Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks

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Abstract

This paper studies competitive equilibria of a production economy with aggregate productivity shocks. There is a continuum of consumers who face borrowing constraints and individual labor endowment shocks. The dynamic economy is described in terms of sequences of aggregate distributions. The existence of competitive equilibrium is proven and a recursive characterization is established. In particular, it is shown that for any competitive equilibrium, there is a payoff equivalent competitive equilibrium that is generated by a suitably defined recursive equilibrium.

Key words: competitive equilibrium, recursive equilibrium, aggregate distribution, heterogeneity, incomplete markets, aggregate shocks

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1. INTRODUCTION

It has been documented by a number of empirical studies that the standard representative agent (or complete markets) model fails to explain many phenomena observed in the data. This leads to interest in models with heterogeneity and incomplete markets.[1] One class of such models, called the Bewley-style model, has drawn special attention. The typical environment of this model features a continuum of consumers making consumption and savings decisions subject to borrowing constraints and idiosyncratic labor endowment shocks. There is only one asset (capital) serving as a buffer against individual shocks. Finally, a single firm makes production decisions subject to aggregate productivity shocks.[2]

Two central open questions are addressed. The first is the existence of a sequential competitive equilibrium. The second question is whether there is a recursive characterization of sequential competitive equilibria. Krusell and Smith [29] and a number of later studies directly pose a recursive equilibrium formulation (henceforth, *KS-recursive equilibrium*) and then proceed with numerical solutions without studying its existence and the relation to sequential competitive equilibrium.

As in Miao [32], this paper reformulates the Bewley-style model along the lines of Hildenbrand [19] and Hart et al [17]. In particular, the dynamic economy is described by sequences of aggregate distributions over consumers' characteristics (individual asset holdings and the realization of endowment shocks) across the population.[3] These sequences of aggregate distributions contain the relevant information for equilibrium analysis and they are the principal objects of study. In particular, given exogenous shocks, aggregate distributions fully determine prices and aggregate quantities such as aggregate capital. It turns out that this reformulation is the key to answering the above questions.

The study of existence of competitive equilibrium begins with a detailed analysis of a typical individual's decision problem. After aggregating individual optimal behavior and deriving the law of motion for aggregate distributions, the existence of a competitive equilibrium is proven by applying the Brouwer-Schauder-Tychonoff Fixed-Point Theorem to a compact space of sequences of aggregate distributions (Theorem 1). This result is established under standard assumptions on preferences and technology and for fairly general individual and aggregate shock processes. For example, these are assumed to satisfy the Feller property, but they need not be stationary or Markovian. However, for technical reasons, I assume that the state space for aggregate shocks is

[1] See the survey by Heaton and Lucas [18].

[2] See [9, 11, 2, 21, 3, 22, 32] for Bewley-style models without aggregate shocks.

[3] Similar formulations are adopted in models of anonymous games [31, 23, 7, 25, 12].

countable.[4]

After imposing the additional assumption that individual and aggregate shocks are time-homogenous Markov processes, I turn to recursive characterizations of competitive equilibria. I define a notion of recursive equilibrium with the state variables consisting of individual asset holdings, the realization of individual shocks, the realization of aggregate shocks, the aggregate distribution, and payoffs (expected discounted utilities). Including the first three as state variables is standard. It is also natural to include the aggregate distribution as a state variable because with incomplete markets and heterogeneous consumers, equilibrium prices generally depend on the distribution of assets across consumers.

Including payoffs as a state variable to make certain decision problems recursive is a technique widely adopted in the literature on sequential games [13, 6, 8] and on dynamic contracts [34, 37, 1]. Here this state variable serves as a device for selecting 'continuation' equilibria when the economy unfolds over time.

Theorem 2 demonstrates that given an initial state, the so defined recursive equilibrium generates a sequential competitive equilibrium. Theorem 3 demonstrates that a recursive equilibrium exists. Moreover, for any sequential competitive equilibrium, there is a payoff equivalent competitive equilibrium that is generated by a recursive equilibrium with the state space including payoffs.

A natural but open question is whether there is a recursive equilibrium with a smaller state space, for example, the KS-recursive equilibrium that excludes expected payoffs as a state variable. In a corresponding model with finitely many agents, Kubler and Schmedders [30] give counter-examples to existence, thus demonstrating that the wealth distribution or the portfolio of asset holdings does not constitute a sufficient endogenous state. The intuition is that equilibrium decisions at any date must be consistent with expectations at the previous date, and that these expectations cannot always be summarized in the wealth distribution. Similar intuition seems relevant for the economy with the continuum of agents studied here. In particular, the future sequences of aggregate distributions must be consistent with expectations in the previous period. However, these expectations may not be summarized in the aggregate distribution if there are multiple competitive equilibria. Under the strong condition that the competitive equilibrium is globally unique for all possible initial values of aggregate distributions and aggregate shocks, Theorem 4 establishes that a KS-recursive equilibrium exists.

The above analysis must surmount two difficulties. First, there is a difficulty associated with

[4]See Bergin and Bernhardt [8] for an analysis of anonymous games with uncountable state space for aggregate shocks.

the presence of aggregate shocks. When they are present, aggregate distributions are generally random measures that may be correlated with individual shocks. As pointed out by Bergin and Bernhardt [7] and illustrated by the example in section 3.2, this creates not only difficulties of tractability but also conceptual problems associated with the meaning of perfect competition. Thus, I follow Bergin and Bernhardt [7] and assume the *conditional no aggregate uncertainty condition.* This requires that, conditional on the history of aggregate shocks, the aggregate distribution at each date be a constant measure. Second, there are subtle technical problems, pointed out by Judd [24], associated with an environment that has a continuum of agents, e.g., measurability and the law of large numbers. This paper deals with these problems in a manner similar to Miao [32].

I now review briefly the related literature. There is a growing literature on numerical analysis of Bewley-style models with aggregate shocks [28, 29, 16, 36]. None of these considers the theoretical issues studied here. As mentioned earlier, this paper is related to the early general equilibrium literature on large economies and also to the literature on anonymous games studied by Schmeidler [33], Mas-Colell [31], Jovanovic and Rosenthal [23], Bergin and Bernhardt [7, 8], and Karatzas et al [25]. The latter relation will be discussed in detail in the concluding section. The paper is also related to Duffie et al [13], Becker and Zilcha [5], Chakrabarti [10], and Kubler and Schmedders [30]. All these papers consider a finite number of heterogeneous consumers.

The remainder of the paper proceeds as follows. Section 2 sets up the model. Section 3 analyzes the existence of a competitive equilibrium. Section 4 studies recursive characterizations of competitive equilibria. Section 5 concludes and discusses an extension of the model. Proofs are relegated to an appendix.

2. THE MODEL

Consider an economy with a large number of infinitely-lived consumers subject to individual endowment shocks and a single firm subject to aggregate productivity shocks. This economy is similar to that studied by Krusell and Smith [29]. Time is discrete and denoted by $t = 0, 1, 2, ...$ Uncertainty is represented by a probability space $(\Omega \times \mathbb{Z}^\infty, \mathcal{F}, P)$ on which all stochastic processes are defined. The state space Ω captures individual shocks, while the state space $\mathbb{Z}^\infty$ captures aggregate shocks. Let $\mathbb{Z}^0 = \mathbb{Z}$, $\mathbb{Z}^{t+1} = \mathbb{Z}^0 \times \mathbb{Z}^t$, and denote by $z^t = (z_0, z_1, ..., z_t) \in \mathbb{Z}^t$ an aggregate shock history at time t. Finally, let $z^\infty = (z_0, z_1, z_2, ...) \in \mathbb{Z}^\infty$ be the complete history and $z^0 = z_0 \in \mathbb{Z}^0$ be a deterministic constant.

Notation. For any Euclidean subspace $\mathbb{D}$, denote by $\mathbb{C}(\mathbb{D})$ the space of bounded and continuous functions on $\mathbb{D}$ endowed with the sup-norm, by $\mathcal{B}(\mathbb{D})$ the Borel σ-algebra of $\mathbb{D}$, and by $\mathcal{P}(\mathbb{D})$

the space of probability measures on $\mathcal{B}(\mathbb{D})$ endowed with the weak convergence topology. For any Euclidean sets $\mathbb{D}$ and $\mathbb{E}$, $\mathcal{B}(\mathbb{D}) \otimes \mathcal{B}(\mathbb{E})$ denotes the product σ-algebra. Finally, any product topological space is endowed with the product topology.

2.1. Consumers

Consumers are distributed on the interval $I = [0, 1]$ according to the Lebesgue measure ϕ. Consumers are ex ante identical in that they have the same preferences and their endowment shock processes are drawn from the same distribution. However, consumers are ex post heterogeneous in the sense that they experience idiosyncratic endowment shocks. The extension to the case of ex ante heterogeneous consumers is outlined in the last section.

Information structure and endowments. Consumer $i \in I$ is endowed with one unit of labor at each date t and a deterministic asset level $a_0^i \in \mathbb{R}_{++}$ at the beginning of time 0. Labor endowment is subject to random shocks represented by a stochastic process $(s_t^i)_{t\geq 0}$ valued in $\mathbb{S} \subset \mathbb{R}_+$, where s_0^i is a deterministic constant. Let $\mathbb{S}^0 = \mathbb{S}$, $\mathbb{S}^{t+1} = \mathbb{S}^0 \times \mathbb{S}^t$, $s^{0i} = s_0^i$, and denote by $s^{ti} = (s_0^i, s_1^i, ..., s_t^i) \in \mathbb{S}^t$ an individual shock history. Let the initial (probability) distribution of asset holdings and endowment shocks be given by

$$\lambda_0(A \times S) = \phi(i \in I : (a_0^i, s_0^i) \in A \times S),\ A \times S \in \mathcal{B}(\mathbb{R}_{++}) \times \mathcal{B}(\mathbb{S}).$$

At the beginning of date t, consumer i observes his labor endowment shock s_t^i and the aggregate productivity shock z_t. His information is represented by a σ-algebra $\mathcal{F}_t^i$ generated by past and current shocks $\{s_n^i, z_n\}_{n=0}^t$.[5] The following assumptions on the shock processes are maintained.

Assumption 1. *$\mathbb{Z} \subset [\underline{z}, \overline{z}] \subset \mathbb{R}_{++}$ is a bounded and countable set endowed with the discrete topology; $\mathbb{S} \subset \mathbb{R}_+$ is compact.*

Assumption 2. *For ϕ-a.e. i,*

(a) given the history $(s^{it}, z^t) = (s^t, z^t)$, (s_{t+1}^i, z_{t+1}) is drawn from the distribution $Q_{t+1}(\cdot, s^t, z^t)$;

(b) $Q_{t+1}(S \times Z, \cdot)$ is measurable for all $S \times Z \in \mathcal{B}(\mathbb{S}) \times \mathcal{B}(\mathbb{Z})$;

(c) Q_{t+1} has the Feller property: $\int h(s', z')Q_{t+1}(ds', dz', \cdot)$ is a continuous function on $\mathbb{S}^t \times \mathbb{Z}^t$ for any real-valued, bounded, and continuous function h on $\mathbb{S} \times \mathbb{Z}$.

[5] Alternatively, one can consider the case where each consumer observes the aggregate shocks after he makes choices so that $\mathcal{F}_t^i$ is generated by $\{s_n^i, z_{n-1}\}_{n=0}^t$, z_{-1} is null.

Remark 1. *It merits emphasis that the state space of aggregate shocks is assumed to be countable, which avoids measurability problems that may arise in dynamic programming. See [8] for the treatment when this space is uncountable.*

Consumption Space. There is a single good. A *consumption plan* $c^i \equiv (c_t^i)_{t=0}^{\infty}$ for consumer i is a nonnegative real-valued process such that c_t^i is $\mathcal{F}_t^i$-measurable.[6] Denote by $\mathcal{C}^i$ the set of all consumption plans for consumer i.

Budget and borrowing constraints. An *asset accumulation plan* $(a_{t+1}^i)_{t\geq 0}$ for consumer i is a real-valued process such that a_{t+1}^i is $\mathcal{F}_t^i$-measurable.

