UNSOLVED ECONOMETRIC PROBLEMS IN NONLINEARITY, CHAOS, AND BIFURCATION

by

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Summary:

In an attempt to resolve the controversies that exist within the field of economics regarding nonlinearity, chaos, and bifurcation, we investigate the relevancy to these controversies of a controlled competition among nonparametric econometric tests for nonlinearity and chaos, and we also report on our results with experiments using parametric macroeconomic models to investigate the implications of bifurcation for macroeconomic policy. These experiments are part of an ongoing research project. What we find so far is that existing views on nonlinearity, chaos, and bifurcation in economics are based upon oversimplified views that currently neither can be confirmed nor contradicted with empirical results that are now available. Since these issues are deep and difficult, considerably more research is needed before any serious conclusions on the subject can be stated with confidence. This fact is particularly true regarding the relevancy of nonlinearity, chaos, and bifurcation for macroeconomic stabilization policy.

1. Introduction

There is much controversy about the potential relevancy of nonlinearity, chaos, and bifurcation for econometrics and for macroeconomics. This fact is despite the clear relevancy of nonlinearity, chaos, and bifurcation to many other scientific fields, and the inherent complex nonlinear dynamic nature of economic theory. Even the results of tests for the existence of nonlinearity of economic time series produce surprising amounts of controversy. We report on a competition among empirical tests for nonlinearity and chaos. We also report on an experiment regarding stabilization policy as bifurcation selection. Both the competition and the experiment help to narrow down the sources of the controversies, but many unsolved problems remain before it will become possible to test the relevant hypotheses in a rigorous formal manner. Since it is not possible to run controlled experiments with the economy itself and since the dimension of the economic system is much higher than the dimension of systems with which physical scientists usually work, the techniques in mathematics and statistics that are available to test for and use nonlinear dynamics, chaos, and bifurcation are very difficult to apply to economics in a convincing manner. As a result, at the present time it seems premature to take strong positions on those subjects in economics on empirical grounds.

2. Tests of Nonlinearity and Chaos

The empirical tests that have been developed and applied by econometricians have produced conflicting results and therefore have been relatively unsuccessful at resolving controversies regarding the existence of nonlinearity or chaos in economic data.. In addition, those statistical tests usually have been time series tests that do not condition upon an economic model and hence cannot distinguish between nonlinearity produced from within the economy and nonlinearity of economic time series produced by nonlinear stochastic shocks to the economy from outside the economy, as from the weather. Tests for nonlinearity and chaos that condition upon a dynamic economic model could succeed in isolated the cause to be within the economy, but so far those tests are very difficult to apply and have discouraged attempts. Furthermore, nonlinearity or chaos from within the economy do not necessarily have policy implications, if the nonlinear dynamics produce Pareto optimal solution paths from a system not subject to any form of market failure, while ncorporating nonlinear dynamics into models subject to market failure presents formidable difficulities.

In the first part of this paper, we report on a controlled competition that was designed and run by Barnett, Gallant, et al (1995, 1996ab, 1997) to investigate the problem of robustness of inferences of nonlinearity or chaos across competing tests. In the second part of this paper, we report on our progress on research on nonlinear dynamics within a structural macroeconometric model, with emphasis upon investigation of potential Pareto improving stabilization policy, modeled as bifurcation selection. We now describe the tests that are relevant to the controlled competition.

3. The Correlation Dimension Test

Unfortunately, the fractal dimension of strange attractors produced by chaotic dynamics cannot be computed easily in practice. To remedy this, Grassberger and Procaccia (1983) suggested the concept of *correlation dimension* (or *correlation exponent*) which approximates the Hausdorf dimension of fractal attractors. The basic idea is that of replacing the box-counting algorithm, necessary to compute fractal dimension, with the measurement of correlations between points of a long time series on the attractor. Hence, the correlation dimension (unlike the fractal dimension) is a probabilistic, not a metric, dimension.

To discuss the correlation dimension test for chaos, let us start with the 1-dimensional series, $\{x_t\}_{t=1}^n$, which can be embedded into a series of m-dimensional vectors $X_t = (x_t, x_{t-1}, ..., x_{t-m+1})'$ giving the series $\{x_t\}_{t=m}^n$. The selected value of m is called the *embedding dimension* and each X_t is known as an *m-history* of the series $\{x_t\}_{t=1}^n$. This converts the series of scalars into a slightly shorter series of (m-dimensional) vectors with overlapping entries. In particular, from the sample size n, $N = n - m + 1$ m-histories can be made. Assuming that the true, but unknown, system which generated $\{x_t\}_{t=1}^n$ is ϑ-dimensional and provided that $m \geq 2\vartheta + 1$, then the N m-histories recreate the dynamics of the data generation process and can be used to analyze the dynamics of the system.

The correlation dimension test is based on the *correlation function* (or *correlation integral*), $C(N, m, \epsilon)$, which for a given embedding dimension m is given by:

$$C(N, m, \epsilon) = \frac{1}{N(N-1)} \sum_{m \leq t \neq s \leq N} H(\epsilon - \|X_t - X_s\|)$$

where ϵ is a sufficiently small number, $H(z)$ is the Heavside function, which maps positive arguments into 1, and nonpositive arguments into 0, i.e.,

$$H(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and $\|.\|$ denotes the distance induced by the selected norm. In other words, the correlation integral is the number of pairs (t, s) such that X_t and X_s are near to each other, with nearness defined by ϵ. Intuitively, $C(N, m, \epsilon)$ measures the probability that the distance between any two m-histories is less than ϵ. If $C(N, m, \epsilon)$ is large (which means close to 1) for a very small ε, then the data are very well correlated.

To move from the correlation function to the correlation dimension, one investigates how $C(N, m, \epsilon)$ changes as ϵ changes. One expects $C(N, m, \epsilon)$ to increase with ϵ (since increasing ϵ increases the number of neighboring points that get included in the correlation integral). In fact, Grassberger and Procaccia (1983) have shown that for small values of ϵ, $C(N, m, \epsilon)$ grows exponentially at the rate of D_c such that

$$C(N, m, \epsilon) = \eta e^{D_c},$$

where η is a constant and D_c is the correlation dimension.

If the increase in $C(N, m, \epsilon)$ is slow as ϵ is increased, then most data points are near to each other, and the data are well correlated. Hence, the higher the correlation dimension [and the faster the increase in $C(N, m, \epsilon)$ as ϵ is increased], the less correlated the data are, and the system is regarded stochastic. On the other hand, the lower the correlation dimension [and the slower the increase in $C(N, m, \epsilon)$ as ϵ is increased], the more correlated the data are, and the system is regarded as essentially deterministic, even if fairly complicated.

The correlation dimension can be defined as

$$D_c = \lim_{\epsilon \to 0} \frac{d \log C(N, m, \epsilon)}{d \log \epsilon},$$

that is, by the slope of the regression of $\log C(N, m, \epsilon)$ versus $\log \epsilon$ for small values of ϵ. As a practical matter one investigates the estimated value of D_c as m is increased. If as m increases D_c continues to rise, then the system is stochastic. If, however, the data are generated by a deterministic process (consistent with chaotic behavior), then D_c reaches a finite saturation limit beyond some relatively small m. The correlation dimension can therefore be used to distinguish true stochastic processes from deterministic chaos (which may be low-dimensional or high-dimensional).

While the correlation dimension measure is potentially very useful in testing for chaos, the sampling properties of the correlation dimension are, however, unknown. As Barnett, Gallant, Hinich, Jungeilges, Kaplan, and Jensen (1995, pp. 306) have argued, "if the only source of stochasticity is noise in the data, and if that noise is slight, then it is possible to filter the noise out of the data and use the correlation dimension test deterministically. However, if the economic structure that generated the data contains a stochastic disturbance within its equations, the correlation dimension is stochastic and its derived distribution is important in producing reliable inference."

Moreover, if the correlation dimension is very large as in the case of high-dimensional chaos, it will be very difficult to estimate it without an enormous amount of data. In this regard, Ruelle (1990) argues that a chaotic series can only be distinguished if it has a correlation dimension well below $2 \log_{10} N$, where N is the size of the data set. This suggests that with economic time series the correlation dimension can only distinguish low-dimensional chaos from high-dimensional stochastic processes. See also Grassberger and Procaccia (1983) for more details.

4. The BDS Test

Motivated by the unknown sampling properties of the correlation dimension test statistic, Brock, Dechert, LeBaron, and Scheinkman (1996) devised a statistical test which is known as the BDS test, which tests the null hypothesis of whiteness against an unspecified alternative using a nonparametric technique.

The BDS statistic, based upon the correlation function (but not the correlation dimension), is

$$W(N, m, \epsilon) = \sqrt{N} \, \frac{C(N, m, \epsilon) - C(N, 1, \epsilon)^m}{\widehat{\sigma}(N, m, \epsilon)}$$

where $\widehat{\sigma}(N,m,\epsilon)$ is an estimate of the asymptotic standard deviation of $C(N,m,\epsilon) - C(N,1,\epsilon)^m$. The formula for $\widehat{\sigma}(N,m,\epsilon)$ can be found in Brock, Dechert, et al. (1996). The BDS statistic is asymptotically standard normal under the whiteness null hypothesis.

