Learning to Perfect Manipulation: Implications for Fertility, Savings, and Old-Age Social Security *

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Abstract

In this paper we consider an overlapping generations model with endogenous fertility and two-sided altruism and show the limitations of applying commonly used open loop Nash equilibrium in characterizing equilibrium transfers from parents to children in the form of bequest, and transfers from children to parents as voluntary old-age support. Since in our model children are concerned with parents' old-age consumption, agents have incentives to save less for old age and to have more children so as to strategically induce their children to transfer more old-age support. We formulate such strategic behavior within a sequential multi-stage game and introduce a notion of learning equilibrium to characterize equilibrium manipulative behavior and then study the consequences of such strategic manipulations on private intergenerational transfers, fertility and savings decisions, and on Pareto optimality of equilibrium allocation. We show that the learning equilibrium notion of the paper simplifies computation of subgame perfect equilibrium, subgame perfect equilibrium is the long-run outcome of dynamic learning equilibrium paths (this aids in selecting, sometimes, a unique equilibrium among multiple subgame perfect equilibria), and an open-loop Nash equilibrium involves "incredible" threats from children. We provide an alternative explanation for the existence of publicly provided social security program and examine its role to correct distortions created by strategic manipulation.

Keywords: *two-sided altruism, endogenous fertility, subgame perfect manipulation of children, social security.*

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1 Introduction

In standard pure exchange overlapping generations (OLG) economies agents are assumed to have life-cycle utility functions. These models do not explain private intergenerational transfers within family and have no bearings on the effects of public transfers policies such as social security on private intergenerational transfers, savings and fertility. Moreover, competitive equilibrium fails to be Pareto optimal; however, a suitably designed pay-as-you-go (PAYG) social security program can remove inefficiencies by allowing transfers from children to parents that are necessary for Pareto optimality but would not be possible in a decentralized competitive equilibrium due to lack of individual incentives for such transfers (see for instance, Samuelson [1958]).

In another framework Becker [1974] establishes his "Rotten Kid Theorem" that under certain circumstances when parents care about their children's welfare, children take actions that maximize the joint family income even though children do not care about their parents, provided parents leave positive bequest to their children. One implication of his Rotten Kid Theorem is that a forced transfer between children and parents have no ultimate effect on equilibrium outcome, since parents can off-set this forced intergenerational transfer by suitably adjusting their bequest level.[1] Barro [1974] uses the above kind of intergenerational altruism in an OLG framework and shows that social security has no effect on savings so long as in equilibrium agents leave positive bequest in all periods. Furthermore, since Barro model is equivalent to one with finite number of infinitely lived agents, a competitive equilibrium is Pareto optimal; hence social security is not required for the purpose of attaining Pareto optimality of equilibrium allocation.

Neither strand of above literature explain why transfers from children to parents are observed in many economies, and why the amount of transfers declines with the introduction of public transfer policies; why a PAYG social security program exists, and whether it is possible for the current living generations to legislate a PAYG social security benefits scheme for the current and all future generations such that the future generations will have no incentives to amend it; and if one such program exists, does it lead to optimal allocation?

A few attempts have been made, however, to explain the existence of PAYG social security programs in frameworks that treat fertility exogenously. One type of explanations postulate that there could be economy of scale and other sources of market failures in pension provision (see, Diamond [1977]) or there might be adverse selection/moral hazard problems in private provision of retirement income insurance and these could be mitigated by compulsory participation (see Diamond and Mirrlees [1978]). These can explain introduction of fully funded system but cannot explain the existence of PAYG system.

Among the other type of explanations, Browning [1975] considers a voting model of social security in an OLG framework in which the old outvote the young to enact a PAYG social security system. It is not, however, clear in Browning's framework why then the

[1] See Bernheim, Shleifer and Summers [1985] for a critique of the Rotten Kid Theorem.

old do not use their power to enact a legislation to extract all income from the young. Hansson and Stuart [1989] provide an alternative explanation by modeling PAYG social security legislation as a trade among living generations. They consider an OLG model in which agents are assumed to derive utility not only from their own young age and old-age consumption but also from properly discounted young age and old-age consumption of their parents and of all future generations. They find conditions under which the young and old agents unanimously agree upon a stream of PAYG social security transfers for the current and all future generations such that the resulting allocation is Pareto optimal and that no future generations have incentives to amend the program.

Veall [1986] provides an alternative explanation for PAYG program by considering an OLG model in which each agent is assumed to derive utility not only from his/her own life-cycle consumption, but also from the level of old-age consumption of his/her parents. Due to this consumption externality, elderly may save little to extract the maximum possible gifts from their children; "This can lead to an inferior steady state, where no one is consuming 'enough' in retirement" (Veall [1986, p.250). If a PAYG social security system is introduced such that it transfers from the young to the old at least the amount that the old could extract from their children by saving nothing, such a social security program could restore inter-temporal efficiency of consumption for each agent and Pareto optimality for the whole society. However, once the agents begin to save, the young may like to reduce their social security contribution and have incentive to amend the PAYG social security legislation. Thus such a PAYG system may not be stable. Veall shows that if social security benefits are set at the level of optimal steady-state old-age consumption, then such a legislation will be honored by all future generations and thus is stable. Moreover, the resulting allocation will be Pareto optimal.

If agents expect to receive gifts from their children to support old-age consumption, it is clear that not only savings decisions but also the fertility decisions will be affected; in fact, agents would like to have more children.[2] Empirical analyses of cross country data as well as household survey data predominantly show that social security affects both fertility level and savings rate (see for instance, Nugent [1985] for a summary of these studies). Hence, it is important to relax the exogenous fertility assumption in the above class of models.

In more recent models that study effects of social security on fertility and savings (Barro and Becker [1989], and Raut [1992]) the existence of social security is not explained. Nishimura and Zhang [1992] include fertility choices in Veall's one-sided altruism framework. Following Veall, they view the optimal old-age consumption in the steady-state as PAYG social security benefits. However, when fertility is also a choice variable, it is not possible to implement the optimal steady-state allocation using only a PAYG social security policy instrument; this was possible in Veall's framework because he treated fertility as exogenous; in fact, once such a PAYG social security program is enacted, the free rider's problem will cripple the system since an individual agent will have no incentive to have children (as they do not affect utility but cost money) and would like to depend on others'

[2] This is an alternative formulation of old-age security hypothesis.

children to contribute to social security program. Since every body would do the same, such a social security program is not individually rational. Therefore, viewing optimal steady-state gifts as a form of PAYG social security in Veall's framework loses both normative and positive virtues once fertility is a choice variable.

In this paper, we extend Veall's framework to rectify some of these problems. We assume that agents derive utility not only from their own young age and old age consumption, but also from old-age consumption of their parents and the young age consumption of their children weighted by the number of children. This allows us to endogenize within family transfers in both directions (i.e., from children to parents as gifts and from parents to children as bequest); moreover, in our framework even when parents do not receive any old-age support they have individual incentives to have children.

In the overlapping generations framework, as in real life, household decisions are made sequentially: in any given period, decisions regarding fertility, savings and intergenerational transfers of past generations and of the currently alive old generation that are made in the past are known to the current decision makers. When agents make their decisions they use all available information. Moreover, since agents know that their actions are observed by their children and hence will affect their children's decisions, they will take into account the incentive effects of their decisions on their children, and thus try to manipulate their children to get the best out of them. For instance, if an agent saves more for his retirement, then his children will transfer less income to the agent when he retires. Since the agent knows that his children react that way to his savings decisions, he might find it strategically advantageous to save little and have more children to extract higher transfers from his children. Much of the previous literature in this area ignores the sequential nature of the above overlapping decisions and apply the notion of open loop Nash equilibrium to characterize equilibrium outcomes. Open loop Nash equilibrium makes sense only when agents must commit to entire time paths of decisions without observing anyone else's. In open loop Nash equilibrium, agents take the actions of other agents as given but not their reactions and thus do not take proper account of the incentives that they face. We modify it by assuming that agents take the reaction functions of their children while making their own decisions, and introduce a notion of learning equilibrium. We show that the more relevant but analytically intractable subgame perfect equilibrium is a particular type of learning equilibrium, which can be computed as a fixed point of a function of single variable, that the long-run outcome of a dynamic learning equilibrium path, and that open loop Nash equilibrium requires a certain type of "incredible" threats.