In each period t, consumer i consumes c_t^i and accumulates assets a_{t+1}^i subject to the familiar budget constraint:

$$c_t^i + a_{t+1}^i = (1 + r_t)a_t^i + w_t s_t^i,\ a_0^i \text{ given,} \tag{2.1}$$

where r_t is the rental rate and w_t is the wage rate. For simplicity, assume that all consumers cannot borrow so that:[7]

$$a_{t+1}^i \geq 0 \text{ for all } i \in I. \tag{2.2}$$

Finally, let $\mathbb{A} = [0, \infty)$, and denote by $\mathcal{A}^i$ the set of all asset accumulation plans of consumer i that satisfy the budget constraint (2.1) and the borrowing constraint (2.2). A consumption plan $c \in \mathcal{C}^i$ corresponding to an asset accumulation plan $a \in \mathcal{A}^i$ is called (budget) *feasible.*

Preferences. Consumer i's preferences are represented by an expected utility function defined on $\mathcal{C}^i$:

$$U(c^i) = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t^i)\right],\ (c_t^i) \in \mathcal{C}^i,$$

where $\beta \in (0, 1)$ is the discount factor and $u : \mathbb{R}_+ \to \mathbb{R}$ is the felicity function satisfying:

Assumption 3. *The function u is bounded, continuous, and strictly concave.*

Decision problem. Consumer i's problem is given by:

$$\sup_{(c_t^i, a_{t+1}^i)_{t\geq 0} \in \mathcal{C}^i \times \mathcal{A}^i} U(c^i). \tag{2.3}$$

[6] Because of this measurability, I may write the value of c_t^i at state (ω, z^{∞}) for consumer i simply as $c_t^i(\omega, z^t)$. Similar notation applies to other adapted processes.

[7] The analysis also follows for any exogenous borrowing constraint.

The plans $(c_t^i)_{t\geq 0}$ and $(a_{t+1}^i)_{t\geq 0}$ are optimal if the above supremum is achieved by $(c_t^i, a_{t+1}^i)_{t\geq 0} \in \mathcal{C}^i \times \mathcal{A}^i$.

Allocation. An *allocation* $((c_t^i, a_{t+1}^i)_{t\geq 0})_{i\in I}$ is a collection of consumption and asset accumulation plans $(c_t^i, a_{t+1}^i)_{t\geq 0}$, $i \in I$. An allocation $((c_t^i, a_{t+1}^i)_{t\geq 0})_{i\in I}$ is *admissible* if both $c_t^i = c_t(i, \omega, z^t)$ and $a_{t+1}^i = a_{t+1}(i, \omega, z^t)$ are $\mathcal{B}(I)\otimes\mathcal{F}_t$-measurable where $\mathcal{F}_t$ is the smallest σ-algebra containing $\mathcal{F}_t^i$ for all $i \in I$, $\mathcal{F}_t = \vee_{i\in I}\mathcal{F}_t^i,\ t \geq 0$. This measurability requirement ensures certain integrals are well defined (see [12] for discussion of the difficulties that arise if it is violated). Since both c_t^i and a_{t+1}^i are $\mathcal{F}_t^i$-measurable for all fixed $i \in I$, they are also $\mathcal{F}_t$-measurable. Thus, the essential content of admissibility is that c_t^i and a_{t+1}^i must be $\mathcal{B}(I)$-measurable for each fixed $(\omega, z^t) \in \Omega \times \mathbb{Z}^t$. To ensure that admissible allocations exist, I assume:[8]

Assumption 4. *For each t, $s_t : I \times \Omega \times \mathbb{Z}^\infty \to \mathbb{S}$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable.*

2.2. The Firm

There is a single firm renting capital at (net) rate r_t and hiring labor at wage w_t at date t. It produces output Y_t with the constant-returns-to-scale technology $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$:

$$Y_t = z_t F(K_t, N_t) + (1 - \delta)K_t,$$

where aggregate capital K_t is $\mathcal{F}_{t-1}$-measurable, aggregate labor N_t is $\mathcal{F}_t$-measurable, and $\delta \in (0, 1)$ is the depreciation rate. Capital is transformed from consumers' accumulated assets and aggregate labor supply N_t is given exogenously.

Assumption 5. *(a) F is strictly increasing, strictly concave, and continuously differentiable, and satisfies: $F(0, \cdot) = F(\cdot, 0) = 0$, $\lim_{K\to 0} F_1(K, \cdot) = \lim_{N\to 0} F_2(\cdot, N) = \infty$, $\lim_{K\to\infty} F_1(K, \cdot) \leq \delta$.*

(b) N_t is uniformly bounded, $0 < N_t \leq \widehat{N}$.

Remark 2. *This assumption implies that there is a maximal sustainable capital stock $\widehat{K}$ which is given by the unique solution to the equation $\overline{z}F(K, \widehat{N}) = \delta K$.*

Finally, competitive profit maximization implies that for all $t \geq 0$,

$$r_t = z_t F_1(K_t, N_t) - \delta, \tag{2.4}$$

$$w_t = z_t F_2(K_t, N_t). \tag{2.5}$$

Note that prices r_t and w_t are $\mathcal{F}_t$-measurable.

[8] The proof of the existence of admissible allocations follows from a similar argument in the proof of [32, Lemma 4.1]. So I omit it in the sequel.

2.3. Competitive Equilibrium

I first define (sequential) competitive equilibrium in the standard way.

Definition 1. *A (sequential) competitive equilibrium* $(((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0})$ *consists of an admissible allocation* $((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}$ *and price processes* $(r_t, w_t)_{t\geq 0}$ *such that: (i) Given prices* $(w_t, r_t)_{t\geq 0}$, $(a^i_{t+1}, c^i_t)_{t\geq 0}$ *solves problem (2.3) for* ϕ*-a.e. i. (ii) Given prices* $(w_t, r_t)_{t\geq 0}$, *the firm maximizes profits so that (2.4) and (2.5) are satisfied for all* $t \geq 0$. *(iii) Markets clear, i.e., for all* $t \geq 0$,

$$\int_I s^i_t \phi(di) = N_t, \tag{2.6}$$

$$C_t + K_{t+1} = z_t F(K_t, N_t) + (1-\delta)K_t, \tag{2.7}$$

where $C_t = \int_I c^i_t \phi(di)$ *and* $K_t = \int_I a^i_t \phi(di)$.

To analyze the existence and properties of equilibria, it is important to introduce the notion of aggregate distribution. Such a distribution is defined over the individual states across the population. An individual state is a pair of individual asset holdings and the history of individual shocks. More formally, if individual asset holdings and the shock history at date $t \geq 0$ are a^i_t and s^{ti}, respectively, $i \in I$, then the aggregate distribution, $\lambda_t \in \mathcal{P}(\mathbb{A} \times \mathbb{S}^t)$, is defined by:

$$\lambda_t(A \times B) = \phi(i \in I : (a_t(i), s^t(i)) \in A \times B), \; A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})^{t+1}. \tag{2.8}$$

Thus, $\lambda_t(A \times B)$ is the measure of consumers whose asset holdings and shock histories at date t lie in the set $A \times B$. Note that λ_t is a random measure since $a^i_t = a^i_t(\omega, z^{t-1})$ and $s^i_t = s^i_t(\omega, z^t)$ are random variables.

Any aggregate variable can be written as an expectation with respect to the so defined aggregate distribution; for example,

$$\begin{aligned} K_t &= \int_I a^i_t \phi(di) = \int_{\mathbb{A}\times\mathbb{S}^t} a \lambda_t(da, ds^t), \\ N_t &= \int_I s^i_t \phi(di) = \int_{\mathbb{A}\times\mathbb{S}^t} s \lambda_t(da, ds^t), \\ C_t &= \int_I c^i_t \phi(di) = (1+r_t)K_t + w_t N_t - K_{t+1}. \end{aligned}$$

The last equation follows from integration of equation (2.1). It implies the resource constraint (2.7) by the homogeneity of F and (2.4)-(2.5). Finally, equations (2.4)-(2.5) induce pricing functions $r_t : \mathcal{P}(\mathbb{A} \times \mathbb{S}^t) \times \mathbb{Z} \to \mathbb{R}$ and $w_t : \mathcal{P}(\mathbb{A} \times \mathbb{S}^t) \times \mathbb{Z} \to \mathbb{R}_+$ as follows:

$$r_t(\lambda_t, z_t) = z_t F_1\left(\int_{\mathbb{A}\times\mathbb{S}^t} a\lambda_t(da, ds^t), \int_{\mathbb{A}\times\mathbb{S}^t} s\lambda_t(da, ds^t)\right) - \delta, \tag{2.9}$$

$$w_t(\lambda_t, z_t) \quad = \quad z_t F_2\left(\int_{\mathbb{A}\times\mathbb{S}^t} a\lambda_t(da, ds^t), \int_{\mathbb{A}\times\mathbb{S}^t} s\lambda_t(da, ds^t)\right). \tag{2.10}$$

From the above discussion, conclude that aggregate distributions contain all the relevant information for equilibrium analysis. Henceforth, they will be the focus of study.

3. EXISTENCE OF COMPETITIVE EQUILIBRIUM

I begin by analyzing a single consumer's decision problem. I then discuss aggregation. Finally, I present the existence result. Notice that the model reduces to the case without aggregate shocks when $\mathbb{Z}$ contains only one element. Thus, all results to follow are valid for this case.[9]

3.1. The One-Person Decision Problem

Consider a single consumer's decision problem, given a sequence of aggregate distributions $\mu = \{\lambda_t\}_{t\geq 0}$. So the consumer index is suppressed.

In general, the aggregate distribution at date t is a measurable function of the individual-relevant state ω and the history of aggregate shocks z^t (see (2.8)). However, section 3.2 will show that under some conditions, equilibrium aggregate distributions do not depend on the individual-relevant state ω. Therefore, this subsection assumes that the aggregate distribution λ_t is a function from the set of histories of aggregate shocks $\mathbb{Z}^t$ to $\mathcal{P}(\mathbb{A}\times\mathbb{S}^t)$. Let $\mathcal{P}(\mathbb{A}\times\mathbb{S}^t)^{\mathbb{Z}^t}$ denote the set of such functions endowed with the product (or pointwise convergence) topology. Let $\mathcal{P}_\infty(\mathbb{A}\times\mathbb{S}) \equiv \times_{t=0}^{\infty}\mathcal{P}(\mathbb{A}\times\mathbb{S}^t)^{\mathbb{Z}^t}$. Then μ is an element in $\mathcal{P}_\infty(\mathbb{A}\times\mathbb{S})$.

It is convenient to analyze an individual's consumption and savings decisions by dynamic programming. Let $V_t(a_t, s^t, z^t, \mu)$ denote the maximized expected utility to the consumer at date t, when his asset holdings is a_t and the sequence of aggregate distributions is μ, given the individual shock history s^t and the aggregate shock history z^t. Then, at date $t \geq 0$, the consumer solves the following dynamic programming problem:

$$\begin{aligned} V_t(a_t, s^t, z^t, \mu) \quad = \quad & \sup_{a_{t+1}\in\Gamma(a_t,s_t,z_t,\lambda_t(z^t))} u((1+r_t(\lambda_t(z^t), z_t))a_t + w_t(\lambda_t(z^t), z_t)s_t - a_{t+1}) \\ & +\beta\int_{\mathbb{S}\times\mathbb{Z}} V_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu)Q_{t+1}(ds_{t+1}, dz_{t+1}, s^t, z^t), \end{aligned} \tag{3.1}$$

where

$$\Gamma(a_t, s_t, z_t, \lambda_t(z^t)) = [0, (1+r_t(\lambda_t(z^t), z_t))a_t + w_t(\lambda_t(z^t), z_t)s_t] \neq \varnothing.$$

[9] Aiyagari [2] and Miao [32] study *stationary equilibria* for economies without aggregate shocks.

The associated policy correspondence is defined by $g_{t+1} : \mathbb{A} \times \mathbb{S}^t \times \mathbb{Z}^t \times \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S}) \to \mathbb{A}$, with $g_{t+1}(a_t, s^t, z^t, \mu) \subset \Gamma(a_t, s_t, z_t, \lambda_t(z^t))$. If g_{t+1} is single-valued, it is called a policy function. If $g_{t+1}(a_t, s^t, z^t, \mu)$ is the set of maximizers of problem (3.1), it is called an optimal policy correspondence.

