$ *for all* $j > 1$.

Proof: Apply recursively Lemma 2 for $j = 1, 2, ...$ ■

Lemma 3. *Assume A1 and A2. In any transition from below, for each period* t *there exists* $j \geq 0$ *such that* $z_{t+j+1} > z_{t+j}$.

Proof: Suppose otherwise, so that there exists a period t for which $z_{t+j+1} < z_{t+j}$, for all $j \geq 0$. Using a similar argument as in the proof of Lemma 2, this implies

$$z_{t+j} < \phi(k_{t+j}) \tag{14}$$

for all $j \geq 0$. But this is a contradiction, since $\{z_{t+j}\}$ is monotonically decreasing by assumption, $\{\phi(k_{t+j})\}$ is monotonically increasing by Lemma 1, and both sides converge to the same steady state value $k^*/\tilde{c}^*$. ■

The next theorem introduces an important result. We show that the product βR_t (i.e., the intertemporal marginal rate of substitution of $\tilde{c}$) represents a lower bound for the rate of growth of capital. This bound will be useful for characterizing the evolution of prices in a transition from below.

Theorem 3. *Assume A1 and A2. In any transition from below,* $k_{t+1}/k_t > \beta R_t$, $\forall t$.

Proof: Suppose $k_{t+1}/k_t \leq \beta R_t = \beta\left[\alpha k_t^{\alpha-1} + (1-\delta)\right]$ for some period t. Then, (9) implies $z_{t+1} \leq z_t$. It follows from Corollary 1 that $z_{t+j+1} < z_{t+j}$ for all $j > 1$, but this contradicts Lemma 3. ■

With the previous result in hand we are ready to study the evolution of the distribution of assets.

5 The Dynamics of the Distribution of Assets

We use the coefficient of variation (the standard deviation divided by the mean) to measure inequality in assets across agents. The following theorem is the main result of the paper.

Theorem 4: *Assume A1 and A2. In any transition from below, the coefficient of variation in assets across agents monotonically decreases over time.*

Proof: From Equation (5) we get

$$S.D.(a_{t+1}^i) = \beta R_t S.D.(a_t^i)$$

and therefore:

$$\frac{S.D.(a_{t+1}^i)/k_{t+1}}{S.D.(a_t^i)/k_t} = \left(\frac{k_t}{k_{t+1}}\right)\beta R_t. \tag{15}$$

The proof is concluded since $k_t = \int_0^1 a_t^i di$ (from Theorem 2) and Theorem 3 implies $(k_t/k_{t+1})\beta R_t < 1$. ■

Comments

1. Theorem 4 says that along a transition to the steady state from below, poor agents accumulate assets at a faster rate than rich agents, and thus the distribution of assets becomes more equal over time. When $\bar{c} = 0$, the result does not depend on initial conditions and the intuition is as follows: All agents have the same intertemporal marginal rate of substitution, hence with no consumption requirement they share the same desired rate of growth of consumption. But since aggregate capital grows at a faster rate than individual consumption (as established in Theorem 3), and consumption is linear in lifetime wealth, aggregate capital grows faster than the average wealth portfolio in the economy. This wealth portfolio is a weighted average of aggregate capital and the present value of the labor income flow, which is common to all agents. Hence, labor income must grow at a lower rate than average wealth. In this context, poor agents accumulate assets at a faster rate than rich agents because the share of the labor component in their wealth portfolio is larger, so they need to save more to match the rate of growth of the average agent's wealth. The result also holds for $\bar{c} > 0$ in the range specified in A2 once the stock of capital is sufficiently close to k^*. In this case, even though poor agents have a lower than average rate of growth of consumption, it is still the case that they need to save more to attain a given rate of growth of consumption, since their wealth portfolio includes more labor income which is growing at a slower rate[7].

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[7] We do not study the case in which $\bar{c} < 0$, since a negative consumption requirement is difficult to interpret in our general equilibrium framework. However, the arguments and results for $\bar{c} = 0$ go through without variation for $\bar{c} < 0$. In this case, poor agents

2. An implication of Theorem 4 is that the investment to *assets* ratio is larger for poor agents, hence these agents accumulate assets at a faster rate. However, this result does not imply that the savings rate (defined as the investment to *income* ratio) is also higher for poor agents, since total income also includes a labor component which is relatively larger for poor agents. The numerical simulations reported in the next section suggest that in our model rich agents indeed have higher saving rates than poor agents, which is consistent with empirical evidence for the U.S. economy (see for instance Huggett and Ventura (2000), and the references therein).

