Bifurcations in Macroeconomic Models

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Abstract:

Grandmont (1985) found that the parameter space of even the simplest, most classical models is stratified into bifurcation regions. But in such classical models all policies are Ricardian equivalent and all solutions are Pareto optimal. As a result he was not able to reach conclusions about policy relevance of his dramatic discovery. Barnett and He (1999,2002) subsequently found transcritical, codimension two, and Hopf bifurcation boundaries within the parameter space of the policy relevant Bergstrom and Wymer continuous time dynamic macroeconometric model of the UK economy.

Because of the Lucas critique, there is increasing interest in Euler equation models with GMM estimated deep parameters. He and Barnett's (2002) analysis of the Leeper and Sims (1994) Euler equations macroeconometric model revealed the existence of singularity-induced bifurcation within the model's parameter space. Although known in engineering, singularity-induced bifurcations have not previously been encountered in economics.

The purpose of the paper is to introduce the bifurcation phenomena that we have encountered in the analysis of macroeconometric models. We include and emphasize the concept of singularity-induced bifurcation and its relationship with other forms of bifurcation. We do so for the benefit of economists who might encounter singularity bifurcation in the future, as we believe is likely with other Euler equation models similarly parameterized with deep parameters.

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1. Introduction

Modern mecroeconomics has witnessed the increasing use of dynamic models in the study of economic behavior. Among the widely recognized models are the Bergstrom and Wymer continuous time dynamic macroeconometric model of the UK economy (Bergstrom and Wymer 1976), the Leeper and Sims (1994) model, and the dynamic Leontief systems (Luenberger and Arbel (1977).

Grandmont (1985) found that the parameter space of even the simplest, most classical models is stratified into bifurcation regions. But in such classical models all policies are Ricardian equivalent and all solutions are Pareto optimal. As a result he was not able to reach conclusions about policy relevance of his dramatic discovery. Barnett and He (1999,2002) subsequently found transcritical, codimension two, and Hopf bifurcation boundaries within the parameter space of the policy relevant Bergstrom and Wymer continuous time dynamic macroeconometric model of the UK economy.

Because of the Lucas critique, there is increasing interest in Euler equation models with GMM estimated deep parameters. He and Barnett's (2002) analysis of the Leeper and Sims (1994) Euler equations macroeconometric model revealed the existence of singularity-induced bifurcation within the model's parameter space. Although known in engineering, singularity-induced bifurcations have not previously been encountered in economics. .

Euler equation models represent an important class of economic systems. In addition to the Leeper and Sims model, there is, for example, also the well-known Luenberger (1977) fundamental dynamic Leontief model. Knowledge of the nature of singularity-induced bifurcations is likely to become increasingly important in understanding the dynamics of modern macroeconomic models. Bifurcation analysis of parameter space stratification is a fundamental and frequently overlooked part of understanding model properties, and can provide surprising results, as we have repeatedly found.

The purpose of the paper is to introduce the bifurcation phenomena that we have encountered in the analysis of macroeconometric models. We include and emphasize the concept of singularity-induced bifurcation and its relationship with other forms of bifurcation. We do so for the benefit of economists who might encounter singularity bifurcation in the future, as we believe is likely with other Euler equation models similarly parameterized with deep parameters. The theory of singularity-induced bifurcation is still in the process of developing. Therefore, we use examples to illustrate the effect of the presence of this type of bifurcation on system behaviors.

2. Stability

Many existing dynamic macroeconomic models can be written in the following general form

$$\mathrm{D}\mathbf{x} = \mathbf{f}(\mathbf{x},\boldsymbol{\theta}), \tag{1}$$

where D is the differentiation operator, $\mathbf{x}$ is the state vector, $\boldsymbol{\theta}$ is the parameter vector, and $\mathbf{f}$ is the vector of functions that governs the dynamics of the system. Every component of $\mathbf{f}(\mathbf{x},\boldsymbol{\theta})$ is smooth (infinitely continuously differentiable) in a local region of interest. For example, the well-known Bergstrom continuous time UK macroeconomic model can be written in the form (1) (Barnett and He (1999)). In the language of systems theory, system (1) is the class of first-order autonomous systems.

For system (1), there may exists a point $\mathbf{x}^*$ such that $\mathbf{f}(\mathbf{x}^*,\theta)=\mathbf{0}$. Then $\mathbf{x}^*$ is an equilibrium. When started at x^*, the system will stay there forever. Without loss of generality, we may assume that $\mathbf{x}^*=\mathbf{0}$ (by replacing x with $x-x^*$).

The value of the parameter vector θ can affect the dynamics of (1). Let us assume that θ can take values within a possible set Θ. It can be important to now how the value of the parameter verctor can change the behavior of (1).

