Local Interactions and Global Persistence*

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Abstract

This paper studies the effects of local interactions on employment distributions in a Keynesian-type model with strategic complementarities. It is shown that rational expectations generate symmetric equilibria for any interaction structure except autarky. Under adaptive expectations, the distribution of employment converges to a symmetric rational expectations equilibrium, implying that inequality cannot persist in the long run. On the basis of both analytical and computational results, however, it is shown that symmetric equilibria are unstable in the sense that, in the presence of noise, local interactions can produce globally persistent inequality.

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1 Introduction

Markets are systems in which large numbers of interacting agents operate in parallel. In a standard Arrow-Debreu framework, the interactions between these agents are 'global'. That is, conditional on equilibrium prices, the buyers and sellers of a given good are indifferent as to who to trade with, hence each agent has an equal probability of trading with any other agent in the economy.

The assumption of 'global interactions' plays an important role in macroeconomics. By appealing to laws of large numbers, it enables us to solve important aggregation problems. By allowing for the construction of 'representative agents', the complexities of multi-agent systems can be significantly reduced.

In real world markets, however, there is no doubt that the assumption of global interactions is violated. Even in today's 'global village', there are many reasons why agents are unable or unwilling to trade with each other. Trade volumes still tend to decrease with distance, in spite of substantial reductions in transportation costs. [1] Due to imperfect information and incomplete contracts, agents that are unknown to each other tend to be less likely to trade. [2] Because of legal constraints, agents cannot choose to live and work wherever they want. And even if they could, cultural differences or language barriers may keep them from moving, or may bias them towards consumption of home-produced goods and services.

In this paper we will focus on the consequences, rather than the causes, of local interactions. In particular, we will study the effects of local interactions on the steady state distributions of employment and income in a Cooper and John (1988) type Keynesian model with strategic complementarities.

The setup of the paper is as follows. First, section 2 presents the basic notation and assumptions of the model. Next, section 3 discusses its equilibrium properties under various assumptions. It will be shown that rational expectations generate symmetric equilibria for any interaction topol-

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[1] This is true even when one controls for spatial correlations in income. As Deardorff (1998) notes, for instance: "It has long been recognized that bilateral trade patterns are well described empirically by the so-called gravity equation, which relates trade between two countries positively to both of their incomes and negatively to the distance between them, usually with a functional form that is reminiscent of the law of gravity in physics." (p. 7). For empirical evidence, see, e.g., Bergstrand (1985) and Frankel (1998).

[2] See, e.g. Williamson (1985) on asset specificity, Kranton (1996) on search costs, Fafchamps (1998) on reputation, and Fukuyama (1995) on trust.

ogy, while under adaptive expectations, the distribution of employment converges to a rational expectations equilibrium, implying that spatial inequality cannot persist in the long run. As a special case, section 4 studies a binary version of the model, which turns it into an evolutionary coordination game. Stochastic versions of this game are analyzed in Section 5. The major finding, supported by extensive computational experiments, is that symmetric equilibria are unstable in the presence of noise, implying that local interactions can indeed generate global persistence. Section 6 interprets and concludes.

2 Description of the Model

We consider an economy with two types of agents, workers and firms, and three types of goods. In addition, the economy has a spatial structure, a time structure, and a technological structure, the details of which are described below.

2.1 Space

Agents live on a finite two-dimensional lattice $S \subset \mathbb{Z}^2$. Each site of the lattice is inhabited by one (representative) worker and one (representative) firm, who are indexed by their coordinates $x = (i, j)$.[3]

The interaction structure between agents is represented as a matrix $\mathbf{P}$ with elements $p(x, y)$, denoting the fraction of income spent by the worker at site x on the goods produced at site y. The interaction strength between x and y is assumed to decrease with their distance, i.e. $p(x, y) = \psi(|x - y|)$, where $|x - y|$ denotes the Euclidean metric, and $\psi(\cdot)$ is a monotonically decreasing function, which in geographic space can be interpreted as resulting from the existence of transportation or travel costs.[4] To simplify the model, however, the selection of trade partners is not explicitly modelled, hence $\mathbf{P}$ will be taken as exogenous.[5] In particular, two special cases

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[3] Alternatively, each worker-firm pair could be thought of as a single representative agent, or a social planner. This interpretation will be used in section 4. For now, however, we will treat the worker and the firm as separate entities, possibly representing a unit mass of workers and a unit mass of firms.

[4] Alternatively, one could think of S as a product characteristics space, where the preferences of agents are such that the utility of goods is decreasing in the distance of their characteristics from the goods they produce themselves. Formally, this could be modelled by a Cobb Douglas utility function, the coefficients of which represent the equilibrium fractions of income that each agent would like to spend on each good.

[5] An important extension for future work will be to endogenize the interaction structure by allowing the selection

will be analyzed:

- Global interactions: $p(x,y) = |S|^{-1}, \qquad \forall x,y$
- Local Interactions: $p(x,y) = \begin{cases} 0 & \text{if } y \notin N_x \\ |N_x|^{-1} & \text{if } y \in N_x, \end{cases}$

where N_x indicates the 'trade neighborhood' (set of trade partners) of x, and $|S|$ and $|N_x|$ denote the total number of agents and of 'neighbors', respectively. Note that, for $N_x = S$, the second case reduces to the first.

2.2 Time

Time is discrete, and is structured as follows:

- At the beginning of each period, workers are hired and goods are sold;
- During each period, goods are produced;
- At the end of each period, workers are paid, and firms determine how much labor to hire in the next period.

2.3 Goods

There are three goods in this economy: a labor commodity η_t, a composite market good y_t, and a composite nonmarket or 'home-made' good h_t. All goods are nonstoreable. Notation is as follows:

$\eta_t^d(x)$	demand for labor to be employed during period t at site x
$\eta_t^s(x)$	supply of labor to be employed during period t at site x
$\eta_t(x)$	labor employed during period t at site x
η_t	employment configuration during period t
$\hat{\eta}_t$	employment rate during period t

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of trade partners to depend on differentials in price levels and transportation costs.

$y_t^d(x)$	demand for market goods at the end of period t at site x
$y_t^s(x)$	supply of market goods at the end of period t at site x
$y_t(x)$	market goods sold at the end of period t at site x
$h_t(x)$	nonmarket goods produced and consumed during period t at site x

The labor commodity $\eta_t(x)$ is assumed to have countable support $\Omega = \{0, \frac{1}{n} \ldots \frac{n-1}{n}, 1\}$. This assumption is mainly made for convenience, and may be interpreted so as to mean that labor can only be hired in discrete quantities (e.g., hours). A state of the economy, then, is a spatial employment distribution $\eta_t \in \Omega^S$.

2.4 Technology

Each worker is endowed with one homogeneous unit of labor time per period, of which $\eta_t(x)$ is allocated to market activities (working for the firm), and $1 - \eta_t(x)$ is allocated to nonmarket activities (e.g. farming, household work, crime, or any other activity that enable workers to survive outside the labor market). The production processes for both activities are assumed to be constant returns to scale, with market production assumed to be more efficient. Normalizing the marginal productivity of market work to one, this gives a pair of production functions for each site x:

$$y_t(x) = \eta_t(x) \tag{1}$$

$$h_t(x) = \sigma[1 - \eta_t(x)], \tag{2}$$

where $\sigma < 1$ indicates the marginal productivity of nonmarket activities.

2.5 Workers

From the perspective of consumption, market goods and nonmarket goods are perfect substitutes. Hence, each worker's preferences may be represented by a utility function of the form

$$u_t(x) = \phi(h_t(x) + c_t(x)), \tag{3}$$

where $\phi(\cdot)$ is a monotonically increasing function; $h_t(x)$ is the amount of goods produced and consumed at home, and $c_t(x)$ denotes the amount of consumption goods purchased in the market.

Since nonmarket goods, by definition, have no market value, market goods can only be purchased in exchange for income earned in the market, which is simply the market wage $w_t(x)$ times the number of hours worked for the firm $\eta_t(x)$. To simplify the analysis, I will assume that borrowing or saving is impossible, and that the firm's profits $\pi_t(x)$ are transferred to the worker at the end of each period. [6] At the beginning of each period, then, each worker solves the following problem:

$$\begin{aligned} \max_{h_t(x),c_t(x)} \quad & \phi(h_t(x) + c_t(x)) \\ \text{s.t.} \quad & c_t(x) \le w_t(x)\eta_t(x) + \pi_t(x) \\ & h_t(x) \le \sigma[1 - \eta_(x)] \end{aligned} \tag{4}$$

This gives the following labor supply curve:

$$\eta_t^s(x) \begin{cases} = 0 & \text{if } w_t(x) < \sigma \\ \in \Omega & \text{if } w_t(x) = \sigma \\ = 1 & \text{if } w_t(x) > \sigma, \end{cases} \tag{5}$$

where Ω denotes the support of $\eta_t(x)$.

