Optimum Population — A New Look:
A Hamiltoniam Approach Towards A Dynamic Analysis

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abstract.

The current optimum population models found in economic literature define *static* optimum population used in forming policy i.e. at a given instant what should be the optimum number of people in a (closed) economy. We believe that although this definition is useful, it is very limiting as far as policy implications are concerned. When making policies, we are hardly concerned with what should be the absolute number of people in the country at any given time, for the parameters are ever changing. Especially so, if the optimum population is defined via market conditions. As an effort to improve these models, we define *dynamic* optimum population, or the optimum population growth trajectory. Our model, therefore deals with population growth rather than the absolute level of population. This gives a dynamic goal (and a more realistic one) to the policy makers and discussions can be held in familiar language once again, 'we want to decrease/increase the population growth to .... .'

This paper applies Hamilton's principle of least action to find optimum control trajectories of population and net consumption. This is done using a neoclassical economic growth model using capital stock, net national income and investment in the equations of motion to maximize social welfare. The major difference between our model and any previous work is that welfare is not maximized only *at* one period thus defining a static optimum, but rather *over* an entire range of periods thus defining optimum population and consumption control trajectories which maximize social welfare and are used for policy analysis.

We analyze some of the more popular population policies in light of our model. These policies have been divided into two categories: long-run and short-run. More specifically, in the long-run, policies are ascribed to control the endogenous factor, population growth. In the short-run, the policy implications are to try and control other variables in the economy so that the natural growth of the population, now an exogenous factor, is the optimum growth.

Key Words (s): Optimum Population, Hamiltonian, Social Welfare, Population Policies, Preferences, Neoclassical Economic Growth Model

JEL Classification(s): J10, J13, J18, O11, O12

Introduction

Macro-level economic policy and national population policy share, at the very least, considerable hubris. Both propose methods that attempt to direct or control complex processes that are poorly understood. Keeping this in mind, we hope to move from the ridiculous to the sublime by incorporating a preference for population into a dynamic macro model. In this way we hope to develop a theoretical tool with which to model the population-development process. We also hope to develop the model as a tool to help generate more informed and realistic development planning.

The model extends the static optimum population model developed by Harold Votey.[1] Votey incorporated a preference for children (increased population) into the standard optimum population model.[2] Votey's model elucidated the tradeoff between increased population and material standards of living. In the same way, we hope to develop an understanding of this tradeoff in a dynamic setting.

The problems associated with deriving a social welfare function have received much attention in economics literature.[3] As our point of entry, we assume that a social welfare function can be derived and is known by policy makers. In this paper, we develop a model for use in maximizing social welfare given other constraints on an economy. Specifically, output is constrained by the available inputs and technology. Further, the level of consumption, an explicit argument in the welfare function, determines the level of future periods by determining investment. We use Hamilton's principle of least action to derive dynamic trajectories and conditions needed to insure that social welfare is maximum given these constraints.

The social welfare function has two main arguments: per capita consumption and an explicit expression of the social preference for population. Alternate parametric specifications are hypothesized to consider societies with pronatalist preferences, indifference regarding population, and "anti-natalist" preferences. In all cases, greater consumption is assumed to be preferred.

Any dynamic population-development theory worth its salt should be able to do two things: first, it should model the population-development process well enough to explain the demographic transition in both historical and contemporary terms; second, it should be able to identify important variables that are available for potential policy targets. At the outset, we need to make the important disclaimer that the model presented here only begins the process of dynamic model building. It will require substantial development to deliver on the two criteria specified above.

After deriving the optimum control trajectories that insure welfare maximization, we discuss the implications if the simple model developed. The model suggests two types of policy

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[1]Votey, H. L. (1969) "The Optimum Population and Growth: A New Look," *Journal of Economic Theory* 1969(I) pp. 273-290.
[2]see Sauvy, H. (1968) "Population: Population Theories," *International Encyclopedia of Social Sciences* Vol 12, pp. 349-357 for a discussion of optimal population models.
[3]see for instance: Gorman (1953, 1968); Sen (1970, 1979); Arrow(1959, 1963)

that might be used to keep the economy on optimal trajectories. The first involves direct manipulation of preferences regarding population. The second involves utilizing the tradeoff between increased population and material standards of living, both current and future. These policies are divided into two groups: long run and short run policies, respectively. Some discussion of this is in order.

The demographic transition theory still occupies a central position in population studies. Those of us from developed countries who obsess on the population problems of less developed countries worry about the pace at which these countries pass through the fertility transition. Policy makers still wonder whether it is possible to accelerate the transition. Further, there are those (Singapore and Malaysia, for instance) who worry that over emphasis on limiting population growth has led to a suboptimal rate of growth.

Scholarly study of both the historic transition of developed countries and recent studies of developing countries reveal many anomalies. These have called into question economically deterministic theories of the transition. Instead, more difficult to quantify cultural and social theories have been suggested. A major issue is identifying events which trigger the fertility transition. These anomalies have created a quandary for those attempting to generate effective policies to hasten fertility reduction in developing countries.

In his discussion of the generalized theory of the demographic transition, Coale characterizes both demand-side and supply-side preconditions for the decline of fertility within marriage.[4] Coale doesn't privilege either supply or demand forces. It seems reasonable to assert that supply of knowledge and methods of contraception is a necessary but not a sufficient condition for fertility reduction. Similarly, demand for contraception is necessary but not sufficient to promote regulation of fertility.

Nevertheless, policy makers interested in slowing population growth can be broadly characterized as supply-siders and demand-siders. In the debate a main point of contention is whether a true demand for contraception exists and whether a significant amount of this demand is unmet. Supply-siders assert that the demand is there and policy should be concerned with delivering contraception. Demand-siders point out that supplying contraception is ineffective if demand for contraception is not well formed. Central to this debate is a putative preference revealed by the population regarding children. Depending on the theorist, this preference is treated either as exogenous or endogenously determined.

The "new home economics" presents fertility as the culmination of the family calculus. Preference for children is generally regarded as exogenous. Family size is determined by the interplay of income and the opportunity cost of having children. In this view, fertility is endogenous to extrahousehold factors like employment opportunities for women, and even tax

[4]Coale's three preconditions for the decline of fertility within marriage can be summarized as: 1) Potential parents must recognize that fertility is within their realm of choice; 2) There must be a "demand" for fewer children; (in Coale's words: "Reduced fertility must be advantageous.") 3) There must be the knowledge and availability of an effective contraceptive method (supply). See: Coale, A.J. (1973) in *Proceedings of the IUSSP International Population Conference,* Liege:International Union for the Scientific Study of Population, pp. 53-72.

policy.[5] This theory provides what some consider the economically deterministic explanation of the demographic transition. Population policy directed at fertility reduction suggested by this theory would target the opportunity cost of children. Any policy that raises the opportunity cost will result in a decline in fertility.