3. Our result in Theorem 4 has to be contrasted with Theorem 1 in Chatterjee (1994). He shows in a similar environment that, along a transition to the steady state from below, the inequality in the distribution of *lifetime wealth*: 1) declines when $\bar{c} < 0$; 2) remains constant when $\bar{c} = 0$; and 3) increases when $\bar{c} > 0$. Note that if the minimum consumption requirement is positive, but small enough as to satisfy Assumption A2, the inequality in assets and in lifetime wealth evolve in opposite directions: from Chatterjee (1994), inequality in lifetime wealth (and hence in consumption) increases in a transition from below, but our Theorem 4 states that the coefficient of variation in assets declines. This is an interesting case that should warn us against the mechanical interpretation of changes in the inequality in assets as changes in consumption (or even welfare) inequality, as is sometimes done in policy analysis.

4. The results in Theorems 3 and 4 will follow as long as $\phi'(k) > 0$. This is the only step in our analysis in which we have used the Cobb-Douglas specification for the production function. Under a more general specification,

$$\phi(k) = \frac{\beta[f'(k)k + (1-\delta)k]}{f(k) - \beta f'(k)k + (1-\beta)(1-\delta)k},$$

where f denotes the production function per unit of labor. Could we still show that $\phi'(k) > 0$ using a more general technology? Consider for example a CES production function $f(k) = [\alpha k^{-\rho} + (1-\alpha)]^{-1/\rho}$, with capital share $\alpha \in (0,1)$ and $-1 < \rho \neq 0$, where $1/(1+\rho)$ measures the elasticity of substitution between capital and labor. The Cobb-Douglas case corresponds to the limit when $\rho \to 0$. Assume for simplicity that $\bar{c} = 0$ (similar results

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accumulate assets at a faster rate for two reasons: first, beacuse they have a higher desired rate of growth of consumption, and second beacuse their weath portfolio includes more labor, which grows at a slower rate. The two effects point to a reduction in inequality in assets.

can be obtained in the other cases). We can show that

$$\frac{\phi'(k)M^2}{\beta} = h''(k)k\left[h(k)+(1-\delta)\right] + h'(k)(h(k)+(1-\delta)) - \left(h'(k)\right)^2 k,$$

where $M = h(k) - \beta f'(k) + (1-\beta)(1-\delta)$ and $h(k) = (\alpha + (1-\alpha)k^{\rho})^{-1/\rho}$. To derive the previous expression, we use the fact that $f(k) = h(k)k$. Since $h'(k) < 0$ for all $k \geq 0$, to obtain $\phi'(k) \leq 0$ it suffices to have $h''(k) \leq 0$. Using the previous notation, we obtain: $h''(k) = h'(k)/k(\alpha(\rho-1) - 2(1-\alpha)k^{\rho})/(\alpha+(1-\alpha)k^{\rho})$, thus $\phi'(k) \leq 0$ for all $k \leq (\alpha(\rho-1)/(2-2\alpha))^{1/\rho} = \tilde{k}$. This is only possible if $\rho > 1$, i.e., if the elasticity of substitution between capital and labor is less than 1/2, ruling out the Cobb-Douglas case. With $\rho > 1$, it follows that for $k^* > \tilde{k} > k_0 > 0$ the coefficient of variation in assets may initially increase, but as soon as $k_t > \tilde{k}$ our result holds and inequality monotonically decreases. A similar result was obtained by Caselli and Ventura (2000) in a continuous time version of the model.

6 Simulations

In this section we numerically solve the model and provide several examples illustrating Theorem 4. We also simulate the economy for cases in which the assumptions required for Theorems 3 and 4 do not hold. The parameter values used in all simulations are as follows: $\beta = .99$, $\alpha = .36$, and $\delta = .025$. These values are standard in quantitative studies simulating quarterly data for the U.S. economy (see, for instance, the studies in Cooley (1995)). The method used to solve the model is explained in detail in the Appendix. The initial distribution of assets is arbitrarily chosen and is the same for all the experiments.