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[6] Since the worker takes these profits as given, this has no effect on the worker's decisions.

2.6 Firms

At the end of each period, each firm maximizes expected profits so as to determine the amount of labor it wishes to hire in the next period:

$$\begin{aligned} \max_{\eta_t(x)} \quad & E_{t-1}\pi_t(x) = E_{t-1}y_t(x) - w_t(x)\eta_t(x) \\ \text{s.t.} \quad & y_t^s(x) \leq \eta_t(x) \\ & y_t(x) = \min\{y_t^d(x), y_t^s(x)\}, \end{aligned} \tag{6}$$

where E_{t-1} is the subjective expectations operator, conditional on information at the end of period $t-1$.

The first constraint says that the supply of market goods must be within the production possibility set; the second that firms cannot sell as much as they want. Assuming that the trade structure $\mathbf{P}$ is common knowledge, the firm at x should expect the demand for its goods to be

$$E_{t-1}y_t^d(x) = \sum_{y \in S} p(y,x) E_{t-1}c_t(y), \tag{7}$$

where $p(y,x)$ denotes the proportion of y's market income that is spent at site x, so that

$$\sum_{y \in S} p(y,x) = \sum_{x \in S} p(y,x) = 1. \tag{8}$$

Substituting this into the firm's objective function gives

$$\max_{\eta_t(x)} \quad E_{t-1}\pi_t \quad = \min\left\{ \sum_{y \in S} p(y,x) E_{t-1}c_t(y), \eta_t(x) \right\} - w_t(x)\eta_t(x). \tag{9}$$

This yields the labor demand curve:

$$\eta_t^d(x)\begin{cases} = \sum_y p(y,x)E_{t-1}c_t(y) & \text{if } w_t(x) \leq 1 \\ \in [0, \sum_y p(y,x)E_{t-1}c_t(y)] & \text{if } w_t(x) = 1 \\ = 0 & \text{if } w_t(x) > 1. \end{cases} \tag{10}$$

2.7 Equilibrium

Definition 1 *An equilibrium of this model is a set of allocations η_t, y_t, c_t, and h_t, trade structure $\mathbf{P}$, and prices w_t and π_t, such that, for all $x, y \in S, x \neq y$,*

(i) given expectations $E_{t-1}c_t(y)$, trade structure $\mathbf{P}$, and wages $w_t(x)$, each firm chooses $\eta_t^d(x)$ and $y_t^s(x)$ so as to solve its profit maximization problem;

(ii) taking profits $\pi_t(x)$ and wages $w_t(x)$ as given, each worker chooses $\eta_t^s(x)$, $c_t(x)$, and $h_t(x)$ so as to solve her utility maximization problem;

(iii) labor markets clear: $\eta_t^d(x) = \eta_t^s(x)$;

(iv) goods markets clear: $y_t^d(x) = y_t^s(x)$.

Proposition 1 In any equilibrium, it must be true that, for all x, y,

$$\eta_t(x) = \sum_{y \in S} p(y,x)E_{t-1}\eta_t(y) \tag{11}$$

Proof: Firms know that workers will spend all their income each period:

$$\begin{aligned} E_{t-1}y_t^d(x) &= \sum_{y \in S} p(y,x)E_{t-1}c_t(y), \\ &= \sum_{y \in S} p(y,x)E_{t-1}[w_t(y)\eta_t(y) + \pi_t(y)] \end{aligned} \tag{12}$$

Substituting this into the firm's labor demand curve gives

$$\eta_t^d(x) = \begin{cases} \sum_{y \in S} p(y,x) E_{t-1}[w_t(y)\eta_t(y) + \pi_t(y)] & \text{if } w_t(x) \leq 1 \\ 0 & \text{if } w_t(x) > 1. \end{cases} \tag{13}$$

Note that this demand curve is decreasing in the firm's own real wage rate, but is increasing in the real wage rates of others. However, in equilibrium, the demand for labor is independent of the real wage rate of others. To see this, observe that, under the assumption that the firm's profits are transferred to the worker (or, alternatively, that the marginal propensity to consume is the same for both workers and firms), we have

$$\pi_t(x) = \eta_t(x)\left(1 - w_t(x)\right). \tag{14}$$

This gives the result that

$$\eta_t^d(x) = \sum_{y \in S} p(y,x) E_{t-1}\eta_t(y). \tag{15}$$

From Figure 1 it can be seen that, for any labor demand, there exist wages such that labor markets clear. Hence, the amount of labor actually employed in equilibrium will satify:

$$\eta_t(x) = \sum_{y \in S} p(y,x) E_{t-1}\eta_t(y) \equiv \hat{\eta}_t(x) \tag{16}$$

In game theoretic terms, this equation can be intepreted as a best response function, implying that our definition of equilibrium is consistent with that of Nash equilibrium. Since the function is positively sloped, there exist 'strategic complementarities' in employment: [7] if the employment rate at site y increases, then, for all x such that $p(y,x) > 0$, this creates a higher demand for the goods produced at site x, which in turn increases the demand for labor in x.

When $\eta_t(x) = 0$ or $\eta_t(x) = 1$, the exact determination of the wage rate, and hence, the division

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[7] On the notion of strategic complements, see Cooper and John (1988).

of the surplus $(1-\sigma)$, depends on the distribution of power between firms and workers, which we will take as exogenous. The reason is that, while the division of the surplus does affect the inequality *within* regions, it turns out not to affect the inequality *between* regions, which is what this paper will focus on.

Proposition 2 There exist 3 types of equilibria, which must satisfy, $\forall x$:

$$\begin{array}{lll} (1) & w_t(x) \in [0,\sigma\rangle, & \eta_t(x) = 0 \\ (2) & w_t(x) = \sigma, & \eta_t(x) = \sum_y p(y,x) E_{t-1}\eta_t(y) \\ (3) & w_t(x) \in \langle\sigma, 1], & \eta_t(x) = 1 \end{array}$$

Proof: See figure 1. $\square$

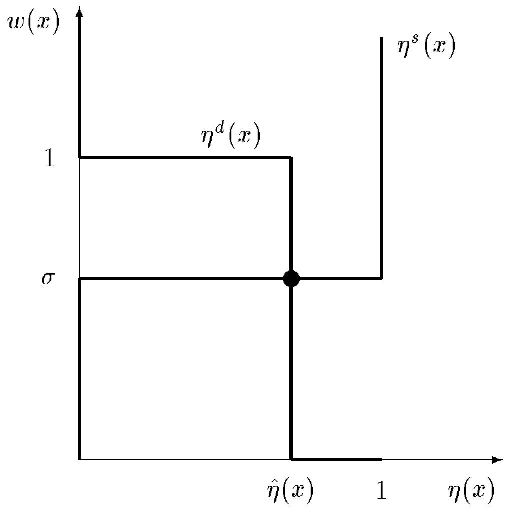

Figure 1: Labor market equilibrium.

3 Equilibrium Properties

In this section, the static and dynamic properties of the equilibrium distribution of employment will be discussed for general support Ω of labor, general interaction structure $\mathbf{P}$, and specific assumptions on expectations. In subsequent sections, the model will be further analyzed for binary state spaces, local interaction structures, and different assumptions concerning the productivity of nonmarket work.

In order to fix expectations, we first need to make further assumptions about the available information to each firm. As a benchmark case, section 3.1 will discuss the equilibrium properties under the assumption of *rational expectations*, which says that subjective expectations equal mathematical equations:

$$E_{t-1}\eta_t(y) = E\eta_t(y).$$

Under this assumption, which implies that firms know the true model of the economy, the model is essentially static, in that there are no temporal correlations between the equilibrium employment levels in different periods. Section 3.2 will discuss a more interesting, dynamic version of the model, in which firms are assumed to have *adaptive expectations*,

$$E_{t-1}\eta_t(y) = \eta_{t-1}(y).$$

Under this assumption, the distribution of employment follows a Markov process, which is shown to converge to a rational expectations equilibrium.

3.1 Equilibrium Properties under Rational expectations

Proposition 3 Let firms have rational expectations: $E_{t-1}\eta_t(y) = E\eta_t(y)$. Then,

a. there exist $|\Omega|$ symmetric equilibria.

b. when $p(y,x) > 0$ for some $y \neq x$, the set of symmetric equilibria is unique.

Proof: Under rational expectations, the equilibrium demand for labor is given by

$$\begin{aligned}
\eta_t(x) &= \sum_{y \in S} p(y,x)\eta_t(y), \quad \forall x, y, t; \\
\eta_t(y) &= \sum_{z \in S} p(z,y)\eta_t(z), \quad \forall y, z, t, \qquad (17) \\
&\dots
\end{aligned}$$

and so on for all $x, y, z \in S$.