One basic property of a system is its stability. If $\mathbf{x}^*$ is an equilibrium of (1), we know that (1) stays at $\mathbf{x}^*$ forever if the system starts at the equilibrium. One would also like to know what would happen if the system starts not exactly at $\mathbf{x}^*$ but in a neighborhood of it. Stability answers that question and related questions.

We now introduce theory regarding stability of a system such as (1) around the equilibrium $\mathbf{x}^*=\mathbf{0}$. For this purpose, let us rewrite (1) as

$$D\mathbf{x} = A(\theta)\mathbf{x} + F(\mathbf{x}, \theta), \qquad (2)$$

Where $A(\theta)$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x}, \theta)$ acquired by differentiating $\mathbf{f}$ with respect to $\mathbf{x}$ and evaluating the resulting matrix at the equilibrium $\mathbf{x}^*=\mathbf{0}$. The matrix $A(\theta)$ is the coefficient matrix of the linear terms, and

$$\mathbf{F}(\mathbf{x},\theta) = \mathbf{f}(\mathbf{x},\theta) - A(\theta)\mathbf{x} = o(\mathbf{x})$$

is the vector of higher order terms. In nonlinear systems theory, the local stability of (1) can be studied by examining the eigenvalues of the coefficient matrix $A(\theta)$, as follows:

(a) If all eigenvalues of $A(\theta)$ have strictly negative real parts, then (1) is locally asymptotically stable in the neighborhood of $\mathbf{x} = \mathbf{0}$.
(b) If at least one of the eigenvalues of $A(\theta)$ has positive real part, then (1) is locally asymptotically unstable in the neighborhood of $\mathbf{x} = \mathbf{0}$.
(c) If all eigenvalues of $A(\theta)$ have nonpositive real parts and at least one has zero real part, the stability of (1) usually cannot be determined from the matrix $A(\theta)$. One needs to analyze higher order terms in order to determine the stability of the system. In most cases, one needs to examine the system behavior along a certain manifold to determine the stability.

Because $A(\theta)$ is a function of the parameter vector θ, stability of (1) could be dependent on θ. Consequently, it is important to know for what parameter values the system (1) is stable and for what values it is not. It is also important to know the nature of the instability, when the system unstable.

The values of θ such that (1) is stable defines stable region S of the parameter space. In order to determine S, we need to find its boundaries. We now examine how to determine the boundary of the stability region. According to the conditions (a)-(c), the boundary could only happen under condition (c), so that $A(\theta)$ has at least one zero eigenvalue. On the boundary, we also need to determine the stability of the system, but finding the boundary provides the crucial step.

We know from matrix theory that $A(\theta)$ has at least one zero eigenvalue if and only if

$$\det(A(\theta)) = 0. \tag{3}$$

In principle, (3) identifies the stability boundary. But when θ is multi-dimensional, it can be difficult to solve for the values of θ that satisfy (3). In some cases, it is possible to reduce (3) into a solvable form such that a closed form solution can be obtained. Otherwise, it might be able to solve (3) numerically. Some interesting cases were reported in Barnett and He (1999, 2000), in which we apply various methods to solve and display stability boundaries characterized by (3).

We need to introduce a concept that is important in identifying boundary points. An equilibrium point $\mathbf{x}^*$ of (1) is called *hyperbolic* if the coefficient matrix $A(\theta)$ has no eigenvalues with zero real parts. For a hyperbolic equilibrium $\mathbf{x}^*$, the asymptotic behavior of (1) is determined by the eigenvalues of $A(\theta)$ according to conditions (a)-(b). The behavior of non-hyperbolic equilibria can be especially interesting.

3. Bifurcations in Macroeconomics

One way of studying system properties, when the values of the system's parameters are not known with certainty, is through bifurcation analysis. Bifurcation refers to a class of phenomena in dynamic systems such that the dynamic properties of the system change when parameters cross a boundary. When the location of a system's parameters is not know with certainty, it is important to know about the existence and location of such bifurcation boundaries and to explore on which side of the boundaries the parameters lie. Bifurcation boundaries have been discovered in many macroeconomic systems. The types of bifurcation boundaries found include Hopf bifurcations in growth models [e.g., Benhabib and Nishimura (1979), Boldrin and Woodford (1990), Dockner and Feichtinger (1991), Nishimura and Takahashi (1992)], pitchfork bifurcations in the tatonnement process [e.g., Bala (1997), Scarf (1960)], and transcritical bifurcations [Barnett and He (1999)]. Bifurcations are especial interesting to dynamic macroeconomic systems, since several well-known models, including Bergstrom and Wymer's (1976) UK model, operate close to bifurcation boundaries between stable and unstable regions of the parameter space..

For small perturbations of parameters, there are no structural changes in the dynamics of a hyperbolic equilibrium, provided the perturbations are sufficiently small. Therefore, bifurcations can occur only in the local neighborhood of non-hyperbolic equilibria.

