Inside and Outside Money, with an Application to the Russian Virtual Economy

Svetlana Boyarchenko and Sergei Levendorskiĭ⋆

Department of Economics, The University of Texas at Austin, 1 University Station C3100, Austin, TX 78712-0301, USA
(e-mail: sboyarch@eco.utexas.edu; leven@eco.utexas.edu)

Received: ; revised:

Summary We analyze the endogenous appearance of money substitutes, their interaction with outside money, and resulting distortions in the price system of an economy with large monopolies and wide-spread informal networks. The economy consists of productive, individually optimizing agents and less productive colluding agents who issue universally acceptable money substitutes. We distinguish equilibria by types of exchange both between agents of one type and between those of different types and show that for small trading frictions, only three types of equilibria can be sustained. A novelty of the analysis is that the agents issuing money substitutes survive by their collusion.

Key words money substitutes; search; collusion; trading frictions.
JEL Classification Numbers: E31, E52, O23.

1 Introduction

The endogenous appearance of money substitutes is one of the important questions of monetary economics. There are several search-theoretic models incorporating

⋆ We are grateful to R. Wright, N. Wallace, B. Ickes, V. Polterovich, M. Castanheira and participants in EERC Workshops (1999, Moscow), CEPR/WDI Annual Conference on Transition Economies (2000, Moscow) and Money, Macro and Finance Millennium Conference (2000, London) for useful discussions and remarks. We are especially thankful to R. Ericson for his suggestion to apply a search-theoretical approach to an economy with structural deficiencies of the Russian economy, and to B. Ickes who kindly provided a preprint of Gaddy and Ickes [11] at an earlier stage of our work. Authors acknowledge a partial support from Economic Education and Research Consortium–Russia (EERC), grant R98-2251.

inside money. The word *money* indicates the object which is used as a tangible medium of exchange among the agents who recognize it as an asset, and the label *inside* shows that this object is supplied by the private sector. In Cavalcanti et al. [6], Cavalcanti and Wallace [7], [8], and Williamson [16] inside money is given a role both as credit and as a tangible medium of exchange. In Cavalcanti et al. [6] and Cavalcanti and Wallace [7], [8], it is assumed that a fraction of the population - a banking sector - has access to a private note-issuing technology, while the rest of the economy - a non-banking sector - uses inside money as a medium of exchange. In Williamson [16] a model with claims on banks as private money is explored. Agents can choose between investing into low or high-return projects, so there may exist welfare dominated equilibria where banks hold low-return assets. Burdett et al. [5] introduce endogenous money as a commodity that can be either consumed or stored and used as a medium of exchange. In this model, there is no role for credit because the trade, if it takes place, is always *quid pro quo*.

In addition to the media of exchange produced in the private sector, all the above models with inside and endogenous money incorporate the exogenous provision by a public sector of fiat currency usually referred to as *outside* or exogenous money. In Burdett et al. [5] and Cavalcanti and Wallace [7], [8], it is shown that an equilibrium can be achieved in an economy with only endogenous or inside money (respectively) in circulation. The former paper also shows that if the supply of exogenous money is sufficiently small, both types of money may coexist. Cavalcanti and Wallace [7], [8] consider only implementable allocations that arise with inside and outside money separately; they do not examine coexistence of both kinds of money. In Cavalcanti et al. [6], coexistence of private and government money is studied only for the case of a discount factor close to one, but no analytical results for endogenous variables are obtained.

The model presented here combines key features of Burdett et al., [5], Cavalcanti et al. [6], and Cavalcanti and Wallace [7], [8]. To be more specific, we consider the case when inside and outside moneys coexist. However, we visualize the note-issuing sector not as a banking sector, but as a coalition of large producers, such as gas, oil, or electricity companies. We rule out the possibility of bankruptcy. A novelty of the analysis is that the agents issuing money substitutes survive by their collusion.

A motivation for our study is the transition Russian economy. There is empirical evidence suggesting that this economy differs significantly from both the command economy with no real monetary transactions, and the market economy, where most of transactions are made with money. Before August 1998, the Russian economy was characterized by wide spread use of barter and money substitutes (up to 60-80 per cent for some industries, see, e.g., Aukutsionek [1]), and many serious distortions were attributed to this fact. Apparently we were confronted with a new type of economy – a partially monetized quasi-market economy[1] – and therefore appropriate macroeconomic models are needed. Our model shows that key features of the Russian economy can be derived from the co-existence of colluding agents with relatively low productivity and more productive individually

[1] This hybrid economy has been called the *virtual economy* (see, for example, Gaddy and Ickes [12]).

optimizing agents. In particular, the model explains naturally wide spread use of barter and money substitutes.[2]

In order to explain the endogenous appearance and circulation of money substitutes in the economy, we take the monetary search approach. Agents are placed in a standard environment (see Kiyotaki and Wright [14]), in which different people have different preferences over a large number of differentiated goods. The economy consists of two types of agents with inherited differences in productivity. Through informal connections, less productive agents can collude (in a sense explained later) in order to maximize their welfare. The collusion makes it possible for these agents to issue universally accepted means of exchange, which we call *notes*. More productive agents have no means to establish informal connections and so they act as individually optimizing agents. The motivation for this setting is a description of the Russian economy as consisting of new or restructured and privatized firms and old, mainly unrestructured enterprises, which manage to survive due to collusion, informal networks and widespread use of IOU's. Notice that despite some positive changes in Russia attributed to the real depreciation of the ruble and high oil prices, the share of loss making enterprises continues to be remarkably large, nearly 40 per cent (see Gaddy and Ickes [13]). It is true that barter went down in recent years, but the problem of the virtual economy as "..the outcome of agents' adapting their behavior to an environment that threatens their survival..." (see Gaddy and Ickes [13]) remains to be acute for today's Russia.

We classify possible types of equilibria for the case of small trading frictions. We prove that there are three classes of equilibria:

- equilibria without exchange between different types of agents and all the money circulating among the more productive agents;
- equilibria where all the money circulates among more productive agents, less productive agents do not use money but notes circulate both between types and among the more productive type;
- equilibria, when both money and notes circulate between types.

In any equilibrium, no means of exchange are used by the colluding sector so that all goods are produced for credit; this reflects the existence of barter chains in the Russian economy. We show that for a sufficiently small level of trading frictions, only the first two of the above classes of equilibria are possible. Moreover, if the difference in productivity across the types of agents is sufficiently large and trading frictions are small, then only equilibria without interaction between types are possible. We will not consider the latter equilibria in this paper. Pure monetary equilibria were examined in Boyarchenko [3] for general utility functions and several types of heterogeneity of agents. Possible pure monetary equilibria in our case are equilibria when there is no trade between types and the economy splits into two disjoint economies. The effect of splitting is demonstrated in Boyarchenko [3]; since in that model there is neither inside money nor collusion, there exists a

[2] Notice that there are models of the Russian economy that analyze its distortions from different viewpoints – see, e.g., Blanchard and Kremer [2], Ericson [9], Ericson and Ickes [10], and Woodruff [17] and the bibliography there. None of these papers uses the approach adopted here or focuses on the key issues in this paper.

cut-off type such that all types with marginal productivity lower than the cut-off do not produce or trade for money.