Define $\mu_t(x) \equiv \frac{\eta_t(x)}{\sum_y \eta_t(y)}$. This gives as the relative employment distribution a system of $|S|$ linear equations in $|S|$ unknowns, which in vector notation can be written as

$$\mu_t = \mathbf{P}\mu_t. \qquad (18)$$

Since both the rows and the columns of $\mathbf{P}$ sum to one, it is a doubly stochastic matrix, which is known to have the uniform distribution as its invariant measure. To see this, suppose that $\mu_t(x) = \mu_t(y) = |S|^{-1}$ for all x, y. Then we must have $\eta_t(y) = \hat{\eta}_t$ for all y, where $\hat{\eta}_t = |S|^{-1} \sum_{x \in S} \eta_t(x)$ denotes the average employment rate (or income level) in the economy as a whole. Substituting this into the labor demand function for x gives $\eta_t(x) = \sum_y p(y,x)\hat{\eta}_t = \hat{\eta}_t$, which confirms that this constitutes an equilibrium. In any symmetric equilibrium, $\eta_t(x) = \hat{\eta}_t$ for all x, implying that the number of equilibrium employment rates equals the cardinality of the set Ω.

To prove (b), observe that the only case in which asymmetric equilibria could exist is when $\mathbf{P} = \mathbf{I}$, in which case each site is autarkic and *any* vector η_t is invariant. This case is ruled out by the assumption that $p(y,x) > 0$ for some $y \neq x$. Note that symmetry implies that all goods markets clear: when all firms hire the same amount of labor, firms are able to predict the demand for their goods exactly, and therefore will be able to avoid any surpluses or shortages. $\square$

To summarize, when firms have rational expectations, each site must have the same level of employment, output, and income at each moment in time along each equilibrium path. Since asymmetric equilibria do not exist, this implies that the rational expectations model cannot generate spatial inequality. Moreover, since the employment levels in different periods are mutually independent, the model is not able to generate temporal correlations either. Along each equilibrium path, the economy is in equilibrium at all times, but it is possible that each period a new equilibrium is selected.

These conclusions are in contrast with the fact that we do observe high serial correlations (even 'unit roots') in a typical employment or GDP timeseries.[8] But they are also disconcerting from a theoretical point of view. What is missing is a theory of equilibrium selection, and an explanation of how it is possible that a large number of agents could possibly coordinate on any given equilibrium.

As evolutionary game theorists have pointed out,[9] the strict rationality assumptions that might be reasonable consistency requirements in one-shot 2-by-2 games, or economies with a small number of "representative agents", become quite untenable when applied to games with a large number of players that repeatedly encounter similar situations. In such cases, of which this model is an instance, it might be more reasonable to assume that agents have *adaptive expectations*, i.e., they base their expectations of others' behavior on the past. The next section will explore the implications of this assumption, which will introduce a dynamic element into the model.

3.2 Equilibrium Properties under Adaptive Expectations

Proposition 4 Let firms have adaptive expectations: $E_{t-1}\eta_t(y) = \eta_{t-1}(y)$. Then

a. a stationary distribution of the adaptive expectations model is a rational expectations equilibrium

b. when $p(x,y) = \psi(|x-y|), \forall x, y$, and $p(y,x) > 0$ for some $y \neq x$, the distribution of employment converges to the rational expectations equilibrium distribution in finite time.

Proof: Under adaptive expectations, the amount of labor employed by a given firm is a weighted average of the amounts of labor employed by its trade partners in the last period:

$$\eta_t(x) = \sum_{y \in S} p(y,x)\eta_{t-1}(y). \tag{19}$$

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[8] Cf. Nelson and Plosser (1982), Blanchard and Summers (1986).

[9] E.g., Samuelson (1997).

Again defining $\mu_t(x) \equiv \frac{\eta_t(x)}{\sum_y \eta_t(y)}$, the temporal evolution of the employment distribution must satisfy

$$\mu_t = \mathbf{P}\mu_{t-1}, \forall t, \tag{20}$$

which exhibits the Markov property:

$$\Pr\ (\mu_t \in A | \mu_{t-1}, \mu_{t-2}, \ldots, \mu_0) = \Pr\ (\mu_t \in A | \mu_{t-1}), \quad \forall A \subseteq \Omega^S. \tag{21}$$

A stationary distribution, or steady state, of this process must satisfy

$$\mu_t = \mathbf{P}\mu_t, \quad \forall t, \tag{22}$$

and therefore must be a rational expectations equilibrium (by proposition 3).

To prove (b), first note that, since $\eta_t(x)$ has countable support, $\mathbf{P}$ can be interpreted as a transition probability matrix. Under the given assumptions, this matrix is irreducible, i.e. S is a connected graph. [10] To see this, note that, if $p(x, y)$ is only a function of the distance between x and y, then if $p(x, y) > 0$ for some $x \neq y$, this implies $p(x, z) > 0$, for all z such that $|x-y| = |x-z|$. Moreover, since $p(x, y) = \psi(|x-y|)$ for all x, y this implies that *all* agents have some possible trade partner at some distance. Finally, since $p(x, y)$ is increasing in the distance, then if $p(x, y) > 0$ for some $x \neq y$, this implies that, on a lattice, x has a positive probability of trading with at least its four nearest neighbors. If all agents have a positive probability of trading with their nearest neighbors then the graph is connected,o i.e. it is possible for each agent to trade either directly or indirectly with any other agent.

In addition to guaranteeing irreducibility, the assumptions on $\mathbf{P}$ also imply that the system is 'positive recurrent', meaning that there exists no closed economies, i.e. no subset of agents that only trade with each other. By the ergodic theorem for Markov chains, irreducibility and positive

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[10] Formally, a graph G consists of a nonempty set $V(G)$ of *vertices*, and a list $E(G)$ of unordered pairs of these vertices, called *edges*. In our case vertices are called 'sites' and an edge of the form xy indicates that x and y are able to trade with each other. In a *connected* graph, any site can be reached from any other site by a path consisting of only a finite number of edges (trades).

recurrence together imply uniqueness of the invariant distribution. On a finite lattice this implies that the economy will converge to its invariant distribution in finite time. □

As an aside, observe that, on a finite lattice with countable state space, $E\hat{\eta}_t = \hat{\eta}_{t-1}$, i.e., the average employment rate is a martingale. While adaptive expectations thus generate some temporal persistence in the short run (which is consistent with the observed random walk behavior of actual employment rates), in the absence of shocks the employment distribution converges to its uniform invariant measure in finite time. Hence, in the long run, spatial inequality cannot persist. Moreover, in terms of absolute employment levels, the system is *nonergodic*: different initial configurations will lead to different steady state levels of employment.

4 Binary Support

We will now focus on the special case where $\eta_t(x)$ has binary support $\Omega = \{0, 1\}$. In other words, workers can be either employed or unemployed, and part time work is not possible. A possible justification for this assumption is that production itself has a binary character, either due to the nature of the good produced, or due to some kind of increasing returns (e.g. startup costs, economies of scale, learning by doing), implying that the optimum amount of production is either 0 or 1.[11] Alternatively, one might think of t as denoting a relatively short period, e.g. a day, so that part-time work is in fact possible, but only by smoothing labor between these periods.

The binary case is interesting because it turns the model into a simple coordination game that can be interpreted as a spin system, i.e., an interacting particle system with state space $\{0, 1\}^S$. In this section we will analyze both the global and local interactions version of this model under the assumption that payoffs are deterministic. A stochastic version will be analyzed in section 5.

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[11] Formally, this could be modelled, for instance, by assuming a Leontief production function: $y_t(x) = \min\{1, \eta_t(x)\}$.

4.1 The Employment Game

Consider a planner version of the model, in which each site is represented by a single agent (e.g., a "social planner") with utility function

$$\begin{aligned} V_t(x) &= [1 - \eta_t(x)]\sigma + w_t(x)\eta_t(x) + \Pi_t(x) \\ &= [1 - \eta_t(x)]\sigma + \eta_t(x) \sum_{y \in S} p(y, x)\eta_t(y). \end{aligned} \tag{23}$$

In binary space, the fraction $p(y, x)$ of income spent by y at x can be equivalently interpreted as the probability that agent x will be "matched" with agent y, who then has the opportunity to purchase goods from x. The assumption that $p(x, y) = |N_x|^{-1}$ for $y \in N_x$ can then be interpreted so as to say that agent x is randomly matched with an agent y in its neighborhood. After being matched, the two agents then play an "Employment Game", the payoff matrix of which is represented in Table 1.