Views that challenge the economic determinism of this theory suggest that fertility is determined (or, in Marxist terms, overdetermined) in a complex sociological, historical, and cultural process. In essence, these theories suggest that historic, cultural, and sociological institutions which are resistant to change arise in one historic setting and exercise some influence on behavior in other settings. Friction between material conditions and these institutions serves to erode the institutions over time. Central to this view is the role of changing consciousness of individuals who realize the obsolescence of institutions and subsequently reject them in favor of institutions more compatible with existing material conditions. Changing of consciousness can be regarded as the shifting of preferences.

Given the different views of the relationship between fertility and development, it isn't odd that population policy is often split between the attempt to reshape preferences regarding children and the attempt to supply contraception to an assumed unmet demand. The former policy is illustrated by the attempts to define and then redefine family-size goals in Malaysia by the edict of the Prime Minister.[6] The latter policy has been attempted in any number of settings.

It is our contention that material conditions as expressed by per capita consumption as well as other factors reshape preferences regarding population in the long run. In the model proposed here, we treat the preference for children as exogenous in the short run and endogenous in the long run. We consider long run policies to bring about economic and sociological changes needed to attain intertemporal welfare maximization, possibly through changing preferences. In the short-run, on the other hand, the assumption is that population growth is an exogenous factor. Thus we can ascribe no realistic policy that will change the population growth to match the optimum growth trajectory. Other policy options are considered that allow the economy to adjust to the exogenous population growth rate. This may be viewed as a temporary or a 'quick fix' to the welfare maximization problem. These short-run policies will provide a solution to a bounded welfare maximization.

While it might seem unrealistic to consider planning both economic growth and population, this is not without precedent. In 1988, Mexican President Carlos Salinas de Gortari announced a medium-term development plan that included a provision linking demographic and economic performance. The provision called for setting the minimum rate of economic growth at twice the rate of population growth.[7] Recently, Malaysia and Singapore have adopted pronatalist population policies based on a fears of declining fertility foreshadowing economic decline.[8] One might well ask whether Mexico's choice of factor is

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[5]Whittington, L. A. et al. (1990) "Fertility and the Personal Exemption: Implicit Pronatalist Policy in the United States," *American Economic Review* 80(3), pp. 545-556.
[6]"Baby Power," *Geographical Magazine*, January 1992, pp. 16-20.
[7]Moreno, L. 1991. "The Linkage Between Population and Economic Growth in Mexico: A New policy Proposal?" *Latin American Research Review* 26(3) pp. 159-60.
[8]"Baby Power," *Geographical Magazine*, January 1992, pp. 16-20.

arbitrary or if it is based on some perceived efficiency. Similarly, one might ask whether the fear of declining per capita income in Malaysia and Singapore is predicated on more than eugenicist grounds. We hope the following model will contribute to the study of the tradeoff between population size and the standard of living of both current and future generations.

The paper is organized in the following way. Part 1 presents the formal structure of the model. The optimal trajectories for population and per capita consumption are derived. In Part 2, the long run policy implications of the model are discussed. Short run policy implications are developed in Part 3.

1. The Optimal Trajectories.

Our goal is to generate a parsimonious model with which to model the tradeoff between material standards of living, as measured by consumption, and population growth. More importantly, we wish to develop this tradeoff in intertemporal terms. While it can be argued that preferences evolve over time (as well as technology), we assume a simple specification for the welfare function that remains invariant over time. The economy is assumed to be closed. Both investment and population changes are generated internally. All inputs are assumed to be fully utilized. While the model is admittedly simplistic, the hope is that bells and whistles might be added in the future.

Our objective is to find trajectories of variables that can be influence through policy so that the social welfare function is maximized at all times. In moving from an initial time t_o to a terminal time t_1, if social welfare is to be maximized at each instant, then an algebraic sum of the cardinal measure[9] of the social welfare must also be maximum over this range. If we measure welfare in units of, say, "happy" and time in "hours", then this allows us to define an objective functional J given by:

$$J = \int_{t_o}^{t_1} \left(h \cdot \left(\frac{C}{N} \right)^{\alpha} \cdot N^{\beta} \right) \cdot dt \qquad \text{<1.1>}$$

where:
h = a non-zero positive constant
C = consumption
N = population size
α = parameter determining the relative preference for per capita consumption
β = parameter determining the relative preference for children (population)

[9]Note that it need not be of cardinal measure for we are not trying to evaluate it. It is interesting to consider, however, that if we do indeed generate a cardinal measure it could be used to do intertemporal welfare evaluation that might eventually be useful in issues like environmental valuation.

The units of the objective functional are "happyhours." The objective functional is a mapping from the set of control trajectories to points on the real number line, the value of which must be maximized in order to maximize social welfare. In moving through the time range, the economy must also follow the equation of motion of the state variable – capital. The equation of motion for $\dot{K}(t)$, or the economic constraint is:

$$\dot{K}(t) = F(K,N) - C - \mu \cdot K. \qquad \text{<1.2>}$$

where:

K = the capital stock
μ = the rate of depreciation of capital

Given the objective functional <1.1> which is to be maximized and the equation of motion <1.2>, we recognize this as a *problem of Lagrange*, a subset of the general control problem. In a nutshell, the general control problem that we are faced with is:

$$\max_{\{C,N\}} \quad J = \int_{t_o}^{t_1} \left(h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta} \right) \cdot dt \qquad \text{<1.3>}$$

$$\text{subject to:} \quad \dot{K}(t) = F(K,N) - C - \mu \cdot K.$$

There are four modern approaches used in engineering economic systems to solve the problem of control. These four approaches are *calculus of variations, dynamic programming*, *maximum principle* also known as *the principle of least action or Hamilton's principle,* and *Mobilization Mathematicas* also know as *putting a mathematician to some use*. We are going to use the third approach because as opposed to the other approaches, it is capable of coping directly with the general constraints on the control variables. Also because it usually suggests the nature of the solution, thus making it easier to analyze the solution in economic terms. The approach is very similar to the static method of Lagrange multipliers used by Votey. This approach demands that a new variable be defined, the Hamiltonian $\mathcal{H}$, which is always maximum in the control variable space. The Hamiltonian is defined as the sum of the intermediate term in the objective functional and the inner product of a time dependent costate variable and the equation of motion. The technique also requires that the rate of change of this Hamiltonian with respect to state variables, in ceteris paribus conditions, be equal to the negative of time rate of change of the costate variable.