The rest of the paper is organized as follows. We specify the model and define equilibrium in Section 2. In Section 3, we characterize equilibria without monetary exchange between types of agents. We show that existence of this class of equilibria depends on the difference in the marginal productivity of agents, trading frictions and the supply of money: when a certain combination of these parameters crosses a certain threshold, this kind of equilibria vanishes. In Section 4, we provide a characterization of equilibria with all kinds of trades between agents and show that these are less viable than the equilibria without monetary exchange between types in the sense that the former disappear at a lower threshold. Section 5 contains our main conclusions. We dot not present proofs of theorems or other technicalities in the paper. An interested reader can find those in Boyarchenko and Levendorskiǐ [4].

2 Model Specification

There is a continuum of infinitely lived agents. The size of population is normalized to one. Agents produce and consume non-storable goods (or services) at discrete points in continuous time. Agents and goods are indexed by points on a circle of circumference two. The agents have idiosyncratic tastes for goods in the sense that at any time, an agent indexed by point i has a demand for a particular variety of goods which lies within the distance $X \in (0, 1/2)$ from point i.[3] When an agent consumes an amount q of a demanded good, she enjoys utility $u(q)$. We assume that u satisfies standard properties: u is smooth, increasing, concave and satisfies the Inada conditions. Agent i derives no utility from any amount of good lying distance $x > X$ from i.

Each agent i can instantaneously produce good j at a fixed distance Z, $2X < Z < 2 - 2X$, clockwise from i. We assume fixed production cost k per unit of good (k may differ among agents). Due to these properties of preferences and the production technology, agents never produce for themselves. They must trade in the exchange sector in order to consume. Trading partners arrive according to a Poisson process with the constant arrival rate α. It is clear that the specification of preferences and production opportunities rules out a double coincidence of wants. Thus there can be no direct barter, and the probability of a single coincidence of wants is X.

There is an exogenous money supply $M \in (0, 1)$. Money is indivisible and perfectly storable. Each agent can carry either one unit of money or none. Indivisibility of money and a unit bound on individual money holdings are strong, but popular assumptions in monetary search models (see, for example, Burdett et al. [5], Cavalcanti et al. [6], Cavalcanti and Wallace [7], [8], and Wallace and Zhou [15]). These assumptions simplify the models, and still allow one to endogenize prices as reciprocals to quantities of good produced for a unit of money. In some situations, such as in Wallace and Zhou [15], indivisibility is used as descriptive of situations the authors want to model. In our model, indivisibility of money and

[3] By a distance we mean the shortest arc between i and a given good.

money substitutes (see below) is assumed for tractability in the first place. On the other hands, since we view the agents, not as individuals but as managers of enterprizes, where all transactions are performed in large currency denominations, and money substitutes are issued in Russia only in large denominations, indivisibility of money is a reasonable assumption.

Agents may issue indivisible and perfectly storable promissory notes (IOU's). Notes are distinguishable in the sense that no counterfeiting is possible. A fraction of agents – we call them *colluding* agents (and the rest the population *private* agents) – collude in the following sense:

(i) they agree on a rule for note-issuing (to be specified below) and a quantity q_n of good to be redeemed for a note issued by any colluding agent;
(ii) they agree to use no means of exchange when they trade with each other and to produce a fixed amount q_c of good if such trade occurs;
(iii) each colluding agent signs a note she issues and writes a date of issue on it;
(iv) all information about a trading history of every colluding agent is recorded and spread among all such agents;
(v) if at some point in time, a colluding agent deviates from any of the agreed rules, other colluding agents make it publicly known that notes issued by the deviator after this moment are not going to be redeemed;
(vi) the rule for note-issuing and quantities q_n and q_c are chosen so as to maximize the welfare of colluding agents subject to incentive compatibility constraints.

Some readers may find it strange that agents collude only either on the trade credit among themselves or trades made with inside money, and do not collude on the price in terms of outside money. Disregarding price setting behavior in terms of outside money makes the model more tractable, and at the same time allows us to emphasize the role of inside money as a device for the transfer of value from more productive to less productive agents.

The note-issuing rule deserves special consideration. On an individual level, a colluding agent would not issue a note unless she enjoys positive utility from consumption of a given good, i.e., an individual agent issues a note with probability X. However, the supply of notes affects all endogenous variables in the economy including the welfare of colluding agents. Thus, it may be optimal to issue a note with probability $x_c > X$ in order to increase the amount of liquidity in the economy and make the outside money less valuable. If this is the case, the "social planner" of colluding agents requires the agent to issue a note even if the agent does not like the good. It may also be optimal to issue a note with probability $x_c < X$ in order to make notes more scarce, increase the transaction value of a note, and receive more goods in exchange for a note. In this case, the "social planner" forbids the issue of a note in some cases when the agent likes the good. So the rule for note-issuing can be summarized as follows:

– (for the case $x_c \leq X$): "never issue a note if a good lies distance $z > X$ from you, and if $z \leq X$, issue a note with probability x_c/X;"
– (for the case $x_c > X$): "always issue a note if $z \leq X$, and if $z > X$, issue a note with probability $(x_c - X)/(1 - X)$."

Private agents have no means to collude, so if such an agent issues a note, no other agent of this type is under obligation to redeem the note. Hence each such note will be redeemed with zero probability, and in no symmetric equilibrium is it optimal for anyone to accept a note issued by a private agent. Thus we may assume that notes of private agents do not circulate at all, and hereafter, we call notes of colluding agents simply *notes*. The agents also differ in their production: private agents are endowed with better production opportunities, so that they suffer lower disutility (cost) of production per unit of good. Namely, we set the marginal cost of production equal to 1 for a private agent, and $k \geq 1$ for a colluding agent.

We assume that any private agent can carry either one unit of money or a note, or neither of these. Therefore each private agent can be in one of three states: a buyer carrying a note, a buyer with a unit of money, or a seller. We denote by V^p_n, V^p_m, V^p_s the value functions of the agent in these states. A colluding agent can carry one or zero units of money. Notice that colluding agents never carry notes, but they can issue them (in other words, notes do not specify a state of a colluding agent). Thus a colluding agent can be in one of two states: a buyer with a unit of money or a trader without money. The corresponding value functions are denoted by V^c_m and V^c. Agents meet pair-wise and at random, and in each meeting decide whether to trade, and how much to produce for a note or a unit of money. When two agents meet they cannot trade unless one agent is a seller and the other is either a buyer with a mean of exchange or a colluding agent entitled to issue a note. Also a trade can take place if both agents are colluding ones and there is a single coincidence of wants.[4]