In section 2, we set up our basic model and discuss the nature of coordination problems that the agents face, and compute the open loop Nash equilibrium. In section 3, we define and compute learning equilibria, and characterize open-loop Nash equilibrium and subgame perfect equilibrium in terms of learning equilibrium. In section 4 we introduce pay-as-you-go social security program and study its effect on population growth, aggregate savings and private intergenerational transfers in subgame perfect equilibrium. Section 5 concludes the paper.

2 Basic Framework

We use the basic Samuelsonian [1958] overlapping generations framework and introduce two-sided altruism to endogenize intergenerational transfers. Let us assume that time is discrete and is denoted as $t = 0, 1, 2, ...$; each person lives for three periods: young, adult, and old. While young he is dependent on his parents for all decisions. We follow the convention that a superscript t refers to an adult of period t and a subscript t refers to time period t. For instance, c_t^t and c_{t+1}^t denote respectively the adult age and old-age consumption of an adult of period t; however, n_t denotes the number of children of an adult of period t, since we assume that only adults can have children, so from the subscript of n_t we can identify which generation it corresponds to. We assume that for all $t \geq 1$, the wage rate w_t and the interest rate r_t are exogenously given.

2.1 Households

We assume that all children are born identical and they all behave identically in a given situation. We would like to derive agent's behavior regarding fertility, savings and intergenerational transfers from utility maximization. We model an individual's concern for his parents and children by assuming that an adult of generation t derives utility from his own life-cycle consumption and from consumption level of his children and parents that he observes during his active life-time (for a justification of these type of utility functions, see Kohlberg [1976], and Pollak [1988]). More specifically we postulate the following utility function:

$$W_t = \delta(n_{t-1})\mathrm{v}(c_t^{t-1}) + \alpha\mathrm{v}(c_t^t) + \beta\mathrm{v}(c_{t+1}^t) + \gamma(n_t)\mathrm{v}(c_{t+1}^{t+1}) \tag{1}$$

Veall [1986] in his exogenous fertility framework and Nishimura and Zhang [1992] in their endogenous fertility framework assumed that $\gamma(n_t) = 0$ and $\delta(n_t) =$constant, for all $t \geq 0$. When there are many siblings, an individual may not care about his parents as intensely as he would do if he were the only child. In the above specification of utility function, we allow the degree of an individual's concern for his parents to depend on the number of siblings. However, much of our results hold if $\delta(.)$ is constant.

In our economy, agents have interdependent utility functions: an agent's utility is affected by the amount of consumption of other family members. Thus, the agents have incentives to transfer part of their income to their parents and children. The decisions that are to be made by a representative adult of period t, $t \geq 1$ are as follows:

An adult of period t earns wage income w_t in the labor market and *expects* to receive a bequest b_t from his parents. These two sources of income constitute his budget during adulthood. Rearing cost per child in period t is $\theta_t > 0$ units of period t good. Given his adulthood budget, he decides the amount of savings s_t, the number of children $n_t \geq 0$, the fraction of income to be transferred to his old parents $a_t \geq 0$; in the next period, he retires and expects to receive $a_{t+1}n_t$ amount of gifts from his children, earns $(1+r_{t+1})s_t$ as return from his physical assets, and decides the amount of bequest $b_{t+1} \geq 0$ to leave for each of his children.

time generation	$t=1$	$t=2$	$t=3$	...	$t-1$	t	...
0	b_1						
1	(a_1, n_1, s_1)	b_2					
2		(a_2, n_2, s_2)	b_3	...			
...				...			
$t-1$					$(a_{t-1}, n_{t-1}, s_{t-1})$	b_t	
t						(a_t, n_t, s_t)	...
...				...			...

Table 1: Time table of actions by overlapping generations of agents

The effects of agent t's action, $\alpha^t = (a_t, n_t, s_t, b_{t+1})$, on the levels of his own life cycle consumption and the levels of consumption of his parents and children in the periods that overlap with his life-cycle, depend on his parent's action, α^{t-1} and his children's action α^{t+1} as follows:

$$\begin{aligned}
c_t^t + s_t + \theta_t n_t &= (1-a_t)w_t + b_t && (2)\\
c_{t+1}^t + n_t b_{t+1} &= (1+r_{t+1})s_t + a_{t+1}w_{t+1}n_t && (3)\\
c_t^{t-1} &= (1+r_t)s_{t-1} - n_{t-1}b_t + a_t w_t n_{t-1} && (4)\\
c_{t+1}^{t+1} &= (1-a_{t+1})w_{t+1} + b_{t+1} - s_{t+1} - \theta_{t+1}n_{t+1} && (5)\\
& c_t^t, c_{t+1}^t \geq 0
\end{aligned}$$

Similarly, the agent $t=0$'s utility function is given by

$$W_0 = \beta \mathrm{v}(c_1^0) + \gamma(n_0)\mathrm{v}(c_1^1)$$

and agent $t=0$ decides the level of bequest b_1, given his past decisions, n_0, s_0, and his children's decisions, α^1. The arguments of his utility function are given by

$$\begin{aligned}
c_1^0 + n_0 b_1 &= (1+r_1)s_0 + a_1 w_1 n_0 && (6)\\
c_1^1 &= (1-a_1)w_1 + b_1 - s_1 - \theta_1 n_1 && (7)\\
& c_1^0 \geq 0
\end{aligned}$$

Note that if $1 > a_t^* > 0$ and $b_t^* > 0$ is an equilibrium combination of gifts and bequest in period t, so is $a_t^* + \epsilon$ and $b_t^* + \epsilon w_t$, for small $\epsilon > 0$; this can lead to gift-bequest war. This could be handled by restricting to open loop Nash equilibria that yield either positive bequest or positive gift within a period but not both. To handle these problems, an open-loop Nash equilibrium is often used, we define it as follows:

A sequence of strategies, $\{\alpha^t\}_{t=0}^{\infty}$ is *feasible* if there exists an associated sequence of non-negative consumption stream $c_1^0, \{c_t^t, c_{t+1}^t\}_1^{\infty}$ such that it satisfies the budget constraints (2)-(7).

Definition 1 A*n open loop Nash equilibrium* is a sequence of feasible strategies$\{\alpha^t\}_0^{\infty}$ such that for given initial condition, n_0, s_0

(i) $a_t > 0 \Rightarrow b_t = 0$ and $b_t > 0 \Rightarrow a_t = 0$

(ii) for any $t \geq 1$, given $\alpha^{t-1} = (a_{t-1}, n_{t-1}, s_{t-1}, b_t)$ and $\alpha^{t+1} = (a_{t+1}, n_{t+1}, s_{t+1}, b_{t+2})$ there does not exist another strategy $\tilde{\alpha}^t$ for agent t such that $\tilde{\alpha}^t$ together with α^{τ}, $\tau \neq t$, $\tau \geq 0$ form a feasible sequence of strategies, and $\tilde{\alpha}^t$ yields higher utility for agent t.

We further distinguish among different types of equilibria. *An open-loop bequest equilibrium* is an equilibrium of the above type that satisfies $a_t = 0$, and $b_t > 0$ for all $t \geq 1$. *An open-loop gift equilibrium* is an equilibrium of the above type that further satisfies $b_t = 0$, and $a_t > 0$ for all $t \geq 1$. Similarly, an *open-loop equilibrium with no transfers* is one in which $b_t = a_t = 0$ for all t ≥ 0. There could be also equilibria in which bequests are operative in some periods and gifts are operative in other periods. In this paper we will analyze only open-loop gift equilibria. It can be seen easily from the first order conditions of the open loop Nash equilibrium that in general there is indeterminacy in the set of such equilibria. This indeterminacy is symptomatic of Nash equilibria with interdependent utility functions. For our purpose, we focus only on steady-state open loop gift equilibria which are determinate.

A *steady-state open loop gift equilibrium* is an open loop gift equilibrium such that $a_t = a^* > 0$, $n_t = n^* > 0$, $s_t = s^* \geq 0$ and $b_t = 0$ for all $t \geq 1$.