The intuition behind the BDS statistic is as follows. The correlation function $C(N,m,\epsilon)$ is an estimate of the probability that the distance between any two m-histories, X_t and X_s, of the series $\{x_t\}$ is less than ϵ. If $\{x_t\}$ were independent, then for $t \neq s$ the probability of this joint event equals the product of the individual probabilities. Moreover, if $\{x_t\}$ were also identically distributed, all of the m probabilities under the product sign would be the same. The BDS statistic therefore tests the null hypothesis that $C(N,m,\epsilon) = C(N,1,\epsilon)^m$, which is equivalent to the null hypothesis of whiteness.

Since the asymptotic distribution of the BDS test statistic is known under the null hypothesis of whiteness, the BDS test provides a direct statistical test for whiteness against general dependence, which includes both nonwhite linear and nonwhite nonlinear dependence. Hence, the BDS test does not provide a direct test for nonlinearity or for chaos, since the sampling distribution of the test statistic is not known under the null hypothesis of nonlinearity, linearity, or chaos. It is, however, possible to use the BDS test to produce indirect evidence about nonlinear dependence by filtering out linearity and testing for remaining structure in the residuals. See Barnett, Gallant, et al. (1997) and Barnett and Hinich (1992) for a discussion of these issues.

5. The Hinich Bispectrum Test

For frequencies ω_1 and ω_2 in the principal domain given by

$$\Omega = \left\{(\omega_1,\omega_2) : 0 < \omega_1 < 0.5, \omega_2 < \omega_1, 2\omega_1 + \omega_2 < 1\right\},$$

the bispectrum, $B_{xxx}(\omega_1,\omega_2)$, is defined by

$$B_{xxx}(\omega_1,\omega_2) = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} C_{xxx}(r,s)\exp[-i2\pi(\omega_1 r + \omega_2 s)].$$

The bispectrum is the double Fourier transformation of the third order moments function and is the third order polyspectrum. The regular power spectrum is the second order polyspectrum and is a function of only one frequency.

The skewness function $\Gamma(\omega_1,\omega_2)$ is defined in terms of the bispectrum as follows

$$\Gamma^2(\omega_1,\omega_2) = \frac{|B_{xxx}(\omega_1,\omega_2)|^2}{S_{xx}(\omega_1)S_{xx}(\omega_2)S_{xx}(\omega_1+\omega_2)}, \tag{1}$$

where $S_{xx}(\omega)$ is the ordinary power spectrum of $x(t)$ at frequency ω. Since the bispectrum is complex valued, the absolute value in Equation (1) designates modulus. Brillinger (1965) proved that the skewness function $\Gamma(\omega_1,\omega_2)$ is constant over all frequencies $(\omega_1,\omega_2) \in \Omega$, if $\{x(t)\}$ is linear, while $\Gamma(\omega_1,\omega_2)$ is flat at zero over all frequencies, if $\{x(t)\}$ is Gaussian. Linearity and Gaussianity can be tested using a sample estimator of the skewness function. But observe that those flatness conditions are necessary but not sufficient for general linearity and Gaussianity, respectively. On the other hand, flatness of the skewness function is both necessary and sufficient for third order nonlinear dependence. The Hinich (1982) "linearity test" tests the null hypothesis that the skewness function is flat, and hence is a test of lack of third order nonlinear dependence. For details of the test, see Hinich (1982).

6. The NEGM Test

The most important tool for diagnosing the presence of sensitive dependence on initial conditions is provided by the dominant Lyapunov exponent, λ, and that sensitive dependence is a fundamental characteristic of chaos. This exponent measures average exponential divergence between trajectories that differ in initial conditions only by an infinitesimally small amount. Positive Lyapunov exponent of a bounded system is an operational definition of chaos.

One early method for calculating the dominant Lyapunov exponent is that proposed by Wolf, Swift, Swinney, and Vastano (1985). This method, however, requires long data series and is sensitive to dynamic noise, such that inflated estimates of the dominant Lyapunov exponent can be obtained, if the sample size is not sufficiently high. Nychka, Ellner, Gallant, and McCaffrey (1992) have proposed a regression method, involving the use of neural network models, to test for positivity of the dominant Lyapunov exponent. The Nychka et al. (1992), hereafter NEGM, Lyapunov exponent estimator is a regression and Jacobian method, unlike the Wolf et al. (1985) direct method.

Assume that the data $\{x_t\}$ are real-valued and are generated by a nonlinear autoregressive model of the form

$$x_t = f(x_{t-L}, x_{t-2L}, ..., x_{t-mL}) + e_t \tag{2}$$

for $1 \leq t \leq N$, where L is the time-delay parameter and m is the length of the autoregression. Here f is a smooth unknown function, and $\{e_t\}$ is a sequence of independent random variables with zero mean and unknown constant variance. The Nychka et al. (1992) approach to estimation of the maximum Lyapunov exponent involves producing a state-space representation of Equation (2), such that

$$X_t = F(X_{t-L}) + E_t, \quad F : \mathbb{R}^m \to \mathbb{R}^m$$

where $X_t = (x_t, x_{t-L}, ..., x_{t-mL+L})'$, $F(X_{t-L}) = (f(x_{t-L}, ..., x_{t-mL}), x_{t-L}, ..., x_{t-mL+L})'$, and $E_t = (e_t, 0, ..., 0)'$. The approach then uses a Jacobian-based method to estimate λ through the intermediate step of estimating the individual Jacobian matrices

$$J_t = \frac{\partial F(X_t)}{\partial X'}.$$

After using several nonparametric methods, McCaffrey et al. (1992) recommended using either thin plate splines or neural nets to estimate J_t. The neural net that they recommended has q units in the hidden layer

$$f(X_{t-L}, \theta) = \beta_0 + \sum_{j=1}^{q} \beta_j \psi(\gamma_{0j} + \sum_{i=1}^{m} \gamma_{ij} x_{t-iL}),$$

where ψ is a known (hidden) nonlinear activation function, which is usually the logistic distribution function $\psi(u) = 1/(1 + \exp(-u))]$. The parameter vector θ is then fit to the data by nonlinear least squares. That is, one computes the estimate $\widehat{\theta}$ to minimize the sum of squares $S(\theta) = \sum_{t=1}^{N} [x_t - f(X_{t-1}, \theta)]^2$, and uses $\widehat{F}(X_t) = (f(x_{t-L}, ..., x_{t-mL}, \widehat{\theta}), x_{t-L}, ..., x_{t-mL+L})'$ to approximate $F(X_t)$.

Since appropriate values of $L, m,$ and q, are unknown, Nychka et al. (1992) recommend selecting that value of the triple (L, m, q) which minimizes the Bayesian Information Criterion (BIC), as defined by Schwartz (1978). As shown by Gallant and White (1992), we can use $\widehat{J}_t = \partial \widehat{F}(X_t)/\partial X'$ as a nonparametric estimator of J_t, when (L, m, q) are selected to minimize BIC. The estimate of the dominant Lyapunov exponent then is

$$\widehat{\lambda} = \frac{1}{2N} \log|\widehat{v}_1(N)|,$$

where $\widehat{v}_1(N)$ is the largest eigenvalue of the matrix $\widehat{T}_N^{\prime}\widehat{T}_N$, and where $\widehat{T}_N = \widehat{J}_N\widehat{J}_{N-1}, ..., \widehat{J}_1$.

7. **The White Test**

White's (1989) has originated a test for nonlinearity. The time series is fitted by a single hidden-layer feed-forward neural network, which is used to determine whether any nonlinear structure remains in the residuals of an autoregressive (AR) process fitted to the same time series. The null hypothesis for the test is linearity in the mean relative to an information set. A process that is linear in the mean has a conditional mean function that is a linear function of the elements of the information set. In applications, that information set usually contains lagged observations on the process.

Under the null hypothesis of linearity in the mean, the residuals obtained by applying a linear filter to the process should not be correlated with any measurable function of the history of the process. White's test uses a fitted neural net to produce the measurable function of the process's history and an AR process as the linear filter. White's method then tests the hypothesis that the fitted function does not correlate with the residuals of the AR process. The resulting test statistic has an asymptotic χ^2 distribution under the null of linearity in the mean.

8. **The Kaplan Test**

The origins of Kaplan's (1994) test are in the chaos literature, although the test is currently being used as a test of linear stochastic process against general nonlinearity, whether or not noisy or chaotic. In the case of chaos, a time series plot of the output of a chaotic system may be very difficult to distinguish visually from a stochastic process. However, plots of the solution paths in phase space (x_{t+1} plotted against x_t and lagged values of x_t) often reveal deterministic structure that was not evident in a plot of x_t versus t. Kaplan's (1994) test is based upon continuity in phase space.

Deterministic solution paths, unlike stochastic processes, have the following property: points that are nearby are also nearby under their image in phase space. Using this fact, he produced a test statistic, which has a strictly positive lower bound for a linear stochastic process, but not for a nonlinear deterministic solution path. By computing the test statistic from an adequately large number of linear processes that plausibly might have produced the data, the approach can be used to test for linearity against the alternative of noisy nonlinear dynamics. The procedure involves producing linear stochastic process surrogates for the data and determining whether the surrogates or a noisy continuous nonlinear dynamical solution path better describe the data. Linearity is rejected, if the value of the test statistic from the surrogates is never small enough relative to the value of the statistic computed from the data.