Evidently, the seller does not produce if she is worse off after the trade. We assume that the buyer (except for the case when two colluding agents meet, or when a colluding agent redeems a note) makes a take-it-or-leave-it offer to the seller, which enables her to extract all the seller's surplus from trade. This means that if a trade takes place, the seller produces her reservation quantity, i.e. the quantity that makes her indifferent between producing and not producing. Therefore a private seller produces amount q_{np} (for a note) or q_{mp} (for a unit of money) given by

$$V^p_n - V^p_s = q_{np}, \text{ and } V^p_m - V^p_s = q_{mp}. \tag{2.1}$$

A colluding agent produces for a private buyer the assigned quantity q_n to redeem a note and q_{mc} for a unit of money, where q_{mc} solves

$$V^c_m - V^c = kq_{mc}. \tag{2.2}$$

A private buyer decides whether to spend her means of exchange given the amount of good the seller agrees to produce. We assume that the buyer trades if she is not worse off after the trade, therefore she spends her unit of money with probability x_{mp} when she meets a private producer, where

$$x_{mp} = \begin{cases} X \text{ iff } u(q_{mp}) \geq V^p_m - V^p_s \\ 0 \text{ otherwise.} \end{cases} \tag{2.3}$$

The case when the buyer meets a colluding producer is treated in the same way: we denote by x_m the probability of spending money and replace x_{mp} and q_{mp} in (2.3) by x_m and q_{mc} respectively. Similarly, the note holder spends her note

[4] When the buyer is a colluding agent and has to issue a note even if she does not like the good, we also use the label single coincidence meeting.

with probability x_{np} (respectively, x_n) in a meeting with a private (respectively, colluding) seller, and

$$x_j = \begin{cases} X \text{ iff } u(q_j) \geq V_n^p - V_s^p \\ 0 \text{ otherwise.} \end{cases} \quad j \in \{np, n\} \tag{2.4}$$

Notice that (2.1), (2.3) and (2.4) together imply that

$$x_{mp} = \begin{cases} X \text{ iff } u(q_{mp}) \geq q_{mp} \\ 0 \text{ otherwise;} \end{cases} \tag{2.5}$$

and

$$x_j = \begin{cases} X \text{ iff } u(q_j) \geq q_{np} \\ 0 \text{ otherwise} \end{cases} \quad j \in \{np, n\}. \tag{2.6}$$

Also, we can derive

$$x_m = \begin{cases} X \text{ iff } u(q_{mc}) \geq q_{mp} \\ 0 \text{ otherwise.} \end{cases} \tag{2.7}$$

Now suppose that a colluding agent meets a private seller. If the former has no money, she issues a note with probability x_c, following the rule assigned by the "social planner." The trading strategy of a colluding agent with money is more complicated because the decision to trade requires choosing the means of exchange. This choice depends on the note-issuing rule x_c. Let z be the distance between the buyer and the seller's good. If $z \leq x_c \leq X$, then the buyer spends her money with probability x_{mc} given by

$$x_{mc} = \begin{cases} x_c \text{ iff } u(q_{mp}) - (V_m^c - V^c) \geq u(q_{np}) \\ 0 \text{ otherwise.} \end{cases}$$

The last equation says that if the agent spends her money, she is not worse off than when she pays with a note, and from (2.2), this is equivalent to

$$x_{mc} = \begin{cases} x_c \text{ iff } u(q_{mp}) - kq_{mc} \geq u(q_{np}) \\ 0 \text{ otherwise.} \end{cases} \tag{2.8}$$

If $x_c < z \leq X$, then the probability of spending money is

$$x_{mc}^+ = \begin{cases} X - x_c \text{ iff } u(q_{mp}) - (V_m^c - V^c) \geq 0 \\ 0 \qquad \text{otherwise.} \end{cases}$$

By using (2.2) once again, we obtain

$$x_{mc}^+ = \begin{cases} X - x_c \text{ iff } u(q_{mp}) - kq_{mc} \geq 0 \\ 0 \qquad \text{otherwise.} \end{cases} \tag{2.9}$$

Notice that the case where the socially optimal (for colluding agents) probability of note-issuing $x_c > X$ can be treated similarly. However we are going to concentrate here on the case $x_c \leq X$, so we omit the alternative case from our consideration below.

Finally, when two colluding agents meet, the seller produces q_c, the agreed amount of good, for the buyer. The "social planner" would like to assign $q_c = q^*$, where q^* maximizes the trading surplus, $u(q) - kq$:

$$u'(q^*) = k; \tag{2.10}$$

However the socially optimal level of production may contradict incentives of agents.[5] Therefore $q_c \le q^*$, and is the largest among those for which the incentive compatibility constraints below are satisfied.

Private agents face no additional incentive compatibility constraints, since they make only individual decisions and do not collude. Colluding agents face three such constraints when they carry a unit of money, and three similar constraints when they have no money. These constraints ensure that an agent finds it optimal to obey the rules imposed by the "social planner". If these constraints are violated, then agents will defect, and the collusion will become impossible.

Denote by V_s (V_m) the value of a colluding agent without (with) a unit of money who is deprived of the privilege of issuing universally accepted notes. The first pair of incentive compatibility constraints expresses the fact that the payoff to a colluding agent, when she redeems a note, is at least as large as the gain from failing to do so:

$$V^c \ge kq_n + V_s. \tag{2.11}$$

$$V^c_m \ge kq_n + V_m. \tag{2.12}$$

Second, we need a pair of non-defection conditions for the inside exchange among colluding agents. These conditions state that there should be no gain from the failure to produce q_c whenever required: $V^c \ge kq_c + V_s$ and $V^c_m \ge kq_c + V_m$. Hence,

$$kq_c = \min\{kq^*, V^1 - V_s,\ V^c_m - V_m\}, \tag{2.13}$$

where q^* solves (2.10).

The last group of constraints consists of conditions of non-defection from the assigned probability of note-issuing x_c. If $x_c = X$, an individually optimal probability, there is no incentive to deviate. If $x_c < X$, and a colluding agent issues a note in violation of this condition, she derives additional instantaneous utility $u(q_{np})$ but her continuation value becomes V_s (if she has no money) or V_m (if she carries a unit of money). Hence, if $x_c < X$, then

$$V^c \ge u(q_{np}) + V_s, \quad V^c_m \ge u(q_{np}) + V_m. \tag{2.14}$$

Note that

$$0 \le V_s \le V^p_s, \tag{2.15}$$

because the marginal production cost for private agents is 1, and for colluding agents, it is $k \ge 1$; $V_s \ge 0$, since the autarky gives 0 for a defector. When a defector with a unit of money trades, she extracts the same surplus as a private

[5] Notice, that in the random matching framework, it is not optimal to produce for credit on an individual level.