	$\eta_t(y) = 0$	$\eta_t(y) = 1$
$\eta_t(x) = 0$	σ, σ	$\sigma, 0$
$\eta_t(x) = 1$	$0, \sigma$	$1, 1$

Table 1: Two-Player Employment Game

The Employment Game has the structure of a "coordination game", the two pure Nash equilibria of which are given by the symmetric strategy profiles $\{0, 0\}$ and $\{1, 1\}$. A mixed strategy $p_y = \Pr\ (\eta_t(y) = 1)$ in this case can be interpreted as a belief on the part of agent x concerning the probability of being matched with some $y \in N_x$ whose workers are employed. Since being employed means having the purchasing power to buy goods, this probability equals the expected demand for x's output:

$$p_y = \Pr\ (\eta_t(y) = 1) = \sum_{y \in S} p(y, x) E_{t-1}\eta_t(y) \equiv \hat{\eta}_t(y), \tag{24}$$

which gives as the expected utility for the representative agent at site x

$$E_{t-1}V_t(x) = [1 - \eta_t(x)]\sigma + \eta_t(x)\hat{\eta}_t(y) \tag{25}$$

Maximizing expected utility gives the best response correspondence for agent x:

$$p_x(p_y) \quad = \quad \Pr\ (\hat{\eta}_t(y) > \sigma) \quad \begin{cases} = 0 & \text{if } \hat{\eta}_t(y) < \sigma \\ \in [0,1] & \text{if } \hat{\eta}_t(y) = \sigma \\ = 1 & \text{if } \hat{\eta}_t(y) > \sigma \end{cases} \tag{26}$$

Since the game is symmetric, a similar correspondence can be obtained for y. As can be seen from Figure 2, the two correspondences intersect in three places, corresponding to three symmetric Nash equilibria, two pure and one mixed.

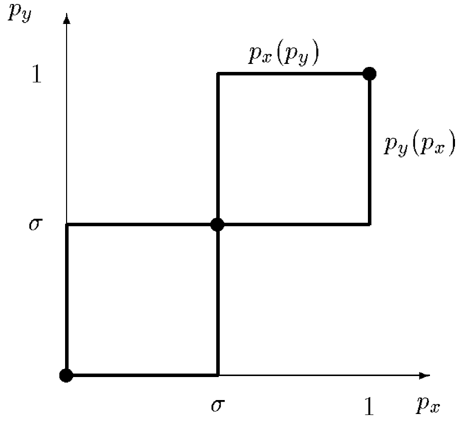

Figure 2: Best Response Correspondences

4.2 Global Interactions

Under global interactions, $N_x = S$, i.e., agents are matched randomly each period with any other agent in the economy. Since agents have to make their decisions before they know who they will be matched with (i.e. before they know who the next buyer will be), the best they can do is to base their predictions on the basis of the average play in the current period. Thus, if the current employment rate is $\hat{\eta}$, the next buyer is expected to be employed with probability $p_x = \hat{\eta}$.

As Blume (1997) notes, this interpretation of mixed strategies as "average play" in a population is consistent with that of Nash (1950: 21-23):

> It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning process. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal ...
>
> We assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed by the 'average member' of the appropriate population. ... Thus the assumption we made in this 'mass-action' interpretation lead to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point.

Under global interactions and adaptive expectations, however, the mixed Nash equilibrium is dynamically unstable. Intuitively, this follows from the fact that the average employment rate is a bounded martingale: $E\hat{\eta}_{t+1} = \hat{\eta}_t$. Since the number of agents is finite, this implies that, even if all agents start out playing the mixed equilibrium strategy $p_x = \sigma$, due to random chance the average employment rate in the next period is likely to differ slightly from σ, and will eventually move to one of the pure Nash equilibria (steady states). [12] Whether this is the Pareto-optimal outcome of full employment or the Pareto-suboptimal outcome of autarky depends on the initial fraction of employed workers, as well as on σ. However, no matter which equilibrium is selected, the implication is again that under global interactions inequality cannot persist.

4.3 Local Interactions

While global interactions constitute one extreme interaction topology, we will now focus on the other extreme, i.e. that of perfectly local interactions, where each agent only interacts with its *nearest neighbors.*[13] A 'nearest neighborhood' of x is defined as the set of agents that are Euclidean distance one removed from x, i.e., $N_x = \{y : |y - x| = 1\}$. This is sometimes referred to as the "Von Neumann neighborhood" of x.

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[12] This also follows directly from proposition 3, which says that $\eta = \{0\}$ and $\eta = \{1\}$ are the only two invariant measures, and from proposition 4, which says that they constitute the absorbing states into which a Markov chain on a countable state space must eventually get trapped.

[13] In future work, we plan to consider more general interaction topologies

In order to ensure that all agents have an equal number of neighbors, a periodic boundary condition is imposed, which turns the economy into a *torus*. To reduce the parameter space, the model is analyzed for the special case $\sigma = 0.5$, under which the equilibrium strategy for each agent reduces to a simple "majority rule": workers will be employed only if the majority of their trade partners were employed in the previous period, and they will not be employed if the majority of their trade partners were unemployed in the previous period. If there is a match, workers are hired with probability one half.

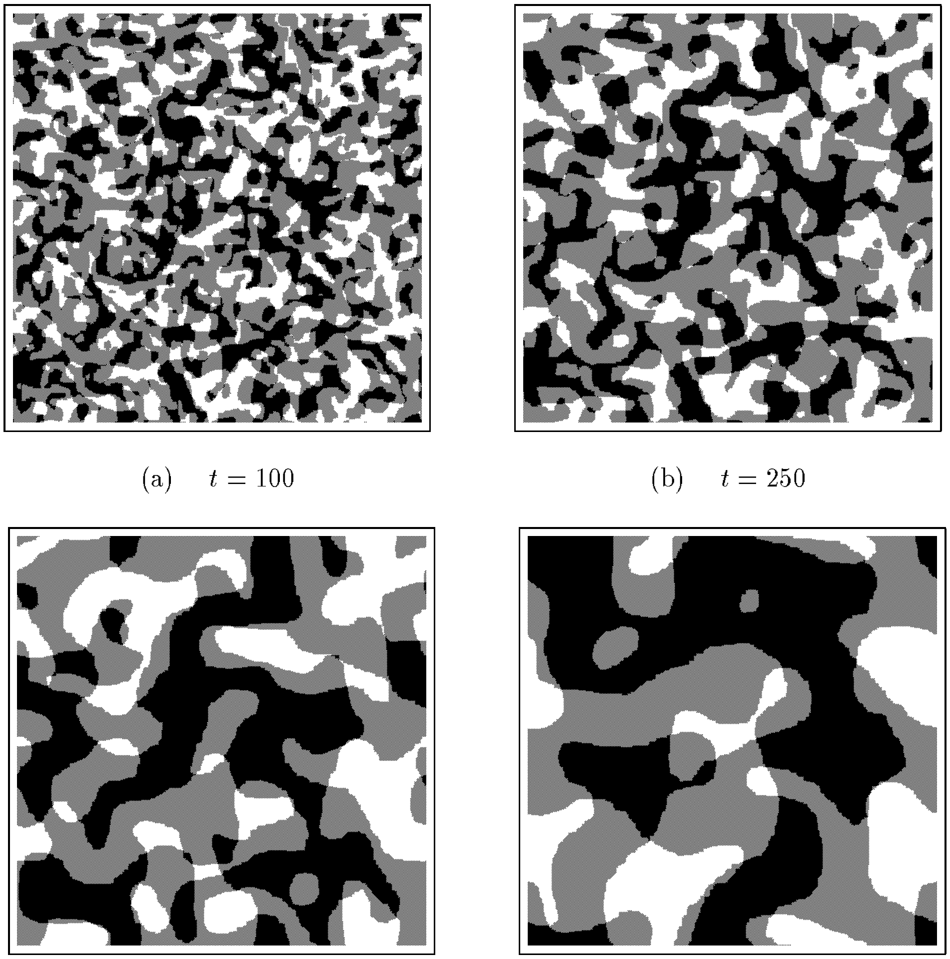
To study the spatio-temporal evolution of this system, the model was implemented on the Cellular Automata Machine (CAM8) at the Santa Fe Institute. Various numerical experiments were carried out on a 512 by 512 array, starting from different random initial configurations (Bernouilli product measures with density $\mu \in [0, 1]$. Figure 3 shows the typical evolution of the local interactions system starting from a symmetric random distribution with density 0.5.

As panel (a) shows, after only a few periods, three regimes emerge: (1) clusters of full employment (represented as black), (2) clusters of unemployment or autarky (white), and (3) 'mixed economies' which appear to be grey, but are in fact 'checkerboard' configurations where agents switch periodically between the two pure strategies.