9. **Evidence of Nonlinearity and Chaos with Economic Data**

Controversies exist regarding the existence of nonlinearity or chaos with economic data and with financial data, although the smaller sample sizes available with economic data have greated deeper differences in the macroeconomic literature than in the financial literature. Barnett and Serletis (2000) list seven studies that have used various economic time series to test for nonlinearity or chaos. Those seven studies provide a broad consensus of support for the proposition that the data generating processes are characterized by a pattern of nonlinear dependence, but there is no consensus at all on whether there is chaos in economic time series. For example, Brock and Sayers (1988), Frank and Stengos (1988), and Frank, Gencay, and Stengos (1988) find no evidence of chaos in U.S., Canadian, and international macroeconomic time series.

On the other hand, Barnett and Chen (1988), claimed successful detection of chaos in the U.S. Divisia monetary aggregates. Their conclusion was further confirmed by DeCoster and Mitchell (1991, 1994). This published claim of successful detection of chaos has generated considerable controversy, as in Ramsey, Sayers, and Rothman (1990) and Ramsey and Rothman (1994), who re-examined the data utilized in Barnett and Chen (1988) and argued that there is no evidence for the presence of chaos. In fact, Ramsey, Sayers, and Rothman (1990) and Ramsey and Rothman (1994) raised similar questions regarding virtually all of the other published tests of chaos.

Further results relevant to this controversy have been provided by Serletis (1995). Building on Barnett and Chen (1988), Serletis (1995) contrasts the random walk behavior of the velocity of money to chaotic dynamics, motivated by the notion that velocity follows a deterministic, dynamic, and nonlinear process which generates output that mimics the output of stochastic systems. In doing so, he tests for chaos using the Lyapunov exponent estimator of Nychka et al. (1992) and reports evidence of chaos in the Divisia L velocity series.

Although from a theoretical point of view, it would be very interesting to obtain empirical verification that macroeconomic series have actually been generated by deterministic chaotic systems, it is fair to say that those series are not the most suitable ones for the calculation of chaos indicators. There are at least two reasons. First of all, the sample sizes are small with regard to the calculations to be performed, since since economic data are usually recorded at best only monthly. Secondly, much economic data probably contain substantial noise, especially for aggregate time series such as GNP.

10. Evidence of Nonlinearity and Chaos with Financial Data

Barnett and Serletis (2000) surveyed the substantial published literature testing for nonlinear dynamics on financial data. For other unpublished work on testing nonlinearity and chaos on financial data, see Abhyankar, Copeland, and Wong (1997, table 1). The analysis of financial time series has led to results which are, as a whole, more reliable than those of macroeconomic series. The reason is the much larger sample sizes available with financial data and the superior quality of that financial data.

Scheinkman and LeBaron (1989) studied United States weekly returns on the Center for Research in Security Prices (CRSP) value-weighted index, employing the BDS statistic, and found strong evidence of nonlinearity and some evidence of chaos. Similar results have been obtained by Frank and Stengos (1989), investigating daily prices (from the mid 1970's to the mid 1980's) for gold and silver, using the correlation dimension and the Kolmogorov entropy. Their estimate of the correlation dimension was between six and seven for the original series and much greater and non-converging for the reshuffled data.

More recently, Serletis and Gogas (1997) test for chaos in seven East European black market exchange rates, using the Koedijk and Kool (1992) monthly data from January 1955 through May 1990. They use three inference methods, the BDS test, the NEGM test, and the Lyapunov exponent estimator of Gencay and Dechert (1992). They find some consistency in inference across methods, and conclude, based on the NEGM test, that there is evidence consistent with a chaotic nonlinear generation process in two out of the seven series: the Russian ruble and East German mark. Altogether, these and similar results suggest that financial series provide a more promising field of research than macroeconomic data for the methods in question.

A notable feature of the literature just summarized is that most researchers, in order to find sufficient observations to implement the tests, use data periods measured in years. The longer the data period, however, the less plausible is the assumption that the underlying data generation process has remained stationary, thereby making the results difficult to interpret. In fact different conclusions have been reached by researchers using high-frequency data over short periods. For example, Abhyankar, Copeland, and Wong (1995) examine the behavior of the U.K. Financial Times Stock Exchange 100 (FTSE 100) index over the first six months of 1993. They used returns from 1 minute up to 60 minutes. Applying the Hinich (1982) bispectral linearity test, the BDS test, and the NEGM test, they find evidence of nonlinearity, but no evidence of chaos.

More recently, Abhyankar, Copeland, and Wong (1997) test for nonlinear dependence and chaos in real-time returns on the world's four most important stock-market indices: the FTSE 100, the Standard & Poor 500 (S&P 500) index, the Deutscher Aktienindex (DAX), and the Nikkei 225 Stock Average. Using the BDS and the NEGM tests, and 15-second, 1-minute, and 5-minute returns from September 1 to November 30, 1991, they reject the hypothesis of independence in favor of a nonlinear structure for all data series, but find no evidence of low-dimensional chaotic processes.

There is other work, using high-frequency data over short periods, that finds order in the apparent chaos of financial markets. For example, the article by Ghashghaie, Breymann, Peinke, Talkner, and Dodge (1996) analyzes all worldwide 1,472,241 bid-ask quotes on U.S. dollar-German mark exchange rates between October 1, 1992 and September 30, 1993. That article applies physical principles and provides a mathematical explanation of how one trading pattern led into and then influenced another. As the authors conclude, "...we have reason to believe that the qualitative picture of turbulence that has developed during the past 70 years will help our understanding of the apparently remote field of financial markets". Nevertheless, despite the large sample sizes and high quality data often available with financial data, the literature on possible nonlinearity or chaos in financial data is not free from controversies. But those controversies are somewhat less dramatic than those produced by tests with macroeconomic data.

11. Sources of the Controversy

As surveyed in the previous two sections, there is little agreement about the existence of chaos or even of nonlinearity in economic and financial data, and some economists continue to insist that linearity remains a good assumption for such data, despite the fact that theory provides very little support for that assumption. It should be noted, however, that the available tests search for evidence of nonlinearity or chaos in data without restricting the boundary of the system that could have produced that nonlinearity or chaos. Hence these tests should reject linearity, even if the structure of the economy is linear, when the economy is subject to shocks from a surrounding nonlinear or chaotic physical environment, as through nonlinear climatological or weather dynamics. Under such circumstances, linearity would seem an unlikely inference.

Since the available tests are not structural and hence have no ability to identify the source of detected chaos, the alternative hypothesis of the available tests is that no natural deterministic explanation exists for the observed economic fluctuations anywhere in the universe. In other words, the alternative hypothesis is that economic fluctuations are produced by supernatural shocks or by inherent randomness in the sense of quantum physics. Considering the implausibility of the alternative hypothesis, one would think that findings of chaos in such nonparametric tests would produce little controversy, while any claims to the contrary would be subjected to careful examination. Yet, in fact the opposite seems to be the case. In the case of nonchaotic nonlinearity, the null hypothesis is even more difficult to take seriously. The null then is that no nonlinear dynamics exists within the economy or within the surrounding systems that shock the system.

It is reasonable to conjecture that the controversies might stem from the high noise level that exists in most aggregated economic time series and the relatively low sample sizes that are available with economic data. However, it is also possible that the controversies are produced by the nature of the tests themselves. The possibility of lack of robustness of competing tests was the subject of Barnett, Gallant, et al. (1995), who used five of the most widely used tests for nonlinearity or chaos with various monetary aggregate data series and acquired results that differed substantially across tests and over sample sizes. These results motivated the design of a competition described in the next section.

12. Single Blind Controlled Competition

Some tests may have higher power against certain alternatives than other tests, without any of the tests necessarily having higher power against all alternatives. Each of the tests may have its own comparative advantages. It also is possible there may be slight differences in the definitions of the null hypotheses among the competing tests. In that case, the tests may be noncomparable. In either of those two possible cases, there may be gains from using more than one of the tests in a sequence designed to narrow down the alternatives.

To explore these possibilities, William Barnett with the assistance of Mark Jensen designed and ran a single blind controlled competition, in which they produced simulated data from various processes having linear, chaotic, or nonlinear nonchaotic signals. They transmitted each simulated data set by e-mail to experts in running each of the statistical tests that were entered into the competition. The e-mailed data included no identification of the generating process, so those individuals who ran the tests had no way of knowing the nature of the data generating processes, other than their sample sizes. There were two sample sizes: a small sample size of 380 and a large sample size of 2000 observations.

Five generating models were used to produce samples of the small and large size. The models were a fully deterministic, chaotic Feigenbaum recursion, a generalized autoregressive conditional heteroskedasticity (GARCH) process, a nonlinear moving average process, an autoregressive conditional heteroskedasticity (ARCH) process, and an autoregressive moving average (ARMA) process. Details of the parameter settings and noise generation method can be found in Barnett, Gallant, et al. (1996). The tests entered into this competition were Hinich's bispectrum test, the BDS test, White's test, Kaplan's test, and the NEGM test of chaos.

The results of the competition are available in Barnett, Gallant, et al. (1997). Those results provide the most systematic available comparison among tests of nonlinearity and indeed do suggest differing powers of each test against certain alternative hypotheses. In comparing the results of the tests, however, one factor seemed to be especially important: subtle differences existed among the definitions of the null hypotheses. Some of the tests were tests of the null of linearity, defined in three different manners, and one test was a test of the null of chaos. Hence there were four null hypotheses that had to be considered in comparing each test's power against alternatives.