3.1. Transcritical Bifurcations

A transcritical bifurcation occurs when a system has a non-hyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point, and when additional transversality conditions also are satisfied [given by the Sotomayor's Theorem in Sotomayor (1973)].

For a one-dimensional system,

$$Dx = G(x,\theta),$$

the transversality conditions for a transcritical bifurcation at $(x, \theta)=(0,0)$ are

$$G(0,0) = G_x(0,0) = 0,\ G_\theta(0,0) = 0,\ G_{xx}(0,0) \neq 0, \text{ and } G_{\theta x}^2 - G_{xx}G_{\theta\theta}(0,0) > 0. \qquad (4)$$

The canonical form of such systems is

$$Dx = \theta x - x^2. \qquad (5)$$

Note that (5) is stable around the equilibrium $x^*=0$ for $\theta < 0$, and unstable for $\theta > 0$. The equilibrium $x^*=\theta$ is stable for $\theta < 0$, and unstable for $\theta > 0$.

The following Figure 1 illustrates the resulting transcritical bifurcation. In Figure 1, the solid line represents stable equilibrium points, while the dashed one shows unstable ones.

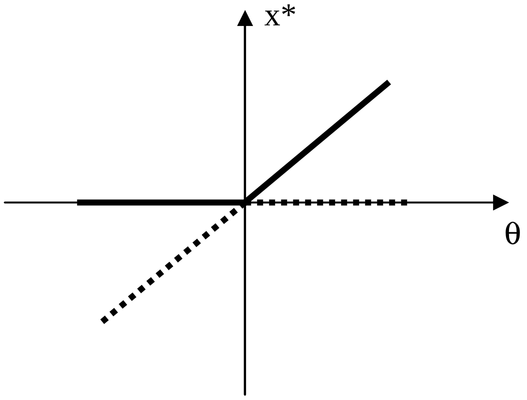

Figure 1. Diagram of Transcritical Bifurcation

Transcritical bifurcations have been found in high-dimensional continuous-time macroeconometric systems. In high dimension cases, transversality conditions have to be verified on a certain manifold. See Guckenheimer and Holmes (1983) for details.

3.2. Pitchfork Bifurcations

The standard one dimensional system with a pitchfork bifurcation is

$$Dx = \theta x - x^3.$$

For each θ>0, this system has three equilibria: $x^*=0$ (unstable), $\pm\sqrt{\theta}$ (stable). For every θ <0, there is only one (stable) equilibrium $x^*=0$. Figure 2 is its bifurcation diagram.

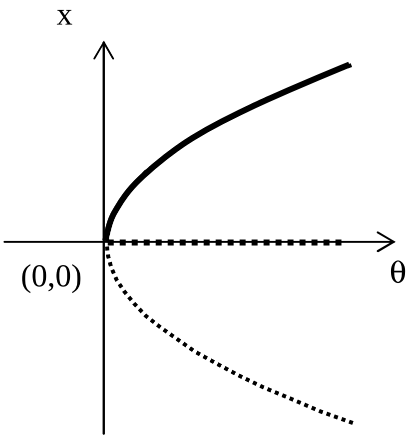

Figure 2. Diagram of Pitchfork Bifurcation.

Transversality conditions can be obtained as the follows. Consider a one-variable, one-parameter differential equation

$$Dx = f(x,\theta).$$

Suppose that there exists an equilibrium x^* and a parameter value θ^* such that (x^*,θ^*) satisfies the following conditions:

$$(a)\frac{\partial f(x,\theta^*)}{\partial x}|_{x=x^*} = 0,$$

$$(b)\frac{\partial^3 f(x,\theta^*)}{\partial x^3}|_{x=x^*} \neq 0,$$

$$(c)\frac{\partial^2 f(x,\theta)}{\partial x \partial \theta}|_{x=x^*,\theta=\theta^*} \neq 0.$$

Then (x^*,θ^*) is a pitchfork bifurcation point. Depending on the signs of the transversality conditions, the equilibrium x^* could change from stable to unstable when the parameter θ crosses θ^*.

Consider the differential equation

$$Dx = \theta x - x^3.$$

We find that $x^*=0$ and $x^*=\pm\sqrt{\theta}$ are equilibria. The Jacobian is $\theta-3x^2$, which is equal to zero when x=0 and θ=0. The transversality conditions are also satisfied at (0,0). Hence the point (0,0) is a pitchfork bifurcation point. Judging by the sign of $\theta-3x^2$, we can see that the equilibrium $x^*=0$ is stable, when θ <0 and unstable when θ >0. The two other equilibria $x^* = \pm\sqrt{\theta}$ are stable for θ >0. In this case, pitchfork bifurcation is said to be supercritical. Otherwise, the pitchfork bifurcation is subcritical.