We denote all steady-state endogenous variables with a $*$, and drop the time scripts. We assume [3] that $w_t = w$, $r_t = r$ and $\theta_t = \theta$ for all $t \geq 1$. Since this stationarity assumption is not critical to the issues of the paper, to simplify exposition, we will maintain this assumption in the rest of the paper. Let us denote by c_1^* and c_2^* respectively the adult age and old-age consumption in the steady-state. Thus, for a steady-state gift equilibrium, we have $c_1^* \equiv (1 - a^*)w - \theta n^* - s^*$ and $c_2^* \equiv (1 + r)s^* + wa^*n^*$. The first order necessary conditions for such an equilibrium simplify to

$$\frac{\mathrm{v}'(c_2^*)}{\mathrm{v}'(c_1^*)} = \frac{\alpha}{\delta(n^*)n^*} \tag{8}$$

[3]One can assume $w_t = w(1 + g)^t$, and study the effect of growth rate g on fertility savings, and old-age transfers. Much of what we do in this paper can be modified easily to incorporate this.

$$\frac{\mathrm{v}(c_1^*)}{\mathrm{v}'(c_1^*)} = \frac{\alpha}{\gamma'(n^*)}\left[\theta - \frac{\beta a^* w}{\delta(n^*)n^*}\right] \tag{9}$$

$$\frac{\mathrm{v}'(c_2^*)}{\mathrm{v}'(c_1^*)} \geq \frac{\gamma(n^*)}{\beta n^*} \tag{10}$$

$$1 + r \leq \frac{\delta(n^*)n^*}{\beta}, (\text{equality if s}^* > 0) \tag{11}$$

In the following example we show the coexistence of unique steady-state open loop gift equilibria of two types: one type with $s^* = 0$ and the other type with $s^* > 0$.

2.2 An Example: (CEM Economy)

The instantaneous utility function satisfies the following:

Assumption A: 1 (constant elasticity of marginal utility (CEM) function)

$$v(c) = \frac{c^{1-\rho}}{1-\rho},\ \rho \neq 1,\ 0 < \rho < \infty \tag{12}$$

where $-\rho$ measures the elasticity of marginal utility.

Assumption A: 2 $\gamma(n) = \gamma_0 n^{1-\gamma_1},\ 0 \leq \gamma_1 < 1$

The significance of this assumption is that parents care about consumption of all children equally. However, the weight they give to such consumption decreases with the number of children whenever $\gamma_1 > 0$.

Assumption A: 3 $\delta(n) = \delta_0 n^{\delta_1 - 1},\ 0 \leq \delta_1 \leq 1$

Two types of steady-state gift equilibria may coexist. Let us first find steady-state gift equilibria with $s^* > 0$. Equation (11) determines the steady-state equilibrium n_s^* uniquely and equations (8) and (9) reduce to the following two linear equations:

$$s = \frac{\mu(w - \theta n_s^*) - w(n_s^* + \mu)a}{1 + r + \mu} \tag{13}$$

$$s = \frac{(1-\rho)\alpha\theta\mu n_s^{*\gamma_1}}{(1+r)\gamma_0(1-\gamma_1)} - \frac{(1+r)\gamma_0(1-\gamma_1)w n_s^{*1-\gamma_1} + (1-\rho)\alpha w\mu}{(1+r)^2\gamma_0(1-\gamma_1)}.n_s^{*\gamma_1} a \tag{14}$$

where $\mu = (\beta(1+r)/\alpha)^{1/\rho}$.

Notice that the intercept of equation (13) is always positive since the child cost, θn_s^* is less than wage income in gift equilibrium. The intercept of equation (14) is positive if $\rho < 1$, in which case the slopes are negative for both lines and we cannot guarantee that they will intersect in the positive orthant. However, if $\rho > 1$, equation (14) will have negative intercept and positive slope. If ρ is sufficiently larger than one, then it will intersect with the line (13), and we have unique steady-state gift equilibrium: We consider a numerical example with parameters, $\delta_0 = .35; \delta_1 = .8$; $\gamma_0 = .3$, $\gamma_1 = .6$; $\rho = 1.5$; $\alpha = .4$; $\beta = .34$; $r = .05$; $w = 10$; and $\theta = .1$. The equilibrium quantities are as follows:

$$(n_s^*, s^*, a^*, U_{max}) = (1.025062190, 1.341247016, .3341720874, -1.241803182)$$

one can easily verify that (10) is satisfied with strict inequality.

Let us now examine if there exists steady-state gift equilibria of the type $s^* = 0$, and we find an equilibrium for the above set of parameters, such an equilibrium could be found. It can be shown easily that (8) and (9) simplify to the following two equations in two unknowns, a and n:

$$(1-a)w - \theta n = \left[\frac{\delta(n)n}{\alpha}\right]^{1/\rho} awn \tag{15}$$

$$(1-a)w - \theta n = \frac{\alpha(1-\rho)}{\gamma'(n)}\left[\theta - \frac{\beta wa}{\delta(n)n}\right] \tag{16}$$

It could be seen readily that the graph of these two non-linear functions intersect only at one point. For the given parameter values the unique solution is given by

$$(n^*, s^*, a^*, U^*) = (1.699710194, 0, .4095616885, -1.140189766)$$

furthermore, the constraints (10) and (11) are satisfied as strict inequalities.

Comparing these two open-loop gift equilibria we find that the equilibrium with zero savings has higher levels of fertility, transfers from children and welfare of a representative agent than the gift equilibrium with positive savings. How reasonable are these equilibria? We begin our enquiry starting with the remarks of the next subsection.

2.3 Remarks on open-loop equilibrium

An open loop Nash equilibrium framework does not fully model the incentives that agents may have to manipulate their parents' or their children's behavior to extract more transfers from them. For instance, since parents make their consumption and fertility decisions prior to their children's, parents may find it strategically advantageous to consume more in their working age, save little on physical assets and possibly have more children so that when they become old they have little income of their own. When the children find that their old parents have little to consume, they will have sympathy for their parents since they care about their parents' consumption; thus they will transfer a larger amount of old-age support than what they would be transferring in the open loop Nash equilibrium. The children in turn can manipulate their children in the same way and be better-off as a result. This process might be self-fulfilling over time. In the next section, we will see that this is true, and we will also point out other problems with the concept open-loop equilibrium.

3 Manipulation and Subgame Perfection

In our formulation, we assume a particular type of information structure in the decision tree of overlapping generations of agents so that we are able to compute and study the properties of subgame perfect equilibrium. More specifically, we divide each time period t into two stages denoted by t and $t.1$ (stage $t.1$ follows stage t) at which the live agents of period t are to make decisions. At stage t, which is the beginning of period t, the agent $t-1$'s decisions $(a_{t-1}, n_{t-1}, s_{t-1})$ as well as all decisions of the previous generations are part of history and are assumed to be observable to the live agents $t-1$ and t. We denote a realization of all these past decisions at stage t by h_t. We assume that given a realization of the history h_t, the agent $t-1$ decides to bequeath b_t to each of his children and each of his children decides the fraction of their income, a_t to be given as gift to their parents. Both agents make their decisions simultaneously and independently. The game moves to stage $t.1$ at which both agents observe the outcome of stage t. We denote a typical realization of these decisions at stage $t.1$ by $h_{t.1}$. Given a realization of the history $h_{t.1}$ at stage $t.1$, we let $t-1$ make no further household decision, agent t, however, decides the number of children and savings (n_t, s_t). See Figure 1 for details of the extensive form representation of our decision tree. Since agent $t-1$ knows that his children will use the information regarding his observable actions, he will choose his actions in each stage that exploits the reactions of his children in most favorable way. Or in other words, parents may find it beneficial to manipulate their children's behavior.

Let us denote by $\mathcal{H}_t$ the set of all possible histories up to time t. We follow the convention of denoting an agent a with a superscript and stage t by a subscript t. Let $S_t^{t-1}(h_t) \subset \Re_+$ be the set of feasible bequest decisions of agent $t-1$ at stage t defined by

$$S_t^{t-1}(h_t) = \left\{ \begin{array}{c} b_t \geq 0 \mid (4) \text{ is satisfied with } c_t^{t-1} \geq 0,\ a_t = 0, \\ \text{and } s_{t-1}, n_{t-1} \text{consistent with } h_t \end{array} \right\}$$

Note that the above set of feasible bequest decisions depend on the history h_t, especially on the agent's own savings and fertility decisions. At stage t, agent $t-1$'s feasible actions are functions of the form $b_t : \mathcal{H}_t \to \Re$, such that $b_t(h_t) \in S_t^{t-1}(h_t)$.