Let q(t) be the costate variable, then the Hamiltonian is defined as:

$$\mathcal{H} = h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta} + q \cdot \dot{K} \qquad \text{<1.4>}$$

$$= h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta} + q \cdot \left(F(K,N) - C - \mu \cdot K \right)$$

where consumption, capital, population and the costate variable are all functions of time. The maximum principle technique, then requires the following conditions:

$$\frac{\partial \mathcal{H}}{\partial \mathrm{N}} = \frac{\partial \mathcal{H}}{\partial \mathrm{C}} = 0 \qquad \text{<1.5>}$$

$$\dot{\mathrm{q}} = -\frac{\partial \mathcal{H}}{\partial \mathrm{K}}. \qquad \text{<1.6>}$$

Calculating these partial derivatives and taking their time derivatives to find the control trajectories of population and consumption, we get the following three equations:

$$\frac{\dot{\mathrm{q}}}{\mathrm{q}} = \frac{\alpha}{\mathrm{C}}\dot{\mathrm{C}} + \frac{(\beta-\alpha-1)}{\mathrm{N}}\dot{\mathrm{N}} - \mathrm{M}\overline{\mathrm{P}}_N \qquad \text{<1.7>}$$

$$\frac{\dot{\mathrm{q}}}{\mathrm{q}} = \frac{(\alpha-1)}{\mathrm{C}}\dot{\mathrm{C}} + \frac{(\beta-\alpha)}{\mathrm{N}}\dot{\mathrm{N}} \qquad \text{<1.8>}$$

and

$$\frac{\dot{\mathrm{q}}}{\mathrm{q}} = \mu - \frac{\partial \mathrm{F}}{\partial \mathrm{K}} \quad \text{where} \quad \dot{\mathrm{q}} = \frac{d\mathrm{q}}{dt} \qquad \text{<1.9>}$$

By turning the set of these three equations into a linear system, we can solve for $\dot{\mathrm{N}}(\mathrm{t})$ and $\dot{\mathrm{C}}(\mathrm{t})$, the optimum growth trajectories for population and consumption (See Appendix A for derivations). These turn out to be:

$$\dot{\mathbf{N}} = \frac{\left(\mu - \frac{\partial \mathbf{F}}{\partial \mathbf{K}} - \mathbf{M}\overline{\mathbf{P}}_N \cdot (\alpha-1)\right)}{(\beta-1)} \cdot \mathbf{N} \qquad \text{<1.10>}$$

$$\dot{\mathbf{C}} = \left(\frac{\left(\mu - \frac{\partial \mathbf{F}}{\partial \mathbf{K}} - \mathbf{M}\overline{\mathbf{P}}_N \cdot (\alpha-1)\right)}{(\beta-1)} + \mathbf{M}\overline{\mathbf{P}}_N \right) \cdot \mathbf{C} \qquad \text{<1.11>}$$

where:

$\frac{\partial \mathrm{F}}{\partial \mathrm{K}}$ is the marginal product of capital;
μ is the constant proportionate rate of capital depreciation;
β is an explicit measure of preference for population;
α is a measure of preference for per capita consumption; and,
$\mathrm{M}\overline{\mathrm{P}}_N$ is the percentage change in the marginal product of labor.

Thus we have successfully found optimal trajectories for population and consumption growth. Following these trajectories ensures that social welfare will be maximum at all times.

2. Long-run Policy Implications: Population Growth is an Endogenous Factor.

The first thing to be noted is that population and consumption growth, in the absence of a change in the marginal product of labor, must be the same. This implies that the ratio of the percentage change of population to the percentage change in consumption must be equal to one. This is to say that the rate of change of population to consumption is exactly one. Equations <1.10> and <1.11> also tell us that per capita consumption C/N, must always be equal to the ratio of the rate of change of population and the rate of change of consumption. This is to say that the percentage change in consumption $\dot{C}/C$ must be equal to the percentage change in population $\dot{N}/N$.

The second important thing to be noted is that relationships exist between the optimum population trajectory and the explicit preference for population (β), the return on capital, and the depreciation rate of capital. Further, the optimum trajectories are insensitive to the magnitude of the marginal product of labor, but rather depend on the percentage change of the marginal product of labor. Let the "natural" growth of population be η, then equation <1.10> states that:

$$\eta = \frac{\left(\mu - \frac{\partial F}{\partial K} - M\bar{P}_N \cdot (\alpha - 1)\right)}{(\beta - 1)} \qquad \text{<2.1>}$$

This is to say that if capital depreciation is, say, four percent and the return on capital five percent and beta is one half, and percentage change in marginal product of the population is zero, then population growth must be exactly two percent in order to insure that welfare is maximum.

What would it mean if the actual population growth rate differed from the theorized optimum? Either the population is not behaving in an optimal manner, or the parameters of the welfare and production functions are incorrect. If it were determined that the population is behaving in a suboptimal way, policies should be initiated to achieve the desired level of population growth. For example, if population is growing faster than the required two percent, then efforts should be made to decrease population growth. If, on the other hand, the parameters of the functions are not correctly specified, then the solution is to more accurately measure these parameters to solve for the welfare maximum.

It should be kept in mind that policymaking is typically a non-neutral exercise. Policy makers might represent the interests of specific constituencies, for example future generations. It might be the case that the population growth level exceeds rates warranted by other exogenous factors, local carrying capacity for instance. In these cases it might behoove the policymaker to change the trajectories of the economy. This could be done in a number of ways. One would be a long run policy targeting β directly. Other measures might make use of the implied tradeoff between the material standard of living and population. In the first case, programs that influence tastes regarding childbearing will directly affect β. These might include education or media campaigns promoting family planning. In the second case, incentives could be given to limit family size by either offering tax breaks to couples who have fewer children or by imposing progressive taxation based on family size. We will consider these in turn.

2.1: Taxation

We did not incorporate taxation into the consumption function, i.e. we only allocated output between consumption and investment. However, we can still discuss its effects because the government might allocate revenues in a variety of ways. Revenue need not be either consumed or invested, it could essentially be held out of circulation as savings and used at a future date. In this case, it might be possible to indirectly influence fertility by reallocating revenues across time.

The idea of controlling population through taxation is certainly nothing new. The underlying assumption of this policy is the backward-bending supply curve of labor and this assumption can easily be traced as far back as Malthus himself. In the second of the three propositions following the theorem of population, Malthus says, "Population invariably increases where the means of subsistence increase"[10]. Further, Malthus also mentions that when the laboring classes have a higher disposable income it will be spent "unproductively." Malthus makes it clear in his writings that unproductive consumption will lower the future output of the nation by throwing labor out of employment[11]. Such Malthusian ideas are the underlying assumptions of many contemporary policies regarding population growth.

Whittington *et al* discuss the influence of the personal exemption on the fertility in the 20th century United States. They show that "the personal exemption has a positive and significant effect on the national birthrate, and this result is robust to a variety of specifications."[12] They state further that "it appears that the United States -- and perhaps other countries as well -- can influence to a degree the fertility decisions of its citizens through deliberate changes in tax policies." [13]

Imposing an additional tax on parents where the tax is proportional to the number of children might help lower fertility to bring it in line with targeted welfare trajectories. The optimal size of the tax, however, will depend on the tradeoff between the preference for children and the preference for consumption.