Bala (1997) explains how pitchfork bifurcation occurs in the tatonnement process. Consider an economy consisting of two goods and two agents. The process consists of two goods and two agents. The agents have CES utility functions parameterized by $\mu \in [0, 1]$. The utility functions and endowments of agents 1 and 2 are

$$\mu^1(x_1, x_2, \mu) = -x_1^{\mu/(\mu-1)} - 2^{1/(\mu-1)} x_2^{\mu/(\mu-1)}$$
$$\mu^2(x_1, x_2, \mu) = -2^{1/(\mu-1)} x_1^{\mu/(\mu-1)} - x_2^{\mu/(\mu-1)}$$

where x_1 and x_2 are the amounts of the two goods which are consumed. Let the price of good 2 be normalized to be 1, and let p denote the price of good 1. The following are the resultant excess demand functions for the economy e_μ:

$$z_1(p, \mu) = \frac{2p^\mu}{2p^\mu + 1} + \frac{p^{\mu-1}}{p^\mu + 2} - 1$$
$$z_2(p, \mu) = \frac{p}{2p^\mu + 1} + \frac{2}{p^\mu + 2} - 1$$

The tatonnement process for the economy e_μ is given by

$$Dp = z_1(p,\mu).$$

Bala (1997) shows that pitchfork bifurcation exists in this system, and, furthermore, that for any $\mu \in (3/4,1)$, the economy e_μ has three equilibria. Chaos also exists in the tatonnement process as shown in Bala and Majumdar (1992).

3.3. Saddle-Node Bifurcations

The standard system with a pitchfork bifurcation is

$$Dx = \theta - x^2.$$

Note that it differs from the basic system for transcritical bifurcation by replacing the first order term with the zero order parameter and from the basic system for pitchfork bifurcation by lowering the orders of both terms. There exists no equilibrium for θ<0. For any given θ>0, this system has two equilibria, $x^* = \pm\sqrt{\theta}$. Figure 3 shows the bifurcation diagram.

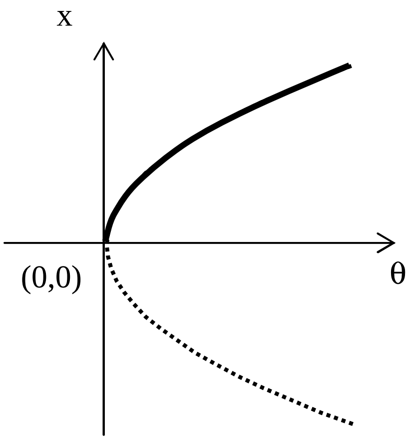

Figure 3. Diagram for Saddle-Node Bifurcation.

Saddle-node bifurcation is generic in the sense that a general system in which A(θ) has a simple zero eigenvalue displays a saddle-node bifurcation under small perturbations.

For a general one-dimensional system,

$$Dx = f(x,\theta).$$

Let x^* be a non-hyperbolic equilibrium, and let θ^* be the corresponding parameter, so that (x^*, θ^*) satisfies

$$\frac{df(x,\theta^*)}{dx}|_{x=x^*} = 0,$$
$$f(x^*,\theta^*) = 0.$$

Then the transversality conditions for saddle-node bifurcations are

$$(a)\frac{\partial f(x,\theta)}{\partial\theta}|_{x=x^*,\theta=\theta^*} \neq 0,$$
$$(b)\frac{\partial^2 f(x,\theta)}{\partial x^2}|_{x=x^*,\theta=\theta^*} \neq 0.$$

Transversality conditions for high-dimensional systems can also be formulated [see Sotomayor (1973)].

The following economic system (Gandolfo 1996) exhibits saddle-node bifurcation

$$Dr = v[F(r,\alpha) - S(r)],$$

where r is the spot exchange rate defined as domestic currency per foreign currency, $v>0$ is the adjustment speed, α is a parameter, and $\partial F/\partial\alpha > 0$. The differential equation indicates the exchange rate adjusts according to the excess demand. In deriving the model, it is assumed that the demand for and supply of foreign exchange come solely from traders and that the supply curve is backward-bending (which is viewed to be normal). Therefore there could exist two points of intersection between the demand curve and the supply curve as well as one point of tangency between the two curves. For this system, it can be verified that the transversality conditions for saddle-node bifurcations are satisfied. Hence the tangent point (r^*, α^*) is a saddle-node bifurcation.

3.4. Hopf Bifurcations

Hopf bifurcations are probably the most studied type of bifurcations. Such bifurcations occur at points at which the system has a non-hyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. Also additional transversality conditions must be satisfied [Hopf Theorem in Guckenheimer and Holmes (1983)].