Similarly, given the history h_t, the set of feasible actions of an adult agent t in stage t,

$S_t^t(h_t) \subset \Re_+$ is defined by

$$S_t^t(h_t) = \left\{a_t \in \Re_+ \mid (2) \text{ is satisfied with } b_t = 0, c_t^t \geq 0\right\}$$

At stage t, agent t's actions are functions, $a_t : \mathcal{H}_t \rightarrow \Re$ such that $a_t(h_t) \in S_t^t(h_t)$. At stage t, the agents t and $t-1$ choose their strategies b_t and a_t simultaneously and non-cooperatively. Once agents t and $t-1$ have chosen their actions in stage t of period t, the history gets updated to $h_{t.1}$, and the game moves to stage $t.1$ at which agent t's set of feasible actions $S_{t.1}^t$ is given by

$$S_{t.1}^t\left(h_{t.1}\right) = \left\{ \begin{array}{c} \left(n_t, s_t\right) \in \Re_+^2 | (2) \text{ is satisfied with } c_t^t \geq 0, a_t, b_t \\ \text{are consistent with the history } h_{t.1} \end{array} \right\}$$

At stage $t.1$, agent t's actions are functions, $(n_t, s_t) : \mathcal{H}_{t.1} \rightarrow \Re^2$ such that $(n_t, s_t)(h_{t.1}) \in S_{t.1}^t(h_{t.1})$.

We denote the game starting at stage t with history h_t as $\Gamma(h_t)$. Figure 1 depicts a part of the extensive form of the game $\Gamma(h_t)$: the tree is shown only up to stage $t+1.1$; the label of a branch describes the action of the agent that it corresponds to; the shaded boxes are the information sets of the agents within a given stage.

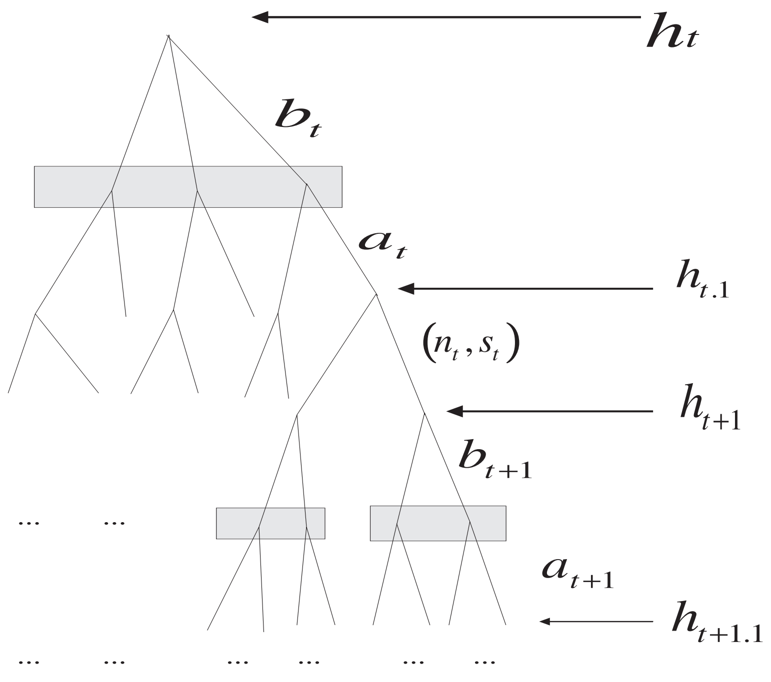


Figure 1: Extensive form representation of the multi-stage game, $\Gamma(h_t)$

In the above set-up, agents in later stages can use very complex punishment rules as their strategies. For instance, an agent $t = 5$ in stage 5 can condition his actions as follows:

> ”he will transfer a certain fraction a_5 of his income to his parents if his parents transferred a certain fraction a_4 of their income to the agent's grandparents, saved certain amount s_4, had certain number of children, n_4, and if his grand parents transferred a certain fraction a_3 of their income to the agent's grand grand parents, ... and so on.”

While these types of strategies may lead to many subgame perfect equilibria, the equilibria that prescribe strategies conditioning on the dead grand parents are hard to execute since it is not possible to objectively verify if the agent's grand parents or grand grand parents did such and such.

Using the Markovian structure of our economy, and the fact that utility functions depend only on parent's old-age and the children's young age consumption, we restrict the set of feasible actions that conditions only on the actions which directly affect an agent's utility. More specifically, note that $S_t^t(h_t)$ does not depend upon history h_t and $S_t^{t-1}(h_t)$ depends only on agent $t-1$'s own past decisions. From equations (2)-(5), and the arguments of the utility function, it is clear that the only information from history that is relevant to decision making of the agents in stage t are agent $t-1$'s own past decisions (s_{t-1}, n_{t-1}) in making his bequest decision b_t, which we represent as functions of the form, $b_t\left(n_{t-1}, s_{t-1}\right)$, and in making agent t's gift decision a_t, which we represent as functions of the form, $a_t\left(n_{t-1}, s_{t-1}\right)$. Similarly at stage $t.1$, the agent t's actions depend on his own past decision a_t and his parent's bequest decision b_t, only through the net effect, $a_t w_t - b_t$ which we represent as functions of the form, $n_t = n_t\left(a_t w_t - b_t\right)$, and $s_t = s_t(a_t w_t - b_t)$. Thus agent t's strategies are functions of the type: $a_t = a_t(n_{t-1}, s_{t-1})$, $n_t = n_t\left(a_t w_t - b_t\right)$, and $s_t = s_t(a_t w_t - b_t)$. When actions at any stage are functions of past actions, they are generally known as *reaction functions.* Putting all the actions and reactions of agent t from all stages of the game together, we note that a *profile of pure strategies* of all agents together is given by,

$$\mathcal{A}_t = \begin{cases} (a_t(n_{t-1}, s_{t-1}), n_t(a_t w_t - b_t), s_t(a_t w_t - b_t), b_{t+1}(n_t, s_t)) & \text{if } t \geq 1 \\ b_1(n_0, s_0) & \text{if } t = 0 \end{cases}$$

where, each component belongs to the relevant strategy spaces specified above. Note that agent t's actions, n_t, s_t, a_t, and b_{t+1} now belong to function spaces, whereas in open loop Nash equilibrium they were non-negative real numbers. Let us denote a subgame starting at h_t by $\Gamma(h_t)$. Note that in our context the subgame $\Gamma(h_t)$ depends effectively only on the components (n_{t-1}, s_{t-1}) of the history h_t; and the subgame $\Gamma\left(h_{t.1}\right)$ depends effectively only on $\left(a_t, b_t\right)$. We use the following characterization of the subgame perfect equilibrium notion.

Definition 2 Let n_0 and s_0 be the initial condition, i.e., history of the initial game $\Gamma(h_1)$. A profile of strategies, $\mathcal{A}_t^* = (a_t^*(n_{t-1}, s_{t-1}), n_t^*(a_t w_t - b_t), s_t^*(a_t w_t - b_t), b_{t+1}^*(n_t, s_t))$, for agent $t \geq 1$, and $\mathcal{A}_0^* = b_1^*(n_0, s_0)$ for agent $t = 0$, is said to be *subgame perfect equilibrium* if at any stage t, and at any history h_t with the last two components (n_{t-1}, s_{t-1}), the pair of actions $b_t^*(n_{t-1}, s_{t-1})$ for agent $t-1$ and $a_t^*(n_{t-1}, s_{t-1})$ for agent t is a Nash equilibrium of the stage t game of the subgame $\Gamma(h_t)$ and at stage $\mathrm{t}.1$, given any history $h_{t.1}$ with the last two components leading to $a_t w_t - b_t$, the actions $n_t^*(a_t w_t - b_t)$, and $s_t^*(a_t w_t - b_t)$ are the optimal actions of agent t, when it is assumed that all the future moves will be made according to the prescription in $\{\mathcal{A}_t^*\}_{\tau=0}^{\infty}$.

Similar to open loop Nash equilibrium, we can define subgame perfect gift equilibrium and subgame perfect bequest equilibrium. However, in the rest of the paper we analyze only the properties of the subgame perfect gift equilibria.