To understand problem (3.1), consider an n-period truncation. At date n, the consumer solves the following problem:

$$V_n^n(a_n, s_n, z^n, \lambda_n(z^n)) = \max_{a' \in \Gamma(a_n, s_n, z_n, \lambda_n(z^n))} u((1 + r_n(\lambda_n(z^n), z_n)a_n + w_n(\lambda_n(z^n), z_n)s_n - a').$$

At date $n-1$, by the principle of optimally, the consumer solves the following problem:

$$\begin{aligned} & V_{n-1}^n(a_{n-1}, s^{n-1}, z^{n-1}, \lambda_{n-1}(z^{n-1}), \lambda_n) = \\ & \max_{a' \in \Gamma(a_{n-1}, s_{n-1}, z_{n-1}, \lambda_{n-1}(z^{n-1}))} u((1 + r_{n-1}(\lambda_{n-1}(z^{n-1}), z_{n-1})a_{n-1} + w_{n-1}(\lambda_{n-1}(z^{n-1}), z_{n-1})s_{n-1} - a') \\ & + \beta \int_{\mathbb{S} \times \mathbb{Z}} V_n^n(a', s_n, z^n, \lambda_n(z^n)) Q_n(dz_n, ds_n, s^{n-1}, z^{n-1}). \end{aligned}$$

In general, at any date $0 \le t \le n$, the consumer solves the problem:

$$\begin{aligned} & V_t^n(a_t, s^t, z^t, \lambda_t(z^t), \lambda_{t+1}, ..., \lambda_n) \\ = & \max_{a' \in \Gamma(a_t, s_t, z_t, \lambda_t(z^t))} u((1 + r_t(\lambda_t(z^t), z_t)a_t + w_t(\lambda_t(z^t), z_t)s_t - a') \\ & + \beta \int_{\mathbb{S} \times \mathbb{Z}} V_{t+1}^n(a', s^{t+1}, z^{t+1}, \lambda_{t+1}(z^{t+1}), \lambda_{t+2}, ..., \lambda_n) Q_{t+1}(dz_{t+1}, ds_{t+1}, s^t, z^t) \end{aligned}$$

Finally, problem (3.1) corresponds to the limiting case as $n \to \infty$.

More formally, let $\mathbb{V}$ denote the set of uniformly bounded and continuous real-valued functions on $\mathbb{A} \times \mathbb{S}^t \times \mathbb{Z}^t \times \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S})$. Let $\mathbb{V}^\infty$ denote the set of sequences $v = (v_0, v_1, v_2, ...)$ of such functions. Note that $\mathbb{V}^\infty$ is a complete metric space if endowed with the norm

$$||v|| = \sup_{(t, a_t, s^t, z^t, \mu)} \left| v_t(a_t, s^t, z^t, \mu) \right|.$$

Then an application of the Contraction Mapping Theorem yields:

Lemma 1. *Given Assumptions 1-5, then there is a unique sequence of functions* $\{V_t\}_{t \ge 0} \in \mathbb{V}^\infty$ *and a unique sequence of continuous policy functions* $\{g_{t+1}\}_{t \ge 0}$ solving (3.1).

3.2. Aggregation and the Law of Motion for Aggregate Distributions

This subsection studies the question of aggregation of individual behavior to form aggregate behavior and derives the law of motion for the aggregate distributions induced by the sequences of individual optimal policy functions $\{g_{t+1}\}_{t\geq 0}$ and individual shocks $(s_t^i)_{t\geq 0}$.

In perfectly competitive markets, each consumer has no influence over prices, and all consumers together determine prices. The continuum formulation and a suitable 'law of large numbers' make this possible. To see this, recall that the aggregate distribution at date t, $\lambda_t(\omega, z^t)$, is defined in (2.8). It is a random measure that depends on the state (ω, z^t). In models without aggregate shocks (e.g., [23], [2] and [32]), perfect competition implies that equilibrium aggregate distributions must be deterministic. The latter can be achieved by assuming a no aggregate uncertainty condition on the shock processes and the underlying probability spaces, introduced in [7, Definition 1] for models of anonymous sequential games. Feldman and Gilles' construction [15, Proposition 2] shows that this condition is not vacuous and their construction is applied directly by Miao [32] to a Bewley-style model without aggregate shocks.

Say that a process $X = (X_t)_{t\geq 0}$, $X_t : I \times \Omega \to \mathbb{D}$, where $\mathbb{D}$ is a Euclidean space and X_t is jointly measurable, satisfies *no aggregate uncertainty* if there exists a nonrandom measure ν such that $\phi(i \in I : X(i,\omega) \in D) = \nu(D)$, $D \in \mathcal{B}(\mathbb{D})$, for P-a.e. ω.[10] Note that whether or not a process X has the no aggregate uncertainty property depends on the underlying probability space. The implication of the no aggregate uncertainty condition is that $\phi(i \in I : X(i,\omega) \in D) = P(\omega \in \Omega : X(i,\omega) \in D)$ if each X^i is drawn from the same distribution. In this case, the measure ν is in fact this common distribution. Thus, the empirical distribution of a sample of random variables $(X_t^i)_{i\in I}$ is the same as the theoretical distribution from which all these random variables are drawn.

To accommodate the case where aggregate shocks are present, I follow [7] and introduce a notion of conditional no aggregate uncertainty. A process $X = (X_t)_{t\geq 0}$, $X_t : I \times \Omega \times \mathbb{Z}^\infty \to \mathbb{D}$, satisfies the *conditional no aggregate uncertainty condition* if given the history of aggregate shocks $z^\infty \in \mathbb{Z}^\infty$, X satisfies the no aggregate uncertainty condition. I now assume:

Assumption 6. *The individual shock process (s_t^i), $s_t : I \times \Omega \times \mathbb{Z}^\infty \to \mathbb{S}$, satisfies the conditional no aggregate uncertainty condition relative to the probability space $(\Omega \times \mathbb{Z}^\infty, \mathcal{F}, P)$.*[11]

This assumption implies that given the history z^∞,

$$\phi(i \in I : s(i, \omega, z^\infty) \in B) = P_z(\omega \in \Omega : s^i(\omega, z^\infty) \in B),\ B \in \mathcal{B}(\mathbb{S}^\infty),$$

[10] Note that this definition is slightly different from [7, Definition 1].

[11] Krusell and Smith [29] make this assumption informally.

where P_z is the conditional measure on Ω given z^∞. Thus, conditional on the history of aggregate shocks z^t, aggregate labor endowments satisfy

$$\int_I s_t^i \phi(di) = \int_{\mathbb{A}\times\mathbb{S}^t} s\lambda_t(da, ds^t) = \int_\Omega s_t^i(\omega, z^t) P_z(d\omega), \forall t \geq 0, \forall i \in I,$$

which is deterministic. This property, along with the labor market clearing condition (2.6), puts a restriction on aggregate labor supply N_t, namely, N_t must depend on z^t only.

To illustrate the potential difficulties involved and the importance of Assumption 6, consider the following example. One might anticipate that consumer i is better off when drawing a good labor endowment shock s_t^i conditional on a history of aggregate shocks z^t. Due to the joint measurability requirement, the family of random variables $(s_t^i)_{i\in I}$ is correlated across i's. If the aggregate distribution of labor endowments conditional on z^t were stochastic, then s_t^i would be correlated with it. Thus, given a high value of s_t^i, the aggregate distribution may be more likely to be concentrated on those consumers with high labor endowment shocks. This would cause aggregate supply to increase and would then lead to a lower wage. Consequently, consumer i could be worse off, contradicting the initial intuition.

Note that this difficulty is not due to cross-sectional correlation of individual shocks, but to the randomness of the aggregate distribution and its correlation with individual shocks. If the sequence of aggregate distributions is deterministic as in models without aggregate shocks, the difficulty disappears. When aggregate shocks are present, it will not arise if aggregate distributions are nonstochastic conditional on the history of aggregate shocks, as assumed before.

Finally, Assumption 6 permits derivation of the law of motion for aggregate distributions, as I now show. Because consumers are ex ante identical, they will choose the same optimal asset accumulation policy. Thus, given the individual state (a_t^i, s^{ti}), the history of aggregate shocks z^t, and the sequence of aggregate distributions μ, let the asset holdings next period be $a_{t+1}^i = g_{t+1}(a_t^i, s^{ti}, z^t, \mu)$ for $\phi-$a.e. i.

Fixing a history of shocks z^{t+1} and using (2.8) and Bayes' Rule, one can derive that for $A \times B \in \mathcal{B}(\mathbb{A})\times\mathcal{B}(\mathbb{S})^{t+2}$,

$$\lambda_{t+1}(\omega, z^{t+1})(A\times B) = \phi(i \in I : (a_{t+1}(i), s^{t+1}(i)) \in A \times B)$$

$$= \int_{\mathbb{A}\times\mathbb{S}^t} \phi(i \in I : (g_{t+1}(a_t^i, s^{ti}, z^t, \mu), s^{t+1,i}) \in A\times B \mid (a_t^i, s^{ti}) = (a_t, s^t))$$

$$\cdot\phi(i \in I : (a_t^i, s^{ti}) \in da_t \times ds^t).$$

$$= \int_{\mathbb{A}\times\mathbb{S}^t} \phi(i \in I : (g_{t+1}(a_t, s^t, z^t, \mu), s^{t+1,i}(z^{t+1})) \in A\times B \mid s^{ti} = s^t)\lambda_t(da_t, ds^t)$$

$$= \int_{\mathbb{A}\times\mathbb{S}^t} \mathbf{1}_A(g(a, s^t, z^t, \mu))\phi(i \in I : s^{t+1}(i, \omega, z^{t+1}) \in B \mid s^{ti} = s^t)\lambda_t(da_t, ds^t)$$

Finally, applying the conditional no aggregate uncertainty condition, one obtains:

$$\begin{aligned}
&\lambda_{t+1}(\omega, z^{t+1})(A \times B) \\
= &\int_{\mathbb{A}\times\mathbb{S}^t} \mathbf{1}_A(g_{t+1}(a_t, s^t, z^t, \mu))P_z(\omega \in \Omega : s^{t+1,i}(\omega, z^{t+1}) \in B \mid s^{ti} = s^t)\lambda_t(da_t, ds^t) \\
= &\int_{\mathbb{A}\times\mathbb{S}^t} \mathbf{1}_A(g_{t+1}(a_t, s^t, z^t, \mu))Q_{t+1}(B, z_{t+1}, s^t, z^t)\lambda_t(da_t, ds^t),
\end{aligned}$$

where I use the fact that

$$Q_{t+1}(B, z_{t+1}, s^t, z^t) = P_z(\omega \in \Omega : s^{t+1,i}(\omega, z^{t+1}) \in B \mid s^{ti} = s^t, z^t).$$

Note that, conditional on the history of aggregate shocks z^{t+1} and the history of individual shocks s^t, $Q_{t+1}(B, z_{t+1}, s^t, z^t)$ does not depend on individual uncertainty. Therefore, if λ_0 is a nonrandom measure, then the conditional no aggregate uncertainty condition implies that conditional on the histories of aggregate shocks the aggregate distribution at each date does not depend on individual uncertainty. Thus, the date t aggregate distribution λ_t can be identified as a mapping from $\mathbb{Z}^t$ to $\mathcal{P}(\mathbb{A} \times \mathbb{S}^t)$.

The above discussion is summarized in the following Lemma:

Lemma 2. *Under the conditional no aggregate uncertainty condition Assumption 6, along a history of aggregate shocks $z^\infty = (z_0, z_1, ...)$, the sequence of aggregate distributions $\mu = (\lambda_t)_{t\geq 0}$ evolves according to*

$$\lambda_{t+1}(z^{t+1})(A \times B) = \int_{\mathbb{A}\times\mathbb{S}^t} \mathbf{1}_A(g_{t+1}(a_t, s^t, z^t, \mu))Q_{t+1}(B, z_{t+1}, s^t, z^t)\lambda_t(da_t, ds^t)(z^t),\ t \geq 0,$$

where λ_0 is given and $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})^{t+2}$.

3.3. The Existence Theorem

I now state one main result of the paper.