agent with a unit of money. After this trade the defector has a continuation value V_m, and a private agent with money V_m^p. Therefore,

$$V_m \leq V_m^p. \tag{2.16}$$

We consider below only stationary equilibria, therefore the distribution of money among the types of agents and the flows of money and notes between the types must be independent of time. This requirement leads to the following steady state conditions. Denote by s_p and s_c the fractions of private and colluding agents respectively, and by N the proportion of private agents carrying notes. Let s_pM_p and s_cM_c denote the fractions of the aggregate money supply, M, which belong to private and colluding agents, respectively. Clearly, $s_p + s_c = 1$ and

$$s_pM_p + s_cM_c = M. \tag{2.17}$$

The remaining two steady state conditions require that the flow of notes being issued equal the flow of notes being destroyed, and that the flow of money from colluding to private agents is equal to the flow in the opposite direction. The flow of notes being issued is equal to

$$s_p(1 - N - M_p)s_c\{(1 - M_c)x_c + M_c(x_c - x_{mc})\};$$

the flow of notes being destroyed is equal to $s_pNs_cx_n$. The inflow of money into the colluding fraction of population is equal to $s_pM_px_ms_c(1 - M_c)$ the outflow of money from colluding agents is equal to

$$s_p(1 - M_p - N)s_cM_c(x_{mc} + x_{mc}^+).$$

Therefore the steady state conditions are

$$(1 - N - M_p)(x_c - M_cx_{mc}) = Nx_n; \tag{2.18}$$

$$M_px_m(1 - M_c) = (1 - M_p - N)M_c(x_{mc} + x_{mc}^+). \tag{2.19}$$

Let r be the rate of time preference and $\rho = r/\alpha$ be a normalized discount rate. Consider a private seller. Her value function satisfies the following Bellman's equation:

$$\rho V_s^p = s_pNx_{np}[V_n^p - V_s^p - q_{np}] + s_pM_px_{mp}[V_m^p - V_s^p - q_{mp}]$$

$$+s_cM_c(x_{mc} + x_{mc}^+)[V_m^p - V_s^p - q_{mp}] + s_c(x_c - M_cx_{mc})[V_n^p - V_s^p - q_{np}].$$

In words, the expected discounted flow of value of the seller equals the sum of all expected net gains from trade. Namely, with probability s_pNx_{np} the seller meets a buyer of her type with a note who likes the seller's good, or with probability $s_c(x_c - M_cx_{mc})$ the seller meets a colluding agent who either has no money or just prefers to pay with a note. As a result of both of such meetings, the seller becomes a note-holder. Similarly, with probability $s_pM_px_{mp}$ (respectively, $s_cM_c(x_{mc} + x_{mc}^+)$) the seller can trade her good for money to a private (respectively, colluding) buyer and become a money holder. By similar reasoning, we can obtain the rest of the

Bellman's equations. Using (2.1) and (2.2), we can replace value functions with q's and write simplified Bellman's equations:

$$V_s^p = 0; \quad V_n^p = q_{np}; \quad V_m^p = q_{mp}; \tag{2.20}$$

$$\rho q_{np} = s_p(1 - M_p - N)x_{np}[u(q_{np}) - q_{np}] + s_c x_n[u(q_n) - q_{np}]; \tag{2.21}$$

$$\rho q_{mp} = s_p(1 - M_p - N)x_{mp}[u(q_{mp}) - q_{mp}] + s_c(1 - M_c)x_m[u(q_{mc}) - q_{mp}]; \tag{2.22}$$

$$\rho V^c = s_p(1 - M_p - N)\min\{x_c, X\}u(q_{np}) - s_p N x_n k q_n + s_c X(u(q_c) - kq_c); \tag{2.23}$$

$$\rho k q_{mc} = s_p(1 - M_p - N)\{-x_{mc}u(q_{np}) + (x_{mc} + x_{mc}^+)[u(q_{mp}) - kq_{mc}]\}; \tag{2.24}$$

Using (2.20), we can derive from (2.15) and (2.16) that $V_s = 0$ and $V_m = V_m^p = q_{mp}$, and rewrite non-defection conditions (2.11)–(2.14) as follows:

$$V^c \geq kq_n; \tag{2.25}$$

$$V^c \geq kq_n + q_{mp} - kq_{mc}; \tag{2.26}$$

$$kq_c = \min\{kq^*, V^c, V^c + kq_{mc} - q_{mp}\}; \tag{2.27}$$

$$V^c \geq u(q_{np}), \quad V^c \geq u(q_{np}) + q_{mp} - kq_{mc}; \tag{2.28}$$

There is one more step left before we can define the equilibrium. As we have mentioned it earlier, the objective of the "social planner" of colluding agents was to maximize their welfare. The welfare consists of the welfare of agents with and without money:

$$W = M_c V_m^c + (1 - M_c)V^c = V^c + M_c(V_m^c - V^c).$$

Using (2.2), we obtain $W = V^c + M_c k q_{mc}$.

Let $\mathbf{g} = \{q_{np}, q_c, q_n, q_{mp}, q_{mc}, V^c, x_{np}, x_c, x_n, x_{mp}, x_m, x_{mc}, x_{mc}^+, M_p, M_c, N\}$. We call $\mathbf{g}$ a stationary equilibrium if

(i) the Bellman's equations (2.20)–(2.24) are satisfied;

(ii) probabilities $x_{np}, x_n, x_{mp}, x_m, x_{mc}, x_{mc}^+$ satisfy optimality conditions (2.5), (2.6), (2.8) and (2.9);

(iii) colluding agents choose x_c, q_c and q_n in order to maximize their welfare;

(iv) incentive compatibility constraints (2.25)–(2.28) are satisfied;

(v) steady state conditions (2.17)–(2.19) hold.

We are going to classify equilibria according to the types of exchange between agents:

- monetary exchange between private agents;
- monetary exchange between private and colluding agents;
- note exchange between private agents;
- note exchange between both types of agents.

The list of possible combinations of patterns of exchange and note-issuing rules is huge. We can make it much shorter if we rule out the case in which there are no trades between types and the economy consists of two separate economies each comprising agents of one type. Since our objective is to explain the use of inside money in the Russian economy, we also want to rule out the case without note exchange between the two types. Such a pattern of exchange is possible if either there remain no notes in circulation at all and the economy is a pure monetary one or some notes remain in circulation among private agents and play the role of additional fiat money for these agents.

In terms of parameters of the model, these restrictions imply: [6]

$$N + M_P < 1, \quad N > 0; \tag{2.29}$$

$$x_c + x_{mc} + x_{mc}^{+} > 0; \tag{2.30}$$

$$x_n + x_m > 0. \tag{2.31}$$

Since we assume that there is note exchange between the two types, we must have $x_c > 0$ or $x_n > 0$. The steady state condition (2.18) implies that

$$x_c > 0 \iff x_n > 0, \tag{2.32}$$

therefore we may impose either of the conditions in (2.32), and then (2.30)–(2.31) are satisfied. Finally, we assume that notes and money are not given away as gifts so that

$$q_{np} + q_n > 0, \quad q_{mp} + q_{mc} > 0. \tag{2.33}$$

Now, we divide equilibria satisfying additional conditions (2.29)–(2.33) into two groups:

– *type-1* equilibria: money is used by private agents only, notes are exchanged both between private agents and between private and colluding agents;
– *type-2* equilibria: notes and money are used in trades between private agents and between private and colluding agents.

Thus, 1 and 2 indicate the number of different types of exchange between private and colluding agents in equilibrium. In Sections 2 and 3, we study type-1 and type-2 equilibria, respectively.

[6] If (2.29) fails, all private agents have either notes or money and do not trade at all, or there are no notes in circulation; if (2.30) fails, all colluding agents pay with neither notes (if $x_c = 0$) nor money ($x_{mc} = x_{mc}^{+} = 0$), and hence, there is no exchange between the types; if (2.31) fails, all private agents pay to colluding agents with neither notes ($x_n = 0$) nor money ($x_m = 0$), and hence, there is no exchange between the types.