3.1 Conditions for subgame perfect gift equilibrium

Let $a_{t+1}(n_t, s_t)$, $n_{t+1}(a_{t+1}w_{t+1} - b_{t+1})$, $s_{t+1}(a_{t+1}w_{t+1} - b_{t+1})$ be the optimal reaction functions of agent $t+1$, and let n_{t-1}, s_{t-1} be any feasible actions of agent $t-1$. Taking these decisions as given, agent t chooses a feasible $\mathcal{A}_t = a_t(n_{t-1}, s_{t-1})$, $n_t(a_t w_t - b_t)$, $s_t(a_t w_t - b_t)$, $b_{t+1}(n_t, s_t)$ that maximizes his utility. For $t > 1$, the first order necessary conditions for his maximization problem are as follows:

At stage t:

$$\delta(n_{t-1})n_{t-1}\mathrm{v}'([1 + r_t)s_{t-1} + a_t w_{tt} n_{t-1}]) - \alpha\mathrm{v}'([(1 - a_t)w_t - s_t - \theta_t n_t]) = 0 \quad (17)$$

$$\begin{aligned} &-\beta\mathrm{v}'(c_t^{t-1})n_{t-1} + \gamma(n_{t-1})\mathrm{v}'(c_t^t)\times \\ &[1 - \theta_t n_t'(b_t - a_t w_t) - s_t'(b_t - a_t w_t)] \leq 0 \text{ and } = 0 \quad \text{if } b_t \quad > 0 \end{aligned} \quad (18)$$

At stage t.1:

$$\begin{aligned} &-\alpha\mathrm{v}'(c_t^t) + \beta\mathrm{v}'(c_{t+1}^t)\left[(1 + r_{t+1}) + w_{t+1}n_t a_{t+1,2}(n_t, s_t)\right]\times \\ &-\gamma(n_t)\mathrm{v}'(c_{t+1}^{t+1})\left[a_{t+1,2}(n_t, s_t)w_{t+1}\right] \leq 0 \text{ and } = 0\text{if } s_t > 0 \end{aligned} \quad (19)$$

$$\begin{aligned} &-\alpha\theta\mathrm{v}'(c_t^t) + \beta\mathrm{v}'(c_{t+1}^t)\left[a_{t+1}(n_t, s_t)w_{t+1} + n_t w_{t+1}a_{t+1,1}(n_t, s_t)\right]\times \\ &+\gamma'(n_t)\mathrm{v}(c_{t+1}^{t+1}) - \gamma(n_t)\mathrm{v}'(c_{t+1}^{t+1})\left[w_{t+1}a_{t+1,1}(n_t, s_t)\right] = 0 \end{aligned} \quad (20)$$

In our framework, a subgame perfect equilibrium with differentiable reaction functions may not exist. Even if we assume that there exists one, it is not possible to compute all subgame perfect gift equilibrium reaction functions from the above first order conditions.[4] Therefore, we explore the above system of equations to find a steady-state local subgame perfect gift equilibrium as follows:

[4]See Kohlberg [1976] for a discussion of such problems in a similar framework.

Let us assume that the wage rate $w_t = w$, the interest rate $r_t = r$, and the cost of raising children $\theta_t = \theta$ for all $t \geq 1$.

Definition 3 A *steady-state local subgame perfect gift equilibrium* is a vector of fertility level, savings amount, and the rate of old-age support to parents, $(n^*, s^*, a^*) \geq 0$ and a vector of reaction functions $(a(n_{t-1}, s_{t-1}), n(a_t w_t - b_t), s(a_t w_t - b_t))$ defined in a neighborhood[5] of (n^*, s^*, a^*) such that

$$a^* = a\left(n^*, s^*\right), \; n^* = n(a^* w)), \; s^* = s(a^* w)$$

and

$$\begin{aligned} a_t(n_{t-1}, s_{t-1}) &= a(n_{t-1}, s_{t-1}) \\ n_t(a_t w_t - b_t) &= n(a_t w_t - b_t) \\ s_t(a_t w_t - b_t) &= s(a_t w_t - b_t) \\ b_t &= b = 0 \text{ for all } t \geq 1 \end{aligned}$$

and that the above satisfies the system of equations (17)-(20) for all $t \geq 1$ with initial condition, $n_0 = n^*$, and $s_0 = s^*$.

It is not possible to compute all the steady-state equilibrium reaction functions from the above first order conditions. Note, however, that conditional on the equilibrium $b_t^* = 0$, the actions (n_t, s_t) at stage $t.1$ and the action b_{t+1} at stage $t+1$ of agent t depend on the history only through his own past action, a_t. This is because given a_t and b_t, parent's old-age consumption is fully determined, and the choice of n_t and s_t cannot affect it. As a consequence of the Envelop Theorem, we note that if agent t chooses a_t, n_t and s_t simultaneously as opposed to recursively, we can treat these actions as scalars instead of reaction functions, and the optimal solution we arrive at this way will be the same as, if we solved the problem recursively and treated these decisions as reaction functions of one's own past decisions instead. With this simplification, and denoting one period lag value and one period forward value of a variable x by x_- and x^+ respectively, the system of equations (17)-(20) for a steady-state subgame perfect equilibrium becomes:

$$\frac{\delta(n_-)n_-}{\alpha} = \frac{\mathrm{v}'\left([1 - a(.)]w - s - \theta n)\right)}{\mathrm{v}'\left((1 + r)s_- + a(.)wn_-\right)} \tag{21}$$

$$\begin{aligned} -\beta \mathrm{v}'((1 + r)s_- + a(.)wn_-)n_- + \gamma(n_-)\times \\ \mathrm{v}'\left([1 - a(.)]w - s - \theta n)\right)\left[1 - \theta n' - s'\right] \leq 0 \text{ and } = 0 \text{ if } b > 0 \end{aligned} \tag{22}$$

[5]The adjective "local" in the definition refers to this neighborhood restriction.

$$-\alpha \mathrm{v}'\left([1-a]w - s - \theta n)\right) + \beta \mathrm{v}'\left((1+r) + wna_2(n,s)\right) - \gamma(n) \times$$
$$\mathrm{v}'\left([1-a(n,s)]w - s^+ - \theta n^+\right)[a_2(n,s)w] \leq 0 \text{ and } = 0 \text{ if } s > 0 \tag{23}$$

$$-\alpha\theta \mathrm{v}'\left([1-a]w - s - \theta n)\right) + \beta \mathrm{v}'\left((1+r) + wna(n,s)\right)[a(n,s)w + a_1(n,s)wn] + \gamma'(n) \times$$
$$\mathrm{v}\left([1-a(n,s)]w - s^+ - \theta n^+\right) - \gamma(n)\mathrm{v}'\left([1-a(n,s)]w - s^+ - \theta n^+\right)[a_1(n,s)w] = 0 \tag{24}$$

Notice that we could solve for $a(.)$ as a function of n_- and s_- from equation (21) treating n and s as given, and then solve for n and s from equations (23)-(24) after plugging in the values of $a(.), a_1(.)$ and $a_2(.)$. This cannot work since n and s in equation (21) are implicit functions of a, and hence it will not be possible to calculate $a(.), a_1(.)$ and $a_2(.)$ from equation (21) alone. This is a curse on subgame perfect equilibrium in overlapping generations models. We introduce our local learning equilibrium concept to handle precisely this computational difficulty of the subgame perfect equilibrium, and show that our equilibrium concept is broad enough to include both open loop and subgame perfect equilibrium as particular types of learning equilibria, and show a way to compute subgame perfect equilibrium as the long-run limit of certain type of learning equilibrium paths.

3.2 Local learning equilibrium and local subgame perfect equilibrium

Let us denote by S the sum of savings in physical capital s and investment in n children θn, i.e., $S = s + \theta n$. Notice that around a steady-state gift equilibrium, S is a function of aw. We postulate that individuals follow an exogenously given rule, $S = \sigma(aw)$. Locally around a steady-state equilibrium values of a^*, n^*, and s^*, the rule $\sigma(aw)$ need be specified as a linear function,

$$\sigma(aw) = s^* + \theta n^* + (a - a^*)w\sigma'(a^*w) \tag{25}$$

We can interpret the behavioral rule $\sigma(aw)$ in equation (25) as a threat strategy that children use against their parents in case parents induce them to choose an a which is different but in a neighborhood of the equilibrium a^*. The nature of the threat is determined by the magnitude of $\sigma'(a^*w)$, and we assume that the nature of threat they apply to their parents is learned from observing their parents' behavior towards their grand parents and from other neighbors in his locality. Notice that conditional on ${\sigma'}^* \equiv \sigma'(a^*w)$, which is a scalar, we can solve for $a(), n(), s()$, and n^*, s^*, a^* from equations (21)-(24). We name this equilibrium as a *steady-state local learning gift equilibrium.* More formally,

Definition 4 *A steady-state local learning gift equilibrium with respect to a given value of* $\sigma'^* = \sigma'(a^*w)$ is a vector of fertility level, savings amount and the rate of gift transfers, (n^*, s^*, a^*) and a vector of reaction functions$(a(n_-, s_-), n(aw), s(aw))$ defined in a neighborhood of (n^*, s^*, a^*) such that