Ronald Lee notes that increased fertility also generates spillover effects which diminish the quality of life for those already alive.[14] He presents an efficient taxation scheme which

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[10]Malthus T.R., (1798) *On the Principles of Population* pp. 127, Barnes & Noble, Inc., (1966).
[11]Malthus T.R., (1798). *Principles Of Political Economy,* Chpt VII., Wells & Lilly, Boston(1821).
[12]Whittington, L. A. et al. (1990) "Fertility and the Personal Exemption: Implicit Pronatalist Policy in the United States," *American Economic Review* 80(3), p. 545.
[13]ibid.
[14]Ronald Lee (1990), "Comment: The Second Tragedy of the Commons," in *Resources, Environment, and Population: Present Knowledge, Future Options,* K. Davis and M.S. Bernstam, eds. New York: Population Council.

captures the external costs imposed on society by added births. Imposing this tax would police the free access to world resources enjoyed by the unborn. However, the taxation scheme, based on a static optimization, doesn't reflect the interests of future generations. A dynamic framework as presented here would aid in determining the optimal tax, given some reasonable representation of future preferences.

Paul Demeny[15]suggests that the cost of having more children should be increased by decentralizing any existing public health and education programs. In this way the localities will be directly responsible for
financing their own schools and health programs. This, in effect, internalizes many of the external costs of fertility or at least localizes them. The effects of this policy option will not differ markedly from the last one, however, the taxation system will be more efficient if it accurately reflects the inherent preference of the population towards children.

If a tax policy matches the rate of population growth to a targeted rate, then aggregated welfare might not be maximized. However, the model is limited with regard to the measurement of welfare losses through taxation schemes. Since we have aggregated population into a single measure, it is impossible to measure distribution effects. This observation reflects Sen's warning: an aggregated welfare function obtained from an infinite set of homothetic utility functions is a welfare function of a "representative" consumer useful only in determining market behavior. It has no bearing on the welfare of any individual because it ignores the distribution.[16] A potential extension of the model might disaggregate population into classes to incorporate distribution.

So far, we have discussed cases in which taxation policies are used to limit fertility. In some cases, the population growth rate might be deemed inadequate for optimal trajectories. Giving incentives to produce more children by tax breaks seems to be especially effective if there is an inherent preference for more children. However, if there is an inherent distaste for more children, care must be taken that the tax breaks promised by the state more than compensate the personal consumption forgone by the parent(s) due to the dependency of the child on their income. If the cost of producing and rearing a child to a parent is more than the tax breaks, and a distaste for children exists, than the policy might be fruitless. In other words, the financial burden of the child must be borne by the state. The tradeoff is made obvious: to incur the cost of every child would mean that the state revenues must be high, which in turn implies either a heavy taxation of people with no children, or a generally very high tax rate and thus low per capita consumption.

2.2: Changing Preferences for Children and the Demographic Transition

Taxation policies and their like take advantage of the tradeoff between people and material standards of living. These allow policy makers to manipulate fertility by adjusting the opportunity costs of having children. An alternate long run policy would be to directly influence tastes. As discussed above, this policy is not without precedent. In Singapore, a family size limitation was imposed through public relations and tax incentives. "Singaporeans

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[15]Paul Demeny, Lecture delivered at Bard College, April 14, 1992.
[16]A.K. Sen (1970), *Collective Choice and Social Welfare*, Holden-Day

responded so obediently to former premier Lee Kuan Yew's appeals for small families that by 1975 the total fertility rate had dropped to two...it continued to decline until by 1985, it had dropped to just 1.6."[17] This resulted in a labor shortage and the need to increase population.

Again, the government might endeavor to change the preferences regarding children. In Singapore, attempts have been made to directly return the economy to an "optimum" trajectory through tax incentives and a new pronatalist public relations campaign. It remains to be seen whether fertility responds in the upward direction to these incentives.

The "Demographic Transition" is defined as a period during which birth and death rates move from a pre − transition level of 40 − 50 per 1000 to a post − transition level that eventually settles down in the neighborhood of 12-14 per 1000[18]. The theory asserts that an onset of the decline of the death rates tends to precede the decline of the birth rates. Further, during the phase of transition when both the rates are declining, death rates fall faster than the birth rates. This initially causes a widening gap between the crude birth rates and death rates and thus increases the growth rate of population. In the later stages of the transition however, this gap is gradually narrowed as the decline in death rates becomes less dominant than the decline in the birth rates until the two converge.

For empirical analysis and projections, the theory divides the countries of the world into two categories: More Developed Countries (MDCs) and Less Developed Countries (LDCs). The obvious flaws of such classification are admitted even by the staunch supporters of this theory. The MDCs consist of Northern America, Europe, Former Soviet Union, Japan, Australia and New Zealand. All the rest of the countries are classified as LDCs.

According to this theory, the fertility revolution is part of a much broader process called "modernization". Modernization involves, on the economic side, a sustained increase in the real output per head, changes in the techniques and scale of production, transportation and distribution of goods as well as monetarization of the economy and shifts towards industrialization of productive resources. On the social and demographic side it involves changes in fertility, mortality and net migration rates. Also it includes changes in spatial distribution of the population, family size and structure as well as educational and public health programs. Broadly speaking, all of these changes have been experienced by the group

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[17]"Baby Power," *Geographical Magazine*, January 1992, pp. 16-20.

[18]See Demeny, Paul, (1986) "The World Demographic Situation", *World Population and U.S. Policy: The Choices Ahead,* Jane Mencken, ed.

[19]"Elements of this transition are apparent in parts of northwestern Europe in the eighteenth century. In the course of the nineteenth century, as modernization gathered increasing headway in its areas of origin, it gradually diffused southward and eastward throughout Europe. By the end of the century, its beginnings can be identified in eastern most Europe, including Russia, and also in Japan. A somewhat similar development was taking place concurrently in overseas areas settled by Europeans, mirroring to some extent the diffusion pattern within Europe. Modernization is apparent first in areas initially settled chiefly by migrants from northwestern Europe − the United States in the first part of the nineteenth century, followed by Canada, Australia, and New Zealand − and subsequently in parts of Latin America, where migration from southern and eastern Europe is especially important. In the twentieth century, and increasingly since World War II, the initial signs of modernization have become more widespread in parts of Asia and northern Africa and, more recently, in sub − Saharan Africa." Easterlin, R.A. and Crimmins, E., *The Fertility Revolution,* p. 4 University of Chicago Press. (1985).

classified as MDC where the transition started around the late eighteenth century[19]. Indeed, the history of MDCs is a major justification for the validity of the theory.