Figure 1 displays the evolution of the coefficient of variation in assets across agents along the transition to the steady state for $\bar{c} = 0$ and $\bar{c} = .1$. These values for $\bar{c}$, and the assumed initial condition for capital, satisfy the assumptions of Theorems 3 and 4. We also include $\bar{c} = -.1$ to illustrate how the results with $\bar{c} = 0$ carry over to $\bar{c} < 0$ trivially. In the three cases, inequality in asset holdings decreases monotonically along the transition. Moreover, the figure reveals that the reduction in the coefficient of variation is less pronounced when $\bar{c} = .1$.

To illustrate the difference between the investment to assets ratio and the saving rate (or the investment to income ratio), Figures 2 and 3 plot the evolution of both ratios for a *rich* agent with initial assets equal to 120% of average assets, and a *poor* agent with an initial level of assets equal to 80% of average assets. The two cases in each graph correspond to the transitions in Figure 1 with $\bar{c} = 0$ and $\bar{c} > 0$. Both ratios monotonically decrease over the transition, but while the poor agent has always a higher investment to assets ratio, it is the rich agent who has a larger saving rate.

Figure 4 displays the dynamics of inequality for a case where $\bar{c} > 0$ satisfies Assumption A2, but where the initial stock of capital does not: $k_0 < k^o < \hat{k}^o$. In this example, the coefficient of variation first increases and then declines towards the steady state, displaying the so-called Kuznets curve. To further illustrate this case, Figure 5 plots the coefficient of variation as a function of the stock of capital. For a large enough initial capital stock, inequality monotonically declines along the transition. This is ensured as long as $k_0 > \max\left\{k^o, \hat{k}^o\right\}$, as required by Assumption A2. But notice that this condition is sufficient, not necessary, as the coefficient of variation also declines starting with k_0 slightly below $\hat{k}^o$. If the initial capital stock is too small, then inequality increases in a range before declining.

Figure 6 displays the dynamics of inequality for $\bar{c}^o < \bar{c} < \bar{c}_{max}$, thus in this case Assumption A2 is also violated.[8] The coefficient of variation in assets is monotonically increasing, and displays a convex-concave pattern. Notice also that in this case the transition to the steady state takes much longer than in the previous examples, since the rate of growth of capital is low.

The evolution of the investment to assets ratio for rich and poor agents consistent with the simulations in Figures 4 and 6 are reported in Figure 7. The top panel shows the case in which inequality display a non-monotonic path: first, the coefficient of variation in assets increases, and then decreases towards the steady state. In this case, the investment to assets ratio is higher for rich agents at the beginning, but eventually poor agents overtake it. The bottom panel reports the case in which the coefficient of variation in assets in monotonically increasing, implying that the investment to assets ratio is always higher for rich agents.

Our last example, Figure 8, displays the dynamics of inequality under $\bar{c} = 0$ and a CES production function with $\rho = 1.1$. In this case the coefficient of variation in assets displays non monotonic dynamics in the beginning of the transition, but as soon as the stock of capital is sufficiently large, it monotonically declines towards the steady state.

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[8] Notice that when $\bar{c}^o < \bar{c}$, Remark 1 after assumption A2 implies that any $k_0 < k^*$ also violates A2.

7 Conclusions

In this paper we use a simple deterministic growth model in discrete time to study the dynamics of the distribution of assets. In particular, we state conditions (in terms of a minimum consumption requirement and the aggregate stock of capital) under which the coefficient of variation in assets across agents monotonically decreases along the transition to the steady state. We complement our theoretical results with numerical examples showing that a rich class of non monotonic dynamics is also possible.

An interesting extension for this research is to provide sufficient conditions under which the aggregate savings rate monotonically decreases in a transition. Our conjecture, based on the numerical results in Figure 3, is that the same assumptions leading to a monotonic inequality in assets also lead to a monotonic savings rate. To the extent of our knowledge, a formal proof of this statement remains to be provided.