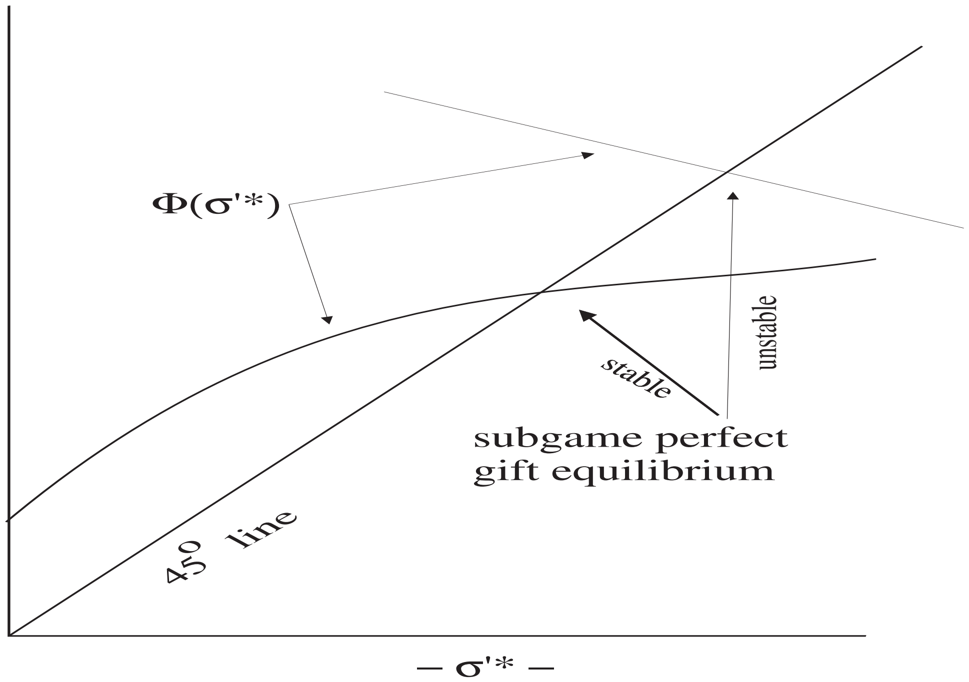


Figure 2: Phase diagram of Learning dynamics

(i) $a(.)$ is a solution of equation (21) with $s + \theta n = \sigma(a(n_-, s_-)w)$ from equation (25),

(ii) $n(aw), s(aw)$ are solutions of equations (23) and (24) after substituting $s^+ + \theta n^+ = \sigma\,(a(n,s)w)$ from equation (25), and

(iii) $n_- = n = n^*, s_- = s = s^*, a^* = a(n^*, s^*)$ and $b = 0$ solve equations (21) -(24) after substituting $s^+ + \theta n^+ = \sigma\,(a(n,s)w)$ from equation (25).

Once we have found a learning equilibrium, reaction functions $a(), n(), ands()$ and the quantities n^*, s^*, a^* with respect to a given value of σ'^*, we can study the local dynamics of the system as follows: beginning with an initial (n_0, s_0) in a neighborhood of (n^*, s^*) we obtain $a_1 = a(n_0, s_0)$, and $(n_1, s_1) = (n(a_1 w), s(a_1 w))$ and so on. But we do not carry out any further analysis along this line.

For a steady-state local learning gift equilibrium corresponding to $\sigma^{'*}$, let us denote by $\Phi(\sigma^{'*}) \equiv \theta n'(a^* w) + s'(a^* w)$. Notice that a local learning gift equilibrium need not be a local subgame perfect gift equilibrium, because $\Phi(\sigma^{'*})$ need not be equal to $\sigma^{'*}$. However, a local subgame perfect gift equilibrium is a local learning gift equilibrium for which $\Phi(\sigma^{'*}) = \sigma^{'*}$. Thus we establish that a local subgame perfect gift equilibrium is a particular type of local learning gift equilibrium and it can be easily computed as a fixed point of the function of a single variable, $\Phi(\sigma'^*)$. We may have multiple steady-state local subgame perfect gift equilibria, which one to select? (see figure 2).We use a learning equilibrium selection criterion as follows:

We assume, more realistically, that children do not bluntly repeat their parent's threat strategy σ'^*, but they learn to update their threat strategy based on their parent's experiences about the relationship between σ'^* and $\Phi(\sigma'^*)$; and suppose this updating is according to a convergent iterative algorithm of finding a fixed point of the map $\Phi(\sigma'^*)$, then such a dynamic learning equilibrium path will converge to a steady-state subgame perfect equilibrium in the long-run. There could be, however, many such algorithms; which learning algorithm will be more appropriate in describing human learning is an empirical question. Let us assume a natural updating rule namely, if generation t's parents used σ'^*_{t-1}, then generation t uses $\sigma'^*_t = \Phi\left(\sigma'^*_{t-1}\right)$. In the long-run the learning equilibrium path will converge to a locally stable fixed points of Φ if the initial generations σ'^*_0 was close to such a fixed point. Let us call a locally stable fixed point of Φ as a *stable steady-state local subgame perfect equilibrium*. Using the learning criterion, we can eliminate unstable equilibria and select only the stable subgame perfect equilibria as reasonable. In our numerical example for CEM economy we will see that this leads to a unique equilibrium selection of steady-state local subgame perfect gift equilibria.

It is important to note that we have been able to reduce an intractable problem of finding subgame perfect equilibrium, involving computation of a fixed point in functions spaces, to a simpler problem.

3.3 Properties of steady-state gift equilibria

In this section we study the properties of steady-state local learning equilibria for which the associated $\sigma'^* > -1$. These results are also true, in particular, for any local subgame perfect gift equilibria for which the associated $\sigma'^* > -1$ (this is true for instance, for the CEM economy in our numerical example that follows). The following proposition shows that the equilibrium reaction of children to parents' higher savings is to reduce old-age support to their parents.

Proposition 1 *Let v(.) be twice continuously differentiable with* $\mathrm{v}''(c) < 0 \;\forall\; c > 0$*, then for all* (n, s) *that lead to positive consumptions in each period, equation (21) has a continuously differentiable solution* $a(n, s)$ *and* $\partial a(n, s)/\partial s < 0$*.*

Proof. Substituting $s + \theta n = \sigma(aw)$ from equation (27) in equation (21), we have an implicit function $\Phi(n, s, a) = 0$ for which

$$\frac{\partial \Phi(.)}{\partial a} = -w\left[\mathrm{v}''(c_1^*)(1+\sigma'^*) + \mathrm{v}''(c_2^*)\delta(n)n^2/\alpha\right] > 0$$

Hence the first part follows from the implicit function theorem. Using the implicit function theorem again, we have

$$\frac{\partial a(n,s)}{\partial s} = -\frac{(1+r)\text{v}''(c_2^*)\delta(n)n/\alpha}{w\left[\text{v}''(c_1^*)(1+\sigma'^*) + \text{v}''(c_2^*)\delta(n)n^2/\alpha\right]} < 0$$

Q.E.D.

While the effect of parents savings is negative on the transfers from children, the corresponding effect of number of children could be ambiguous. To show this, let us denote by $\phi(n) \equiv \delta(n).n/\alpha$ and assume that $\phi(n)$ is an increasing function of n. Proceeding in the same manner as in the proof of above proposition, we can derive that

$$\frac{\partial a(.,.)}{\partial n} = -\frac{\phi'(n)\text{v}'(c_2^*) + [\phi(n)a(.,.)w\text{v}''(c_2^*)]}{[w\text{v}''(c_1^*)(1+\sigma'^*) + \phi(n)wn\text{v}''(c_2^*)]}$$

Note that both bracketed terms in the above are negative and the first term of the numerator is positive. Thus the sign of the right hand side of the above partial derivative will depend on the relative magnitudes of the bracketed terms and the first term on the numerator. In the numerical example for CEM economy that we consider later, the right hand side is unambiguously negative, which means that if parents have more children, they will receive less gifts from each child.

The following proposition finds condition under which a local learning equilibrium is also a open-loop gift equilibrium in the steady-state.

Proposition 2 *A steady-state local learning gift equilibrium corresponding to $\sigma'^* = \infty$ is also a steady-state open-loop gift equilibrium*

Proof. Notice above that if $\sigma'^* = \infty$, then both $a_1(n,s)$ and $a_2(n,s)$ are zero for equilibrium reaction function $a(.)$, and thus it follows from equations (21)-(24) and equation (25) that the equilibrium conditions for the steady-state local learning equilibrium is the same as the conditions for steady-state open-loop gift equilibrium.