Theorem 1. *Given Assumptions 1-6, there exists a sequential competitive equilibrium. Moreover, the set of equilibrium sequences of aggregate distributions are compact.*

The idea of the proof can be described as follows. Consider a sequence of aggregate distributions $\mu = \{\lambda_t(z^t)\}_{t\geq 0} \in \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S})$ along a history of aggregate shocks z^∞. Denote by $\mathcal{P}^0_\infty(\mathbb{A} \times \mathbb{S})$

the set of all such sequences satisfying the labor market clearing condition

$$\int_{\mathbb{A}\times\mathbb{S}^t} s\lambda_t(da, ds^t) = N_t,\ t \geq 0.$$

A sequence of optimal asset accumulation policies $\{g_{t+1}\}_{t\geq 0}$ can be derived from Lemma 1. Define a new sequence of aggregate distributions $\widetilde{\mu} = \{\widetilde{\lambda}_t(z^t)\}_{t\geq 0}$ by: $\widetilde{\lambda}_0(z^0) = \lambda_0(z_0)$,

$$\widetilde{\lambda}_{t+1}(z^{t+1})(A\times B) = \int_{\mathbb{A}\times\mathbb{S}^t} \mathbf{1}_A(g_{t+1}(a_t, s^t, z^t, \mu))Q_{t+1}(B, z_{t+1}, s^t, z^t)\lambda_t(da_t, ds^t), \tag{3.2}$$

where $A\times B \in \mathcal{B}(\mathbb{A})\times\mathcal{B}(\mathbb{S})^{t+2}$, $t\geq 0$. Furthermore, define a map $\psi : \mathcal{P}^0_\infty(\mathbb{A}\times\mathbb{S}) \to \mathcal{P}^0_\infty(\mathbb{A}\times\mathbb{S})$ by $\psi(\mu) = \widetilde{\mu}$. Then the fixed point of ψ, $\mu^* = (\lambda^*_0, \lambda^*_1, \lambda^*_2, ...)$, induces a sequential competitive equilibrium $(((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0})$. Specifically, for any histories of shocks (s^i_t, z_t), let

$$\begin{aligned} a^i_{t+1} &= g_{t+1}(a^i_t, s^i_t, z_t, (\lambda^*_\tau)_{\tau\geq t}),\ c^i_t = (1+r_t)a^i_t + ws^i_t - a^i_{t+1}, \\ r_t &= z_tF_1(K_t, N_t) - \delta,\ w_t = z_tF_2(K_t, N_t), \\ K_t &= \int_{\mathbb{A}\times\mathbb{S}^t} a\lambda^*_t(da, ds^t),\ \int_{\mathbb{A}\times\mathbb{S}^t} s\lambda^*_t(da, ds^t) = N_t, \end{aligned}$$

where $a^i_0, s^i_0, z_0, \lambda^*_0 = \lambda_0$ are given.

However, $\mathcal{P}^0_\infty(\mathbb{A}\times\mathbb{S})$ is not a compact set since $\mathbb{A}$ is not compact. To apply the Brouwer-Schauder-Tychonoff Fixed-Point Theorem [4, Corollary 16.52], one needs the domain of ψ to be compact. Thus, I construct another compact set so that ψ is a self-map in this domain.

The set is constructed as follows. Because of Assumption 5 and the resource constraint, one can restrict attention to the set of sequences of aggregate distributions $\{\lambda_t\}_{t\geq 0}$'s such that $K_t = \int_{\mathbb{A}\times\mathbb{S}^t} a\lambda_t(da, ds^t) \leq \widehat{K}$. Then let

$$\widehat{\mathcal{P}}(\mathbb{A}\times\mathbb{S}^t)(z^t) = \left\{\lambda(z^t) \in \mathcal{P}(\mathbb{A}\times\mathbb{S}^t) : \int_{\mathbb{A}\times\mathbb{S}^t} a\lambda(z^t)(da, ds^t) \leq \widehat{K},\ \int_{\mathbb{A}\times\mathbb{S}^t} s\lambda(z^t)(da, ds^t) = N_t(z^t)\right\},$$

$$\widehat{\mathcal{P}}_\infty(\mathbb{A}\times\mathbb{S}) = \times^\infty_{t=0} \times_{z^t\in\mathbb{Z}^t} \widehat{\mathcal{P}}(\mathbb{A}\times\mathbb{S}^t)(z^t).$$

Lemma 3. *$\widehat{\mathcal{P}}_\infty(A\times S)$ is a compact and convex subset of a locally convex Hausdorff space.*

Finally, apply the Brouwer-Schauder-Tychonoff Fixed-Point Theorem to the map $\psi : \widehat{\mathcal{P}}_\infty(\mathbb{A}\times\mathbb{S}) \to \widehat{\mathcal{P}}_\infty(\mathbb{A}\times\mathbb{S})$. Any fixed point induces a competitive equilibrium.

4. RECURSIVE CHARACTERIZATIONS

To permit a recursive characterization of sequential competitive equilibria, I make two stationarity assumptions:

Assumption 7. $Q_{t+1}(S \times Z, s^t, z^t) = Q(S \times Z, s_t, z_t)$ *for all* $t \geq 0$ *and* $S \times Z \in \mathcal{B}(\mathbb{S}) \times \mathcal{B}(\mathbb{Z})$.

Assumption 8. *Aggregate labor endowments at any date* $t \geq 0$ *is given by a measurable function* $N : \mathbb{Z}^t \to (0, \widehat{N}]$.

Given these assumptions, the economy is the same as that studied by Krusell and Smith [29]. These two assumptions also implies that past histories of individual shocks do not affect current decisions. Thus, the aggregate distribution of asset holdings and individual shocks at date t, λ_t, can be defined by

$$\lambda_t(A \times B) = \phi(i \in I : (a_t^i, s_t^i) \in A \times B), \; A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}). \tag{4.1}$$

The set of all aggregate distributions is denoted by $\mathcal{P}_\infty(\mathbb{A} \times \mathbb{S}) = \times_{t=0}^{\infty} \mathcal{P}(\mathbb{A} \times \mathbb{S})^{\mathbb{Z}^t}$.

Under Assumptions 1-8, the pricing functions (2.4)-(2.5) become $r : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{Z} \to \mathbb{R}$, $w : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{Z} \to \mathbb{R}_+$,

$$r(\lambda, z) = zF_1\left(\int_{\mathbb{A}\times\mathbb{S}} a\lambda(da, ds), \int_{\mathbb{A}\times\mathbb{S}} s\lambda(da, ds)\right) - \delta, \tag{4.2}$$

$$w(\lambda, z) = zF_2\left(\int_{\mathbb{A}\times\mathbb{S}} a\lambda(da, ds), \int_{\mathbb{A}\times\mathbb{S}} s\lambda(da, ds)\right). \tag{4.3}$$

Moreover, a typical consumer's decision problem at date t (3.1) can be formulated by the following dynamic programming:

$$\begin{aligned} V(a_t, s_t, z_t, (\lambda_\tau)_{\tau \geq t}) = & \sup_{a' \in \Gamma(a_t, s_t, z_t, \lambda_t)} u((1 + r(\lambda_t, z_t))a_t + w(\lambda_t, z_t)s_t - a') \\ & + \beta \int_{\mathbb{S}\times\mathbb{Z}} V(a', s', z', (\lambda_\tau)_{\tau \geq t+1}) Q(ds', dz', s_t, z_t). \end{aligned} \tag{4.4}$$

This problem is studied in Lemma 4 below.

To derive a recursive characterization, it is important to select state variables. A current state must be a sufficient statistic for the future evolution of the system. With incomplete markets and heterogeneous consumers, equilibrium prices generally depend on the distribution of assets across the consumers. Thus, it is natural to include the aggregate distribution as a state variable. The question is whether it constitutes a sufficient endogenous aggregate state. To answer this question, I define a notion of equilibrium correspondence in the next subsection.

4.1. Equilibrium Correspondence

I first provide a lemma characterizing an equilibrium sequence of aggregate distributions.

Lemma 4. *Let Assumptions 1-8 hold. Then:*

(i) There is a unique continuous and bounded function $V : \mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S}) \to \mathbb{R}$ *and a unique continuous policy function* $g : \mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S}) \to \mathbb{A}$ *solving problem (4.4).*

(ii) Any equilibrium sequence of aggregate distributions $(\lambda_t)_{t\geq 0}$ *is characterized by the following equations: for* $t \geq 0,\ A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$,

$$\int_{\mathbb{A}\times\mathbb{S}} s\lambda_t(z^t)(da, ds) = N(z^t), \tag{4.5}$$

$$\lambda_{t+1}(z^{t+1})(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(g(a_t, s_t, z_t, (\lambda_\tau)_{\tau\geq t}))Q(B, z_{t+1}, s_t, z_t)\lambda_t(z^t)(da_t, ds_t), \tag{4.6}$$

where λ_0 *is given.*

Equation (4.5) is the labor market clearing condition. Equation (4.6) says that the evolution of $(\lambda_t)_{t\geq 0}$ must be consistent with consumers' optimal behavior. It embodies rational expectations.

I now define an *equilibrium correspondence* $\mathcal{E} : \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \to \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S})$, where $\mathcal{E}(z, \lambda)$ is the set of equilibrium sequences of aggregate distributions associated with an initial aggregate state (z, λ). Theorem 1 shows that $\mathcal{E}(z, \lambda)$ is nonempty and compact so that the correspondence $\mathcal{E}$ is well defined.

Lemma 5. *Under Assumptions 1-8, the equilibrium correspondence* $\mathcal{E}$ *is upper hemicontinuous.*

Because the equilibrium correspondence is generally not single-valued, there may be multiple equilibrium trajectories that are consistent with a given initial aggregate distribution and a given initial value of aggregate shock. This implies that the current aggregate distribution is typically not a sufficient (endogenous) statistic for the future evolution of the aggregate distributions (or prices). This motivates the need for additional state variables.

Before I turn to recursive characterizations in the next subsection, I define another correspondence. Let

$$\begin{aligned} \mathbb{X} &= \{(z, \lambda, v) \in \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S})) : \\ \exists \mu &\in \mathcal{E}(z, \lambda),\ v(\cdot, z, \lambda) = V(\cdot, z, \mu))\} \end{aligned}$$

Define a correspondence $\varphi : \mathbb{X} \to \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S})$ by

$$\varphi(z, \lambda, v) = \{\mu \in \mathcal{E}(z, \lambda) : v(\cdot, z, \lambda) = V(\cdot, z, \mu)\}.$$

Thus, the correspondence φ assigns to any point $(z, \lambda, v) \in \mathbb{X}$ the set of equilibrium sequences of aggregate distributions μ with the property that the expected payoff to consumer i is $v(a, s, z, \lambda)$ when the initial state is $(a_0^i, s_0^i, z_0, \lambda_0) = (a, s, z, \lambda)$.

Lemma 6. *Under Assumptions 1-8, the correspondence φ is upper hemicontinuous.*

4.2. Recursive Equilibria

Inspired by the literature on sequential games [13, 6, 8], I include the expected payoffs as an additional endogenous state variable and define a recursive equilibrium as follows.

Definition 2. *A recursive (competitive) equilibrium $((f, T^v, G), (r, w))$ consists of a measurable policy function $f : \mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S})) \rightarrow \mathbb{A}$, a measurable mapping $T^v : \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S})) \rightarrow \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}))$, a measurable mapping $G : \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S})) \times \mathbb{Z} \rightarrow \mathcal{P}(\mathbb{A} \times \mathbb{S})$, and measurable pricing functions $r : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{Z} \rightarrow \mathbb{R}$ and $w : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{Z} \rightarrow \mathbb{R}_+$ such that:*

(i) Given the pricing functions r and w, the policy function f solves the following problem

$$v(a, s, z, \lambda) = \sup_{a' \in \Gamma(a,s,z,\lambda)} u((1 + r(\lambda, z))a + w(\lambda, z)s - a') + \beta \int_{\mathbb{S} \times \mathbb{Z}} v'(a', s', z', \lambda')Q(ds', dz', s, z),$$

where $v \in \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}))$ and

$$v'(\cdot) = T^v(z, \lambda, v)(\cdot) \in \mathbb{C}(\mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S})) \text{ and } \lambda' = G(z, \lambda, v, z').$$

(ii) The firm maximizes profits so that r and w satisfy (4.2)-(4.3).

(iii) The sequence of aggregate distributions induced by G is such that labor markets clear: $\int_{\mathbb{A} \times \mathbb{S}} s\lambda_t(da, ds) = N(z^t)$, $\forall z^t \in \mathbb{Z}^t$, where $\lambda_{t+1} = G(z_t, \lambda_t, v_t, z_{t+1})$ and λ_0 is given.