Hopf bifurcation requires the presence of a pair of purely imaginary eigenvalues. Hence the dimension of a system needs to be at least two. The transversality conditions, which are rather lengthy, are given in Glendinning (1994). The basic requirements are (1) the occurrence of a pair of purely imaginary eigenvalues and (2) that the system crosses the stability boundary with nonzero zero. The canonical form of such systems is

$$Dx = -y + x(\theta - (x^2 + y^2)),$$
$$Dy = x + y(\theta - (x^2 + y^2)).$$

It has a pair of conjugate eigenvalues $\theta+i$ and $\theta-i$. The eigenvalues are purely imaginary when $\theta=0$, which is the bifurcation point.

The Hopf bifurcation boundaries could be determined numerically. Consider the case of $\det(A(\theta))\neq 0$, when $A(\theta)$ has at least one pair of purely imaginary eigenvalues. If $A(\theta)$ has exactly one such pair, and if some additional transversality conditions hold, this point is on a Hopf bifurcation boundary.

To find Hopf bifurcation points, let $p(s)=\det(sI-A)$ be the characteristic polynomial of A, and express it as

$$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \ldots + c_{n-1} s^{n-1} + s^n ,$$

Construct the following (n-1) by (n-1) matrix

$$S = \begin{bmatrix} c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & 0 & \ldots & 0 \\ 0 & c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & \ldots & 0 \\ & & \ldots & & & & & \ldots & \\ 0 & 0 & \ldots & 0 & c_0 & c_2 & c_4 & \ldots & 1 \\ c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & \ldots & 0 \\ & & \ldots & & & & & \ldots & \\ 0 & 0 & \ldots & 0 & c_1 & c_3 & \ldots & \ldots & c_{n-1} \end{bmatrix}$$

Let S_0 be obtained by deleting rows 1 and n/2 and columns 1 and 2, and let S_1 be obtained by deleting rows 1 and n/2 and columns 1 and 3. Then the matrix $A(\theta)$ has exactly one pair of purely imaginary eigenvalues [see, e.g., Guckenheimer et al. (1997)] if

$$\det(S) = 0, \qquad \det(S_0)\det(S_1) > 0. \tag{6}$$

If $\det(S) \neq 0$ or if $\det(S_0)\det(S_1)<0$, then $A(\theta)$ has no purely imaginary eigenvalues. If $\det(S)=0$ and $\det(S_0)\det(S_1)=0$, then $A(\theta)$ may have more than one pair of purely imaginary eigenvalues. Therefore, the second condition for a bifurcation boundary is

$$\det(S) = 0, \qquad \det(S_0)\det(S_1) \neq 0. \tag{7}$$

The condition (7) could be used to find candidates for bifurcation boundaries, and then the candidate segments could be checked to determine which are true boundaries. Since solving (7) analytically is impossible with realistic cases, a numerical procedure was provided in Barnett and He (1999) to find bifurcation boundaries. The stability of (7) at parameter values on the bifurcation boundary can be analyzed in the same manner as for transcritical bifurcations. Figure 4 shows the bifurcation diagram for Hopf bifurcations.

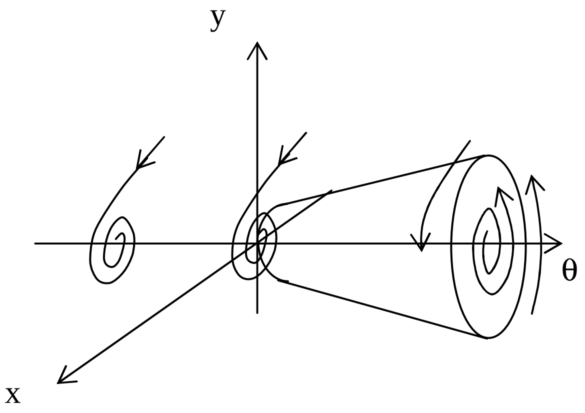

Figure 4. Diagram for Hopf Bifurcation

4. Singularity-Induced Bifurcations

In Section 3, we reviewed some well-documented bifurcation regions encountered in macroeconomic models. We devote this section to a recently discovered surprising bifurcation region found in the Leeper and Sims (1994) bifurcation model: singularity-induced bifurcation.

Some macroeconomic models, such as the widely recognized dynamic Leotief model and the Leeper and Sims model, have the form

$$E\mathbf{x}(n+1) = A\mathbf{x}(n) + \mathbf{f}(n), \tag{8}$$

in which $\mathbf{x}(n)$ is the state vector, $\mathbf{f}(n)$ is the vector driving variables, n is time, and E and A are constant matrices of appropriate dimensions. The most significant aspect of (8) is the possibility that the matrix E could be singular. If E is always invertible, then (8) will be in the discrete-time form of (1).