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A Appendix

Proof of Theorem 1. The case of $\bar{c} = 0$ is the standard case in the literature and has been studied, for instance, in Harris (1987, Theorem 2.6, p. 43). For the case of $\bar{c} > 0$ fix k_0 satisfying A1 and define $\underline{c}(k_0) = k_0^\alpha - \delta k_0 - \bar{c}$. Then $\underline{c}(k_0)$ is the consumption corresponding to a policy that keeps the initial stock of capital k_0 constant over time. Under A1 this policy is feasible. Choose $M > (\log(\tilde{c}^*) - \log(\underline{c}(k_0)))/(1-\beta)$ and define $c_M(k_0) \equiv \exp\left(\log(\underline{c}(k_0)) - \beta M\right)$. Next, let the mapping T on $C(K)$, the space of bounded, continuous, strictly increasing real-valued functions on $K = [k_0, k^*]$ equipped with the sup norm, be given by

$$\begin{aligned} (Tv)(k) &= \max\{u(\tilde{c}) + \beta v(k')\} \\ \text{s. to } &(\tilde{c}, k') \in \Gamma(k) \end{aligned} \tag{16}$$

where $\Gamma(k) = \{(\tilde{c}, k') : \tilde{c} + k' \leq k^\alpha + (1-\delta)k - \bar{c},\ \tilde{c} \geq c_M(k_0), k' \in K\}$.

Theorem 4.6 in Stokey and Lucas (1989) states that T defines a contraction and that it has a unique fixed point $v \in C(K)$, provided that u is bounded on the non-empty, continuous correspondence $\Gamma(k)$. Clearly, u on $\Gamma(k)$ is bounded above and below. Also, $\Gamma(k)$ is non-empty by assumption A1, and $\Gamma(k)$ can be shown to be continuous. Thus Theorem 4.6 applies. The existence of a continuous, single-valued policy function g such that $k' = g(k)$ follows from standard arguments (e.g., Theorem 4.8 in Stokey and Lucas (1989)). In addition, g is strictly increasing and strictly concave, hence the constraint $k' \in K$ does not bind for any $k \in K$ such that $k \neq k^*$. The correspondence between the solution of the planner's problem in sequence form and in its recursive formulation is established in Theorems 4.2-4.5 in Stokey and Lucas (1989). Thus points *i)* to *iv)* of Theorem 1 follow directly from the previous facts if we prove that $\tilde{c}$ is always larger than $c_M(k_0)$.

For future reference, let $v(k^*) = \log(\tilde{c}^*)/(1-\beta)$. Consider the policy of keeping the stock of capital constant over time. The present value of utility associated to such policy is given by $\underline{v}(k_0) = u(\underline{c}(k_0))/(1-\beta)$ when $k = k_0$. Suppose now that it is optimal to have $\tilde{c} = c_M(k_0)$, thus $v(k_0) = u(c_M(k_0)) + \beta v(k') = u(\underline{c}(k_0)) + \beta(v(k') - M)$. It follows that $v(k_0) - \underline{v}(k_0) = \beta(v(k') - M - \underline{v}(k_0))$. If $k' > k_0$, then we have that $v(k_0) - \underline{v}(k_0) < \beta(v(k^*) - M - \underline{v}(k_0))$, because v is in $C(K)$. Since $v(k^*) - \underline{v}(k_0) = (\log(\tilde{c}^*) - \log(\underline{c}(k_0)))/(1-\beta) < M$, this contradicts the fact that $\tilde{c} = c_M(k_0)$ is optimal. If $k' = k_0$, a similar argument shows that the same conclusion holds for all $M > 0$. Thus we conclude in both cases that it is not optimal to choose $\tilde{c} = c_M(k_0)$, completing the argument for k_0 because in (16) $k' \in K$. Next, under A1 it is possible to keep constant over time any stock of capital for all $k \leq k^*$ and still have $\underline{c}(k) > 0$. Furthermore, since $\underline{c}'(k) > 0$ for all $k \leq k^*$, then present value of the utility associated to such policy is also strictly increasing in k. It follows that $M > (\log(\tilde{c}^*) - \log(\underline{c}(k)))/(1-\beta)$ for all $k \geq k_0$. Hence similar arguments can be used to rule out that it is optimal to choose $\tilde{c} = c_M(k_0)$ for all $k \in K$. That is, given k_0, $c_M(k_0)$ represents a uniform lower bound for consumption, and the proof is concluded. ■