Since the tests do not all have the same null hypothesis, differences among them are not due solely to differences in power against alternatives. Hence one could consider using some of them sequentially in an attempt to narrow down the inference on the nature of the process. For example, the Hinich test and the White test could be used initially to determine whether the process lacks third order nonlinear dependence and is linear in the mean. If either test rejects its null, one could try to narrow down the nature of the nonlinearity further by running the NEGM test to see if there is evidence of chaos. Alternatively, if the Hinich and White tests both lead to acceptance of the null, one could run the BDS or Kaplan test to see if the process appears to be fully linear. If the data leads to rejection of full linearity but acceptance of linearity in the mean, then the data may exhibit stochastic volatility of the ARCH or GARCH type.

In short, the available tests provide useful information, and such comparisons of other tests could help further to narrow down alternatives. But ultimately we are left with the problem of isolating the nature of detected nonlinearity or chaos to be within the structure of the economy. This final challenge remains unsolved, especially in the case of chaos.

13. Testability of Chaos within the Economy

There has been considerable criticism of the existing research on chaos, as for example in Granger's (1994) review of Benhabib's (1992) book. However, it is unwise to take a strong opinion (either pro or con) in that area of research. Contrary to popular opinion within the profession, there have been no published tests of chaos within the structure of the economic system. The published tests of chaos in economic data test nonparametrically for evidence of chaos in the data without conditioning upon an economic model. As discussed above, if chaos is found, the test has no way of determining whether or not the source of the chaos is from within the structure of the economy or perhaps is from within the chaotic weather systems that surround the planet. To determine whether the source of chaos in economic data is from within the economic system, a model of the economy must be constructed. The null hypothesis then is that the parameters of the model are within the subset of the parameter space that supports the chaotic bifurcation regime of the dynamic system. Currently, however, we do not have the mathematical tools to find and characterize that subset, when more than three parameters exist. Hence, with any usable model of any economy, the set that defines the null hypothesis cannot be located. No one can test a null hypothesis that cannot be located and defined.

Since the hypothesis of chaos within the economic system has not been tested, we may instead wish to consider whether or not chaos is plausible on philosophical ground. On that basis, the question would be whether the economy should be viewed as having evolved naturally, as in the natural sciences, or was the product of intentional human design by economic 'engineers.' Systems intentionally designed by engineers to be stable should be stable and not chaotic, if designed optimally. Nature, however, was not designed by human beings, and is chaotic. The weather, for example, will never converge to a steady state. Which view is more appropriate to understanding the dynamics of actual economies is not clear.

But whether or not the economy is nonlinear or even chaotic, the implications of policy are unclear if the dynamics are nevertheless Pareto optimal. Since nonlinear or even chaotic solutions can be Pareto optimal, the case for policy intervention may be unrelated to whether or not the economy's dynamics are linear, nonlinear, or chaotic. Hence the most interesting results on nonlinearity or chaos would be acquired from within a model that possesses some forms of market failure, so that Pareto improving policy intervention is possible.

In the rest of this paper, we report on an initial attempt to explore the implications of unstable dynamics within a structural model having policy relevance. In that model, stable and unstable bifurcation regions of the parameter space exist, and dynamics produced from within the unstable region are not Pareto optimal. Stabilization policy designed to bifurcate the model from the unstable region into the stable region can be welfare improving.

14. An Application Conditionally upon an Economic Structure

Continuous time econometrics has been very important for dynamic disequilibrium modeling. The specification of econometric models in continuous time, rather than discrete time, has several advantages such as the characterization of the interaction between the variables during the unit observation period, more accurate representation of the partial adjustment processes in dynamic disequilibrium models, the independence of the unit of the observation period, and the capability of forecasting the continuous time path of the variables. An informative discussion of the advantages is provided by Bergstrom (1996). Since the development of the first continuous time macroeconometric model by Bergstrom and Wymer (1976), there has been a significant growth in the use of continuous time econometric methods in macroeconomic modeling. Economy-wide continuous time models have been developed for many of the leading industrial countries of the world. See Bergstrom et al. (1992). The idea is to model a system by a set of second order differential equations. An important feature of the continuous time modeling approach is that the estimator uses a derived, discretized model that is satisfied by the observations generated by the differential equation system irrespective of the observation interval of the sample, so that the properties of the parameter estimators of the differential equation system can be derived from the sampling properties of the discretized model. A recent survey was given by Bergstrom (1996).

Most research on continuous time models focuses on estimation and model building for various countries' economies. Continuous time economic models have been built, for example, for the United Kingdom in Bergstrom and Wymer (1976) and Knight and Wymer (1978), for the United States in Donaghy (1993), for the Netherlands in Nieuwenhuis (1994), and for Italy in Tullio (1972) and Gandolfo and Padoan (1990). A complete list of different models is provided in Bergstrom (1996). With these models available, the next stage of research is naturally performance analysis. It is important to understand the structural properties of the continuous time economic models. There are several papers dealing with stability of continuous time models. In particular, Bergstrom et al. (1992) and Donaghy (1993) examine the stability of the models for the United Kingdom and the United States economies, respectively. They notice that for the estimated parameter values, these models are slightly unstable. Bergstrom et al. (1994) analyze the effect of monetary and fiscal feedback controls on the stability of the UK model and find that simple fiscal policy feedbacks cannot stabilize the system. They further obtain a stabilizing controller, based on optimal control theory. Nieuwenhuis and Schoonbeek (1997) investigates the relationship between the stability of the continuous time models and the structure of the matrices appearing in the models. Their results are obtained by analyzing the dominant-diagonal structures of the matrices. Gandolfo (1992) considers sensitivity analysis of continuous time models and their use in investigating bifurcations. Wymer (1997) suggests the study of singularities and bifurcations of continuous time models.

This paper describes our recent effort in stability analysis of the Bergstrom et al. (1992) continuous time macroeconometric model of the United Kingdom. We use the approach developed by Barnett and He (1999) for finding the stability boundaries. Barnett and He (1999) is one of the papers produced by this ongoing project, which we anticipate will produce a series of papers, since progress in this direction requires solving a number of currently unsolved problems. We also describe some further preliminary results from the project, which is beginning to shed some light on the difficult question of whether stabilization policy can reasonably be expected to bifurcate an unstable model such that the model becomes stable.

15. The Model

In this ongoing project,we are using consider the Bergstrom, Nowman and Wymer (1992) continuous time macroeconometric model of the United Kingdom. The model is particularly well suited to this experiment, since it contains adjustment speeds that produce Keynesian rigidities and hence possible Pareto improving policy intervention. In addition, since the model consists of a system of second order differential equations, it can produce interesting dynamics and possesses enough equations and parameters to permit it to be fitted plausibly to the UK data.

To introduce the model, a set of variables are first defined.

Endogenous variables

C real private consumption

E_n real non-oil exports

F real current transfers abroad

I volume of imports

K amount of fixed capital

K_a cumulative net real investment abroad (excluding changes in official reserve)

L employment

M money supply

P real profits, interest and dividends from abroad

p price level

Q real net output

q exchange rate (price of sterling in foreign currency)

r interest rate

w wage rate

Exogenous variables

d_x dummy variable for exchange controls ($d_x = 1$ for 1974-79, $d_x = 0$ for 1980 onwards)

E_o real oil exports

G_c real government consumption

p_f price level in leading foreign industrial countries

p_i price of imports (in foreign currency)

r_f foreign interest rate

T_1 total taxation policy variable ($(Q+P)/T_1$ is real private disposable income)

T_2 indirect taxation policy variable (Q/T_2 is real output at factor cost)

t time

Y_f real income of leading foreign industrial countries

Then the dynamic behavior of the UK economy is described by the following 14 differential equations.

Model

$$D^2 \log C = \gamma_1(\lambda_1 + \lambda_2 - D\log C) + \gamma_2 \log\left[\frac{\beta_1 e^{-\{\beta_2(r - D\log p) + \beta_3 D\log p\}}(Q+P)}{T_1 C} \right] \quad (3)$$

$$D^2 \log L = \gamma_3(\lambda_2 - D\log L) + \gamma_4 \log\left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{L} \right] \tag{4}$$

$$D^2 \log K = \gamma_3(\lambda_1 + \lambda_2 - D\log K) + \gamma_6 \log\left[\frac{\beta_5 (Q/K)^{1+\beta_6}}{r - \beta_7 D\log p + \beta_8} \right] \tag{5}$$

$$\begin{aligned} D^2 \log Q = {} & \gamma_7(\lambda_1 + \lambda_2 - D\log Q) \\ & + \gamma_8 \log\left[\frac{\{1 - \beta_9 (qp/p_i)^{\beta_{10}}\}(C + G_c + DK + E_n + E_o)}{Q} \right] \end{aligned} \tag{6}$$

$$\begin{aligned} D^2 \log p = {} & \gamma_9(D\log(w/p) - \lambda_1) \\ & + \gamma_{10} \log\left[\frac{\beta_{11}\beta_4 T_2 w e^{-\lambda_1 t}\{1 - \beta_5 (Q/K)^{\beta_6}\}^{-(1+\beta_6)/\beta_6}}{p} \right] \end{aligned} \tag{7}$$