3 Type-1 equilibria: no monetary trades between the types

3.1 Equilibria specification

Suppose an equilibrium without monetary trades between the two types exists. Then steady state conditions (2.17) and (2.19) become redundant. Further, in (2.18), we must set $M_c x_{mc} = 0$, and solve (2.18) for N:

$$N = \frac{(1 - M_p)x_c}{x_c + x_n}.$$

By introducing $\gamma = x_c/X$, we can write

$$N = \frac{(1 - M_p)\gamma}{1 + \gamma}. \tag{3.34}$$

Since we assume that there is trade between types, with notes as the means of exchange, it must be that $x_c > 0$, and (2.32) gives $x_n > 0$, hence $x_n = X$, and from (2.6), we obtain

$$u(q_n) \geq q_{np}. \tag{3.35}$$

Using (2.33), (3.35), (2.6) and (2.21), we obtain

$$q_n > 0, \;\; q_{np} > 0. \tag{3.36}$$

Even though several equilibrium price patterns may result in the absence of monetary trades between private and colluding agents, it is possible to show that only the case examined below is feasible in this model. Namely, equilibrium prices are such that it is not optimal for private agents to spend money when trading with colluding agents but colluding agents are willing to spend money in some cases. In a steady state, this pattern of exchange leads to all the money accumulated by the private agents: $M_c = 0$, $M_p s_p = M$. Since we rule out autarky for the private agents, this case can be taken into consideration only if $M < s_p$ and hence, $M_p = M/s_p < 1$. In the rest of this Section, we fix $M_p \in (0, 1)$. Further, by (2.7)–(2.9) we must have that

$$x_m = 0, \quad u(q_{mc}) < q_{mp}, \tag{3.37}$$

and

$$x_{mc} + x^+_{mc} > 0, \quad u(q_{mp}) \geq q_{mc}. \tag{3.38}$$

Notice that we must assume that

$$q_{mc} < q_{mp}; \tag{3.39}$$

if $q_{mc} \geq q_{mp}$, there is no reason why private money holders should not accept q_{mc} units of good from colluding agents if they accept $q_{mp} \leq q_{mc}$ from private sellers.

Let $h \equiv \rho/X$, we will call this parameter trading friction. Set $B(\gamma) \equiv s_p(1 - M_p)/(1+\gamma)$, $\kappa_m \equiv (x_{mc} + x^+_{mc})/X$, $\gamma_{m1} \equiv x_{mc}/X$. Using (2.6) and (3.34), we can rewrite Bellman's equations (2.21)–(2.23) as

$$hq_{np} = B(\gamma)\frac{x_{np}}{X}(u(q_{np}) - q_{np}) + s_c(u(q_n) - q_{np}); \tag{3.40}$$

$$hq_{mp} = B(\gamma)\frac{x_{mp}}{X}(u(q_{mp}) - q_{mp}); \tag{3.41}$$

$$hV^c = B(\gamma)[\min\{1, \gamma\}u(q_{np}) - \gamma k q_n] + s_c(u(q_c) - kq_c); \tag{3.42}$$

$$hkq_{mc} = B(\gamma)\{\kappa_m(u(q_{mp}) - kq_{mc}) - \gamma_{m1}u(q_{np})\}. \tag{3.43}$$

Since $M_c = 0$, i.e. there are no money holders among colluding agents, the "social planner" maximizes V^c. It is possible to show that incentive compatibility constraints (2.25)–(2.28) become redundant in this setting. However, if the colluding agents ever issue notes, the following participation constraint must be satisfied:

$$\min\{1, \gamma\}u(q_{np}) - \gamma k q_n \geq 0. \tag{3.44}$$

Thus in this Section, we are looking for a solution to (3.40)–(3.43), satisfying (3.34)–(3.39) and (3.44). In Boyarchenko and Levendorskiĭ [4], we prove that there are two equilibrium scenarios: either

$$\gamma = 1,\ x_{mc} = X,\ q_{mc} = 0,\ u(q_n) = q_{np} = q_{mp}, \tag{3.45}$$

where q_{mp} is a (unique) positive solution to an equation

$$\left(1 + \frac{2h}{s_p(1 - M_p)}\right) q = u(q), \tag{3.46}$$

or

$$\gamma \in (0, 1),\ x_{mc} = 0,\ \kappa_m = 1 - \gamma. \tag{3.47}$$

In the first case, colluding agents issue notes with the individually optimal frequency, and they would have preferred spending a unit of money to paying with a note. Money has no value for these agents, which is indicated by $q_{mc} = 0$. In the second case, the "social planner" restricts note issuing, and colluding agents would have paid with money only when note printing was not allowed. In what follows, we are going to study the case of small trading frictions, because it admits analytical solutions. In particular, in Subsection 3.3, we show that if the h is small, then alternative (3.47) is non-optimal, and hence, equilibrium is either determined by (3.45)–(3.46), or it does not exist at all.

3.2 Low productivity colluding agents

If the number of single coincidence meetings per unit of time is large the trading friction, h, is small. This means that agents can be patient and wait for a better trading opportunity to arrive. A private buyer with a note will not be willing to trade with colluding sellers, unless colluding agents can redeem a note for a sufficiently high amount q_n. Because they suffer higher production cost than private agents, this q_n may be too high a production level for colluding agents. Therefore it may be optimal for colluding agents not to trade at all. In this case, it remains for private agents to trade only among themselves. The economy splits into two disjoint sectors, a case which we do not consider in this paper.

Theorem 3.1 *For $k > 1$ fixed, there exists $h_0 > 0$ such that if $h \in (0, h_0)$ then type-1 equilibria do not exist.*

3.3 High productivity colluding agents

If $k = 1$, i.e. colluding agents are as productive as private agents, a problem described in the previous Subsection does not arise and an equilibrium with interacting types exists for arbitrary small $h > 0$.

Theorem 3.2 *Let $k = 1$. Then there exists $h_0 > 0$ such that for $h \in (0, h_0)$*

a) a type-1 equilibrium exists, and it is unique;
b) the equilibrium amount of good produced by private sellers $q_{np} = q_{mp}$, can be found as a unique positive solution to

$$hq = \frac{s_p(1 - M_P)}{2}[u(q) - q]; \tag{3.48}$$

c) colluding agents issue notes with probability $x_c = X$, the equilibrium amount of notes in circulation equals $s_p(1 - M_P)/2$, and q_n, the optimal amount of good redeemed for a note, is determined from

$$u(q_n) = q_{np}; \tag{3.49}$$

Let q_* denote the unique positive solution to

$$u(q) = q; \tag{3.50}$$

then it is possible to show that

$$q^* < q_n < q_{np} = q_{mp} < q_*; \tag{3.51}$$

Moreover as $h \to +0$, q_n and $q_{np} = q_{mp}$ converge to q_*. Thus we see that

- colluding agents exercise their monopoly power and redeem for a note less than what private agents produce for the same note or for a unit of money;
- as the trading friction vanishes ($h \to 0$), so does the monopoly power as measured by $1 - q_{np}/q_n$.