(iv) The law of motion for aggregate distributions G is generated by the individual optimal policy f, i.e., for all $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$,

$$G(z, \lambda, v, z')(A \times B) = \int_{\mathbb{A} \times \mathbb{S}} \mathbf{1}_A(f(a, s, z, \lambda, v))Q(B, z', s, z)\lambda(da, ds).$$

Remark 3. *If individual shocks and aggregate shocks are independent, then* $Q(B, z', s, z)$ *does not depend on* z' *so that* G *does not depend on* z'*. In this case,* $\lambda' = G(z, \lambda, v)$*. Note that requirement (iv) embodies rational expectations. It is justified by the analysis in section 3.2 and Lemmas 2 and 4.*

The following theorem shows that given an initial state, a recursive equilibrium generates a sequential competitive equilibrium.

Theorem 2. *Let Assumptions 1-8 hold. Given the initial state* $((a_0^i, s_0^i)_{i\in I}, z_0, \lambda_0, v_0)$*, a recursive equilibrium* $((f, T^v, G), r, w)$ *generates a sequential competitive equilibrium* $(((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}$*,* $(r_t, w_t)_{t\geq 0})$ *in which consumer* i*'s expected discounted utilities are given by* $v_0(a_0^i, s_0^i, z_0, \lambda_0)$*.*

The dynamics of the sequential competitive equilibrium $(((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0})$ is described as follows. Given the initial state $((a_0^i, s_0^i)_{i\in I}, z_0, \lambda_0, v_0)$, the interest rate and the wage rate are given by $r_0 = r(\lambda_0, z_0)$ and $w_0 = w(\lambda_0, z_0)$, respectively. Consumer i accumulates assets $a_1^i = f(a_0^i, s_0^i, z_0, \lambda_0, v_0)$ and consumes the remaining wealth $c_0^i = (1 + r_0)a_0^i + w_0 s_0^i - a_1^i$. At date 1, when the realizations of individual shocks and aggregate shocks are $(s_1^i)_{i\in I}$ and z_1, the date 1 state $((a_1^i, s_1^i)_{i\in I}, z_1, \lambda_1, v_1)$ is determined by the mappings (f, G, T^v). In particular, $\lambda_1 = G(z_0, \lambda_0, v_0, z_1)$, $v_1 = T^v(z_0, \lambda_0, v_0)$. Then the date 1 prices are given by $r_1 = r(\lambda_1, z_1)$ and $w_1 = w(\lambda_1, z_1)$. Under these prices, consumer i accumulates assets $a_2^i = f(a_1^i, s_1^i, z_1, \lambda_1, v_1)$ and consumes the remaining wealth $c_1^i = (1+r_1)a_1^i + w_1 s_1^i - a_2^i$. The state then moves to date 2, and so on. Finally, the expected payoff to consumer i in the equilibrium $(((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0})$ is given by $v_0(a_0^i, s_0^i, z_0, \lambda_0)$.

Does a recursive equilibrium exist? Can any sequential competitive equilibrium be generated by a recursive equilibrium? The following theorem answers these questions.

Theorem 3. *Under Assumptions 1-8, for any competitive equilibrium* $(((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}$*,* $(r_t, w_t)_{t\geq 0})$ *with the sequence of aggregate distributions* μ^**, there exists a payoff equivalent competitive equilibrium that is generated by a recursive equilibrium.*

This theorem implies that a recursive equilibrium exists. Moreover, any payoff implied by a sequential competitive equilibrium can be generated by a recursive equilibrium.

The key to the proof of the theorem is to construct an equilibrium sequence of aggregate distributions $\overline{\mu} = (\lambda_t)_{t\geq 0}$ such that its law of motion satisfies (iv) in Definition 2. This is achieved by taking a measurable selection ξ from the correspondence φ. Then λ_{t+1} is obtained as the second component of $\xi(z_t, \lambda_t, v_t)$. The payoff $v_{t+1}(a_{t+1}, s_{t+1}, z_{t+1}, \lambda_{t+1})$ is obtained as the continuation utility at date $t+1$, $V(a_{t+1}, s_{t+1}, z_{t+1}, \xi(z_t, \lambda_t, v_t))$, implied by the equilibrium sequence of aggregate distributions $\xi(z_t, \lambda_t, v_t)$ when the economy starts at date t. This reflects rational expectations formed at the previous date. Moreover, v_{t+1} serves as a device to select the 'continuation' equilibrium $\xi(z_{t+1}, \lambda_{t+1}, v_{t+1})$ when the economy starts at date $t+1$. Finally, since the dynamics of the constructed equilibrium is stationary, the mappings (f, T^v, G) can be constructed so that a recursive equilibrium is obtained.

Turn to another recursive characterization proposed by Krusell and Smith [29], which assumes that the aggregate distribution does constitute a sufficient endogenous (aggregate) state.

Definition 3. *A KS-recursive (competitive) equilibrium $((v, h, H), (r, w))$ consists of a value function $v : \mathbb{A}\times\mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \to \mathbb{R}$, a measurable policy function $h : \mathbb{A}\times\mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \to \mathbb{A}$, a measurable mapping $H : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{Z}^2 \to \mathcal{P}(\mathbb{A} \times \mathbb{S})$, and measurable pricing functions $r : \mathcal{P}(\mathbb{A} \times \mathbb{S})\times\mathbb{Z} \to \mathbb{R}$ and $w : \mathcal{P}(\mathbb{A} \times \mathbb{S})\times\mathbb{Z} \to \mathbb{R}_+$ such that:*

(i) Given the function H and the pricing functions r and w, v and h solve the problem:

$$v(a, s, z, \lambda) = \sup_{a' \in \Gamma(a,s,z,\lambda)} u((1 + r(\lambda, z))a + w(\lambda, z)s - a') + \beta \int_{\mathbb{S}\times\mathbb{Z}} v(a', s', z', \lambda')Q(ds', dz', s, z), \tag{4.7}$$

subject to $\lambda' = H(\lambda, z, z')$.

(ii) The firm maximizes profits so that r and w satisfy (4.2)-(4.3).

(iii) The sequence of aggregate distributions induced by H is such that labor markets clear: $\int_{\mathbb{A}\times\mathbb{S}} s\lambda_t(da, ds) = N(z^t)$, $\forall z^t \in \mathbb{Z}^t$, where $\lambda_{t+1} = H(\lambda_t, z_t, z_{t+1})$ and λ_0 is given.

(iv) The law of motion for aggregate distributions H is generated by the individual optimal policy h, i.e., for all $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$,

$$H(\lambda, z, z')(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(h(a, s, z, \lambda))Q(B, z', s, z)\lambda(da, ds).^{12} \tag{4.8}$$

It is straightforward to show that a KS-recursive equilibrium generates a sequential competitive equilibrium. Does a KS-recursive equilibrium exist? One possible approach to proving the existence of a KS-recursive equilibrium is the following. Given an arbitrary mapping $H : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \times \mathbb{Z}^2 \to \mathcal{P}(\mathbb{A} \times \mathbb{S})$, let the optimal policy for (4.7) be given by $a' = \widetilde{h}(a, s, z, \lambda; H)$.

[12] This condition can be justified by a similar analysis to that in section 3.2.

Then following similar arguments in section 3.2, one can derive a new law of motion for aggregate distributions

$$\widetilde{H}(\lambda, z, z')(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(\widetilde{h}(a, s, z, \lambda; H))Q(B, z', s, z)\lambda(da, ds),$$

where $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$. This induces a map Ψ on the space of all H functions defined by $\Psi(H) = \widetilde{H}$. Finally, a fixed point of Ψ induces a recursive equilibrium.

Another approach is to start with a given function $h : \mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}(\mathbb{A} \times \mathbb{S}) \to \mathbb{A}$. Then the next period aggregate distribution is given by

$$\lambda'(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(h(a, s, z, \lambda))Q(B, z', s, z)\lambda(da, ds).$$

Now substitute the above expression for λ' into (4.7). Let the optimal policy be $a' = \widetilde{h}(a, s, z, \lambda; h)$. This induces a map Φ on the space of all h functions defined by $\Phi(h) = \widetilde{h}$.

The problem with both approaches is that it seems to be hard to find sufficient structure for the space of functions H or h and the mapping Ψ or Φ so that a suitable fixed point theorem can be applied.

However, if the competitive equilibrium is unique for any aggregate distribution and for any realization of aggregate shocks, then a KS-recursive equilibrium exists.

Theorem 4. *Under Assumptions 1-8, if the equilibrium correspondence is single-valued, then there exists a KS-recursive equilibrium.*

The condition that the competitive equilibrium is globally unique is very strong because it is typically the case that there are multiple equilibria for an incomplete markets economy. It is an open question whether a KS-recursive equilibrium exists without this condition.

5. CONCLUDING REMARKS

This paper has described the Bewley-style model with aggregate shocks in terms of sequence of aggregate distributions. The existence of competitive equilibrium is proven and a recursive characterization is established.

To conclude, I first discuss the implication for calibration of the no aggregate uncertainty condition. I then analyze the relation with anonymous games, as promised in the introduction. Finally, I outline an extension of the model.

5.1. Implication for Calibration

The no aggregate uncertainty condition imposes a restriction on the shock processes. This restriction must be considered when calibrating the model. To illustrate, consider an environment studied in Krusell and Smith [29].

Let the aggregate shock z_t take two values z_g and z_b representing good technology and bad technology respectively. Let the individual shock s_t^i take two values 0 and 1 representing unemployed status and employed status respectively. Thus, $\mathbb{Z} = \{z_g, z_b\}$ and $\mathbb{S} = \{0, 1\}$. Assume that individual shocks (s_t^i) and aggregate shocks (z_t) are correlated and that for ϕ-a.e. i, (s_t^i) and (z_t) follow jointly a Markov process with a transition matrix $(\pi_{zsz's'})$, where $z, z' \in \mathbb{Z}$ and $s, s' \in \mathbb{S}$. The interpretation is that given the aggregate and individual shocks (z, s), $\pi_{zsz's'}$ is the probability that the aggregate and individual shocks tomorrow take the value (z', s').

The aggregate distribution of employment shocks at date t, $\nu_t \in \mathcal{P}(\mathbb{S})$, is defined by

$$\nu_t(s) = \phi(i \in I : s_t^i = s), \; s = 0, 1.$$

Thus, $\nu_t(s)$ is the mass of consumers whose employment status is $s = 0, 1$. Furthermore, by the labor market clearing condition one can derive that

$$N(z^t) = \int_I s_t^i \phi(di) = \int_{\mathbb{S}} s\nu_t(ds) = \nu_t(1).$$

Note that ν_t is the marginal distribution of the aggregate distribution λ_t defined in (4.1). Under the conditional no aggregate uncertainty condition Assumption 6, it follows from Lemma 4 that given the history of aggregate shocks z^{t+1}, $(\nu_t)_{t \geq 0}$ must satisfy:

$$\nu_{t+1}(s)(z^{t+1}) = \pi_{z_t 0 z_{t+1} s} \nu_t(0)(z^t) + \pi_{z_t 1 z_{t+1} s} \nu_t(1)(z^t). \tag{5.1}$$

This equation can also be rewritten in a recursive form:

$$\nu'(s) = \pi_{z0z's}\nu(0) + \pi_{z1z's}\nu(1).$$

Equation (5.1) constitutes all the relevant restrictions under the conditional no aggregate uncertainty condition Assumption 6. In particular, equation (5.1), together with the exogenously given employment data, imposes a restriction on the transition matrix $(\pi_{zsz's'})$. Thus when parameterizing the model in order to solve it numerically, one must take this restriction into account.