Interestingly, while the mixed strategy itself is unstable, these mixed economies are stable given the specific local interaction topology. Moreover, the mixed regimes behave just like the pure strategy regimes in terms of their spatial evolution. An important difference is, however, that the pure strategy regimes do not interact; they need to be "mediated" by a mixed strategy regime.

Eventually, the regime that starts to dominate by chance takes over the entire lattice. Pockets of unemployment or full employment which are completely surrounded by the mixed regime gradually disappear, and *vice versa*. The process by which this happens is called *curvature-driven surface tension*: since clusters that span a curved boundary will on average have unequal frequencies of the two strategies, the common strategy will continue to be favored, at a rate which decreases with the curvature of the boundary.

Ultimately, the process always converges to one of the three steady states. Again, there is sensitivity to initial conditions: from random initial distributions with density $\mu \neq 0.5$, the initially



(a) $t = 100$ (b) $t = 250$

(c) $t = 1000$ (d) $t = 3,000$

Figure 3: Evolution of the Majority Rule Model

predominant regime ends up dominating with overwhelming probability.

5 Stochastic Employment Games

In this section, we extend the previous model by allowing for uncertainty with respect to the value of nonmarket production, σ, which will now be allowed to vary both in time and in space. We will investigate two special cases:

(1) $\sigma_t(x) \sim U[0, 1]$

(2) $\sigma_t(x) \sim$ logistic

These distributional assumptions are chosen mainly for their simplicity, and because they may be appealing from an econometric point of view. Moreover, they allow the model to be directly related to the statistical mechanics literature. As section 5.1 discusses, under the first assumption our model reduces to an interacting particle system called the *Voter Model*, while under the second assumption, the model bears close relationship to the Ising model from statistical mechanics. The latter model was first introduced to economics by Blume (1993) and Brock (1993), and has been further studied by Brock and Durlauf (1995, 2000). The Ising model is discussed in section 5.2.

5.1 The Voter Model

Suppose that $\sigma_t(x) \sim U[0,1]$, or that agents believe this is the case. Then the solution to the expected utility maximization problem is

$$\Pr\ (\eta_t(x) = 1) = \Pr\ (\sigma_t(x) < \hat{\eta}_t(y)) = \hat{\eta}_t(y) \tag{27}$$

This specification implies that firms have linear best response functions, hence that *any* symmetric strategy profile constitutes a Nash equilibrium.

In this special case, the model is formally equivalent to a well-known interacting particle system called the Voter Model, which was introduced independently by Clifford and Sudbury (1973) and by Holley and Liggett (1975). The latter authors, who gave the model its name, considered a population of interacting voters, who determine their political position $\eta_t(x) \in \{0,1\}$ by choosing a random 'friend' $y \in S$, and adopting his or her position with probability $p(y,x)$. Hence, η_t can be regarded as a process in which the value $\eta_t(x)$ is replaced by $\eta_{t-1}(y)$ with probability $p(y,x)$.

An interesting property of the Voter model is that, in one and two dimensions, the system converges to states with all zeros or all ones (i.e., pure strategy equilibria), under *any* interaction topology that constitutes a connected graph. This is true even on infinite lattices, while in infinite spaces with three or more dimensions additional steady states exist. The proof of this claim is given in Durrett (1988: 21-26), and will be sketched rather informally below.

Define $L(m_t)$ as the location of a given amount of income m, which initially satisifies $L(m_0) = x$.

Then the path followed by this income across locations is a symmetric random walk with the same transition probabilities as the employment/income path of a given location:

$$\begin{aligned} p(y,x) &= \Pr\ (L(m_t) = x | L(m_{t-1}) = y) \\ &= \Pr\ (\eta_t(x) = 1 | \eta_{t-1}(y) = 1) \end{aligned} \tag{28}$$

Under the assumption that $p(y,x) = \psi(|y-x|)$, where $\psi(\cdot)$ is a monotonically increasing function, this random walk is positive recurrent. Next consider two sites, x and y which start out with unequal incomes: $\eta_t(x) = 1, \eta_t(y) = 0$, say. Both incomes follow symmetric random walks in two-dimensional space, hence the distance between the locations of these incomes must also follow a symmetric random walk. In one and two dimensions, symmetric random walks are recurrent, which implies that the distance must eventually be 0 with probability 1. Now observe that the paths followed by locations of incomes and by incomes of locations are the same in distribution (i.e. are each other's *dual process*). This implies that the distance between two individual incomes must eventually be 0 as well, which in turn implies that inequality cannot persist in two dimensions. [14]

Figure 4 illustrates the evolution of the finite, two dimensional case with nearest neighbor interactions.

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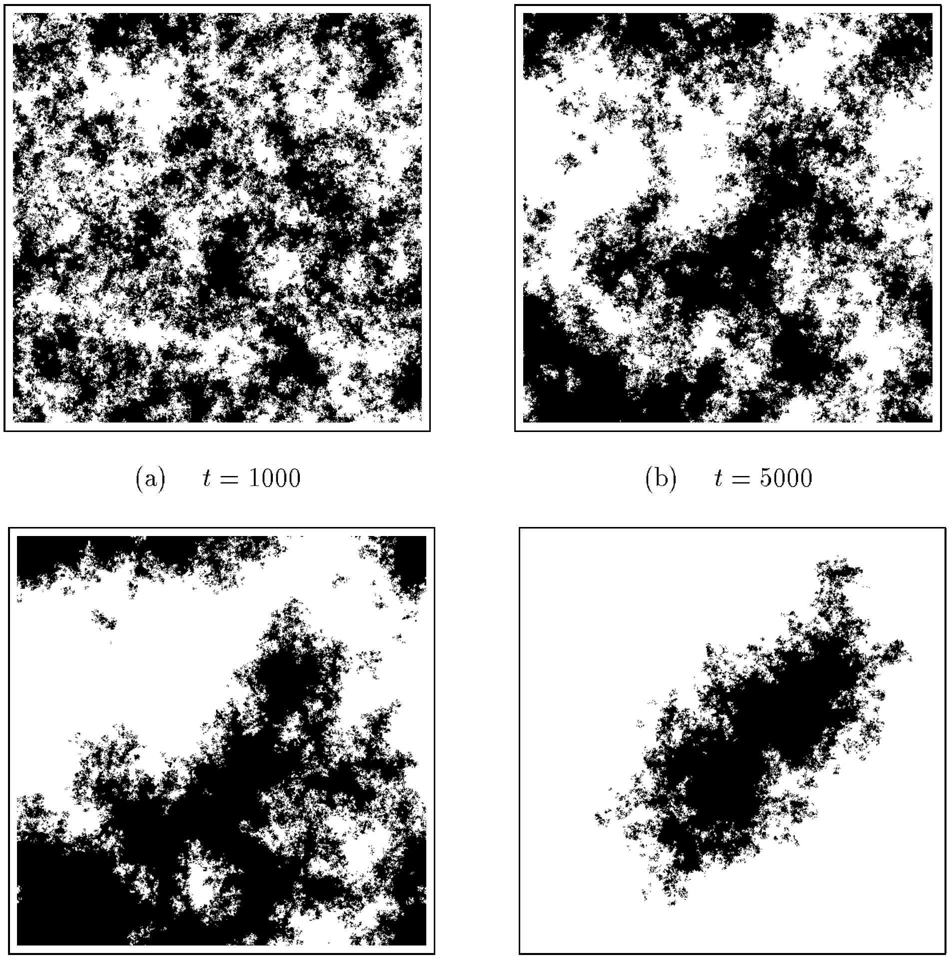
[14] In three or more dimensions, however, random walks are not recurrent and hence this may not happen. These results generalize to any homogeneous model in which x adopts the position of y with some probability $p(y)$, with the classification being determined by the recurrence or transience of the random walk with kernel p.