Proof of Lemma 1: To see the first part, compute

$$\phi'(k) = \frac{\beta\left\{(1-\delta)k^{\alpha}(1-\alpha)^2 - \bar{c}\left[\alpha^2 k^{\alpha-1} + (1-\delta)\right]\right\}}{\left[(1-\alpha\beta)k^{\alpha} + (1-\beta)(1-\delta)k - \bar{c}\right]^2},$$

and notice that $\phi'(k) > 0$ when $\bar{c} = 0$. Thus, the result follows because in a transition from below $k_t < k_{t+1}$. For the case of $\bar{c} > 0$, notice that $\phi'(k) > 0$ if and only if $\varphi(k) > \bar{c}$, so the result follows from A2 (see Remark 2 after this assumption). The limit $\lim_{j\to\infty} \phi(k_{t+j})$ is obtained from:

$$\begin{aligned}
\phi(k^*) &= \frac{\alpha\beta(k^*)^{\alpha-1} + \beta(1-\delta)}{(1-\alpha\beta)(k^*)^{\alpha-1} + (1-\beta)(1-\delta) - \bar{c}/k^*} \\
&= \frac{1-(1-\delta)\beta + \beta(1-\delta)}{\left(\frac{1-\alpha\beta}{\alpha\beta}\right)\left[1-(1-\delta)\beta\right] + (1-\beta)(1-\delta) - \bar{c}/k^*} \\
&= \frac{1}{\frac{1-(1-\delta)\beta}{\alpha\beta} - \delta - \bar{c}/k^*} = \frac{1}{(k^*)^{\alpha-1} - \delta - \bar{c}/k^*} = \frac{k^*}{\tilde{c}^*},
\end{aligned}$$

where we have used the definitions of k^* and $\tilde{c}^*$ from Theorem 1. To see the second part, divide both sides of (9) by k_{t+1}, and use the definition of z_t to obtain:

$$\left(\frac{k_{t+1}}{k_t}\right) z_t = \beta\left[\alpha k_t^{\alpha-1} + (1-\delta)\right] z_{t+1}. \tag{17}$$

Similarly, dividing both sides of (10) by k_t, and using the definition of z_t we get:

$$\left(\frac{k_{t+1}}{k_t}\right) = \left[k_t^{\alpha-1} + (1-\delta) - \frac{\bar{c}}{k_t}\right]\frac{z_{t+1}}{z_{t+1}+1}. \tag{18}$$

Combining (17) and (18) we obtain

$$z_{t+1} = \left(\frac{k_t^{\alpha-1} + (1-\delta) - \frac{\bar{c}}{k_t}}{\beta\left[\alpha k_t^{\alpha-1} + (1-\delta)\right]}\right) z_t - 1,$$

which gives us the expression in (12), as desired. ■

Numerical method

The numerical method we use for the simulations is based on dynamic programming. We define the correspondence $\Gamma(k) = \{(c, k') : c + k' = k^{\alpha} + (1-\delta)k, c \geq \max\{0, \bar{c}\}, k' \geq 0\}$, and starting from an arbitrary (differentiable, increasing and concave) function v_0 of the state k, we define a mapping T as

$$(Tv)(k) = \max{}_{c,k' \in \Gamma(k)} \log(c_t - \bar{c}) + \beta v_0(k'),$$

and successive mappings T^n as $T^1 v = Tv$, $T^2 v = T(Tv)$... The first order condition implicit in the n-mapping is given by:

$$\frac{1}{k^{\alpha} + (1-\delta)k - k' - \bar{c}} = \beta T^{n-1} v_0'(k'). \tag{19}$$

We compute $1/(k^{\alpha} + (1-\delta)k - k' - \bar{c})$ and $v_0'(k')$ on a grid of points, and we use linear interpolation to approximate the value of these functions when k' is not in the grid. We iterate until this decision rule has approximately converged. Thus, with this method the corresponding version of the first order condition holds exactly at points in the grid (see Huggett (1993) for further details). In practice we use a grid with 800 points evenly spaced (in some examples we increased the number of points up to 2,000). Once we obtain the decision rule for capital, we simulate the transition towards the steady state over 1,000 periods. To compute the coefficient of variation over this transition we exploit the recursive structure in Equation (15).