$$\begin{aligned} D^2 \log w = {} & \gamma_{11}(\lambda_1 - D\log(w/p)) + \gamma_{12} D\log(p_i/qp) \\ & + \gamma_{13} \log\left[\frac{\beta_4 e^{-\lambda_1 t}\{Q^{-\beta_6} - \beta_5 K^{-\beta_6}\}^{-1/\beta_6}}{\beta_{12} e^{\lambda_2 t}} \right] \end{aligned} \tag{8}$$

$$D^2 r = -\gamma_{14} Dr + \gamma_{15}\left[\beta_{13} + r_f - \beta_{14} D\log q + \beta_{15}\frac{p(Q+P)}{M} - r \right] \tag{9}$$

$$\begin{aligned} D^2 \log I = {} & \gamma_{16}(\lambda_1 + \lambda_2 - D\log(p_i I/qp)) \\ & + \gamma_{17} \log\left[\frac{\beta_9 (qp/p_i)^{\beta_{10}}(C + G_c + DK + E_n + E_o)}{(p_i/qp)I} \right] \end{aligned} \tag{10}$$

$$D^2 \log E_n = \gamma_{18}(\lambda_1 + \lambda_2 - D\log E_n) + \gamma_{19} \log\left[\frac{\beta_{16} Y_f^{\beta_{17}} (p_f/qp)^{\beta_{18}}}{E_n} \right] \tag{11}$$

$$D^2 F = -\gamma_{20} DF + \gamma_{21}[\beta_{19}(Q + P) - F] \tag{12}$$

$$D^2 P = -\gamma_{22} DP + \gamma_{23}\{[\beta_{20} + \beta_{21}(r_f - D\log p_f)]K_a - P\} \tag{13}$$

$$D^2 K_a = -\gamma_{24} DK_a + \gamma_{25}\{[\beta_{22} + \beta_{23}(r_f - r) - \beta_{24} D\log q - \beta_{25} d_x](Q + P) - K_a\} \tag{14}$$

$$\begin{aligned} D^2 \log M = {} & \gamma_{26}(\lambda_3 - D\log M) + \gamma_{27} \log\left[\frac{\beta_{26} e^{\lambda_3 t}}{M} \right] \\ & + \gamma_{28} D\log\left[\frac{E_n + E_o + P - F}{(p_i/qp)I} \right] + \gamma_{29} \log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I} \right] \end{aligned} \tag{15}$$

$$D^2 \log q = \gamma_{30} D \log(p_f/qp) + \gamma_{31} \log\left[\frac{\beta_{27} p_f}{qp} \right] + \gamma_{32} D \log\left[\frac{E_n + E_o + P - F}{(p_i/qp)I} \right]$$
$$+ \gamma_{33} \log\left[\frac{E_n + E_o + P - F - DK_a}{(p_i/qp)I} \right] \tag{16}$$

where D is the differential operator, $Dx = dx/dt$, $D^2x = d^2x/dt^2$, $\beta_i, i = 1, 2, ..., 27$, $\gamma_j, j = 1, 2, ..., 33$, and $\lambda_k, k = 1, 2, 3$, are structural parameters that can be estimated from historical data. A set of their estimates using quarterly data from 1974 to 1984 are given in Table 2 of Bergstrom et al. (1992). We use those point estimates. The exact interpretations of these 14 equations are omitted here, because they can be found in Bergstrom et al. (1992).

Equations (3)-(16) are nonlinear. To study the steady-state behavior, it was assumed in Bergstrom et al. (1992) that the exogenous variables satisfy the following conditions.

$$d_x = 0$$

$$E_o = 0$$

$$G_c = g^*(Q + P)$$

$$p_f = p_f^* e^{\lambda_4 t}$$

$$p_i = p_i^* e^{\lambda_4 t}$$

$$r_f = r_f^*$$

$$T_1 = T_1^*$$

$$T_2 = T_2^*$$

$$Y_f = Y_f^* e^{((\lambda_1 + \lambda_2)/\beta_{17})t}$$

where g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^* and λ_4 are constants.

Under the assumption of the exogenous variables, it can be shown that $C(t)$, ..., $q(t)$ in equations (3)-(16) change at constant rates in equilibrium. In what follows, we study the behavior of the system of differential equations (3)-(16) near equilibria. For this purpose, let the variables $y_1(t)$, $y_2(t)$, ..., $y_{14}(t)$ be defined as follows:

$$y_1(t) = \log\{C(t)/C^* e^{(\lambda_1 + \lambda_2)t}\}$$

$$y_2(t) = \log\{L(t)/L^* e^{\lambda_2 t}\}$$

$$y_3(t) = \log\{K(t)/K^* e^{(\lambda_1 + \lambda_2)t}\}$$

$$y_4(t) = \log\{Q(t)/Q^* e^{(\lambda_1 + \lambda_2)t}\}$$

$$y_5(t) = \log\{p(t)/p^* e^{(\lambda_3-\lambda_1-\lambda_2)t}\}$$

$$y_6(t) = \log\{w(t)/w^* e^{(\lambda_3-\lambda_2)t}\}$$

$$y_7(t) = r(t) - r^*$$

$$y_8(t) = \log\{I(t)/I^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_9(t) = \log\{E_n(t)/E_n^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{10}(t) = \log\{F(t)/F^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{11}(t) = \log\{P(t)/P^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{12}(t) = \log\{K_a(t)/K_a^* e^{(\lambda_1+\lambda_2)t}\}$$

$$y_{13}(t) = \log\{M(t)/M^* e^{\lambda_3 t}\}$$

$$y_{14}(t) = \log\{q(t)/q^* e^{(\lambda_1+\lambda_2+\lambda_4-\lambda_3)t}\}$$

where $C^*, L^*, K^*, Q^*, p^*, w^*, r^*, I^*, E_n^*, F^*, P^*, K_a^*, M^*, q^*$ are functions of the vector (β, γ, λ) of 63 parameters in equations (3)-(16) and the additional parameters g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, λ_4. Then an equilibrium of the system (3)-(16) corresponds to zero values of $y_i(t) = 0, i = 1, 2, ..., 14$. The set of equations satisfied by $y_i(t), i = 1, 2, ..., 14$, can be obtained from equations (3)-(16).

$$\begin{aligned} D^2 y_1 = &-\gamma_1 D y_1 + \gamma_2\{\log(Q^* e^{y_4} + P^* e^{y_{11}}) \\ &- \log(Q^* + P^*) - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1\} \end{aligned} \tag{17}$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4\left\{\frac{1}{\beta_6}\log\left[\frac{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}}{(Q^*)^{-\beta_6} e^{-\beta_6 y_4} - \beta_5 (K^*)^{-\beta_6} e^{-\beta_6 y_3}}\right] - y_2\right\} \tag{18}$$

$$\begin{aligned} D^2 y_3 = &-\gamma_5 D y_3 + \gamma_6 (1+\beta_6)(y_4 - y_3) \\ &+ \gamma_6\{\log[r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8] - \log[y_7 + r^* - \beta_7(D y_5 + \lambda_3 - \lambda_1 - \lambda_2) + \beta_8]\} \end{aligned} \tag{19}$$

$$\begin{aligned} D^2 y_4 = &-\gamma_7 D y_4 + \gamma_8\left\{\log\left[\frac{1 - \beta_9 (q^* p^*/p_i^*)^{\beta_{10}} e^{\beta_{10}(y_5+y_{14})}}{1 - \beta_9 (q^* p^*/p_i^*)^{\beta_{10}}}\right]\right\} \\ &+ \gamma_8\{\log(C^* e^{y_1} + g^*(Q^* e^{y_4} + P^* e^{y_{11}}) + K^* e^{y_3}(D y_3 + \lambda_1 + \lambda_2) + E_n^* e^{y_9})\} \\ &- \gamma_8\{\log(C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*) - y_4\} \end{aligned} \tag{20}$$

$$
\begin{aligned}
D^2y_5 \;=\; &\gamma_9(Dy_6 - Dy_5) + \gamma_{10}\left\{y_6 - y_5 - \frac{1+\beta_6}{\beta_6}\log\left[1 - \beta_5\left(\frac{Q^*}{K^*}\right)^{\beta_6} e^{\beta_6(y_4-y_3)}\right]\right\} \\
&+ \gamma_{10}\left\{\frac{1+\beta_6}{\beta_6}\log\left[1 - \beta_5\left(\frac{Q^*}{K^*}\right)^{\beta_6}\right]\right\}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
D^2y_6 \;=\; &\gamma_{11}(Dy_5 - Dy_6) - \gamma_{12}(Dy_5 + Dy_{14}) + \gamma_{13}\left\{\frac{1}{\beta_6}\log[(Q^*)^{-\beta_6} - \beta_5(K^*)^{-\beta_6}]\right\} \\
&- \gamma_{13}\left\{\frac{1}{\beta_6}\log[(Q^*)^{-\beta_6}e^{-\beta_6 y_4} - \beta_5(K^*)^{-\beta_6}e^{-\beta_6 y_3}]\right\}
\end{aligned} \tag{22}
$$