Q.E.D.

Although, a threat to parents by the children of the type $\sigma'^* = \infty$ leads to open-loop Nash equilibrium but it is incredible since it sounds like:

> ”if his parents choose levels of fertility and saving different from that are prescribed by the open loop Nash-equilibrium level n^*, s^* and thus induce him to transfer more (resp. less) amount than that is prescribed by open-loop Nash equilibrium, he will consume nothing (resp. consume everything that he has, and if necessary he will borrow against his children) during his adult age.”

A feasible steady-state allocation is said to be *Pareto Optimal* if there does not exist another feasible steady-state allocation that gives higher utility to a representative agent. [6]

Proposition 3 *Consider an economy that has a steady-state local learning gift equilibrium* $(s^*, n^*, a^*(.,.))$ *with* $s^* = 0$ *and no bequest constraint, (10), holds as a strict inequality, and suppose further that the equilibrium satisfies:*

$$\beta - \left(\frac{\gamma(n^*)}{n^*}\right) \cdot \left(\frac{v'(c_1^*)}{v'(c_2^*)}\right) \equiv \mu > 0 \text{ and } \delta(n^*) < \mu$$

then all agents can be made better-off with a suitably designed pay-as-you-go social security program. Hence such an equilibrium is not Pareto optimal.

Proof. Consider a pay-as-you-go social security program which marginally taxes all adult agents and redistributes the revenues equally among their old parents. Suppose for the moment that agents do not change their fertility and savings decisions in response to introduction of such a social security program. The utility gains of a representative agent is $n^*\beta \mathrm{v}'(c_2^*)$ from the increased consumption in the old-age. The utility loss is given by $\alpha \mathrm{v}'(c_1^*) + \gamma(n)\mathrm{v}'(c_1^*)$, where the first term corresponds to welfare loss due to fall in own adult-age consumption and the second term corresponds to the welfare loss due to reduction in children's adult-age consumption. Thus the net gain is

$$\begin{aligned}
\triangle U &= n^*\beta \mathrm{v}'(c_2^*) - \alpha \mathrm{v}'(c_1^*) - \gamma(n^*)\mathrm{v}'(c_1^*) \\
&= n^*\beta \mathrm{v}'(c_2^*) - \delta(n^*)n^*\mathrm{v}'(c_2^*) - \gamma(n^*)\mathrm{v}'(c_1^*) \\
&= n^*\left(\beta \mathrm{v}'(c_2^*) - \gamma(n^*)\mathrm{v}'(c_1^*)\right) - \delta(n^*)n^*\mathrm{v}'(c_2^*) \\
&> 0
\end{aligned}$$

In deriving the above we have used equation (21) and the fact that equation (10) is a strict inequality by assumption.

It is clear that if the agents optimally adjust their fertility and savings decisions, the gains in utility will be even higher.

Q.E.D.

Social security can improve Pareto efficiency of a steady-state subgame perfect gift equilibrium provided no bequest condition[7] (10) is a strict inequality. If the no-bequest condition is an equality, introduction of social security cannot improve Pareto efficiency.

[6] This is a modified version of Pareto Optimality, modified to take into account the problem of comparing non-existing individuals' utilities under two different feasible steady-state allocations. See Raut [1990] for a discussion of this problem and the related literature on this issue.

[7] This is a stronger condition for no-bequest in the subgame perfect equilibrium since (10) implies (22).

3.4 The CEM Example Continued

Let the utility function be a CEM function as in (12). For this utility function, we have the following explicit solution $a(n,s)$ of equation (21):

$$a(n,s) = \frac{(\delta_0/\alpha)^{1/\rho} n^{\delta_1/\rho} (w - [s^* + \theta n^* - a^* \sigma'^*]) - (1+r)s}{w\left(n + \left(1 + \frac{\sigma'^*}{w}\right)(\delta_0/\alpha)^{1/\rho} n^{\delta_1/\rho}\right)} \tag{26}$$

One can easily verify that both $a_1(.)$ and $a_2(.)$ are negative for this reaction function.

For various values of σ'^* we solved the local learning steady-state equilibria numerically using the Maple V Software. For all the values of σ'^* we found two equilibria one with $s = 0$ and the other one with $s > 0$, and the former equilibrium always produced higher utility and rate of gift transfers of a representative agent. Furthermore, we found that when $\sigma'^* \longrightarrow \infty$, the local learning equilibrium of each type tends to the steady-state open loop gift equilibrium of the corresponding type given in the previous section. In the first row of table 2 we present two steady-state local learning gift equilibria corresponding to $\sigma'^* = 0$, (other parameters are as in the previous numerical example).

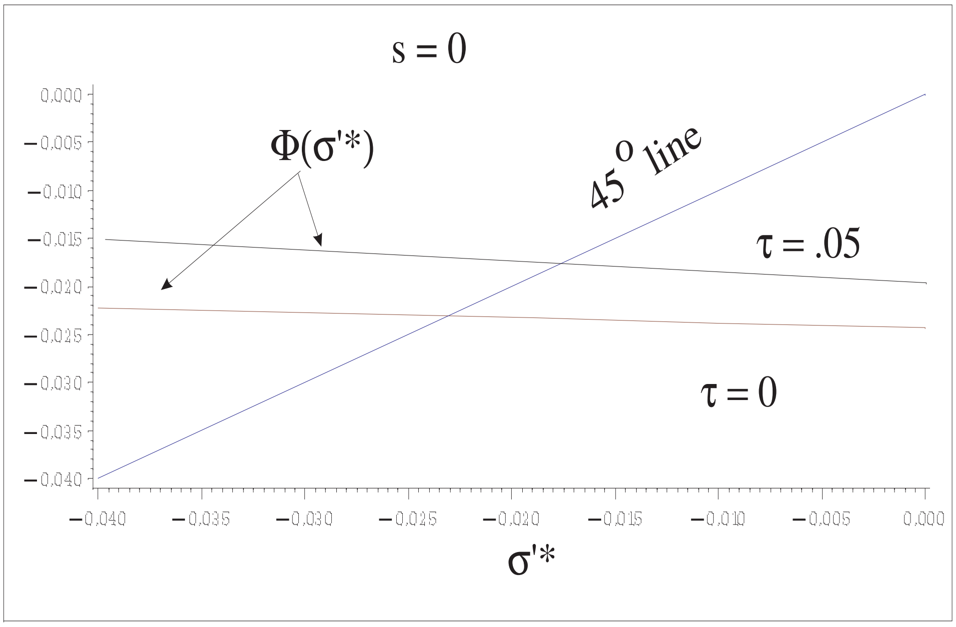
Table 2: Steady-state local learning and subgame perfect gift equilibria

τ	σ'^*	$(n^*, s^*, a^*, U_{\max})$
0	0	$(1.598904972, 0, .4168212214, -1.150134237)$ $(.8658794251, 1.477940857, .3265849827, -1.270158580)$
0	-.02308703065*	$(1.598801259, 0, .4168289337, -1.150145147)$
0.05	-.01729425139*	$(1.339780353, 0, .3879101535, -1.182775844)$

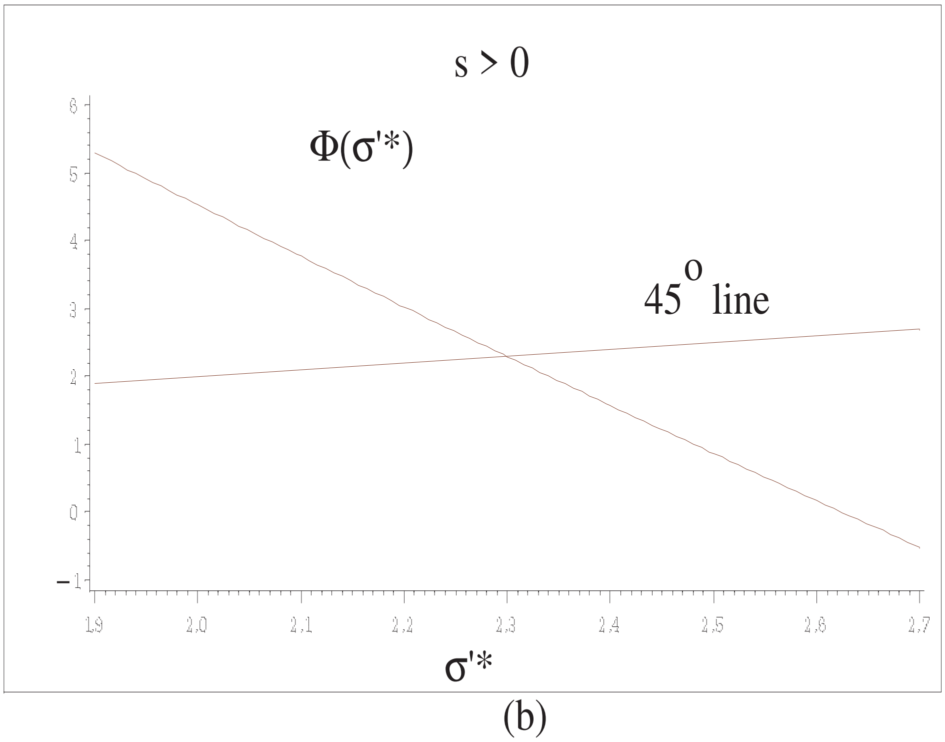
A row with * corresponds to the stable subgame perfect equilibrium.