The next theorem shows that equilibrium exits even when the trading friction is small and colluding agents are less productive than private agents. For this to be true, $k - 1$ must be small. The theorem also gives necessary and sufficient conditions for an admissible level of $k - 1$.

Theorem 3.3 *Let q_{np} be a unique positive solution to (3.48). Then*
a) if

$$k < u(q_{np})/u^{-1}(q_{np}), \tag{3.52}$$

then there exists $h_0 > 0$ such that for $h \in (0, h_0]$, a type-1 equilibrium exists, and it is unique. The equilibrium amount of goods produced in a round of trade and that of notes in circulation are determined as in Theorem 3.2.
b) if

$$k > u(q_{np})/u^{-1}(q_{np}),$$

then there exists $h_0 > 0$ such that for $h \in (0, h_0]$, a type-1 equilibrium does not exist.

If $k > 1$ but satisfies (3.52), then claims made after Theorem 3.2 remain valid. Moreover, the equilibrium quantities q_{np} and q_n, and the welfare of private agents $V = s_p(1+M_p)q_{np}/2$ are independent of the difference $k-1$. However, the cost kq_n, which a colluding agent suffers while redeeming a note, increases, and the value function of colluding agents, V^c, decreases with k growing. The loss of welfare is proportional to $k-1$:

$$V^c = h^{-1}s_c(u(q^*) - q^*)+$$

$$\frac{q_*(1+u'(q_*))}{u'(q_*)} - k_1\left(s_c q^* + q_*\frac{s_p(1-M_p)}{2}\right) + O(h), \tag{3.53}$$

where $k_1 = (k-1)/h$.[7]

To summarize, when the trading friction and the difference in the marginal productivity of agents are small, only colluding agents have to bear the cost of their low productivity, their monopoly power notwithstanding. At the same time, prices observable by private agents are independent of $k-1$. Even though Theorem 3.3 provides an insight into the existence of type-1 equilibria, we are going to formulate one more existence theorem below. This will allow us to compare type-1 and type-2 equilibria in the next Section. By Theorem 3.3, $q_{np} \to q_*$ as $h \to +0$, therefore from (3.52), we deduce the following result.

Theorem 3.4 *a) Let*

$$k_1(1-M_p) < \frac{2(1+u'(q_*))}{u'(q_*)s_p}. \tag{3.54}$$

Then there exists $h_0 > 0$ such that if $h \in (0, h_0]$ and $k \le 1 + k_1 h$, then a type-1 equilibrium exists.
b) Let

$$k_1(1-M_p) > \frac{2(1+u'(q_*))}{u'(q_*)s_p}. \tag{3.55}$$

Then there exists $h_0 > 0$ such that if $h \in (0, h_0]$ and $k \ge 1 + k_1 h$, then a type-1 equilibrium does not exist.

It is important to notice that when the product $k_1(1-M_p)$ crosses a certain threshold, namely, the RHS in (3.54) and (3.55), a type-1 equilibrium vanishes.

3.4 Dependence on the supply of money

Suppose that the government gives an additional amount of money to some private sellers. Then some of private sellers become buyers, fractions s_p and s_c of private and colluding agents do not change, but the fraction M of agents carrying money and the fraction $M_p = M/s_p$ of private agents carrying money increase. After some transition period, the economy arrives to the new steady state. If the increase in the money supply is not very large, the type of equilibrium remains the same, and in the new steady state

---

[7] $O(h)$ is the standard notation for any function $f(h)$, which decays as fast as h, as $h \to 0$: $|f(h)| \le Ch$, where C is independent of h.

1) the amount of notes $s_p(1-M_p)/2$ in circulation decreases but the total amount of liquidity $s_pM_p + s_p(1-M_p)/2 = s_p(1+M_p)/2$ increases: 2 units of money are needed to replace 1 note;
2) the product $k_1(1-M_p)$ decreases, hence the economy moves away from the threshold where it can lose stability and split into two disjoint economies;
3) the welfare of private agents

$$V(M) = \frac{s_p + M}{2} q_* + O(h)$$

increases, provided M is not large.

Moreover, we can state the following important welfare result.

Theorem 3.5 *If the trading friction is small, and colluding agents do not differ in productivity from private agents, then their welfare decreases with M growing.*

This result implies that when colluding agents have the same marginal productivity as private agents, the former would prefer the supply of outside money being reduced to the level, which is necessary to sustain the equilibrium. The above remains true if $k-1$ is positive but small. In general, for a given $k>1$, there exists a minimal positive level $M_* = M_*(k)$ such that if $M < M_*(k)$, the economy splits into two separate economies (M_* can be found from (3.54)), and an optimal level $M_{**}(k) \geq M_*(k)$, which maximizes the welfare of colluding agents and can be obtained from (3.53). It is possible to show that the supply of money maximizing the welfare of colluding agents is smaller than the supply of money optimal for private agents.

4 Type-2 equilibria: all types of trade

4.1 Main results

Due to more complex pattern of exchange, it is more difficult to obtain analytical results, and constructions become long. So, we start with the description of the main results.[8] In the preceding Section, we have reduced the initial problem to a relatively simple equation (which, for example, in the case of the Cobb-Douglas utility function, admits an explicit solution so that it is possible to obtain analytic expressions for all endogenous variables); here we manage to reduce the initial problem to a constrained maximization of a rather complicated function on an interval $(0,1)$. From the computational point of view, to solve such a problem and calculate equilibrium values of endogenous variables is easy. The corresponding procedures are robust and do not require much computation time. A closed form solution is however impossible even for the simplest utility functions.[9]

We study the case of moderately picky agents, small trading frictions and moderate money supply. The main results can be summarized as follows:

---

[8] For a full version, including numerical examples and their discussion, see Boyarchenko and Levendorskiĭ [4].

[9] In numerical examples, we consider the Cobb-Douglas utility $u(q) = dq^{\beta}$, where $d>0$ and $\beta \in (0,1)$.

1. Colluding agents issue notes below the individually optimal level.[10]
2. A colluding money holder pays with a note whenever she is allowed and as a result, colluding agents accumulate more money per capita than private ones.[11]
3. If the trading friction vanishes faster than the difference $k-1$, the economy loses stability and splits.
4. Type-2 equilibria are more fragile than type-1 equilibria in the sense that the threshold (mentioned after Theorem 3.4) for the former is lower than that for the latter.

4.2 Specification of type-2 equilibria

In this Subsection, we sketch the reduction of the initial problem to a constrained optimization problem. This reduction involves tedious algebra: for the details see Boyarchenko and Levendorskiĭ [4].

To simplify the study of type-2 equilibria, we make two assumptions:

$$h/X \le \epsilon, \tag{4.56}$$

where $\epsilon > 0$ is sufficiently small, i.e. agents are not very picky, and

$$M < s_c, \tag{4.57}$$

which means that the fraction of colluding agents with money is bounded away from one. Now (4.57) and (2.17) imply that

$$M_c \le M/s_c < 1. \tag{4.58}$$

These two conditions exclude "bad" equilibria with $x_c \sim 1$, when too many notes are being issued and equilibrium quantities are small (one can construct such equilibria if $h/X >> 1$).