$$
D^2y_7 \;=\; -\gamma_{14}Dy_7 + \gamma_{15}\left[\beta_{15}\frac{p^*e^{y_5}(Q^*e^{y_4} + P^*e^{y_{11}})}{M^*e^{y_{13}}} - \beta_{15}\frac{p^*(Q^* + P^*)}{M^*} - \beta_{14}Dy_{14} - y_7\right] \tag{23}
$$

$$
\begin{aligned}
D^2y_8 \;=\; &\gamma_{16}(Dy_5 + Dy_{14} - Dy_8) + \gamma_{17}[(1+\beta_{10})(y_5 + y_{14}) - y_8] \\
&+ \gamma_{17}\{\log[C^*e^{y_1} + g^*(Q^*e^{y_4} + P^*e^{y_{11}}) + K^*e^{y_3}(Dy_3 + \lambda_1 + \lambda_2) + E_n^*e^{y_9}]\} \\
&- \gamma_{17}\{\log[C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*]\}
\end{aligned} \tag{24}
$$

$$
D^2y_9 \;=\; -\gamma_{18}Dy_9 - \gamma_{19}\{\beta_{18}(y_5 + y_{14}) + y_9\} \tag{25}
$$

$$
D^2y_{10} \;=\; -\{\gamma_{20} + 2(\lambda_1 + \lambda_2)\}Dy_{10} - (Dy_{10})^2 + \gamma_{21}\beta_{19}\left\{\frac{Q^*e^{y_4} + P^*e^{y_{11}}}{F^*e^{y_{10}}} - \frac{Q^* + P^*}{F^*}\right\} \tag{26}
$$

$$
\begin{aligned}
D^2y_{11} \;=\; &-\{\gamma_{22} + 2(\lambda_1 + \lambda_2)\}Dy_{11} - (Dy_{11})^2 \\
&+ \gamma_{23}\{\beta_{20} + \beta_{21}(r_f^* - \lambda_4)\}\left[\frac{K_a^*e^{y_{12}}}{P^*e^{y_{11}}} - \frac{K_a^*}{P^*}\right]
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
D^2y_{12} \;=\; &-[\gamma_{24} + 2(\lambda_1 + \lambda_2)]Dy_{12} - (Dy_{12})^2 \\
&+ \gamma_{25}[\beta_{22} + \beta_{23}(r_f^* - r^* - y_7) - \beta_{24}(Dy_{14} + \lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^*e^{y_4} + P^*e^{y_{11}}}{K_a^*e^{y_{12}}} \\
&- \gamma_{25}[\beta_{22} + \beta_{23}(r_f^* - r^*) - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^* + P^*}{K_a^*}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
D^2y_{13} \;=\; &-\gamma_{26}Dy_{13} - \gamma_{27}y_{13} \\
&+ \gamma_{28}\left\{\frac{E_n^*e^{y_9}Dy_9 + P^*e^{y_{11}}Dy_{11} - F^*e^{y_{10}}Dy_{10}}{E_n^*e^{y_9} + P^*e^{y_{11}} - F^*e^{y_{10}}} + Dy_5 + Dy_{14} - Dy_8\right\} \\
&+ \gamma_{29}\{\log[E_n^*e^{y_9} + P^*e^{y_{11}} - F^*e^{y_{10}} - K_a^*e^{y_{12}}(Dy_{12} + \lambda_1 + \lambda_2)]\} \\
&- \gamma_{29}\{\log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)] \; + y_5 + y_{14} - y_8\}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
D^2 y_{14} = & -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14}) \\
& + \gamma_{32}\left[\frac{E_n^* e^{y_9} Dy_9 + P^* e^{y_{11}} Dy_{11} - F^* e^{y_{10}} Dy_{10}}{E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}}} + Dy_5 + Dy_{14} - Dy_8 \right] \\
& + \gamma_{33}\{\log[E_n^* e^{y_9} + P^* e^{y_{11}} - F^* e^{y_{10}} - K_a^* e^{y_{12}}(Dy_{12} + \lambda_1 + \lambda_2)]\} \\
& - \gamma_{33}\{\log[E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)] + y_5 + y_{14} - y_8\} \qquad (30)
\end{aligned}
$$

Equations (17)-(30) form an autonomous system with equilibrium 0 for any parameter values of $\{\beta_i, \gamma_j, \lambda_k\}$. System (17)-(30) might have other equilibria. However, as a first step, we focus on the properties of the trajectories of the system (17)-(30) near the equilibrium 0.

16. Linearization of Macroeconometric Equations

Consider an ordinary differential equation

$$Dx(t) = f(x(t)) \tag{31}$$

where $x \in R^n$ is the state vector and the mapping $f(.) : R^n \to R^n$ is continuously differentiable (with respect to each argument). Suppose that $x^* \in R^n$ is a constant vector satisfying

$$f(x^*) = 0.$$

Then x^* is an equilibrium of the system. Let A be the Jacobian matrix of $f(x)$ evaluated at x^*

$$A = \frac{\partial f(x)}{\partial x}|_{x=x^*}.$$

Then the following system

$$Dy = Ay \tag{32}$$

is the linearized system of (31) around the equilibrium x^*. The advantage of linearization is that the local stability behavior of trajectories of the nonlinear system (31) in a neighborhood of the equilibrium x^* can be studied through that of its linearization (32). Briefly, if all eigenvalues of A have negative real parts, then (31) is stable in the neighborhood of x^*, in the sense that every trajectory approaches x^* as $t \to \infty$, when the initial state $x(0)$ is sufficiently close to x^*. If at least one of the eigenvalues of A has positive real part, then (31) is unstable in the neighborhood of x^*. In this case, there exists an initial state $x(0)$, arbitrarily close to x^*, for which $x(t)$ does not approach x^* as $t \to \infty$. If all eigenvalues of A have nonpositive real parts and at least one has zero real part, the stability of (31) usually cannot be determined from the matrix A. Then one needs to analyze higher order terms in order to determine the stability of the system. In that case, one needs to examine the system behavior along certain manifold to determine the stability; see Khalil (1992). In this paper, we only consider local stability around the equilibrium $x^* = 0$.

In many problems such as the continuous time macroeconomic system (17)-(30), the function $f(x)$, and consequently the coefficient matrix • of the corresponding linearized system (32), depend on parameters. In this case, write (32) in the following form

$$Dy = A(\theta)y, \tag{33}$$

where $\theta \in \Theta$ is the vector of parameters taking values in the parameter space Θ. Since θ may change eigenvalues of • (θ), the stability of (32) might depend on θ.

To determine stability boundaries of the continuous time macroeconometric system (17)-(30), we consider its linearization. The parameters θ are chosen to be those that were estimated by

Bergstrom et al. (1992):

$$\theta = [\beta_1, ..., \beta_{27}, \gamma_1, ..., \gamma_{33}, \lambda_1, \lambda_2, \lambda_3]',$$

where $\theta \in R^{63}$ is a 63-dimensional column vector. The feasible region Θ is specified by the parameter bounds given in Table 2 of Bergstrom et al. (1992).

The linearized system of (17)-(30) is

$$D^2 y_1 = -\gamma_1 D y_1 + \gamma_2 \left[\frac{Q^* y_4 + P^* y_{11}}{Q^* + P^*} - \beta_2 y_7 + (\beta_2 - \beta_3) D y_5 - y_1 \right] \tag{34}$$

$$D^2 y_2 = -\gamma_3 D y_2 + \gamma_4 \left[\frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} - y_2 \right] \tag{35}$$

$$D^2 y_3 = -\gamma_5 D y_3 + \gamma_6 \left[(1 + \beta_6)(y_4 - y_3) - \frac{y_7 - \beta_7 D y_5}{r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_8} \right] \tag{36}$$

$$\begin{aligned} D^2 y_4 = {} & -\lambda_7 D y_4 + \gamma_8 \left[-y_4 - \frac{\beta_9 (q^* p^* / p_i^*)^{\beta_{10}}}{1 - \beta_9 (q^* p^* / p_i^*)^{\beta_{10}}} \beta_{10} (y_5 + y_{14}) \right] \\ & + \gamma_8 \left[\frac{C^* y_1 + g^* (Q^* y_4 + P^* y_{11}) + K^* D y_3 + K^* (\lambda_1 + \lambda_2) y_3 + E_n^* y_9}{C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n^*} \right] \end{aligned} \tag{37}$$

$$D^2 y_5 = \gamma_9 (D y_6 - D y_5) + \gamma_{10} \left[(1 + \beta_6) \frac{\beta_5 (Q^*/K^*)^{\beta_6}}{1 - \beta_5 (Q^*/K^*)^{\beta_6}} (y_4 - y_3) + y_6 - y_5 \right] \tag{38}$$

$$D^2 y_6 = \gamma_{11} (D y_5 - D y_6) - \gamma_{12} (D y_5 + D y_{14}) + \gamma_{13} \frac{(Q^*)^{-\beta_6} y_4 - \beta_5 (K^*)^{-\beta_6} y_3}{(Q^*)^{-\beta_6} - \beta_5 (K^*)^{-\beta_6}} \tag{39}$$

$$\begin{aligned} D^2 y_7 = {} & -\gamma_{14} D y_7 + \gamma_{15} (-\beta_{14} D y_{14} - y_7) \\ & + \gamma_{15} \left\{ \frac{\beta_{15}}{M^*} [(Q^* + P^*) p^* (y_5 - y_{13}) + p^* (Q^* y_4 + P^* y_{11})] \right\} \end{aligned} \tag{40}$$