The model (8) in continuous time has the following form.

$$E(\mathbf{x},\boldsymbol{\theta})D\mathbf{x} = \mathbf{F}(\mathbf{x},\boldsymbol{\theta}). \tag{9}$$

Singularity-induced bifurcation occurs when the rank of $E(\mathbf{x},\boldsymbol{\theta})$ changes, such as from an invertible matrix to a singular one. In such cases, the dimension of the dynamic part of the system changes accordingly. To see this point, for any given form of (9), we can always perform appropriate coordinate transformation so that (9) is equivalent to the following form.

$$\begin{aligned} E_1(x_1, x_2, \theta)Dx_1 &= F_1(x_1, x_2, \theta) \\ 0 &= F_2(x_1, x_2, \theta) \end{aligned}$$

For this reason, the system (9) is often referred to as a differential-algebraic system.

The structural properties of the dynamics for (9) are substantially more complex than those for (1). Standard forms are available in bifurcation analysis of (1), but no canonical forms are available for (9). When E=I, (9) becomes (1). In that case bifurcations can be classified according to the canonical forms obtained from transforming A. The values that E may take create a large number of possibilities.

We use the following examples to demonstrate the complexity of bifurcation behavior of (9).

Example 1. Consider the following system modified from the canonical system for transcritical bifurcation.

$$Dx = \theta x - x^2 \tag{10}$$

$$0 = x - y^2 \tag{11}$$

The equilibria now become (0,0) and $(\theta,\pm\sqrt{\theta})$. In this case, (10)-(11) is stable around the equilibrium $(x^*,y^*) = (0,0)$ for $\theta < 0$, and unstable for $\theta > 0$. The equilibrium $(x^*,y*) = (\theta,\pm\sqrt{\theta})$ is undefined when $\theta < 0$ and unstable when $\theta > 0$. Figure 5 shows the three-dimensional bifurcation diagram for this system,

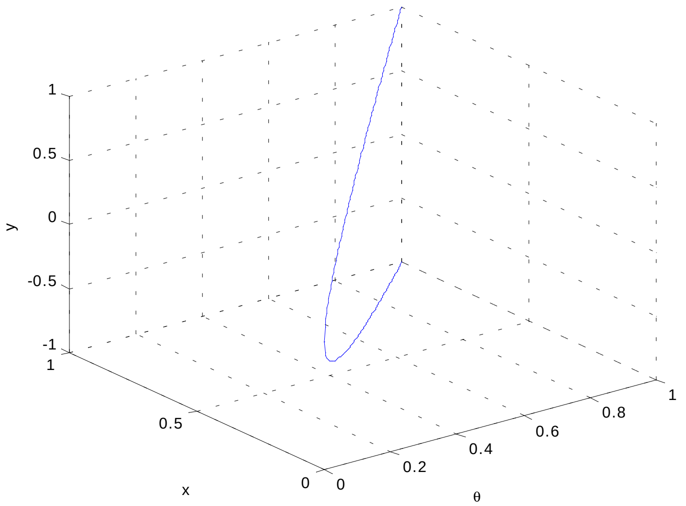

Figure 5. Bifurcation Diagram for (10)-(11) for θ>0.

Example 2. The following system is modified from the canonical system for saddle-point bifurcation.

$$Dx = \theta - x^2 \tag{12}$$
$$0 = x - y^2 \tag{13}$$

The equilibria are $(\sqrt{\theta},\pm\sqrt[4]{\theta})$, which are defined only for $\theta > 0$. In this case, (12)-(13) is stable around both equilibria.

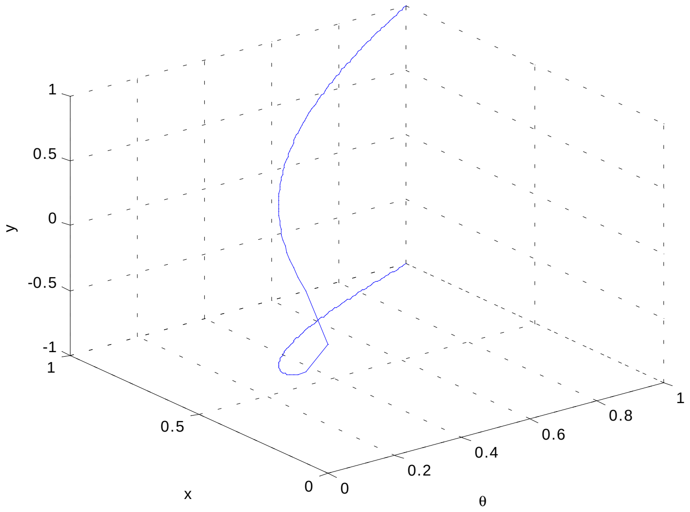

Figure 6. Bifurcation Diagram for the System (12)-(13).

The form of matrix E is fixed to be

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

in both Example 1 and Example 2. However, in some systems such as the Leeper and Sims model, the matrix E is also parameterized. The following example demonstrates bifurcation in such cases.