It is possible to show that private agents always use money and notes in trades with each other. This implies in particular, that $x_n = x_m = X$ and

$$u(q_{mc}) \ge q_{mp}. \tag{4.59}$$

Also we can prove that colluding agents issue notes whenever they are allowed to do this. These results make it possible to simplify the steady state conditions (2.18)–(2.19). Fix $\gamma \in (0,1)$ then using $x_c/X = \gamma$ and $(x_{mc} + x^+_{mc})/X = 1-\gamma$ we obtain:

$$(1 - N - M_c)\gamma = N,$$
$$M_p(1 - M_c) = (1 - M_p - N)M_c(1 - \gamma).$$

[10] If agents are too specialized in consumption, there may exist "bad" equilibria when too many notes circulate and the production level is too low.

[11] It is shown in Boyarchenko [3], that in a pure monetary equilibrium, agents with higher productivity accumulate more money per capita. The effect of concentration of all the money in a group of high productivity agents was also demonstrated in Wallace and Zhou [15].

Using (2.17) and the last two equations, we find

$$M_c = (M - s_p M_p)/s_c, \tag{4.60}$$

$$N = \frac{(1 - M_p)\gamma}{1+\gamma}, \tag{4.61}$$

where $M_p = M_p(\gamma)$ is a positive solution to the equation

$$2\gamma s_p M_p^2 + M_p(1 + \gamma(s_c - s_p - 2M)) + M(\gamma - 1) = 0. \tag{4.62}$$

It is easy to verify that for any $\gamma \in (0,1)$, there exists a unique positive root of equation (4.62), such that $M_p \in (0, M)$. Moreover $M_p = M_p(\gamma)$ is continuous on $[0,1)$. Now from (2.17) and (4.60) it follows that $M_c > M$, i.e. colluding agents accumulate more money per capita than private ones. For $\gamma \in [0,1)$, set $B(\gamma) = s_p(1 - M_p(\gamma))/(1+\gamma)$, and notice that since $M_p \le M < 1$, $B(\gamma)$ is bounded away from 0 uniformly in h. Keeping in mind that M_p and M_c are uniquely defined by a choice of γ, and using (4.61), we rewrite the Bellman's equations (2.21)–(2.23) as follows

$$hq_{np} = B(\gamma)(u(q_{np}) - q_{np}) + s_c(u(q_n) - q_{np}); \tag{4.63}$$

$$hq_{mp} = B(\gamma)(u(q_{mp}) - q_{mp}) + s_c(1 - M_c(\gamma))(u(q_{mc}) - q_{mp}); \tag{4.64}$$

$$hV^c = B(\gamma)\gamma(u(q_{np}) - kq_n) + s_c(u(q_c) - kq_c); \tag{4.65}$$

$$hkq_{mc} = B(\gamma)(1-\gamma)(u(q_{mp}) - kq_{mc}). \tag{4.66}$$

Next, it is possible to show that there exists $h_0 > 0$ such that for any $h \in (0, h_0]$, in equilibrium, colluding agents produce for each other the "socially optimal" amount of good $q_c = q^*$; incentive compatibility constraints (2.25)–(2.28) are satisfied; and

$$u(q_n) = q_{np}. \tag{4.67}$$

Now, for a given $\gamma \in [0,1)$, we can define step by step:

- $M_p = M_p(\gamma)$ from (4.62);
- $M_c = M_c(\gamma)$ from (4.60);
- $N = N(\gamma)$ from (4.61);
- $q_{np} = q_{np}(\gamma)$ from

$$hq_{np} = B(\gamma)(u(q_{np}) - q_{np}) \tag{4.68}$$

- $q_n(\gamma) = u^{-1}(q_{np}(\gamma))$;
- $q_{mc} = q_{mc}(\gamma, q_{mp})$ as a function of q_{mp} and γ:

$$q_{mc} = k^{-1}\left(1 + \frac{h}{B(\gamma)(1-\gamma)}\right)^{-1} u(q_{mp}). \tag{4.69}$$

By substituting (4.69) into (4.64), we obtain an equation

$$hq_{mp} = B(\gamma)(u(q_{mp}) - q_{mp}) +$$

$$s_c(1 - M_C(\gamma))(u(q_{mc}(\gamma, q_{mp})) - q_{mp}). \tag{4.70}$$

For γ fixed, (4.70) can be written in the form

$$Aq = v(q),$$

where A is a positive constant, and v satisfies the same conditions as u, namely, v is increasing, concave and satisfies the Inada conditions (the verification of all these properties is straightforward). Hence, (4.70) has a (unique) positive solution $q_{mp} = q_{mp}(\gamma)$, and after that, (4.69) defines q_{mc} as a function of γ.

Thus we obtain all endogenous variables as functions of γ. Recall that colluding agents choose γ to maximize their welfare $W = V^c + M_c k q_{mc}$. We see that the colluding agents maximize a function

$$\Phi(\gamma) = h^{-1}B(\gamma)\gamma(u(q_{np}) - kq_n) + M_c(\gamma)kq_{mc}(\gamma)$$

on $(0, 1)$. Notice that Φ is the welfare of colluding agents net of the gain from trade with each other. We are able to prove that optimality conditions and incentive compatibility constraints are satisfied by construction, and the only remaining constraint is (4.59).

In Boyarchenko and Levendorskiǐ [4], we provide necessary and sufficient conditions for existence of type-2 equilibria. In particular, we show that there exist $C > 0$ and $h_0 > 0$ such that if $h \in (0, h_0]$, $X \geq Ch$, and $k \geq 1 + Ch$, then type-2 equilibria do not exist and hence no non-degenerate equilibria exist.[12] Thus, if the trading friction vanishes faster than the difference $k - 1$, then the economy loses stability and splits into two disjoint parts.

4.3 Some asymptotic analysis

Here we derive asymptotic formulas for equilibrium quantities as $h \to 0$, which allow us to find approximate conditions of existence of equilibria, approximate formulas for value functions, and compare type-2 and type-1 equilibria. By the result stated at the end of the previous Subsection, an equilibrium does not exist if the difference $k - 1$ is sufficiently large. Therefore to obtain existence result, we may assume that $k = 1 + k_1 h$, where $k_1 = O(1)$ as $h \to 0$. For simplicity, we assume that k_1 is a non-negative constant, and in the end, formulate existence conditions in terms of k_1.

First, we obtain the asymptotics for the objective function of colluding agents. As $h \to +0$,

$$\Phi(\gamma) = \Phi_0(\gamma) + O(h), \tag{4.71}$$

where

$$\Phi_0(\gamma) = q_* \left(\gamma \frac{u'(q_*) + 1}{u'(q_*)} - k_1 \gamma B(\gamma) + M_c(\gamma) \right). \tag{4.72}$$

By using the above approximation, we can simplify the study of type-2 equilibria. Introduce a function

$$F(\gamma) = \frac{k_1 s_p (1 - M_p(\gamma))}{1 + \gamma} + \frac{1}{1 - \gamma}.$$

[12] In Section 3, we have shown that under these conditions, type-1 equilibria do not exist as well.