5.2. Relation with Anonymous Games

The Bewley-style model with ex identical consumers can be described as an anonymous sequential game. Let Assumptions 1-8 hold. The set of players is $I = [0, 1]$. A player's characteristics are described by the individual states (a_t^i, s_t^i). His action at date t is consumption choice $c_t^i \in \mathbb{R}_+$. A *distributional strategy* at date t, τ_t, is a measurable mapping from $\mathbb{Z}^t$ to $\mathcal{P}(\mathbb{A}\times\mathbb{S}\times\mathbb{R}_+)$. The interpretation is that given the history of aggregate shocks z^t, the marginal distribution of τ_t on $\mathbb{A}\times\mathbb{S}$ gives the aggregate distribution λ_t, and the conditional distribution of τ_t on $\mathbb{R}_+$ gives a mixed strategy for individuals in state (a,s). Following [31, 23, 7, 8], the equilibrium notion is defined in terms of distributional strategies. An *equilibrium for the anonymous sequential game* is a distributional strategy $\tau = \{\tau_t\}_{t\geq 0}$ such that: (i) The aggregate distribution at date $t+1$ (the marginal distribution of τ_{t+1}) must be consistent with the date t distributional strategy τ_t and the transition of individual state,

$$
\begin{aligned}
& \lambda_{t+1}(z^{t+1})(A\times B) = \tau_{t+1}(z^{t+1})(A\times B\times\mathbb{R}_+) \\
= & \int_{\mathbb{A}\times\mathbb{S}\times\mathbb{R}_+} \mathbf{1}_A((1+r(\lambda_t(z^t),z_t))a + w(\lambda_t(z^t),z_t)s - c))Q(B,z_{t+1},s,z_t)\tau_t(da,ds,dc)(z^t),
\end{aligned}
$$

where $A\times B\in\mathcal{B}(\mathbb{A})\times\mathcal{B}(\mathbb{S})$. (ii) Almost all players optimizes under the measure τ_t for all $z^t\in\mathbb{Z}^t$ at each date t,

$$
\begin{aligned}
& \tau_t\left(\{(a,s,c)\in\mathbb{A}\times\mathbb{S}\times\mathbb{R}_+ : c\in\Gamma(a_t,s_t,z_t,\tau_t),\ \text{and for all } \widetilde{c}\in\Gamma(a_t,s_t,z_t,\tau_t),\right. \\
& u(c) + \beta E\left[W((1+r(\lambda_t,z_t))a_t + w(\lambda_t,z_t)s_t - c, s_{t+1}, z_{t+1},\tau)|s_t,z_t\right] \\
& \left.\geq u(\widetilde{c}) + \beta E\left[W((1+r(\lambda_t,z_t))a_t + w(\lambda_t,z_t)s_t - \widetilde{c}, s_{t+1}, z_{t+1},\tau)|s_t,z_t\right]\}\right) = 1,
\end{aligned}
$$

where W is given by the following dynamic programming problem:

$$
\begin{aligned}
W(a_t,s_t,z_t,\tau) \quad = & \sup_{c_t\in\Gamma(a_t,s_t,z_t,\lambda_t)} u(c_t) \\
& +\beta\int_{\mathbb{S}\times\mathbb{Z}} W((1+r(\lambda_t,z_t))a_t + w(\lambda_t,z_t)s_t - c_t, s', z',\tau)Q(ds',dz',s_t,z_t).
\end{aligned}
$$

(iii) The aggregate distribution at each date t must satisfy the labor market clearing condition:

$$
\int_{\mathbb{A}\times\mathbb{S}} s\lambda_t(da,ds)(z^t) = N(z^t),\ \forall z^t\in\mathbb{Z}^t.
$$

This equilibrium notion extends Jovanovic and Rosenthal [23] to allow for aggregate shocks. Note that it does not fit into the framework studied by Bergin and Bernhardt [7, 8] where intertemporal savings behavior is not considered. Thus, the existence and characterization results in [7, 8] cannot be applied directly. However, I conjecture that similar results can be obtained by modifying their analysis.

The more important point to be emphasized is that this equilibrium notion is different from the competitive equilibrium studied here so that it admits different interpretations. In particular, in anonymous games individual policies do not play any role; it is the *fraction* of consumers who take actions that matters. Moreover, prices do not play any role in anonymous games but they are important objects of study in general equilibrium.[13] Finally, anonymous games are usually formulated for the case of ex ante identical agents. The existing anonymous game formulation such as that given above is not suitable for the Bewley-style model with ex ante heterogeneous agents.

My formulation follows from the early general equilibrium literature, notably Hildenbrand [19] and Hart et al [17]; I extend the latter to dynamic economies. It is generally far from trivial to deduce the logical relation between the competitive equilibrium and the equilibrium for the anonymous game based on distributional strategies.[14]

5.3. Ex Ante Heterogeneous Consumers

So far, I have assumed that all consumers are ex ante identical. I now extend the analysis to the case where consumers are ex ante heterogeneous. I will only outline key arguments and omit the detailed proof.

Suppose that there are J types of ex ante identical consumers. Let I be partitioned into J disjoint measurable sets I_j such that $I = \cup_{j=1}^{J} I_j$, $q_j = \phi(I_j) > 0$, $\sum_{j=1}^{J} q_j = 1$, and let all consumers in the set I_j be endowed with utility function u^j and discount factor β^j. Furthermore, denote by Q^j the transition function of $(s_t^i, z_t)_{t\geq 0}$ for type j consumers $i \in I_j$. Suppose Assumptions 1-8 are satisfied for each type of consumers.

Define the date t aggregate distribution for type j consumers as

$$\lambda_t^j(A \times B) = \phi(i \in I_j : (a_t^i, s_t^i) \in A \times B), \ A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}).$$

Then the economy-wide aggregate distribution at date t is given by

$$\begin{aligned} \overline{\lambda}_t(A \times B) &= \phi(i \in I : (a_t^i, s_t^i) \in A \times B) \\ &= \sum_{j=1}^{J} \phi(I_j)\phi(i \in I_j : (a_t^i, s_t^i) \in A \times B) \\ &= \sum_{j=1}^{J} q_j \lambda_t^j (A \times B). \end{aligned}$$

Let $\overrightarrow{\lambda}_t = (\lambda_t^1, ..., \lambda_t^J)$ be the vector of aggregate distributions at date t, and let $\overrightarrow{\mu} = (\overrightarrow{\lambda}_t)_{t\geq 0}$ be the sequence of these vectors.

[13] See [25] for a strategic market game model of competitive price formation.

[14] There is a parallel relation between Schmeidler's [33] formulation and Mas-Collel's [31] formulation. See [27] for further discussions.

Now consider a typical type j consumer's decision problem. We can derive a dynamic programming problem similar to (4.4). The difference is that the state variables are $(a_t, s_t, z_t, (\overrightarrow{\lambda}_\tau)_{\tau \geq t})$, instead of $(a_t, s_t, z_t, (\lambda_t)_{t \geq \tau})$. Since all type j consumers are ex ante identical, they choose the same policy function g^j. One can apply a similar argument to that in section 3.2 to derive the evolution of the aggregate distribution for type j consumers:

$$\lambda_{t+1}^j(z^{t+1})(A \times B) = \int_{\mathbb{A} \times \mathbb{S}} \mathbf{1}_A(g^j(a_t, s_t, z_t, (\overrightarrow{\lambda}_\tau)_{\tau \geq t}))Q^j(B, z_{t+1}, s_t, z_t)\lambda_t(da_t, ds_t)(z^t), \ t \geq 0,$$

Given a sequence of the vectors of aggregate distributions $(\overrightarrow{\lambda}_t)_{t \geq 0}$, define a new sequence of the vectors of aggregate distributions $(\widetilde{\overrightarrow{\lambda}}_t)_{t \geq 0}$ by: $\widetilde{\lambda}_0^j(z^0) = \lambda_0^j(z_0)$,

$$\widetilde{\lambda}_{t+1}^j(z^{t+1})(A \times B) = \int_{\mathbb{A} \times \mathbb{S}} \mathbf{1}_A(g^j(a_t, s_t, z_t, (\overrightarrow{\lambda}_\tau)_{\tau \geq t}))Q^j(B, z_{t+1}, s_t, z_t)\lambda_t^j(da_t, ds_t)(z^t), \quad (5.2)$$

for $t \geq 0$ and $j = 1, ..., J$. Finally, define a mapping $\Theta : (\overrightarrow{\lambda}_t)_{t \geq 0} \mapsto (\widetilde{\overrightarrow{\lambda}}_t)_{t \geq 0}$. A fixed point induces a sequential competitive equilibrium.

A recursive equilibrium can be derived as in section 4.2. The difference is that each state variable must take into account the vector of all types of consumers; for example, the state variable of assets a should be the vector $(a^1, ..., a^J)$.

A. Appendix

Proof of Lemma 1:

Define an operator T on $\mathbb{V}^\infty$ as follows. For $v \in \mathbb{V}^\infty$, let t^{th} component of $Tv(a_t, s^t, z^t, \mu)$ be the expression

$$\begin{aligned}(Tv)_t(a_t, s^t, z^t, \mu) \quad = \quad & \max_{a_{t+1} \in \Gamma(a_t, s_t, z_t, \lambda_t(z^t))} u((1 + r_t(\lambda_t(z^t), z_t))a_t + w_t(\lambda_t(z^t), z_t)s_t - a_{t+1}) \\ & + \beta \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu) Q_{t+1}(ds_{t+1}, dz_{t+1}, s^t, z^t), \qquad (A.1)\end{aligned}$$

I first show that $Tv \in \mathbb{V}^\infty$. It is immediate that each $(Tv)_t$ is bounded. To show continuity of $(Tv)_t$, I apply the Maximum Theorem. Consider a sequence $(a_{t+1}, a_t, s^t, z^t, \mu)^n \to (a_{t+1}, a_t, s^t, z^t, \mu)$. Since $\mathbb{Z}$ is countable, $(z^t)^n = z^t$ for all n large enough. By (2.9)-(2.10) and the definition of weak convergence, $r_t(\lambda_t^n((z^t)^n), (z_t)^n) \to r_t(\lambda_t(z^t), z_t)$, $w_t(\lambda_t^n((z^t)^n), (z_t)^n) \to w_t(\lambda_t(z^t), z_t)$. Thus, Γ is a continuous correspondence. Moreover, the first term on the right-hand side of (A.1) is continuous in $(a_{t+1}, a_t, s^t, z^t, \mu)$ since u is continuous.

Turn to continuity of the second term. For n sufficiently large,

$$\begin{aligned} & \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}((a_{t+1})^n, (s^t)^n, s_{t+1}, (z^t)^n, z_{t+1}, \mu^n) Q_{t+1}(ds_{t+1}, dz_{t+1}, (s^t)^n, (z^t)^n) \\ = \; & \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}((a_{t+1})^n, (s^t)^n, s_{t+1}, z^{t+1}, \mu^n) Q_{t+1}(ds_{t+1}, dz_{t+1}, (s^t)^n, z^t). \end{aligned}$$

Thus, it is sufficient to show that the following expression converges to zero:

$$\begin{aligned} & \Bigg| \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}((a_{t+1})^n, (s^t)^n, s_{t+1}, z^{t+1}, \mu^n) Q_{t+1}(ds_{t+1}, dz_{t+1}, (s^t)^n, z^t) \\ & - \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu) Q_{t+1}(ds_{t+1}, dz_{t+1}, s^t, z^t) \Bigg| \\ \leq \; & \Bigg| \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}((a_{t+1})^n, (s^t)^n, s_{t+1}, z^{t+1}, \mu^n) Q_{t+1}(ds_{t+1}, dz_{t+1}, (s^t)^n, z^t) \qquad (A.2) \\ & - \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu) Q_{t+1}(ds_{t+1}, dz_{t+1}, (s^t)^n, z^t) \Bigg| \\ & + \Bigg| \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu) Q_{t+1}(ds_{t+1}, dz_{t+1}, (s^t)^n, z^t) \\ & - \int_{\mathbb{S}\times\mathbb{Z}} v_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu) Q_{t+1}(ds_{t+1}, dz_{t+1}, s^t, z^t) \Bigg|. \end{aligned}$$

Since $((a_{t+1})^n, (s^t)^n, s_{t+1}, z^{t+1}, \mu^n) \to (a_{t+1}, s^{t+1}, z^{t+1}, \mu)$, there is a compact set $D \subset \mathbb{A} \times \mathbb{S}^{t+1} \times \mathbb{Z}^{t+1} \times \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S})$ such that $((a_{t+1})^n, (s^t)^n, s_{t+1}, z^{t+1}, \mu^n) \in D$ for all n large enough, and

$(a_{t+1}, s^{t+1}, z^{t+1}, \mu) \in D$. Since v_{t+1} is continuous, it is uniformly continuous on D. Thus, for every $\varepsilon > 0$, there exists $N > 1$ such that for all $n > N$, $s_{t+1} \in \mathbb{S}$, and $z^{t+1} \in \mathbb{Z}^{t+1}$,

$$\left| v_{t+1}((a_{t+1})^n, (s^t)^n, s_{t+1}, z^{t+1}, \mu^n) - v_{t+1}(a_{t+1}, s^{t+1}, z^{t+1}, \mu) \right| < \varepsilon.$$

This implies that the first absolute value in (A.2) vanishes as $n \to \infty$. The second absolute value also vanishes by the Feller property.