By inductive reasoning, the theory assumes that LDCs will follow the same pattern of fertility decline as that experienced by the European countries in the 19th century and thus makes population projections accordingly. However, these predictions are not in harmony with the empirical evidence. Teitelbaum[20] lists eight factors[21] that help explain why LDCs have been impeded from achieving the predicted fertility decline. In addition he also lists eight factors[22] that allegedly will accelerate the process of decline of the fertility rates in LDCs.

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[20]Teitelbaum, M., "The Relevance of Demographic Transition Theory for the Developing Countries," *Science,* 188(4187) May 2, 1975. pp. 420 – 425. (1975).

[21]A brief summary of the eight factors (given by Teitelbaum) that will impede LDCs from achieving a fertility decline similar to that experienced by the European countries in the 19th. century are as follows:

1 : *Pace & source of mortality decline* in European countries were gradual and the same is not true for many developing countries.

2 : *Fertility levels before decline* in most developing countries today is much higher than in pretransitional Europe.

3 : *International migration* in 19th. century Europe consisted of "exporting" tens of millions of their citizens to the "new" continents. The same opportunity is not available to the developing nations of today.

4 : *Rate of Population growth* due to the previously mentioned factors in developing nations is much greater than that experienced by 19th. century European countries during their transition. Doubling time of population in LDCs is about 20 years, for the 19th century European countries, the average was about 90 years. This may slow any natural tendencies to completion of the demographic transition.

5 : *Momentum for further growth* is far greater for developing countries than was for the 19th. century European countries due to the developing countries higher fertility rates. This has also resulted in a much "younger" age structure in the developing nations than was for European countries.

6 : *Opportunities for occupational and rural-to-urban mobility* are very limited for developing nations as compared to the 19th. century European countries due to extra ordinarily rapid urban growth in the developing nations. This curbs the hypothesized effects of urban alternative life styles and reproductive values on the increased rural population.

7 : *Fewer opportunities for female participation in the labor force* in nonagricultural sector in the developing nations as compared to the 19th. century European countries.

8 : *Difficulties of providing universal education* in developing countries due to doubling of school age cohorts in only two or three decades compounded with the difficulty of expanding educational facilities at the same rate. This defers the goal of universal education along with its effects on fertility behaviors.

[22]Eight factors (by Teitelbaum) that will allow LDCs to make a more rapid fertility decline than that experienced by the 19th. century European countries:

1 : *Pace of social & economic development* in developing countries has been much more rapid than in the 19th. century European countries and "causes" or expedites the decline in marital fertility.

One set of policies suggested by this school of thought are directed towards fertility control through sociological development. Although it takes us out of the domain of economics proper, it behooves us to discuss some of these sociological factors at least in terms of beta(β) in our model.

The first step towards decreasing fertility levels in the LDCs might be to educate people. This might be general education, which is believed by some to change the relationship of the individual to the environment (consciousness).[23] The education might be more specifically about the advantages of small family units. This in turn will affect the tastes and preferences of the people over a period of time such that the decline in the fertility levels will be voluntary i.e. without any taxation incentives given by the government. The Princeton European Fertility Study stresses the importance of subnational linguistics and cultural barriers impeding fertility decline. Clearly then, as pointed out by Teitelbaum also, the role of the central government in LDCs should include greater integration of the society as well as promotion of both general education and education about fertility control in multiple languages. This is not an impossible task given that newspaper publications, radio and TV communications are already well established in LDCs. Improvement in transportation, telecommunication and postal systems will also help in the same direction.

If we believe that child replacement is a motivation for higher fertility, voluntary fertility control through a change in tastes will be implemented in the economy only if people have the assurance that every child born has a good chance of survival. This in turn declares a prerequisite for voluntary fertility control, that is high life expectancy, especially of infants. This is a difficult task for any government, but in LDCs, where western technology is imported readily, including medical technology (which in fact is responsible for rapid mortality decline in

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2 : *Methods of fertility control* are more effective and safe today than those available to the 19th. century European countries. Access and easier availability of fertility control methods will, consequently, have greater demographic effects.

3 : *Greater latitude of deferment of marriage and increased nonmarriage* was a factor in fertility decline in the 19th. century development. Fertility in developing nations may be reduced via changes in both in marriage practices and fertility within marriages.

4 : *Increased legitimacy of the small family norm* has been established due to the European transition. Although small families were not very common in the 19th. century European countries itself, but its adoption in the developing nations can cause acceleration of the process of cultural and intellectual diffusion which in turn will have demographic effects.

5 : *Increased interest & planning capability of government* in developing nations will result in fertility decline. By contrast, the leaders in the 19th. century were not as aware of population growth being a variable in the economic development. Already governments representing 81% of the population of the developing nations have declared their population growths to be too excessive.

6 : *Government ability to permeate subnational linguistic/cultural barriers* in developing nations, unlike its possibility in the 19th. century European countries, will also decrease the fertility rates. This being so because of availability of improved administrative and technological infrastructure.

7 : *International assistance* via United Nations and other agencies means that LDCs may call upon substantial outside resources and expertise.

8 : *More rapid pace of fertility decline* has been observed in those countries that <u>have</u> experienced fertility decline in recent decades as compared to 19th. century European countries. This may be due to a combination of factors given above.

[23] J. Cleland and C. Wilson (1987), "Demand Theories of the Fertility Transition," *Population Studies* 41(1) pp. 5-30.

LDCs compared to 19th. century Europe) this is not an impossible task. Thus a government, by providing improved medical facilities, can increase the infant life expectancy which in turn will allow for voluntary fertility decline.

Another step towards voluntary fertility control is to provide a substitute for more children as a source of income in old age--- another possible reason for the preference for more children. This could be provided through transfer payments such as adoption of social security and increased implementation of pension plans in the government and private sector. Compared to the 19th. century European countries, where this facility was not available, such a policy might contribute towards a faster decline in the fertility rates in LDCs.

The second important sociological change, again pointed out by Teitelbaum, is to discourage early marriages. Premature nuptials are commonplace in LDCs and consequently affect the fertility rates. Policy implications lie in the judiciary system that make marriages illegal before a certain age. This may lead to a deferment of marriages, which removes at least one factor contributing towards a slower fertility decline in LDCs.

2.3: Other Policies.

Depending upon whether a country wants to increase or decrease its population growth, it can import or export its labor through immigration. Though this may not always be possible due to economic conditions; job opportunities in the importing countries may not seem attractive to the foreign labor class. Also the political realities of today necessarily limit international migration. Therefore, it may not be one of the easiest policies to adopt.

Migration policy, given the ease with which it can be adjusted, is clearly one way in which policy makers can directly adjust population growth in the short run. This leads us to consider other short run policies, not necessarily directed at fertility control that might be used to optimize social welfare.