We can state the following analogue of Theorem 3.4, which holds provided $F(\gamma)$ is increasing on $[0, 1)$.

Theorem 4.1 *a) Let*

$$k_1(1 - M) < \frac{1}{u'(q_*)s_p}, \tag{4.73}$$

then there exists $h_0 > 0$ *such that f* $h \in (0, h_0]$ *and* $k \leq 1 + k_1 h$, *then a type-2 equilibrium exist.*

b) Let

$$k_1(1 - M) > \frac{1}{u'(q_*)s_p}. \tag{4.74}$$

Then there exists $h_0 > 0$ *such that if* $h \in (0, h_0]$ *and* $k \geq 1 + k_1 h$, *then a type-2 equilibrium does not exist.*

Notice that the expression on the LHS in (4.73) and (4.74) is similar to the one in (3.54)–(3.55). We see that the RHS in (4.73) and (4.74) is smaller than the one in (3.54)–(3.55), i.e. $2(1 + u'(q_*))/(u'(q_*)s_p)$. By Theorem 3.4 and Theorem 4.1, both type-1 and type-2 disappear when the product $k_1(1 - M)$ crosses a certain critical value. This critical value is lower for type-2 equilibria, this means that type-2 equilibria are more fragile than type-1 equilibria.

We can use approximation (4.71)–(4.72) to compare the welfare of colluding agents in both types of equilibria.

Theorem 4.2 *There exists* $h_0 > 0$ *such that if* $h \in (0, h_0]$ *and a type-2 equilibrium exists, then a type-1 equilibrium exists as well, and the welfare of colluding agents in the latter equilibrium is larger than in the former.*

The reason a type-2 equilibrium is inferior is constraint (4.59), which must be satisfied if type-1 is ever to get money from private agents. To satisfy this constraint, colluding agents have to produce sufficiently large amount of good for a unit of money. It is possible to show that to achieve the same level of welfare as in a type-1 equilibrium, colluding agents must issue notes in large quantities. If this is the case, then the relative value of holding money increases and private agents start to require even more of a good in exchange for a unit of money.

5 Conclusion

We have constructed a benchmark model reflecting basic features of the Russian virtual economy, which generates equilibria with wide-spread use of money substitutes. There are two types of agents in the economy: a type of low productivity agents who can collude and issue universally accepted notes (inside money), and a type of high productivity agents who have no means to collude and act as individually optimizing agents. There is genuine (outside) money in the economy as well. Within each type, agents specialize in consumption. With this structure, we have shown that the existence and essential properties of equilibria depend on the trading frictions, the difference in marginal productivity of agents, and the supply of money in the economy. In general, the equilibria can be classified according to two criteria: types of exchange between agents and the probability of note issuing assigned by the "social planner" of colluding agents. The list of possible combinations of these two types of characterization is huge. We have classified possible equilibria when the trading friction is small, and agents do not differ significantly in productivity.

If colluding and private agents trade with each other there may exist one or both of the following equilibria:

1) an equilibrium without money flows between the types because colluding agents find it optimal to produce only zero quantities of good for money (type-1 equilibrium), and
2) an equilibrium where both money and notes are used in trades between the types (type-2 equilibrium).

In the former equilibrium, colluding agents issue money substitutes with the individually optimal probability. In the latter equilibrium, the "social planner" restricts note issuing below the individually optimal level, making notes more scarce and increasing the transaction value of a note. As a result, colluding agents pay with money only when they are not allowed to pay with a note for a good they like.

We have shown that if the trading friction is sufficiently small and there is no difference in productivity of agents, then type-2 equilibria are inferior from the point of view of colluding agents agents. Therefore colluding agents would not use money unless forced by some exogenous (from the point of view of the model) factor, like the necessity to pay taxes with money (to treat this situation consistently, our model should be modified). This agrees with previous findings by N. Wallace and others about superiority of inside money to outside money. If the trading friction is not very small, and colluding agents are fairly productive, then type-2 equilibria are superior for them, but the supply of money maximizing welfare of colluding agents is smaller than that for private agents. If the existing money supply is at or above the optimal level (from the point of view of colluding agents), there is a strong incentive for these agents to get rid of new money, should the new money arrive.[13]

[13] This result was obtained when one of the popular explanations for the widespread use of money substitutes in the Russian economy was the insufficient money supply. After the money supply in the economy had increased significantly due to high prices of oil , Mr. Greff - the minister of economic development - still complained (in June 2000) that the economy was unable to make a proper use of the money. We believe that this was a good verification of the model.

References

1. Aukutsionek, S.: Industrial barter in Russia. Communist Economies Economic Transformation **10**, 179-88 (1998)
2. Blanchard, O., Kremer, M.: Disorganization. Quarterly Journal of Economics **112**, 1091-1126 (1997)
3. Boyarchenko, S.I. : A monetary search model with heterogeneous agents. Working paper, available at http://www.eco.utexas.edu/ sboyarch/hetm5.pdf (2000)
4. Boyarchenko, S.I., Levendorskiĭ, S.Z. : Search-money-and-barter models of financial stabilization. Working Paper # 332, The Davidson Institute Working Paper Series (2000)
5. Burdett, K., Trejos, A., Wright, R.: Cigarette money. Journal of Economic Theory **99**, 117-142 (2001)
6. Cavalcanti, R., Erosa, A., Temzelides, T.: Private money and reserve management in a random matching model. Journal of Political Economy **107**, 929–945 (1999)
7. Cavalcanti, R., Wallace, N.: A model of private bank-note issue. Review of Economic Dynamics **2**, 104–136 (1999)
8. Cavalcanti, R., Wallace, N.: Inside and outside money as alternative media of exchange. Journal of Money, Credit, and Banking **31**, 443-457 (1999)
9. Ericson, R.E.: The revenge of the 'virtual economy'. The Harriman Review, Special Issue, December, 3–6 (1998)
10. Ericson, R.E., Ickes, B.W.: A model of Russia's 'virtual economy'. Review of Economic Design **6**, 185-214 (2001)
11. Gaddy, C., Ickes, B.W.: To restructure or not to restructure: informal activities and enterprise behavior in transition. Working Paper #134, The Davidson Institute Working Paper Series (1998)
12. Gaddy, C., Ickes, B.W.: Russia's virtual economy. Foreign Affaires, September-October , 53–67 (1998)
13. Gaddy, C., Ickes, B.W.: Russia's virtual economy. Washington DC: Brookings Institution Press 2002
14. Kiyotaki, N., Wright, R.: A search-theoretic approach to monetary economics. American Economic Review **83**, 63–77 (1993)
15. Wallace, N., Zhou, R.: A model of currency shortage. Journal of Monetary Economics **40**, 555–572 (1997)
16. Williamson, S.: Private money. Journal of Money, Credit, and Banking **31**, 469–491 (1999)
17. Woodruff, D.: Money unmade: barter and the fate of Russian capitalism. Ithaca London: Cornell University Press 1999