Next, T is a contraction by a straightforward application of the Blackwell Theorem adapted to the space $\mathbb{V}^\infty$ (see [14, Lamma A.1]). Finally, applying the Contraction Mapping Theorem and the Maximum Theorem yields the desired results. ■

Proof of Lemma 3:

I first show $\widehat{\mathcal{P}}(\mathbb{A} \times \mathbb{S}^t)$ is compact. Then $\widehat{\mathcal{P}}_\infty(\mathbb{A} \times \mathbb{S}^t)$ is also compact under the product topology. For any $\lambda \in \widehat{\mathcal{P}}(\mathbb{A} \times \mathbb{S}^t)$ and $a^0 > 0$,

$$\widehat{K} \geq \int_{\mathbb{A} \times \mathbb{S}^t} a\lambda(da, ds^t) \geq \int_{[a^0, \infty] \times \mathbb{S}^t} a\lambda(da, ds^t) \geq a^0 \lambda([a^0, \infty) \times \mathbb{S}^t).$$

This implies that for any $\varepsilon > 0$, there exists an a^0 large enough such that $\lambda([a^0, \infty) \times \mathbb{S}^t) < \varepsilon$. Thus, $\widehat{\mathcal{P}}(\mathbb{A} \times \mathbb{S}^t)$ is tight and hence relatively compact (see [4, Theorem 14.22]). Furthermore, $\widehat{\mathcal{P}}(\mathbb{A} \times \mathbb{S}^t)$ is closed with respect to the weak convergence topology. It follows that $\widehat{\mathcal{P}}(\mathbb{A} \times \mathbb{S}^t)$ is compact. ■

Proof of Theorem 1:

I verify that the map $\psi : \widehat{\mathcal{P}}_\infty(\mathbb{A} \times \mathbb{S}) \to \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S})$ defined in section 3.3 satisfies the conditions of the Brouwer-Schauder-Tychonoff Fixed Point Theorem ([4, Corollary 16.52]). I first show that ψ maps from $\widehat{\mathcal{P}}_\infty(\mathbb{A} \times \mathbb{S})$ into itself. Let $\mu = (\lambda_0, \lambda_1, ...) \in \widehat{\mathcal{P}}_\infty(\mathbb{A} \times \mathbb{S})$. Then $\psi(\mu) = \widetilde{\mu} = (\widetilde{\lambda}_0, \widetilde{\lambda}_1, ...)$ is defined as in (3.2). It follows from (3.2) and Assumption 5 that

$$\begin{aligned}
\int_{\mathbb{A} \times \mathbb{S}^t} a\widetilde{\lambda}_{t+1}(da, ds^{t+1}) &= \int_{\mathbb{A} \times \mathbb{S}^t} g_{t+1}(a_t, s^t, z^t, \mu)\lambda_t(da_t, ds^t) \\
&\leq \int_{\mathbb{A} \times \mathbb{S}^t} \left[(1 + r_t(\lambda_t, z_t))a_t + w_t(\lambda_t, z_t)s_t\right] \lambda_t(da_t, ds^t) \\
&= (1 + r_t(\lambda_t, z_t))K_t + w_t(\lambda_t, z_t)N_t \\
&= (1 - \delta)K_t + z_t F(K_t, N_t) \\
&\leq (1 - \delta)\widehat{K} + \overline{z}F(\widehat{K}, \widehat{N}) = \widehat{K}.
\end{aligned}$$

Thus, $\psi(\mu) \in \widehat{\mathcal{P}}_\infty(\mathbb{A} \times \mathbb{S})$.

Finally, I show that ψ is continuous. Fix a history of aggregate shocks z^∞. Let the sequence of aggregate shocks $\mu^k \to \mu$ $(k \to \infty)$, $\mu^k, \mu \in \widehat{\mathcal{P}}_\infty(\mathbb{A}\times\mathbb{S})$. Obviously, $\widetilde{\lambda}_0^k = \lambda_0^k \to \lambda_0 = \widetilde{\lambda}_0$. For any $t \geq 0$, it follows from (3.2) that for any bounded and continuous function $h : \mathbb{A}\times\mathbb{S}^{t+1} \to \mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{A}\times\mathbb{S}^{t+1}} h(a_t, s^{t+1})\widetilde{\lambda}_{t+1}^k(da_t, ds^{t+1}) \\
= &\int_{\mathbb{A}\times\mathbb{S}^t}\int_{\mathbb{S}^{t+1}} h(g_{t+1}(a_t, s^t, z^t, \mu^k), s^{t+1})Q_{t+1}(ds^{t+1}, z_{t+1}, s^t, z^t)\lambda_t^k(da_t, ds^t)
\end{aligned}$$

converges to

$$\begin{aligned}
&\int_{\mathbb{A}\times\mathbb{S}^t}\int_{\mathbb{S}^{t+1}} h(g_{t+1}(a_t, s^t, z^t, \mu), s^{t+1})Q_{t+1}(ds^{t+1}, z_{t+1}, s^t, z^t)\lambda_t(da_t, ds^t) \\
= &\int_{\mathbb{A}\times\mathbb{S}^{t+1}} h(a_t, s^{t+1})\widetilde{\lambda}_{t+1}(da_t, ds^{t+1}),
\end{aligned}$$

where I have used the facts that λ_t^k converges to λ_t weakly and that g_{t+1} is continuous in a_t, s^t, and μ^k by Lemma 1. ■

Proof of Lemma 4:

(i) Let $\mathbb{W}$ denote the set of uniformly bounded and continuous real-valued functions on $\mathbb{A}\times\mathbb{S}\times\mathbb{Z}\times\mathcal{P}_\infty(\mathbb{A}\times\mathbb{S})$, where $\mathcal{P}_\infty(\mathbb{A}\times\mathbb{S}) = \times_{t=0}^\infty \mathcal{P}(\mathbb{A}\times\mathbb{S})^{\mathbb{Z}^t}$. Let $\mathbb{W}^\infty$ denote the set of sequences $\overline{W} = (W, W, W, ...)$ of such functions. Note that $\mathbb{W}^\infty$ is a complete metric space if endowed with the norm

$$\left|\left|\overline{W}\right|\right| = \sup_{(a,s,z,\mu)} \left|W(a,s,z,\mu)\right|.$$

Let the pricing functions $r : \mathcal{P}(\mathbb{A}\times\mathbb{S})\times\mathbb{Z} \to \mathbb{R}$ and $w : \mathcal{P}(\mathbb{A}\times\mathbb{S})\times\mathbb{Z} \to \mathbb{R}_+$ be defined as in (4.2)-(4.3).

Next, let $\overline{W} = (W, W, ...) \in \mathbb{W}^\infty$. Given any sequence of aggregate distributions $(\lambda_t)_{t\geq 0}$, rewrite problem (A.1) as

$$\begin{aligned}
(T\overline{W})_t(a_t, s_t, z_t, (\lambda_\tau)_{\tau\geq t}) \quad = &\sup_{a_{t+1}\in\Gamma(a_t, s_t, z_t, \lambda_t)} u((1 + r(\lambda_t, z_t))a_t + w(\lambda_t, z_t)s_t - a_{t+1}) \\
&+\beta\int_{\mathbb{S}\times\mathbb{Z}} W(a_{t+1}, s_{t+1}, z_{t+1}, (\lambda_\tau)_{\tau\geq t+1})Q(ds_{t+1}, dz_{t+1}, s_t, z_t),
\end{aligned}$$

where I have applied Assumptions 7-8. Since the expression on the right side of the above equation is a time invariant function of $(a_t, s_t, z_t, (\lambda_\tau)_{\tau\geq t})$, the operator T maps a sequence of time invariant function to another sequence of time invariant function. Thus, the fixed point of T is a sequence of time invariant function, denoted by $(V, V, ...)$ where $V : \mathbb{A}\times\mathbb{S}\times\mathbb{Z}\times\mathcal{P}_\infty(\mathbb{A}\times\mathbb{S}) \to \mathbb{R}$ is continuous. The corresponding sequence of optimal policies is also time invariant, denoted by

$(g, g, ...)$ where $g : \mathbb{A} \times \mathbb{S} \times \mathbb{Z} \times \mathcal{P}_\infty(\mathbb{A} \times \mathbb{S}) \to \mathbb{R}$. Moreover, g is continuous by the Maximum Theorem.

Part (ii) follows from Theorem 1 and the surrounding discussions. ■

Proof of Lemma 5:

Using a similar argument surrounding Lemma 3, one can restrict the range of the correspondence $\mathcal{E}$ to be a compact space. By Theorem 1, $\mathcal{E}$ is closed-valued. Thus, to show that $\mathcal{E}$ is upper hemicontinuous, it suffices to show that $\mathcal{E}$ has a closed graph by the Closed Graph Theorem [4, Theorem 16.12].

Let $(z, \lambda)^n$ be a sequence converging to (z, λ). Let $((\lambda_t)_{t \geq 0})^n \in \mathcal{E}(z^n, \lambda^n)$ $(\lambda_0 = \lambda)$ be a sequence of equilibrium sequence of aggregate distributions that converges to $(\lambda_t)_{t \geq 0}$. Then for any bounded and continuous function f on $\mathbb{A} \times \mathbb{S}$,

$$\int_{\mathbb{A} \times \mathbb{S}} f(a, s) \lambda_1^n(z^1)(da, ds) = \int_{\mathbb{A} \times \mathbb{S}} \int_{\mathbb{S}} f(g(a_0, s_0, z^n, (\lambda_\tau^n)_{\tau \geq 0}), s') Q(ds', z_1, s_0, z_0) \lambda_0^n(da_0, ds_0)$$

converges to $\int_{\mathbb{A} \times \mathbb{S}} f(a, s) \lambda_1(z^1)(da, ds)$. Since g is continuous, the expression on the RHS of the above equation converges to

$$\int_{\mathbb{A} \times \mathbb{S}} \int_{\mathbb{S}} f(g(a_0, s_0, z, (\lambda_\tau)_{\tau \geq 0}), s') Q(ds', z_1, s_0, z_0) \lambda_0(da_0, ds_0).$$

Thus it equals $\int_{\mathbb{A} \times \mathbb{S}} f(a, s) \lambda_1(z^1)(da, ds)$. This implies that

$$\lambda_1(z^1)(A \times B) = \int_{\mathbb{A} \times \mathbb{S}} \mathbf{1}_A(g(a_0, s_0, z, (\lambda_\tau)_{\tau \geq 0}) Q(B, z_1, s_0, z_0) \lambda_0(da_0, ds_0).$$

Similarly, one can derive that for any $t \geq 1$, λ_t satisfies (4.6).

Finally, because V is continuous, $(\lambda_t)_{t \geq 0}$ satisfies the dynamic programming equation (4.4). Further, $(\lambda_t)_{t \geq 0}$ clearly satisfies (4.5). Thus, by Lemma 4, $(\lambda_t)_{t \geq 0}$ is an equilibrium sequence of aggregate distributions, i.e., $(\lambda_t)_{t \geq 0} \in \mathcal{E}(z, \lambda)$. ■

Proof of Lemma 6:

By a similar argument to that in Lemma 5, it suffices to show that φ has a closed graph. This follows immediately from its definition and the fact that V is continuous and $\mathcal{E}$ is upper hemicontinuous established in Lemma 5. ■

Proof of Theorem 2:

I show that the tuple $(((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0})$ described below the statement of Theorem 2 in the main text constitutes a competitive equilibrium. First, it is clear that given prices (r_t) and (w_t), the firm maximizes profits. Second, I verify the market clearing condition. Integrating with respect to the measure ϕ yields:

$$\begin{aligned} C_t + K_{t+1} &= \int_I c^i_t \phi(di) + \int_I a^i_{t+1}\phi(di) \\ &= (1+r_t)\int_I a^i_t \phi(di) + w_t \int_I s^i_t \phi(di) \\ &= (1+r_t)K_t + w_t N_t \\ &= z_t F(K_t, N_t) + (1-\delta)K_t, \end{aligned}$$

where the last equality follows from the construction of r_t and w_t and the homogeneity of F. Finally, given the constructed sequence of aggregate distributions $(\lambda_t)_{t\geq 0}$, by part (i) in Definition 2 and the principle of optimality, one can show that for any consumer i, $(a^i_{t+1}, c^i_t)_{t\geq 0}$ is optimal. Moreover, the implied expected discounted utilities are given by $v_0(a^i_0, s^i_0, z_0, \lambda_0)$. ■

Proof of Theorem 3:

By Lemma 4, there exists continuous functions V and g solving the dynamic programming problem (4.4). Moreover, the first period expected payoffs to consumer i implied by the equilibrium $(((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}, (r_t, w_t)_{t\geq 0})$ are given by $V(a^i_0, s^i_0, z_0, \mu^*)$.