3. Short-run Policy Implications: Population Growth is an Exogenous Factor.

Admittedly, attempts to change behavior regarding fertility involve medium to long term policies. In all of the policies given above, the crucial underlying assumption was that these policies can bring about changes in the population growth prior to any other changes in the economy. It might be more interesting to consider the short run policy environment. In this case we assume that fertility is exogenous and constant. Policy will be directed toward maximizing welfare by adjusting the rate of population increase through macroeconomic instruments that change the rate at which population increase is traded off against consumption.

Of critical importance in the short run is the assessment of the rate of change of welfare over time. While welfare might be optimized at one point in time, if optimal welfare is decreasing over time this suggests at least political unsustainability. Assessing the direction of change of welfare will give policymakers long-range targets while they go about satisfying short term goals.

Even if we assume that preferences do not change, and hence α and β remain the same, in reality we cannot expect the return on capital and the percentage change in the return on labor to remain the same. In fact we may even enter period two, when the 'constant' depreciation rate of capital may have changed to a new 'constant'. If any of these factors change prior to obtaining the desired population growth, we will have to reevaluate the new trajectory. In extreme cases the new calculated growth may even demand exact reversal of previously employed policies. Such a wild goose chase may lead to economic and political chaos. Even in countries like China, where the central government has very persuasive methods available to control population trajectories, it is hardly possible that the Utopian condition of bringing about these changes fast enough, exists.

The question now becomes which economic factors are to be controlled so that the natural growth of population imitates the optimum trajectory for population.

Taking η as exogenous population growth, then we can rewrite equation <2.1> as:

$$\frac{\partial \mathbf{F}}{\partial \mathbf{K}} = \mathbf{MP}_K = \boldsymbol{\mu} - \boldsymbol{\eta} \cdot (\boldsymbol{\beta} - \mathbf{1}) - \mathbf{M\overline{P}}_N \cdot (\boldsymbol{\alpha} - \mathbf{1}) \qquad \text{<3.1>}$$

and inserting this in equation <1.11> for the optimum trajectory for consumption, we get:

$$\dot{\mathrm{C}} = (\eta + \mathrm{M\overline{P}}_N) \cdot \mathrm{C} \qquad \text{<3.2>}$$

Note that if the percentage change in marginal product of population is zero, then consumption must grow at the same rate as population, an observation also made earlier.

In order to analyze the relationship between different variables, we will choose specific values for certain constants under realistic assumptions. Note that if $\alpha > 0$, then the society quite greedily views consumption as a "more is better" good. Hence any value of $\alpha > 0$ would be realistic but it would make our model much easier to understand if we make a simplifying assumption that $\alpha = 1$. Choosing alpha equal to one, equation <2.1> becomes:

$$\eta = \frac{\left(\mu - \frac{\partial \mathrm{F}}{\partial \mathrm{K}}\right)}{(\beta - 1)} \qquad \text{<3.3>}$$

From the equation above we can calculate the real rate of return on capital that will insure that η, the natural growth of population is the optimum growth. For the moment, however, we are not interested in this exact value, but rather whether the level of investment needed to optimize welfare is obtainable in a free market setting. Specifically, if the real rate of return net of depreciation is negative investment determined by a market will not be likely to invest to the extent necessary.

We will analyze various scenarios on a case by case basis. One last equation that we need for our analysis is the $\dot{\mathrm{W}}$ equation. The correct value of the return on capital will insure that we follow the optimum trajectory which will maximize social welfare. The sign of $\dot{\mathrm{W}}$ will tell us if the maximum welfare (whatever it is) is increasing or decreasing over a period of time. If welfare is decreasing, we can assume that this trajectory is inherently unsustainable.

$$\dot{W} = h[\, N^{(\beta-1)} \cdot \dot{C} + C \cdot (\beta\text{-}1) \cdot N^{(\beta-2)} \cdot \dot{N}]^{24} \qquad \text{<3.4>}$$

3.1 Consumption is "MIB", Population Growth is Negative and there is a preference for more children.

When $\beta - \alpha > 0$, there is a preference for more children. Since we have assumed that $\alpha = 1$, then we have the inequality $(\beta - 1)$ is greater than zero. Also since η is less than zero, then from equation <3.3>, we get the

condition that $\left(\mu - \frac{\partial F}{\partial K}\right)$ must be negative. Hence:

$$\mu < \frac{\partial F}{\partial K}. \qquad \text{<3.5>}$$

When the return on capital, or the marginal product of capital is high, it means that less capital is being employed relative to labor. Since the population is decreasing, a high marginal product of capital or one greater than the constant proportional depreciation of capital will give an incentive to increase investment. The increase in investment will increase the stock of capital in future periods i.e. it accumulates faster than it depreciates. This will generate increased output in the next period due to an increase of capital usage. However, a higher level of investment implies that the net consumption in the first period has decreased. The maximization of social welfare is dependent on consumption per capita and not net national consumption. Since the population itself has decreased, the consumption-population ratio is such that welfare remains maximum. We claimed, on the basis of common sense, that an increase in investment will decrease the net consumption; but do our equations predict the same thing? Plugging the appropriate signs, generated by the conditions given in this section, in equation <1.11>, we see that the percentage change in consumption is indeed less than zero[25].

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[24] $W = h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta}$,
$\dot{W} = h[\, N^{(\beta-\alpha)} \cdot \dot{C} \cdot \alpha \cdot C^{(\alpha-1)} + C^{\alpha} \cdot (\beta\text{-}\alpha) \cdot N^{(\beta-\alpha-1)} \cdot \dot{N}]$ and substituting $\alpha = 1$

we get
$\dot{W} = h[\, N^{(\beta-1)} \cdot \dot{C} + C \cdot (\beta\text{-}1) \cdot N^{(\beta-2)} \cdot \dot{N}]$

[25] $\dot{C} = \left(\frac{\left(\mu - \frac{\partial F}{\partial K} - M\bar{P}_N \cdot (\alpha - 1)\right)}{(\beta - 1)} + M\bar{P}_N \right) \cdot C$

where $\alpha = 1$, and $\beta - 1 > 0$ therefore

$\frac{\dot{C}}{C} = \frac{\left(\mu - \frac{\partial F}{\partial K}_N\right)}{(\beta - 1)} = \frac{(-)}{(+)} = (\,-\,)$

In fact, equation <3.2> shows that the sign of the optimum consumption trajectory is dictated by the sign of η. We must always be very careful in saying whether η is the exogenous growth of population or if it is the optimum control trajectory of population.

In a famous lecture[26], Keynes argued that the effect of a declining population is an era in which demand for capital tends to be much lower than expected. This leads to a state of excess supply (or underconsumption) that is not very easily corrected and if the demand for capital is not increased, the results could be disastrous. Keynes thus advocated that in the face of declining population, "the demand for a net increase of capital goods is thrown back into being wholly dependent on an improvement in the average level of consumption or on a fall in the rate of interest." Thus, the condition <3.5> imposed by our model is compatible with the Keynesian remedy.