$$\begin{aligned} D^2 y_8 = {} & \gamma_{16} (D y_5 + D y_{14} - D y_8) + \gamma_{17} [(1 + \beta_{10})(y_5 + y_{14}) - y_8] \\ & + \gamma_{17} \left[\frac{C^* y_1 + g^* (Q^* y_4 + P^* y_{11}) + K^* (\lambda_1 + \lambda_2) y_3 + K^* D y_3 + E_n^* y_9}{C^* + g^* (Q^* + P^*) + K^* (\lambda_1 + \lambda_2) + E_n^*} \right] \end{aligned} \tag{41}$$

$$D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19} \{ \beta_{18} (y_5 + y_{14}) + y_9 \} \tag{42}$$

$$D^2 y_{10} = -[\gamma_{20} + 2(\lambda_1 + \lambda_2)] D y_{10} + \frac{\gamma_{21} \beta_{19}}{F^*} [Q^* (y_4 - y_{10}) + P^* (y_{11} - y_{10})] \tag{43}$$

$$D^2 y_{11} = -[\gamma_{22} + 2(\lambda_1 + \lambda_2)] D y_{11} + \gamma_{23} [\beta_{20} + \beta_{21} (r_f^* - \lambda_4)] \frac{K_a^*}{P^*} (y_{12} - y_{11}) \tag{44}$$

$$
\begin{aligned}
D^2 y_{12} &= -[\gamma_{24} + 2(\lambda_1 + \lambda_2)]Dy_{12} + \gamma_{25}\left[-\beta_{24}\frac{Q^* + P^*}{K_a^*}Dy_{14} - \beta_{23}\frac{Q^* + P^*}{K_a^*}y_7 \right] \\
&\quad + \gamma_{25}\left\{ [\beta_{22} + \beta_{23}(r_f^* - r^*) - \beta_{24}(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)]\frac{Q^*(y_4 - y_{12}) + P^*(y_{11} - y_{12})}{K_a^*} \right\}
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
D^2 y_{13} &= -\gamma_{26}Dy_{13} - \gamma_{27}y_{13} \\
&\quad + \gamma_{28}\left[\frac{E_n^* Dy_9 + P^* Dy_{11} - F^* Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8 \right] \\
&\quad + \gamma_{29}\left[\frac{E_n^* y_9 + P^* y_{11} - F^* y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^* Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8 \right]
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
D^2 y_{14} &= -\gamma_{30}(Dy_5 + Dy_{14}) - \gamma_{31}(y_5 + y_{14}) \\
&\quad + \gamma_{32}\left[\frac{E_n^* Dy_9 + P^* Dy_{11} - F^* Dy_{10}}{E_n^* + P^* - F^*} + Dy_5 + Dy_{14} - Dy_8 \right] \\
&\quad + \gamma_{33}\left[\frac{E_n^* y_9 + P^* y_{11} - F^* y_{10} - K_a^*(\lambda_1 + \lambda_2)y_{12} - K_a^* Dy_{12}}{E_n^* + P^* - F^* - K_a^*(\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8 \right]
\end{aligned}
\tag{47}
$$

or in matrix form

$$\dot{x} = A(\theta)x \tag{48}$$

where

$$x = [y_1 \quad Dy_1 \quad y_2 \quad Dy_2 \quad \ldots \quad y_{14} \quad Dy_{14}]' \in R^{28},$$

and $A(\theta) \in R^{28\times 28}$ is the coefficient matrix. For the estimated values of $\{\beta_i\}$, $\{\gamma_j\}$, and $\{\lambda_k\}$ in Table 2 of Bergstrom et al. (1992), all the eigenvalues of $A(\theta)$ are stable (having negative real parts) except for three

$$s_1 = 0.0033, \;\; s_2 = 0.0090 + 0.0453i, \;\; s_3 = 0.0090 - 0.0453i,$$

where $i = \sqrt{-1}$ is the imaginary unit. However, the real parts of the unstable eigenvalues are (relatively) so small that it is unclear whether they are caused by errors in estimation or by the structural properties of the system itself.

Note that the system (17)-(30), or the linearized system (34)-(47), operates in locally unstable region. We are interested in locating the stable region and the boundary. Our approach is to first find a stable sub-region of Θ and then expand the sub-region to find its boundary. To this end, we next find a parameter vector $\theta^* \in \Theta$ such that (48) is stable. From this θ^* we will find the stable region of θ and the stability boundaries. We use the gradient method to find a θ^* such that all eigenvalues of $A(\theta^*)$ have strictly negative real parts.

To find such a θ^*, we consider the following problem of minimizing the maximum real part of eigenvalues of the matrix $A(\theta)$:

$$\min_{\theta\in\Theta} R_{\max}(A(\theta)) \tag{49}$$

where

$$R_{\max}(A(\theta)) = \max_i \{\text{real}(\lambda_i) : \lambda_1, \lambda_2, \ldots, \lambda_{28} \text{ are eigenvalues of } A(\theta)\}.$$

Since the dimension of A is 28, which is relatively high, it is infeasible to have a closed-form expression for $R_{\max}(A(\theta))$. We use the gradient method to solve the minimization problem (49). The details of the procedure that we use are provided in Barnett and He (1999). After several iterations (20 iterations in this case), the algorithm arrived at θ^*, which is a point located in the stable region. Again see Barnett and He (1999) for determination of that starting point. Starting from this stable point, we found the stable region of the parameter space and the stability boundaries.

17. Determination of Stability Boundaries

In this section we describe the Barnett and He (1999) method of finding the stability boundaries. Since the linearized system (48) only determines the local stability of (17)-(30), we are dealing with local stability.

The system (17)-(30) can be written as

$$Dx = A(\theta)x + F(x,\theta) \tag{50}$$

where $F(x,\theta) = O(x^2)$ includes terms of higher orders.

On one hand, we have seen in the previous section that $A(\theta)$ has three eigenvalues with strictly positive real parts for the set of parameter values given in Table 2 of Bergstrom et al. (1992). On the other hand, all eigenvalues of (48) have strictly negative real parts for $\theta = \theta^*$. Since eigenvalues are continuous functions of entries of $A(\theta)$, there must exist at least one eigenvalue of $A(\theta)$ with zero real part on the stability boundary. Two types of stability boundaries may occur according to the way unstable eigenvalues are created.

Transcritical bifurcation boundary

A transcritical bifurcation occurs when the system has an equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and additional transversality conditions are satisfied (see for example Gandolfo (1996) for the exact conditions).

When $det(A(\theta)) = 0$, $A(\theta)$ has at least one zero eigenvalue. The first condition we use to find the transcritical bifurcation boundary is

$$det(A(\theta)) = 0. \tag{51}$$

Note that $A(\theta)$ in the linearized system (48) is a sparse matrix. Analytical forms of stability boundaries can be obtained for most parameters. To demonstrate the feasibility of this approach, we consider finding the stability boundaries for β_2 and β_5.

Theorem 1. The stability boundary for β_2 and β_5 is determined by

$$1.36\beta_2\beta_5 + 21.78\beta_5 - 2.05\beta_2 - 10.05 = 0. \tag{52}$$

Proof. See Barnett and He (1999). □

The boundary (52) is illustrated as the dashed line in Figure 1.

Hopf bifurcation boundary

A Hopf bifurcation occurs at points at which the system has a pair of purely imaginary but non-zero eigenvalues, and additional transversality conditions are satisfied. For the Hopf Theorem, see Guckenheimer and Holmes (1983). Consider the case of $det(A(\theta)) \neq 0$ but with $A(\theta)$ having at least one pair of purely imaginary eigenvalues (with zero real parts and non-zero imaginary parts).

To find Hopf bifurcation boundaries, let $p(s) = det(sI - A(\theta))$ be the characteristic polynomial of $A(\theta)$, and express it as

$$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + ... + c_{n-1} s^{n-1} + s^n$$

where $n = 28$ for the system (48). Construct the following $(n-1)$ by $(n-1)$ matrix

$$S = \begin{bmatrix} c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & 0 & \ldots & 0 \\ 0 & c_0 & c_2 & \ldots & c_{n-2} & 1 & 0 & \ldots & 0 \\ & \ldots & & & & & & \ldots & \\ 0 & 0 & \ldots & 0 & c_0 & c_2 & c_4 & \ldots & 1 \\ c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & & \ldots & 0 \\ 0 & c_1 & c_3 & \ldots & c_{n-1} & 0 & 0 & \ldots & 0 \\ & \ldots & & & & & & \ldots & \\ 0 & 0 & & \ldots & 0 & c_1 & c_3 & \ldots & c_{n-1} \end{bmatrix}$$

Let S_0 be obtained by deleting rows 1 and n/2 and columns 1 and 2, and let S_1 be obtained by deleting rows 1 and n/2 and columns 1 and 3. Then the following theorem of Guckenheimer et al. (1997) gives a condition for $A(\theta)$ to have exactly one pair of purely imaginary eigenvalues.