Example 3. Consider the system

$$Dx = ax - x^2. \tag{14}$$

$$\theta Dy = x - y^2 \tag{15}$$

in which a>0. For every θ, the equilibria are (0,0) and (a, $\pm\sqrt{a}$). In this case, (14)-(15) is unstable around the equilibrium $(x^*,y^*) = (0,0)$ for any value of θ. The equilibrium $(x^*,y^*) = (a, +\sqrt{a})$ is unstable for θ <0, and stable for θ >0, although the value of the equilibrium does not depend on θ at all. The third equilibrium $(x^*,y^*) = (a, -\sqrt{a})$ is unstable for θ >0 and stable for θ <0.

The effect of adding the second dynamic equation is more visible if we consider the system (14)-(15) in phase plan.

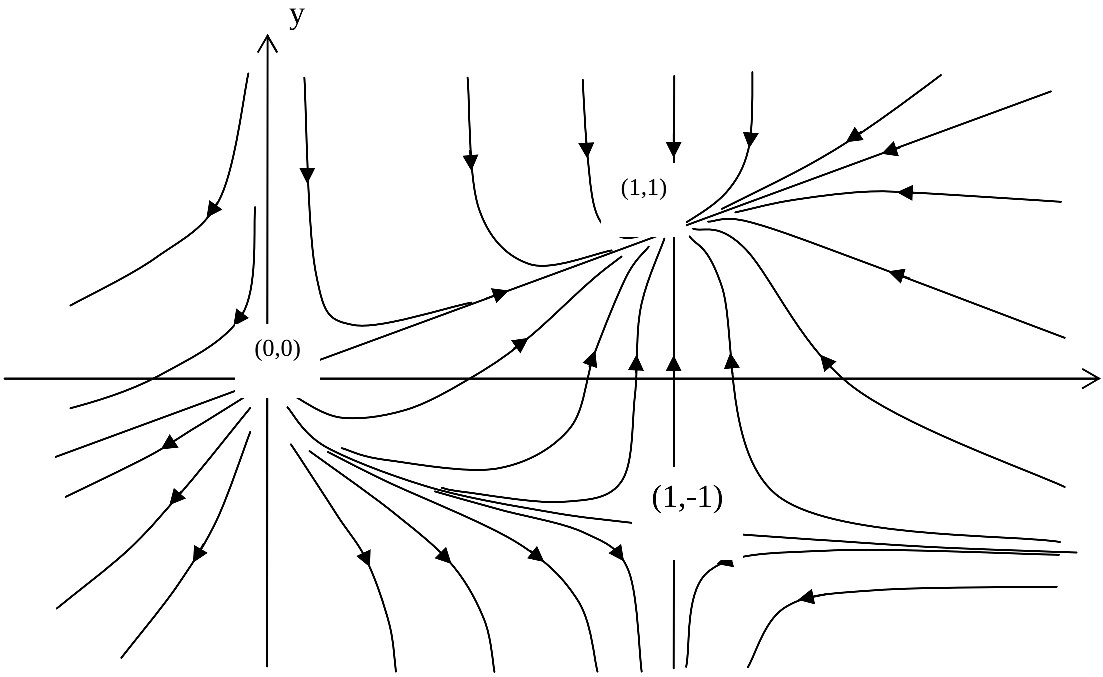

Figure 7. Phase Portrait of (14)-(15) for θ>0

Figure 7 clearly shows the stability of the equilibrium point (1,1) and the instability of (1, -1) and (0,0). It displays two-dimensional dynamics for any θ≠0. However, when θ=0, the system behavior degenerates into the movement along the curve $x-y^2=0$, as shown in Figure 8.

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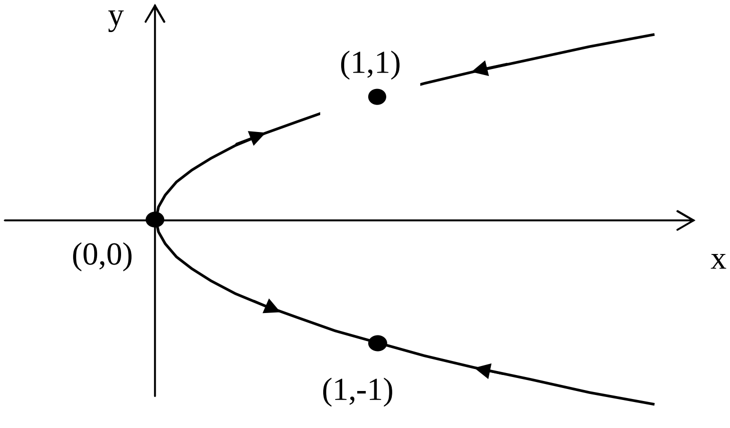

Figure 8. Phase Portrait of (14)-(15) for θ=0

Example 4. If the second equation in (14)-(15) is changed to be linear, such that

$$Dx = ax - x^2 \tag{16}$$

$$\theta Dy = x - y \tag{17}$$

then for every θ the equilibria are (0,0) and (a,a). In this case, (15)-(16) is unstable around the equilibrium $(x^*,y^*) = (0,0)$ for any value of θ. The equilibrium $(x^*,y^*)=(a,a)$ is unstable for $\theta < 0$ and stable for $\theta > 0$. Again the value of the equilibrium does not depend on θ at all. Figures 9 and 10 show the phase portraits for (16)-(17) for $\theta > 0$ and for $\theta = 0$, respectively.