Given exogenous conditions on population growth, and social preferences, welfare optimization requires labor intensive techniques of production compounded with an interest rate that insures an increase in investment. However, a closer look at equation <3.4> tells us that even though we are maintaining maximum welfare in the short run, the welfare itself is decreasing[27]. The reason for the decrease in the maximum welfare is self evident: there is a preference for more children, but population growth is negative, negative population growth requires negative consumption growth for optimization of welfare, but consumption is "MIB." Therefore any prolonged period of decrease in welfare forces us to look at our equations as dictating a long run intervention. The population growth must now be taken as an endogenous factor requiring intervention. These long-run policies, lest we forget, were passionately discussed in the last section.

3.2 Consumption is "MIB", Population Growth is Positive and there is a distaste for more children.

Distaste for more children means that $(\beta - \alpha) < 0$ which implies that $(\beta - 1) < 0$. Since η is now positive, we get the exact same condition on the relationship of μ and $\frac{\partial F}{\partial K}$. Once again

$\left(\mu - \frac{\partial F}{\partial K}\right)$ must be negative and hence:

$$\mu < \frac{\partial F}{\partial K}. \qquad \text{<3.6>}$$

Thus we require a low capital labor ratio and an interest rate for which investment is positive and has a reasonable market solution. However, this time the percentage change in consumption is greater than zero. (This can be readily checked by the reader plugging in appropriate signs in equation <1.11>) This is so because an increasing population will require

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[26] Keynes, J.M., "Some Economic Consequences of a Declining Population," *Eugenics Review* (Feb. 16th, 1937)

[27] $\dot{W} = h[N^{(\beta-1)} \cdot \dot{C} + C \cdot (\beta-1) \cdot N^{(\beta-2)} \cdot \dot{N}]$, h, N and C are all positive, $\dot{N} < 0$, $\dot{C} < 0$, and $(\beta-1) > 0$. Therefore $\dot{W} = h[((+) \cdot (-)) + ((+) \cdot (+) \cdot (+) \cdot (-))] = h[(-) + (-)] = -$

an increasing level of consumption that yields per capita consumption that maximizes the social welfare. Increasing levels of consumption as well as investment necessitates that net national income be constantly increased through labor intensive production.

It is unclear whether under the given scenario and the optimization conditions, the social welfare is increasing or decreasing. The sign of $\dot{W}$ depends on the relative sizes of $\dot{C}$, $\dot{N}$, C/N & (β-1)[28]. In the special case when the percentage change in the marginal product of labor is zero, i.e. $\overline{MP}_N = 0$, then equation <3.2> yields the condition $\dot{C} = \eta \cdot C$. By substituting $\eta \cdot C$ for $\dot{C}$ in <3.4> we see that welfare increases only if β takes on a value between 0 and 1[29].

Thus in the short-run we can maximize the social welfare by maintaining an interest rate dictated by the inequality <3.6>. Further, this maximum welfare will be increasing (for $\overline{MP}_N$ =0) if the preference for population is such that $0 < \beta < 1$. If β is not within this range, then long run policies to control the population growth must be adopted, otherwise, even though the welfare is maximum at each instant, it will go on decreasing over a period of time.

3.3 Consumption is "MIB", Population Growth is Negative and there is a distaste for more children.

Since $(\beta - \alpha) < 0$ and $\eta < 0$, we get an unusual relationship that

$\left(\mu - \frac{\partial F}{\partial K}\right) > 0$. Thus:

$$\mu > \frac{\partial F}{\partial K} \qquad \text{<3.7>}$$

In this case the return on capital must be less than the depreciation rate of capital. This implies a negative net return on investment, i.e. the real rate of interest $i = \frac{\partial F}{\partial K} - \mu < 0$. Due to a rapid decline in population the interest rates on capital will be very low and labor wages very high. Since capital is cheap the economy must be extremely capital intensive. The net national output, generated primarily due to relatively high capital employment, must mostly be consumed. However, equation <3.2> demands that for welfare maximization, the percentage change in net consumption must also be negative. Once again we rely on Keynes to support this absurdity[30].

Since according to equation <3.2> we can not have an improvement in the average level of consumption, we must rely on a drastic fall in the rate of interest to the extent that it is actually negative (condition <3.4.7> i.e μ > marginal product of capital). This model then claims that since the population is decreasing and has a propensity to go on decreasing while it

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[28] $\dot{W} = h[\,N^{(\beta-1)} \cdot \dot{C} + C \cdot (\beta\text{-}1) \cdot N^{(\beta-2)} \cdot \dot{N}] = h \cdot N^{(\beta-1)} \cdot [\,\dot{C} + \frac{C}{N} \cdot (\beta\text{-}1)\dot{N}] = h \cdot N^{(\beta-1)} \cdot [(+) + (+)(-)(+)] = (+)[(+) + (-)]$

[29] If $\overline{MP}_N = 0 \rightarrow \dot{C} = \eta \cdot C$. Therefore $\dot{W} = h \cdot N^{(\beta-1)} \cdot [\,\dot{C} + \frac{C}{N} \cdot (\beta\text{-}1)\dot{N}] = h \cdot N^{(\beta-1)} \cdot [\eta \cdot C + \eta \cdot C \cdot (\beta\text{ -}1)] = h \cdot N^{(\beta-1)} \cdot \eta \cdot C[1 + (\beta\text{ -}1)] = h \cdot N^{(\beta-1)} \cdot \eta \cdot C[\,\beta\,]$ Given that η is greater than zero, $\dot{W}$ will be positive only if $\beta > 0$. However, we also know that $(\beta - 1) < 0$, therefore $0 < \beta < 1$.

[30] Keynes advocates that in the face of a declining population, the demand for an increase of capital goods wholly depends on an improvement in the average level of consumption or on a fall in the rate of interest.

also prefers that per capita consumption be increasing, it must also somehow increase, or at the very least, maintain its net national output. Since the amount of labor needed to maintain the production is decreasing, the only way (ignoring technology) to maintain production is to have an unusually high level of capital employment. Thus our model claims that we need a negative interest rate to to have such a high level of capital employment. The noncompatibility of this remedy with everyday common sense can be resolved by the fact that we are looking at a short run model where our objective is to maximize the social welfare through consumption-population ratios. Note also that in this scenario, the welfare would be increasing if β is less than zero[31]. In the long-run we need to direct our policies towards population growth.