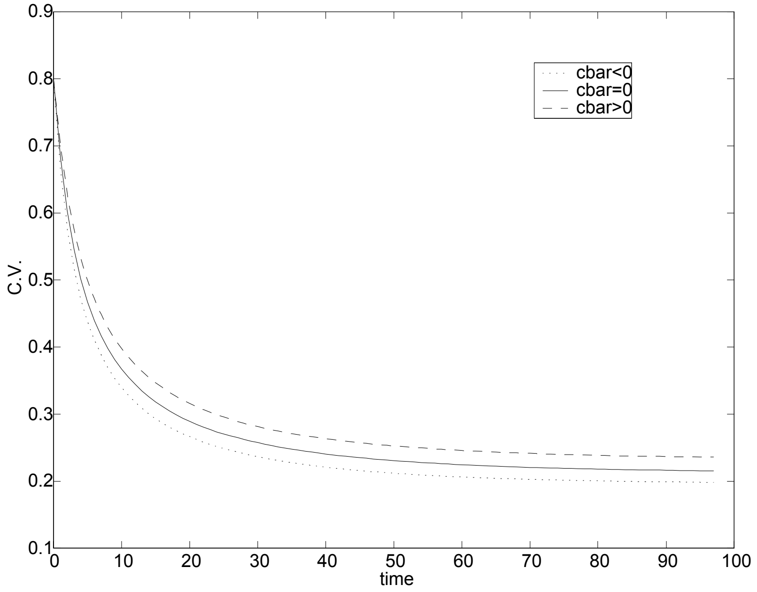

Figure 1: Evolution of Inequality under A2 for different values of $\bar{c}$.

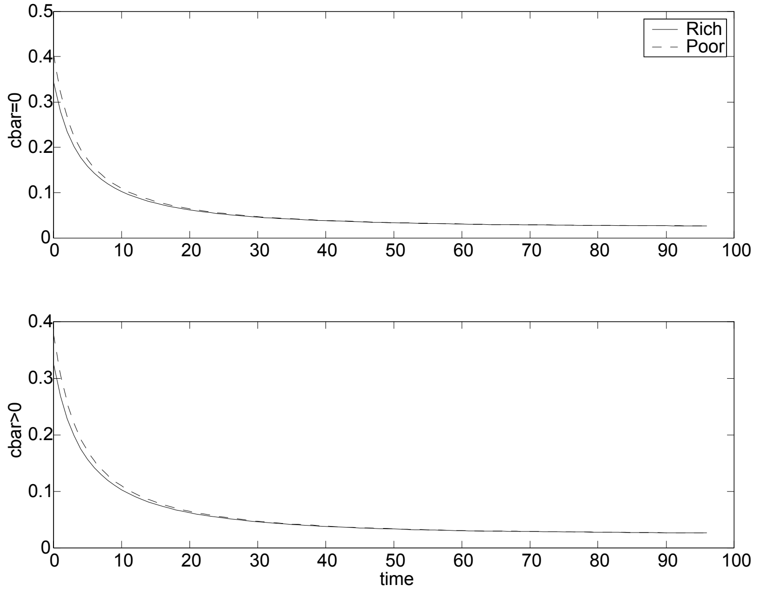

Figure 2: Evolution of the Investment to Assets ratio under A2.

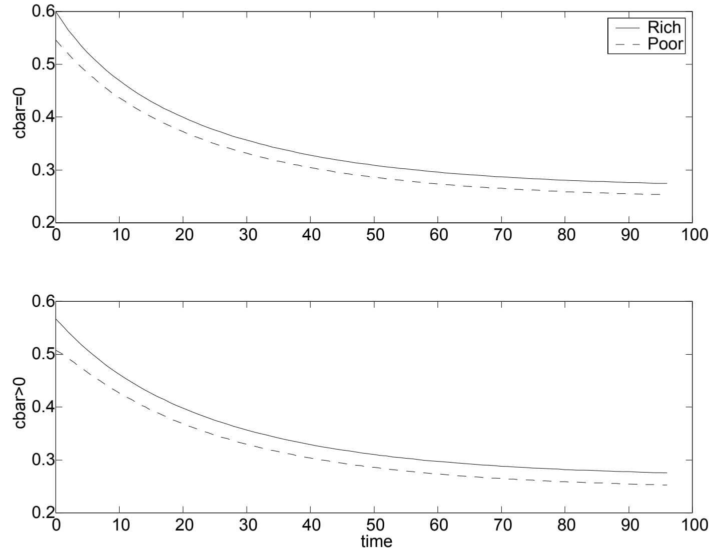

Figure 3: Evolution of the Saving Rate under A2.