In panels (a) and (b) of figure 3 corresponding to the cases $s = 0$ and $s > 0$, we have plotted the graph of $\Phi(\sigma'^*)$ around its fixed points. τ in this figure represents social security tax rate that we will consider later.

We chose initial value $\sigma_0'^*$ close to the fixed point and simulated the learning equilibrium path $\{\sigma_t'^*\}_{t\geq 1}$ we found that in the case of $s = 0$, $\{\sigma_t'^*\}_{t\geq 1}$ converged to a stable local subgame perfect gift equilibrium, $\sigma_s'^* = -0.2308703065$, the equilibrium allocation is shown in the second row of table 2. For the case $s > 0$ none of the sequences of learning equilibria that we considered converged. This was the case also for $\tau = .05$. Thus for a wide class of CEM economies (at least for the wide range of parameter values that we considered), learning mechanism selects a unique steady-state subgame perfect gift equilibrium in the long-run.



(a)



(b)

Figure 3: Phase diagram of learning equilibrium for numerical CEM economy

4 Social security and its effects

We have seen in our previous results that parents do have incentives to manipulate to receive a higher percentage of their children's income transferred to them. In this section we examine the effects of introducing a pay-as-you-go social security which can directly reduce or even eliminate the need for manipulation to effect old-age supports. We study the effects of social security on the rate of voluntary old-age transfers, fertility, savings, both at the individual level and the aggregate level. We illustrate these effects using CEM economy for analytical simplicity.

Let us suppose that a pay as you go social security program is introduced so that, apart from making decisions regarding savings, fertility, bequest and old-age gifts to parents as specified in our previous model, the agents of every generation t pays τw_t as social security taxes when he is adult, and receives $\tau n_t w_{t+1}$ as social security benefits when he is old. Although, the benefits $\tau n_t w_{t+1}$ depends on agent t's number of children n_t, but he takes it as an externality. For a given value of σ'^* the steady-state local learning equilibrium $a(n, s)$ for our CEM economy is given by

$$a(n,s) = \frac{(\delta_0/\alpha)^{1/\rho} n^{\delta_1/\rho}(w - \tau w - [s^* + \theta n^* - a^* \sigma'^*]) - (1+r)s - n\tau w}{w\left(n + \left(1 + \frac{\sigma'^*}{w}\right)(\delta_0/\alpha)^{1/\rho} n^{\delta_1/\rho}\right)} \tag{27}$$

For all non-negative values of σ'^*, which are reasonable values in our case, it is clear from equation (27) that $\frac{\partial a}{\partial \tau} < 0$, and it is not necessarily equal to -1. That is social security does not perfectly crowd-out private transfers. It is not possible to analytically derive the effect on fertility and savings. However, for various values of σ'^*, we found that the effect of τ on fertility is always negative. Thus we may conclude that if a society consists of two groups of people, for one group bequest being operative and for the other group, old-age gift transfers being operative, and thus each individual in the first group has positive savings and the second group has zero savings. Now suppose a PAYG is introduced. Let us suppose that the agents in the first category fully off-set the program's effect of forced transfers from children to parents by transferring an equal amount to their children (as in Barro [1974] for instance), without changing any other decisions.[8] The agents in the other category, however, will reduce their voluntary old-age gift transfers, will have less children, and will continue to have zero savings. Thus the effect of such social security on the aggregate economy would be to reduce the population growth, and total savings; the per capita savings, however, will be increased. The effect on savings rate will depend on whether the total savings declines more than GDP or not. In a more general setting with endogenous human capital formation, it may be possible to establish that the negative effect of social security on fertility level of the second category of agents leads to more investment in human capital of their children

[8] In our set-up, this Ricardo-Barro neutrality effect of social security may not hold for agents with operative bequests. I have not examined it either in this paper.

through "quality-quantity trade-off " (which is a universally observed phenomenon) and thus may have positive effect on long-run growth.

We have assumed in our main analysis that parents cannot leave a debt to be paid by their children, i.e., we have assumed $s_t \geq 0$, not negative. Suppose we allow s_t to be negative, i.e., some parents are net dissavers in the society (which is observed for some agents in less developed countries), then social security may reduce the magnitude of dissavings of these agents and the aggregate savings might even go-up as a result.

In our view, one of the motives for introduction of social security is to overcome the incentives to throw oneself to the mercy of the younger generation in old-age. Our view of social security is different from the social insurance view put forward by Diamond-Mirrlees [1978] and others. The purpose of social security is clearly more to force people to save for their retirement since we all know that we would not be able to let the elderly live miserably if they do not save for their retirement. Our view of social security is close to the social conscience view except that in our context the social conscience is extended to the family members only.

In our model, similar to Veall [1986], social security benefits and taxes are endogenously determined. As in the Hansson and Stuart model, a social security tax-benefits stream for the current as well as all future generations that is implied by the subgame perfect gift equilibrium could be legislated by the living generations in period $t = 1$ and no future generations will have incentives to change it.

5 Conclusion

In this paper we have considered a pure exchange overlapping generations model with two-sided limited altruism in the sense that agents care not only about their own life-cycle consumption, but they also care about their parents' old-age consumption and their children's adult-age consumption. In our economy agents decide their levels of fertility, savings, and transfers of resources to parents and children. We argue that the commonly used open-loop Nash equilibrium does not fully take into account the incentives that agents may have to manipulate their children's or parents' behavior to effect higher rate of transfers.

We use more appropriate sequential multi-stage game in extensive form to model the manipulative behavior of agents and the notion of subgame perfect equilibrium to characterize the optimal manipulative behaviors. Our analysis is locally around steady-state equilibria. We show that there may exist multiple subgame perfect equilibria, and it is generally not possible to select a particular subgame perfect equilibrium as more reasonable description of actual behavior than other equilibria; furthermore, computation of subgame perfect equilibria in overlapping generations framework has been problematic since it involves computation of fixed points in function spaces, and thus studying the general properties of subgame perfect equilibria has been extremely cumbersome. We introduce a notion of local learning equilibrium, rationalizing it to describe a form of bounded rational human behavior in the sense that children learn certain behaviors from their parents or neighbors. We demonstrate that the computation of local subgame perfect equilibrium reduces to an easier problem of finding a particular type of local learning equilibrium, which is, indeed, a fixed point of a function of a single variable. We also show that a set of local learning equilibrium paths converge to local subgame perfect equilibrium in the long-run; we name such subgame perfect equilibrium as stable. Using this as a reasonable equilibrium selection criterion, we demonstrate that for a class of CEM (i.e., constant elasticity of marginal utility) economies this criterion selects a unique local subgame perfect equilibrium out of two local subgame perfect gift equilibria.

We further show that for all types of equilibria, the equilibrium rate of old-age support to parents depend negatively with their parents' savings and the number of children (the latter is true for CEM economy). The total amount of transfers from children may go down as parents choose more children, and this is in contrast to the traditional view of old-age security view for child bearing. Thus with a manipulative behavior of choosing less savings, and more or less children, depending on the economy, agents can extract higher rate of old-age support from children. For the CEM economy, the stable local subgame perfect equilibrium has zero savings, higher fertility and old-age support and it is Pareto superior as compared to the other subgame perfect equilibrium with positive savings. Thus by manipulation, individuals can effect higher rate of old-age supports from children and for this a social security may not be necessary. In our view, one of the reasons for introduction of social security is to overcome the incentives to throw oneself to the mercy of the younger generation in old-age. We have also examined the effect of social security on subgame perfect equilibrium rate of population growth and aggregate savings rate.

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