(a) $t = 1000$ (b) $t = 5000$

(c) $t = 10,000$ (d) $t = 50,000$

Figure 4: Evolution of the Voter Model

5.2 The Ising Model

Consider next the case where $\sigma_t(x) = \frac{1}{2} + \mu\epsilon_t(x)$, where $\epsilon_t(x)$ is a logistically distributed random utility term, and μ represents the impact of this error term on agent's decisions. Under these assumptions, each agent has the following best response function:

$$\begin{aligned} Pr(\eta_{t+1}(x) = 1) &= \text{Pr }(\hat{\eta}_t(y) > \sigma + \mu\epsilon_t(x)) \\ &= \text{Pr }(\epsilon_t(x) < \frac{1}{\mu}(\hat{\eta}_t(y) - \sigma)) \\ &= \frac{1}{1 + e^{-2\beta(\hat{\eta}_t(y)-\sigma)}}, \end{aligned} \tag{29}$$

where $\beta = \frac{1}{2\mu}$ is chosen so as to make the model formally equivalent to an Ising model with state space $\{0,1\}$. While in statistical mechanics β is commonly interpreted as the "inverse temperature", in economics it has been termed the "intensity of choice" (Brock and Durlauf 1995), for the reason that, as β increases, agents become 'intenser' in the sense that they are less sensitive to noise. This implies that, as $\beta \to \infty$, the best response function for the stochastic case reduces to the majority rule best response correspondence of the deterministic model:

$$Pr(\eta_{t+1}(x) = 1) \to \begin{cases} 0 & \text{if } \hat{\eta}_t(y) < \sigma \\ 0.5 & \text{if } \hat{\eta}_t(y) = \sigma \\ 1 & \text{if } \hat{\eta}_t(y) > \sigma. \end{cases} \tag{30}$$

5.2.1 Ising Model with Global Interactions

Under adaptive expectations, the best response function is

$$Pr(\eta_{t+1}(x) = 1) = \frac{1}{1 + e^{-2\beta(\hat{\eta}_t(x)-0.5)}} \tag{31}$$

Assuming global interactions, this implies:

$$\hat{\eta}_{t+1}(\hat{\eta}_t) = \frac{1}{1 + e^{-2\beta[\hat{\eta}_t-0.5]}} \tag{32}$$

The derivative of this function is

$$\frac{\partial \hat{\eta}_{t+1}(\hat{\eta}_t)}{\partial \hat{\eta}_t} = 2\beta \left(1 + e^{-2\beta[\hat{\eta}_t - 0.5]}\right)^{-2} \tag{33}$$

The mixed Nash equilibrium is stable when the best response function cuts the 45-degree line from above, i.e. when

$$\begin{aligned} \frac{\partial \hat{\eta}_{t+1}(0.5)}{\partial \hat{\eta}_t} &< 1 \\ \beta &< 2 \end{aligned} \tag{34}$$

As illustrated in Figure 5, for $\beta \geq 2$ the mixed equilibrium becomes unstable and two additional steady states emerge. Note that these are not pure strategy Nash equilibria; rather, they correspond to steady state distributions where either a majority of workers is employed or a majority is unemployed. However, since each worker is employed with equal probability, the inequality that may occur at a given point in time is not persistent.

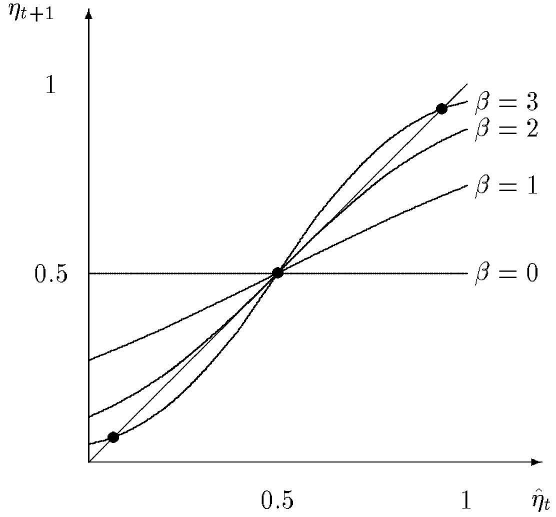


Figure 5: Best Response Functions for the Ising Model

5.2.2 Generalizing the Interaction Structure

So far we have assumed that the transition probabilities are independent of agents' own current state. Implicitly, this implies that agents do not consume any of the goods produced at their own site. This may seem quite unrealistic in the sense that, even in today's globalized economy, much if not most trade still seems to be taking place within rather than between regions (e.g., net exports rarely constitute more than 50% of a country's GDP). Also, at the individual level, the probability of being employed appears to be affected at least to some extent by individuals' own employment history, in addition to the employment rate in the neighborhood.

To capture the more general case where agents consume some fraction α of their own goods, the trade structure may be redefined as a matrix $\mathbf{P}$ with elements:

$$p(y,x) = \begin{cases} \alpha & \text{if } y = x \\ \frac{1}{4}(1-\alpha) & \text{if } y \in N_x \\ 0 & \text{if } y \notin N_x \end{cases} \tag{35}$$

This gives the following best response function:

$$\Pr\left(\eta_{t+1}(x) = 1\right) = \frac{1}{1+e^{-2\beta[\alpha\eta_t(x)+(1-\alpha)\hat{\eta}_t-0.5]}} \tag{36}$$

Under global interactions, $\eta_t(x) = \eta_t$, hence the steady state level of employment is independent of α. However, conditioning on each agent's current state, we obtain a different result:

$$\begin{aligned} \hat{\eta}_{t+1}(\hat{\eta}_t) &= \hat{\eta}_t \cdot \Pr\left(\hat{\eta}_{t+1} = 1 \mid \hat{\eta}_t = 1\right) \quad + \quad (1-\hat{\eta}_t)\cdot \Pr\left(\hat{\eta}_{t+1} = 1 \mid \hat{\eta}_t = 0\right) \\ &= \hat{\eta}_t \left(1+e^{-2\beta[\alpha+(1-\alpha)\hat{\eta}_t-0.5]}\right)^{-1} + \quad (1-\hat{\eta}_t)\left(1+e^{-2\beta[(1-\alpha)\hat{\eta}_t-0.5]}\right)^{-1}. \end{aligned} \tag{37}$$

According to this equation, a phase transition is predicted to take place for

$$\begin{aligned} \frac{\partial\hat{\eta}_{t+1}(0.5)}{\partial\hat{\eta}_t} &= \left(1+e^{-\alpha\beta}\right)^{-1} + \left(1+e^{-\alpha\beta}\right)^{-2}\beta(1-\alpha)e^{-\alpha\beta} \\ &\quad + \left(1+e^{\alpha\beta}\right)^{-1} + \left(1+e^{\alpha\beta}\right)^{-2}\beta(1-\alpha)e^{\alpha\beta} \quad \geq 1, \end{aligned} \tag{38}$$

which reduces to

$$\beta(1-\alpha) \geq 1 + e^{-\alpha\beta} \tag{39}$$

This inequality suggests that, as α increases, larger values of β are required for clustering to occur. Intuitively this makes sense, because larger values of α imply that agents become less likely to change their current state, hence a larger 'intensity of choice' (β) is needed in order for spatial clustering to occur.

5.2.3 Ising Model with Local Interactions

With our more general interactions topology, a "nearest neighborhood" trade structure may be defined as

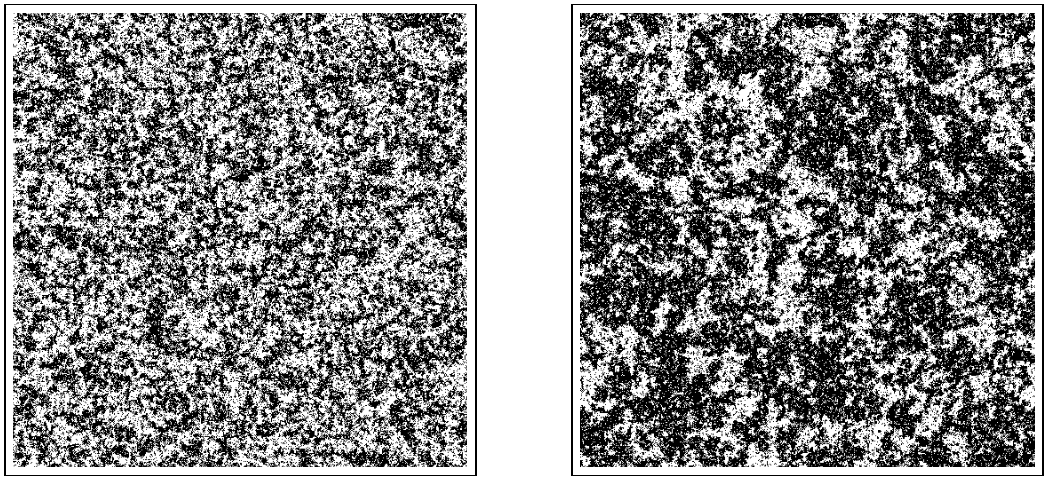
$$p(y,x) = \begin{cases} \alpha & \text{if } |y-x| = 0 \\ \frac{1}{4}(1-\alpha) & \text{if } |y-x| = 1 \\ 0 & \text{if } |y-x| > 1 \end{cases} \tag{40}$$

The local interactions model was first simulated for the case $\alpha = 0.2$, under which each agent has an equal probability of trading within her own region as she has with any of the agents from the four neighboring regions. The results are shown in Figure 6.