Theorem 2. The matrix $A(\theta)$ has precisely one pair of pure imaginary eigenvalues if

$$det(S) = 0, \quad det(S_0) * det(S_1) > 0.$$

If $det(S) \neq 0$ or if $det(S_0) * det(S_1) < 0$, $A(\theta)$ has no pure imaginary eigenvalues. □

If $det(S) = 0$ and $det(S_0) * det(S_1) = 0$, then A may have more than one pair of pure imaginary eigenvalues. Therefore, the condition for the Hopf bifurcation boundary is

$$det(S) = 0, \quad det(S_0) * det(S_1) \geq 0. \tag{53}$$

We will use (53) to determine candidates of the Hopf bifurcation boundaries and determine segments of true boundaries. In principle, the approach outlined in the proof of Theorem 1 can also be applied to find Hopf bifurcation boundaries. However, in most cases, analytical formula of $det(S)$ is not available. A numerical procedure for finding the bifurcation boundaries is provided by Barnett and He (1999). We use that procedure in the next section.

18. Sections of the Stable Region

In this section, Barnett and He's (1999) numerical procedure is used to find explicit stability boundaries for several sets of parameters. In order to be able to view the boundaries, we only consider two or three parameters, so that the regions that we display are two and three dimensional sections of the stable region, which is in a much higher dimensional space. The procedure is applicable to any number of parameters.

Section I: β_2 and β_5

We first find the stability boundaries for β_2 and β_5 for the system (48). The result is illustrated in Figure 1 in which the dashed line is given by $det(A(\theta)) = 0$. The solid line is the set of parameter pairs satisfying (53). The shaded area shows the stable region. All other regions produce an unstable system. It can also be seen from Figure 1 that the segment of the dashed line defining the stable region is the boundary of transcritical bifurcation boundaries while the other segment of the same line is not a stability boundary at all. Similarly, the segment of the solid line defining the stable region is a Hopf bifurcation boundary. The other part of the solid line is not a stability boundary. The stability behavior of (48) along the stability boundaries is unclear and is a subject of ongoing research.

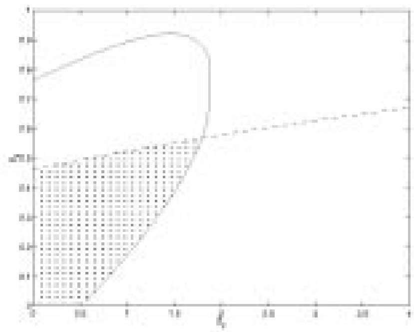


Fig. 1. Stability boundaries for β_2, β_5.

Of particular interest is the intersection point of the two stability boundaries. That point is approximately $(\beta_2, \beta_5) = (1.785, 0.566)$. At this point the coefficient matrix has three eigenvalues with zero real parts: $s_1 = 0.0000, s_2 = -0.0000 + 0.0336i, s_3 = -0.0000 - 0.0336i$. See Barnett and He (1999) for the trajectories of x and phase portraits of (x_1, x_{10}, x_{27}) when the parameters cross the two boundaries.

Section II: β_2, β_5, and β_{15}

We now add the parameter β_{15} to (β_2, β_5) to produce a three dimensional section. Use again the Barnett and He (1999) numerical procedure , we find the surface of the stability boundary for β_2, β_5, and β_{15} as shown in Figure 2.

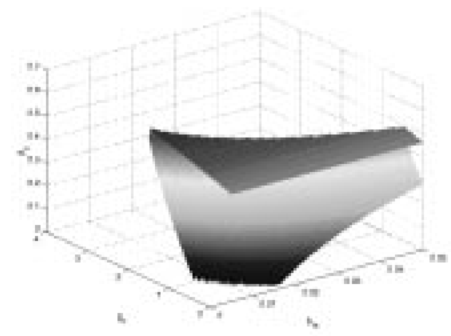


Fig. 2. Stability boundary for $\beta_2, \beta_5, \beta_{15}$.

Section III: γ_8 and β_{15}

In this case, we find stability boundaries for the parameters γ_8 and β_{15}. The result is illustrated in Figure 3, in which only Hopf bifurcation boundaries exist. The shaded area shows the stable region of the resulting 2-dimensional section.

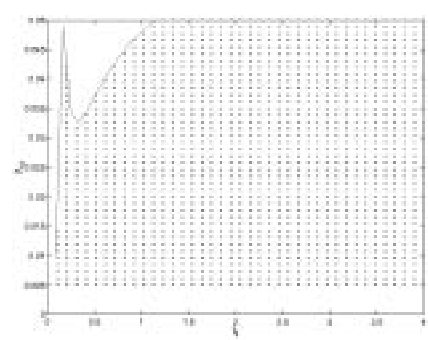
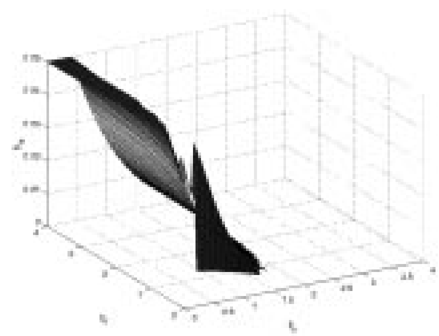


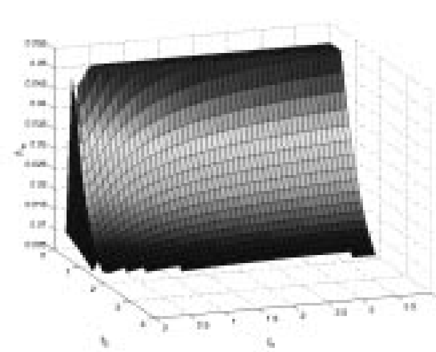
Fig. 3. Stability boundary for γ_8,β_{15}.

Section IV: γ_8, β_{15}, and β_2

We now introduce β_2 a third dimension to produce the three dimensional stability boundary for γ_8, β_{15}, and β_2. Similar to the 2-dimensional section III, only Hopf bifurcation boundaries exist. The following figure illustrates the boundary viewed from two different directions.



(a)



(b)

Fig. 4. Stability boundary for γ_8, β_{15}, and β_2.

The next step in this experiment is to introduce a possible stabilization policy. Since the point estimates of the parameters without policy are in the unstable region, one might hope that a plausible choice of stabilization policy might bifurcate the system into the stable region. This experiment could be conducted by first adjoining to the model a feedback stabilization policy based upon plausible macroeconomic reasoning. Since such a policy equation would itself have parameters, the result would be to increase the dimension of the parameter space. The above two and three dimensional sections could be produced again, but now through the higher dimensional stable set resulting from adjoining a stabilization policy equation to the model. If the policy is successful, then the point estimates of the parameters of the original model now will be inside the stable region.

As suggested by this possible approach, stabilization policy can be viewed as a form of bifurcation selection. But it is well known in the engineering literature that bifurcation selection is a particulary difficult area of nonlinear dynamics, and hence macroeconomic reasoning may be inadequate to assure that the system will successfully bifurcate from instability to stability. We have so far tried this experiment with the fiscal policy feedback rule proposed by Bergstrom et al (1992). We have produced two sections through the resulting higher dimensional stable region, and we have found that the policy tends to move the stable region away from the point estimates of the model's parameters. Hence that policy seems to be potentially counterproductive. But the parameter space dimension is high, and two sections through the stable set are not adequate to support our tentative conclusion. Considering the computational cost and time involved in producing these sections, we shall need to complete considerably more work before we shall be able to provide a confident conclusion about that stabilization policy.

We also produced a stabilization policy derived from optimal control theory. That policy does successfully bifurcate the system to stability. But the resulting feedback rule is very complicated and so difficult to interpret in terms of economics that the policy would not likely be usable by a governement. In addition, policies produced by optimal control theory are sensitive to specification error of the model used to derive the policy and are potentially time inconsistent.

19. Conclusions

The single blind controlled competition demonstrated that the competing tests are often not directly comparable, since their null hypotheses often are not the same, and the tests have power against different alternatives. Under these circumstances, it is tempting to suggest using all of the tests as a means of narrowing down the nature of the dynamics in the data. But this presents pretesting problems and cannot resolve the problems associated with isolating the source of the dynamics to be within the economic system, since inference regarding nonlinear dynamics in economic time series using univariate time series methods cannot distinguish between nonlinear produced from within the system and nonlinearity induced into the data from nonlinear shocks originating outside the economic system.

To address these problems, we have begun work on experiments within the structure of macroeconomic models. We proposed a procedure for determining the stability boundaries within the plausible range of parameter values for the Bergstrom, Nowman and Wymer continuous time macroeconometric model. This paper reports on the first results from an ongoing research project along with tentative unfinished results. In a sense, that part of this paper comprises a progress report on a complex ongoing project that is likely to produce numerous papers.

At a later date we plan to apply our approach to the more difficult problem of exploring bifurcation policy in a modern stochostic dynamic general equilibrium growth model. In short, the current results are only a first step, but are critical as motivation for the future research we now contemplate.

The primary conclusions from these two related projects are that testing, isolating, and using nonlinearity in macroeconomics requires careful understanding and application of difficult literatures in engineering, mathematics, and statistics. Taking strong positions a priori about the existence of nonlinearity or chaos or advocating stabilization policies based upon easily interpreted macroeconomic reasoning, when the system is unstable, nonlinear dynamic without policy, may be unjustified. A deeper level of sophistication is needed to produce results that can be substantiated in a convincing manner, and much more research in these difficult areas is needed before the implications for econometrics and macroeconomics can be determined.

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