Step 1. Since the correspondence φ is upper hemicontinuous by Lemma 6, it follows from [19] that there exists a measurable selection ξ from φ. I use ξ to construct a recursive equilibrium with the expanded state space. Let $v_0(a_0, s_0, z_0, \lambda_0) = V(a_0, s_0, z_0, \mu^*)$.

Step 2. Let $\mu^1 = (\lambda_0, \xi(z_1, \lambda_1, v_1))$ where $\lambda_1 = \xi_2(z_0, \lambda_0, v_0)$, the second component of sequence of distributions $\xi(z_0, \lambda_0, v_0)$, and $v_1(a_1, s_1, z_1, \lambda_1) = V(a_1, s_1, z_1, \xi(z_0, \lambda_0, v_0))$. Claim that μ^1 is a sequence of aggregate distributions arising from an equilibrium with the expected payoffs given by $V(a_0, s_0, z_0, \mu^*)$.

By construction, $\xi(z_1, \lambda_1, v_1)$ is an equilibrium sequence of aggregate distributions for an economy starting from date 1 with the initial data $((a^i_1, s^i_1)_{i\in I}, z_1, \lambda_1)$. Moreover, the expected payoff satisfies $V(a_1, s_1, z_1, \mu^1) = V(a_1, s_1, z_1, \xi(z_1, \lambda_1, v_1))$. By the definition of ξ, $V(a_1, s_1, z_1, \xi(z_1, \lambda_1, v_1)) = v_1(a_1, s_1, z_1, \lambda_1)$. Thus, $V(a_1, s_1, z_1, \mu^1) = V(a_1, s_1, z_1, \xi(z_0, \lambda_0, v_0))$.

At date 0, given the sequence of aggregate distributions μ^1 the consumer solves the dynamic programming problem

$$V(a_0, s_0, z_0, \mu^1) = \sup_{a_1 \in \Gamma(a_0, s_0, z_0, \lambda_0)} u((1 + r(\lambda_0, z_0))a_0 + w(\lambda_0, z_0)s_0 - a_1) \qquad \text{(A.3)}$$

$$+\beta \int_{\mathbb{S}\times\mathbb{Z}} V(a_1, s_1, z_1, \mu^1) Q(ds_1, dz_1, s_0, z_0).$$

The optimal policy g induces an aggregate distribution at date 1,

$$\widetilde{\lambda}_1(z^1)(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(g(a_0, s_0, z_0, \mu^1)) Q(B, z_1, s_0, z_0) \lambda_0(da_0, ds_0).$$

On the other hand, since $V(a_1, s_1, z_1, \mu^1) = V(a_1, s_1, z_1, \xi(z_0, \lambda_0, v_0))$, the date 0 dynamic programming problem (A.3) is the same as that when the sequence of aggregate distributions is given by $\xi(z_0, \lambda_0, v_0)$. In particular, $V(a_0, s_0, z_0, \mu^1) = V(a_0, s_0, z_0, \xi(z_0, \lambda_0, v_0)) = v_0(a_0, s_0, z_0, \lambda_0) = V(a_0, s_0, z_0, \mu^*)$. Since u is strictly concave, it follows from a standard argument that V is strictly concave in a. Thus, the optimum in (A.3) is unique so that $g(a_0, s_0, z_0, \mu^1) = g(a_0, s_0, z_0, \xi(z_0, \lambda_0, v_0))$. Finally, since $\xi(z_0, \lambda_0, v_0)$ is an equilibrium sequence of aggregate distributions, $\xi_2(z_0, \lambda_0, v_0)$ must be consistent with individual optimal behavior so that $\widetilde{\lambda}_1 = \xi_2(z_0, \lambda_0, v_0) = \lambda_1$. Thus, μ^1 is an equilibrium sequence of aggregate distributions.

Step 3. Let $\mu^2 = (\lambda_0, \xi_2(z_0, \lambda_0, v_0), \xi(z_2, \lambda_2, v_2))$ where $\lambda_2 = \xi_2(z_1, \lambda_1, v_1)$ and $v_2(a_2, s_2, z_2, \lambda_2) = V(a_2, s_2, z_2, \xi(z_1, \lambda_1, v_1))$. Claim that μ^2 is a sequence of aggregate distributions arising from an equilibrium with expected payoffs given by $V(a_0, s_0, z_0, \mu^*)$.

By construction, $\xi(z_2, \lambda_2, v_2)$ is an equilibrium sequence of aggregate distributions for an economy starting from date 2 with the initial data $((a_2^i, s_2^i)_{i\in I}, z_2, \lambda_2)$. Moreover, the expected payoff satisfies $V(a_2, s_2, z_2, \mu^2) = V(a_2, s_2, z_2, \xi(z_2, \lambda_2, v_2))$. By the definition of ξ, $V(a_2, s_2, z_2, \xi(z_2, \lambda_2, v_2)) = v_2(a_2, s_2, z_2, \lambda_2)$. Thus, $V(a_2, s_2, z_2, \mu^2) = V(a_2, s_2, z_2, \xi(z_1, \lambda_1, v_1))$.

At date 1, given the sequence of aggregate distributions μ^2 the consumer solves the dynamic programming problem

$$\begin{aligned} V(a_1, s_1, z_1, \mu^2) \quad = & \sup_{a_2 \in \Gamma(a_1, s_1, z_1, \lambda_1)} u((1 + r(\lambda_1, z_1))a_1 + w(\lambda_1, z_1)s_1 - a_2) \\ & + \beta \int_{\mathbb{S}\times\mathbb{Z}} V(a_2, s_2, z_2, \mu^2) Q(ds_2, dz_2, s_1, z_1). \end{aligned} \tag{A.4}$$

The optimal policy induces an aggregate distribution:

$$\widetilde{\lambda}_2(z^2)(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(g(a_1, s_1, z_1, \mu^2)) Q(B, z_t, s_1, z_1) \lambda_1(da_1, ds_1).$$

Because $V(a_2, s_2, z_2, \mu^2) = V(a_2, s_2, z_2, \xi(z_1, \lambda_1, v_1))$, the dynamic programming problem (A.4) is the same as that when the sequence of aggregate distributions is $\xi(z_1, \lambda_1, v_1)$. Thus, $V(a_1, s_1, z_1, \mu^2) = V(a_1, s_1, z_1, \xi(z_1, \lambda_1, v_1))$. Moreover, following similar reasoning in step 2, $g(a_1, s_1, z_1, \mu^2) = g(a_1, s_1, z_1, \xi(z_1, \lambda_1, v_1))$ and $\widetilde{\lambda}_2 = \lambda_2 = \xi_2(z_1, \lambda_1, v_1)$.

At date 0, the consumer solves the dynamic programming

$$\begin{aligned} V(a_0, s_0, z_0, \mu^2) &= \sup_{a_1 \in \Gamma(a_0, s_0, z_0, \lambda_0)} u((1 + r(\lambda_0, z_0))a_0 + w(\lambda_0, z_0)s_0 - a_1) \\ &\quad + \beta \int_{\mathbb{S}\times\mathbb{Z}} V(a_1, s_1, z_1, \mu^2) Q(ds_1, dz_1, s_0, z_0). \end{aligned} \tag{A.5}$$

The optimal policy induces an aggregate distribution

$$\overline{\lambda}_1(z^1)(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(g(a_0, s_0, z_0, \mu^2)) Q(B, z_1, s_0, z_0) \lambda_0(da_0, ds_0).$$

Because $V(a_1, s_1, z_1, \mu^2) = V(a_1, s_1, z_1, \xi(z_1, \lambda_1, v_1)) = v_1(a_1, s_1, z_1, \lambda_1) = V(a_1, s_1, z_1, \xi(z_0, \lambda_0, v_0))$, the dynamic programming problem (A.5) is the same as that when the sequence of aggregate distribution is $\xi(z_0, \lambda_0, v_0)$. In particular, $V(a_0, s_0, z_0, \mu^2) = V(a_0, s_0, z_0, \xi(z_0, \lambda_0, v_0)) = v_0(a_0, s_0, z_0, \lambda_0) = V(a_0, s_0, z_0, \mu^*)$. Thus, following similar reasoning in Step 2, $g(a_0, s_0, z_0, \mu^2) = g(a_0, s_0, z_0, \xi(z_0, \lambda_0, v_0))$ and $\overline{\lambda}_1 = \lambda_1 = \xi_2(z_0, \lambda_0, v_0)$.

Step 4. Proceeding in this way, one can construct a sequence of sequences of aggregate distributions $(\mu^n)_{n\geq 1}$, each of which arises from an equilibrium with expected payoffs given by $V(a_0, s_0, z_0, \mu^*)$. This sequence $(\mu^n)_{n\geq 1}$ converges to a limit

$$\overline{\mu} = (\lambda_0, \xi_2(z_0, \lambda_0, v_0), \xi_2(z_1, \lambda_1, v_1), \xi_2(z_2, \lambda_2, v_2), ...)$$

in $\mathcal{E}(z_0, \lambda_0)$ in the product topology. Thus, $\overline{\mu}$ is an equilibrium sequence of aggregate distributions. Moreover, it arises from an equilibrium with expected payoffs given by $V(a_0, s_0, z_0, \mu^*)$.

Define the mappings,

$$\begin{aligned} f(a, s, z, \lambda, v) &= g(a, s, z, \xi(z, \lambda, v)), \\ G(z, \lambda, v, z')(A \times B) &= \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(g(a, s, z, \xi(z, \lambda, v)) Q(B, z', s, z) \lambda(da, ds), \end{aligned}$$

and

$$T^v(z, \lambda, v)(a', s', z', \lambda') = V(a', s', z', \xi(z, \lambda, v)),$$

where λ' is the second component of the sequence $\xi(z, \lambda, v)$. Then $((f, T^v, G), (r, w))$ is a recursive equilibrium. Finally, in the competitive equilibrium generated by this recursive equilibrium, consumer i has the expected discounted utilities $V(a_0^i, s_0^i, z_0, \mu^*)$. ■

Proof of Theorem 4:

By Theorem 1, given the initial state (a, s, z, λ), there is an equilibrium sequence of aggregate distributions $\mu^* = (\lambda, \lambda_1^*, \lambda_2^*, ...)$. Since the equilibrium is unique, the equilibrium correspondence

$\mathcal{E}$ is a single-valued mapping. I now use $\mathcal{E}$ to construct a KS-recursive equilibrium. Define

$$\begin{aligned} h(a,s,z,\lambda) &= g(a,s,z,\mathcal{E}(z,\lambda)),\ v(a,s,z,\lambda) = V(a,s,z,\mathcal{E}(z,\lambda)), \\ H(\lambda,z,z')(A\times B) &= \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(h(a,s,z,\lambda))Q(B,z',s,z)\lambda(da,ds), \end{aligned}$$

To show such a construction $((v,h,H),(r,w))$ constitutes a KS-recursive equilibrium, it suffices to show that the evolution of aggregate distributions possesses stationarity so that (4.7) holds. To this end, consider the economy that starts at date $t=1$, given the initial aggregate state (z_1,λ_1^*). Because of Assumptions 7-8, this economy is the same as that starting at date 0 with the initial state (z_1,λ_1^*) replacing (z_0,λ_0). Thus, since the equilibrium is unique, $\mathcal{E}(z_1,\lambda_1^*) = (\lambda_1^*,\lambda_2^*,...)$ constitutes an equilibrium sequence of aggregate distributions for the economy starting at date 1. Moreover, λ_1^* and $\lambda_0^*=\lambda$ are linked by the relation:

$$\lambda_1^* = H(\lambda,z_0,z_1).$$

Finally, by the construction of $((v,h,H),(r,w))$ and (4.4), one obtains (4.7) as desired. ■

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