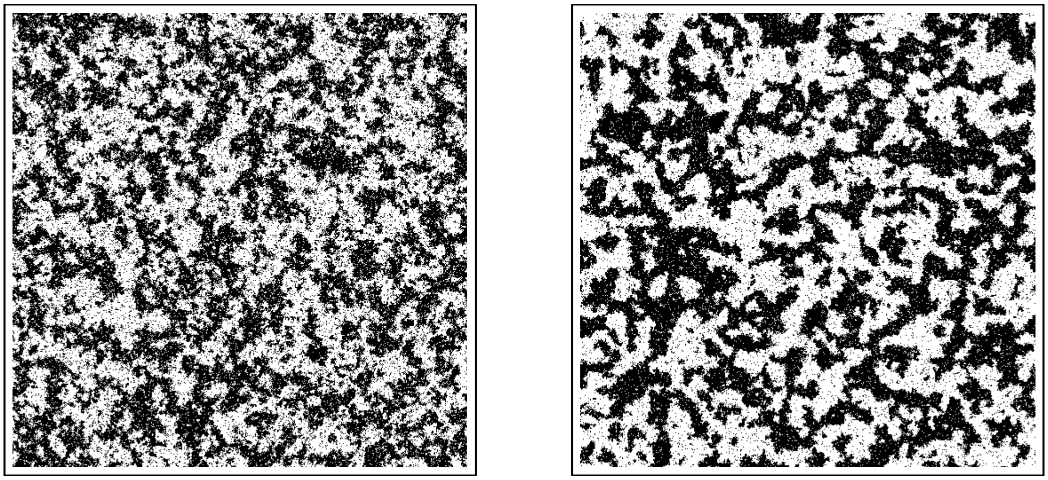
In statistical mechanics, a global interactions model is often used as a so-called 'mean field approximation' (MFA) to a local interactions model. In this case, this appears to be a bad approximation, to the extent that for β as large as 2.8 the system is still ergodic with an invariant distribution close to random noise. Apparently, the intensity of choice is not strong enough for clustering to occur. As β increases, however, clusters do start to form, to the point that, for approximately $\beta > 3.1$, the economy converges to either in a near-full-employment or a near-autarktic state. This process of clustering is similar to that of the deterministic model (see Figure 2) and is shown for $\beta = 3.5$ in Figure 7.

The most interesting observation here is that, for β neither 'too low' nor 'too high' relative to α, there exists a 'phase separation regime', in which clusters of employment and unemployment emerge that are able to become quite large, but never take over the entire lattice.

This observation turns out to be quite general, as shown in Figure 8.

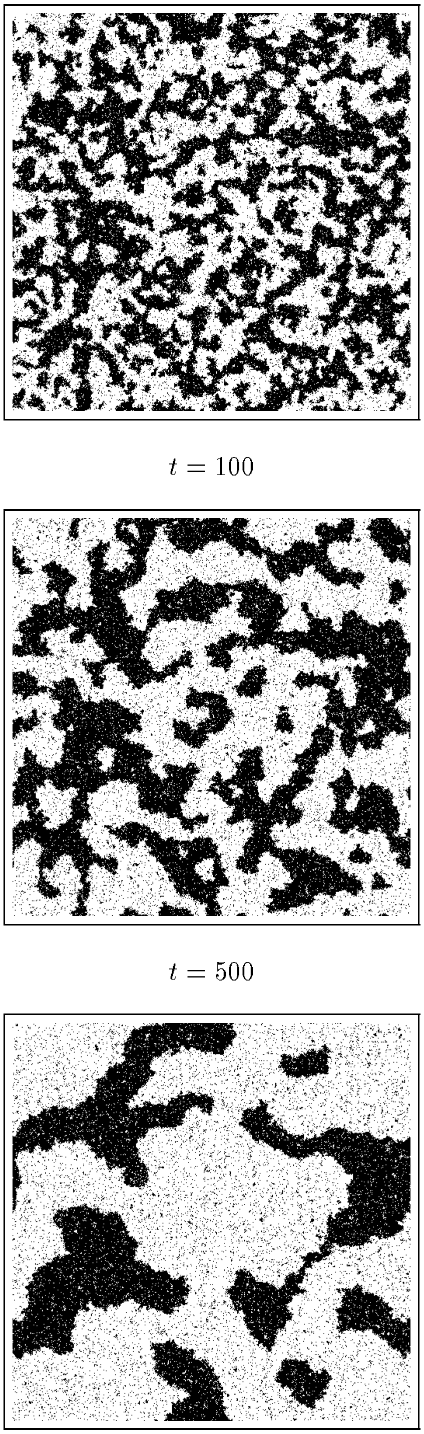


$\alpha = 0.2, \beta = 2.8, t = 30,000$ $\alpha = 0.2, \beta = 3, t = 50,000$



$\alpha = 0.2, \beta = 3.1, t = 100$ $\alpha = 0.2, \beta = 3.5, t = 100$

Figure 6: Ising Model ($\alpha = 0.2$, varying β)



$t = 100$

$t = 500$

$t = 3000$

Figure 7: Clustering of the Ising Model ($\alpha = 0.2$, $\beta = 3.5$)

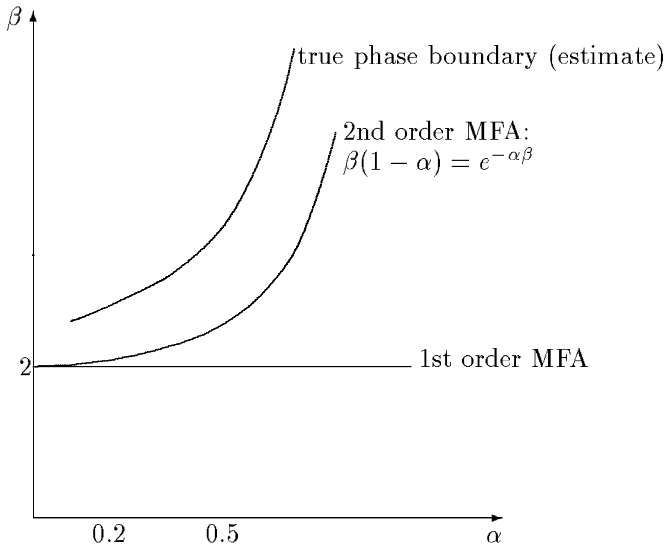


Figure 8: Ising Model: phase diagram

Interestingly, the actual phase boundary (estimated on the basis of a large number of numerical experiments) appears to be quite close to the analytical prediction of equation 39, which could be called a 'second order MFA'. For any level of α, there appears to exist a value of β such that, for β 'large enough' the system undergoes a transition from an 'ergodic' to a 'clustering' regime.

While a phase separation regime does appear to exist for any level of α, it is not always easy to observe. When α is relatively small, as in the $\alpha = 0.2$ case, the clusters that persist are so small that the phase separation regime is hard to distinguish from the ergodic regime. When α is too large, on the other and, the clusters are so big that it becomes hard to distinguish them from the clusters in the clustering regime.

For intermediate levels of α, however, the phase separation regime can be observed quite clearly. As is shown in Figure 9 for $\alpha = 0.5$, the general pattern is the same as for the $\alpha = 0.2$ case, but this time the phase separation regime is clearly distinguishable. For $\beta = 3.5$, the invariant distribution is ergodic, with close to zero spatial persistence, while for $\beta = 4$ the sensitivity to neigbors is large enough as to allow for curvature-driven-surface-tension clustering to occur. For slightly lower values of β, however, medium sized clusters emerge that are able to persist indefinitely. [15] In Figure 10 the same phase transition is shown for β fixed at 3.9. As α increases, a transition takes place