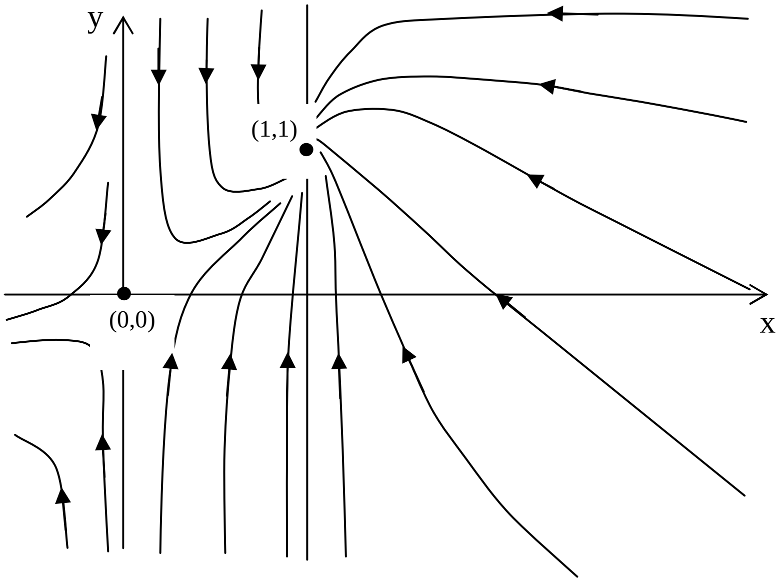

Figure 9. Phase Portrait of (16)-(17) for θ>0.

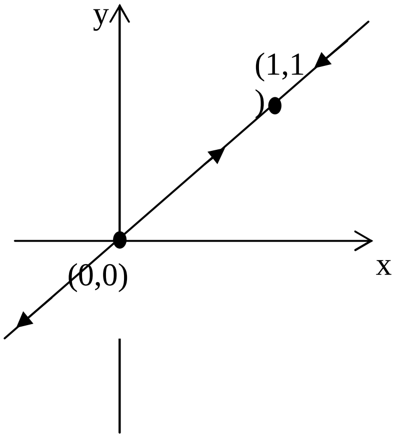

Figure 10. Phase Portrait of (16)-(17) for θ=0.

Again, Figures 9 and 10 demonstrate the drastic changes of dynamical properties, when the parameter traverses the bifurcation boundary. When $\theta=0$, the variable y in (16)-(17) is just a replica of the variable x in (16)-(17). The real independent dynamics is just one-dimensional. However, when $\theta\neq0$, the system moves into a two-dimensional space. The variable y follows x with some deviation error. The error asymptotically diminishes to zero.

Changes in the dynamical properties of (9) can reflect more than a simple change of the rank of E. In fact, even with the same rank of E, the order of the dynamical part of (9) could still vary, when parameters take different values, as illustrated in the following example.

Example 5. Consider the following system

$$\begin{aligned} Dx_1 &= x_3 \\ Dx_2 &= -x_2 \\ 0 &= x_1 + x_2 + \theta x_3 \end{aligned} \tag{18}$$

For any $\theta\neq0$, solving from the last equation results in

$$\begin{aligned} Dx_1 &= -(x_1 + x_2)/\theta \\ Dx_2 &= -x_2 \end{aligned} \tag{19}$$

which is stable at the equilibrium (0,0) for $\theta > 0$ and unstable at equilibrium (0,0) for $\theta < 0$.

Solving from the last equation of (18) when $\theta=0$, we obtain

$$\begin{aligned} x_1 &= -x_2 \\ x_3 &= x_2 \\ Dx_2 &= -x_2 \end{aligned} \tag{20}$$

for any $t>0$. Note the difference of the order of dynamics in (20) from that of (19)!

5. Conclusion

In this paper, we have provided a summary of some well-documented bifurcation phenomena in macroeconomic models. Most notably, we have introduced singularity-induced bifurcations, which have not previously been encountered in economics and which He and Barnett (2002) surprisingly recently discovered in the Leeper and Sims Euler equations macroeconometric model. Although many interesting results have been obtained in the existing literature, bifurcation theory in economic dynamics is far from complete.

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