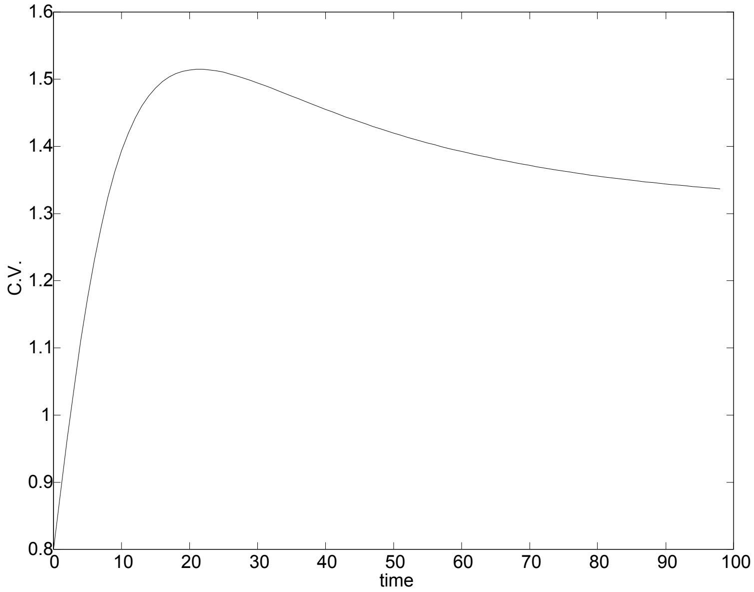

Figure 4: Evolution of Inequality when k_0 does not satisfy A2: $k_0 < k^o$.

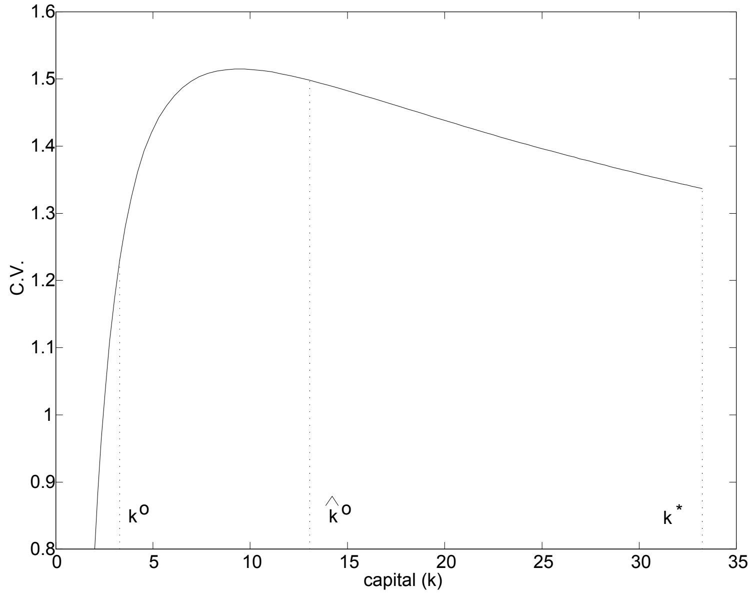

Figure 5: Inequality as a function of k when $0 < \bar{c} < \min\{\bar{c}^o, \bar{c}_{\max}\}$.