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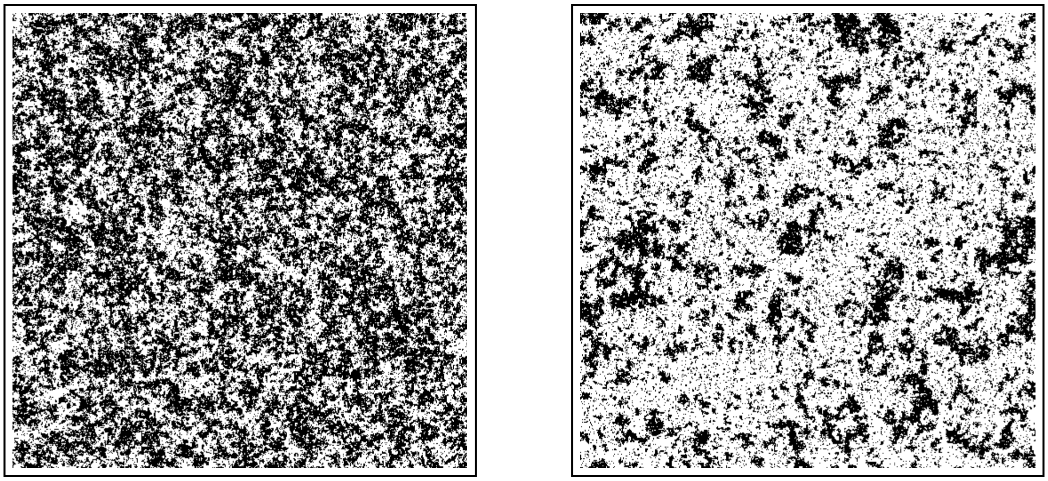
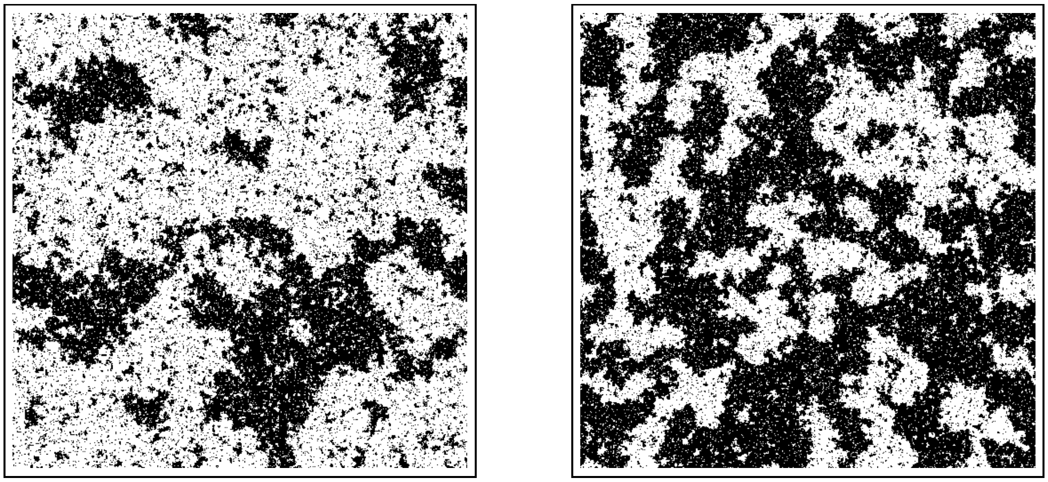
[15] The time periods of $t = 50,000$ and $t = 100,000$ indicate the number of updates after which the simulation was stopped; these times are quite arbitrary in that the configuration did not seem to change much either before or after this time.



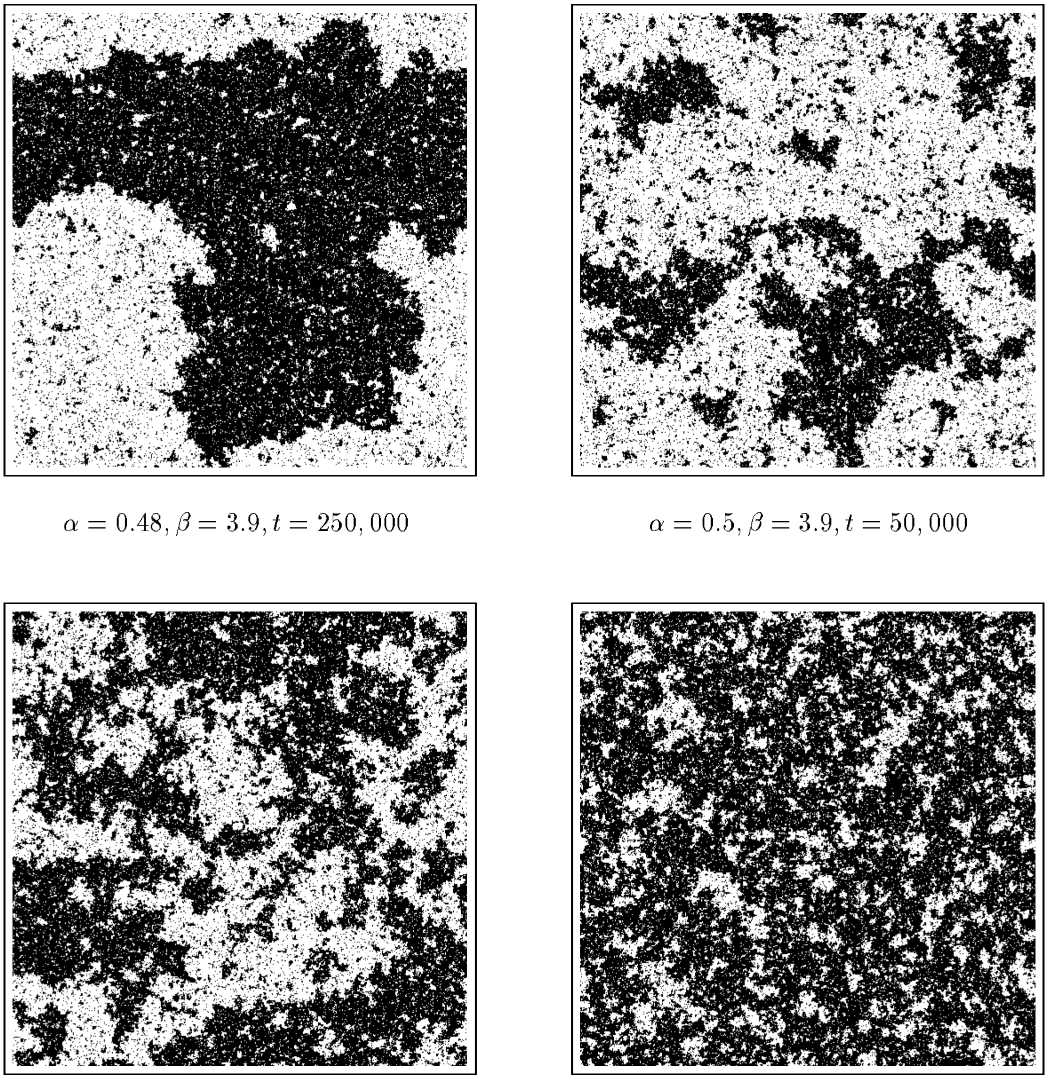
$\alpha = 0.5, \beta = 3.5, t = 100,000$ $\alpha = 0.5, \beta = 3.8, t = 100,000$



$\alpha = 0.5, \beta = 3.9, t = 50,000$ $\alpha = 0.5, \beta = 4, t = 2,000$

Figure 9: Ising Model ($\alpha = 0.5$, varying β)

from clustering to randomness with obvious spatial persistence occuring both at $(\alpha, \beta) = (0.5, 3.9)$ and $(\alpha, \beta) = (0.51, 3.9)$.



$\alpha = 0.48, \beta = 3.9, t = 250,000$

$\alpha = 0.5, \beta = 3.9, t = 50,000$

$\alpha = 0.51, \beta = 3.9, t = \infty$

$\alpha = 0.52, \beta = 3.9, t = \infty$

Figure 10: Ising Model ($\beta = 3.9$, varying α)

6 Interpretation and Conclusion

In light of previous analytical results, the occurrence of global persistence may seem surprising, and may even be considered to be due to the specificities of the software, or length of the simulation procedure. The reason why inequality can persist, however, is in fact quite intuitive, and can be explained as follows:

While the probability of becoming unemployed conditional on living in a full employment economy is very small, in the presence of noise some workers do occasionally become unemployed even if this is an "irrational" decision on the part of firms. When a given worker becomes unemployed, this decreases the expected demand for goods produced in the neighborhood of this worker, which in turn increases the probability that neighboring firms will stop hiring workers as well. Once those neighbors stop hiring workers, however, the neighbors of those neighbors are more likely to do so, etc., so that a cluster of unemploymed workers may start to grow within an economy that originally was at full employment.

Of course, the probability that this happens at any given location is extremely small, since workers who become unemployed by chance will tend to be hired again as soon as firms find out that there is in fact sufficient demand for their output. However, the probability that an unemployment cluster starts to grow *somewhere* within a full employment cluster increases with the size of the employment cluster, and increases with α (which makes agents "fall back" less quickly). This implies that, when the intensity of choice (β) is large enough (relative to α) to allow for large clusters, but small enough to allow for noise, it is possible for the process to continue indefinitely, growing clusters within clusters within clusters.

So far, the analysis has been constrained to the two extreme cases of perfectly global and perfectly local trade. Plans for future work include the study of more general trade networks, the structure of which may be estimated on the basis of actual bilateral trade flow data, and the exploration of the implications of 'globalization' for the persistence of inequality both in space and in time.

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