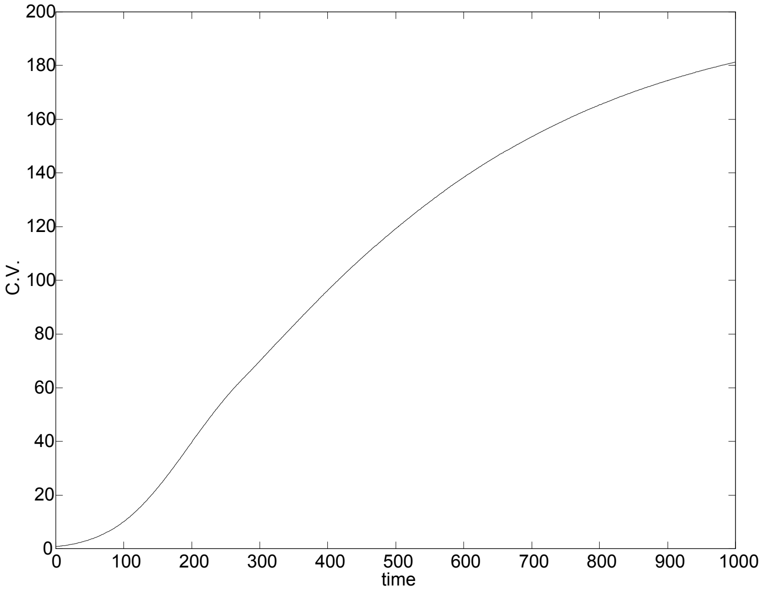

Figure 6: Evolution of Inequality when $\bar{c}$ does not satisfy A2: $\bar{c}^o < \bar{c} < \bar{c}_{\max}$.

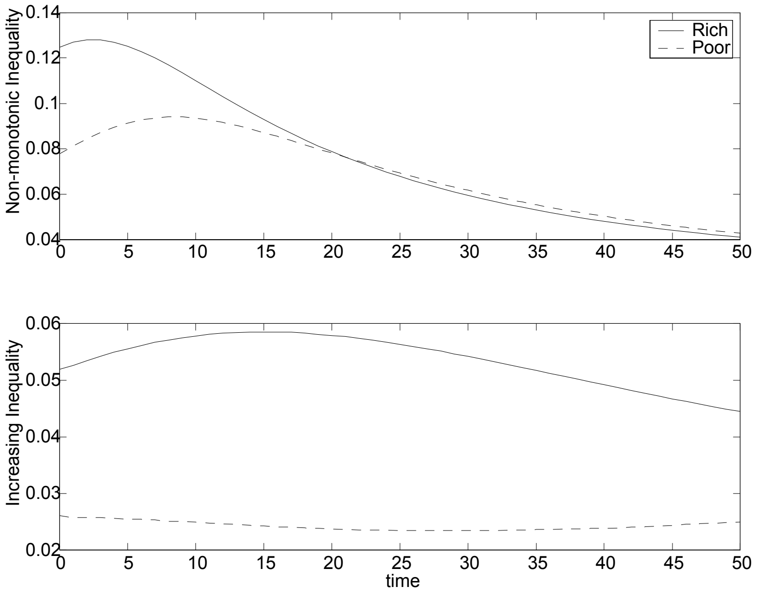

Figure 7: Evolution of the Investment to Assets ratio when A2 is violated.

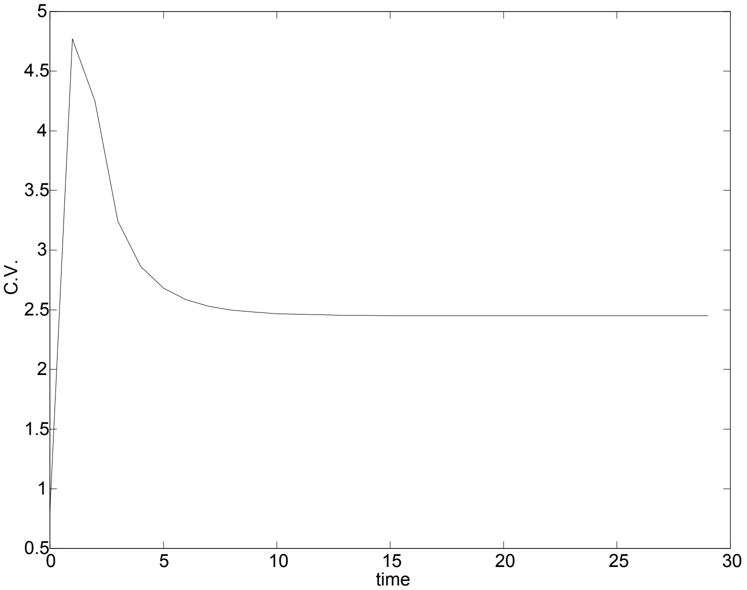

Figure 8: Evolution of Inequality for $\bar{c} = 0$ and CES technology with $\rho > 1$.