3.4 Consumption is "MIB", Population Growth is Positive and there is a preference for more children.

This case scenario is that $(\beta - \alpha) > 0$, $\eta > 0$ hence $\left(\mu \;-\; \frac{\partial F}{\partial K}\right) > 0$

thus:

$$\mu \;>\; \frac{\partial F}{\partial K}. \qquad \text{<3.8>}$$

Once again we get the unusual requirement that the return on capital be less than the depreciation rate of capital. However, here we see that the percentage change in consumption must be greater than zero (see equation <3.2>). This brings about a dilemma. How is it possible, given that population is increasing at a positive exponential rate, to maintain increasing consumption in the same proportion while also the returns on capital be very low? In the economic region, low marginal productivity of capital will be obtained if there is an unusually high employment of capital which means that interest rates are low. In this case the level of capital must be so high that the rate of return is actually negative. Then despite the increasing supply of labor, the real wages must also be increasing in order to insure the increasing level of consumption. This implies that consumption as well as investment are both high and increasing. However, in our model, net national output was distributed among consumption and investment which implies a trade-off between the two. Then there must be some other factor affecting the production function such that the net national output is high enough to allow both, a high investment as well as a high level of consumption. Since we are dealing with a closed economy, the high level of output needed to support a high investment and consumption, given that the return on capital is negative, must be due to an extremely high return of labor employment. This is to say that the production is in the region where marginal product of capital is negative and the marginal product of labor is very high. (N.B. We are claiming that the marginal product of labor be very high, *not* the percentage change in marginal product of labor, which we actually assumed to be zero.)

In a rational market – based society, we cannot expect there to be further investment once real interest rates become negative. Yet our model demands a negative interest rate to

[31]If $M\overline{P}_N = 0$, then from equation <3.3.4> we get $\dot{W} = h \cdot N^{(\beta-1)} \cdot \eta \cdot C[\,\beta\,]$ which will be positive if $\beta < 0$, given that η is less than zero.

support this high investment of capital. Government would have to provide this level of investment.

Western nations of the nineteenth century and developing nations of the twenty first century(LDCs), broadly speaking, are examples of the scenario where there is a population explosion and there is a preference for more children. Moreover, records show that the average real wage in Western European countries was rising in the nineteenth century[32]. Our model demands an exogenous factor to explain this phenomenon. The technological innovation of the Industrial Revolution is one factor; one we have not dealt with explicitly in this model. The other is subsidized investment. When we consider the publicly subsidized railroad expansion in the United States during the 19th century, it could be argued that the net return to investment might well have been negative over a long period of time.

As far as developing nations facing positive population growth and a revealed preference for children, the optimal solutions (in the short run) require that the level of investment must be high enough that the return is driven into the negative range. Clearly then, the short run solution to the population problem in the developing nations lies in heavily subsidized investment. Technology transfer (migration of capital, human or physical) type might also work to maintain welfare maximization. Of course, the appropriateness of the investment vis a vis labor intensity might be very important. This does not in any way mean that no long-run policies should be adopted to control the population growth.

4. Conclusion.

In this paper we have used *Hamilton's Principle* to find optimum control trajectories to insure that social welfare is maximum given production constraints. These optimum control trajectories are given by equations <1.10> and <1.11> and derivations are shown in Appendix A. We then analyzed these trajectories in light of α and β; the preference profiles of the population. In order to test our model we chose some realistic values of these constants and analyzed the implications under different scenarios. We also discussed some of the popular policy options in reference to population problems. Throughout this paper we have stressed that population growth is to be taken as an endogenous factor in the long-run and as an exogenous factor in the short-run.

By making this distinction in the long-run and the short-run policy options, we have avoided a lot of confusion that could have arisen. For example it is unrealistic to try to change the economy's preference profile regarding children `overnight'. Thus in the short-run we can address the economic growth problem by focusing on migration, the interest rate, capital and technology. At the same time we recognize that such conditions cannot be maintained indefinitely and so we must also focus on population growth.

In effect, our model allows us to accept the fact that the population growth in a developing nation, say Sudan, is not going to start decreasing in the near future. Giving that it won't, can we still ensure welfare maximization (and economic development) in Sudan? The

[32]Lee, R., "Population in Preindustrial England: An Econometric Analysis", *Quarterly Journal of Economics* 87(4), pp. 581 – 607, (1973).

answer is certainly yes as long as we do not forget that adjusting capital stock is a temporary fix. Eventually we must try to control population also.

The model immediately suggests extensions. One important extension would be to incorporate technological change explicitly. If we adopt Boserup's view that technological innovation occurs due to population pressure, we could model technology itself as a function of N or population density.[33] Another extension would be to incorporate a time dimension in the specification of β. β might be a function of population size, population density, or a variety of other "exogenous" variables: education levels; dominant religion; or, family structure, for instance. As mentioned above, population itself might be disaggregated into labor classes. We might relax the assumption of full utilization of resources to consider the effects of dependency ratios on welfare maximization. Further, capital could be disaggregated into types.

Finally, the most important extension of the model would be to develop an empirical specification that might be used to estimate values of α, β, and the influence of other variables on them. We might be able to model the demographic transition empirically in order to understand its structure. Understanding the population-development interaction might assist development planning in important ways. Sustainable development planning might be greatly aided by such a tool.

Development is a process that unfolds over time. As such, static theoretical models are limited in their ability to model it. Dynamic modeling has rarely been applied to development planning. To our knowledge, it has never been applied to the population-development interaction. We offer this model as a meager first step.

[33] Ester Boserup, *Population and Technological Change,* Chicago:University of Chicago Press, (1981) Chapters 1-3.

APPENDIX

We want to maximize welfare when moving from period t_0 to t_1. The welfare function and the equation of motion are:

$$W = h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta} \qquad \dot{K}(t) = F(K,N) - C - \mu \cdot K,$$

where K, N and C are all functions of time. This is a general control problem in which the objective functional is maximized when the Hamiltonian is maximum for all t between t_o and t_1, where the Hamiltonian $\mathcal{H}$ is:

$$\begin{aligned} \mathcal{H} &= h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta} + q \cdot \dot{K} \\ &= h \cdot \left(\frac{C}{N}\right)^{\alpha} \cdot N^{\beta} + q \cdot \Big(F(K,N) - C - \mu \cdot K\Big), \end{aligned} \qquad \langle A.1 \rangle$$

and q, the costate variable is also a function of time. In order to find the optimum trajectories of the control variables, that is population and consumption, the principal of least action requires for interior solutions that the rate of change of the Hamiltonian with respect to control variables alone be zero at all times and that the time rate of change of the costate variable be equal to the negative partial derivative of the Hamiltonian with respect to the state variable, capital. Algebraically these conditions are:

$$\frac{\partial \mathcal{H}}{\partial N} = \frac{\partial \mathcal{H}}{\partial C} = 0 \qquad \langle A.2 \rangle$$

$$\dot{q} = -\frac{\partial \mathcal{H}}{\partial K} \qquad \langle A.3 \rangle$$

We now evaluate these partial derivatives.

$$\frac{\partial \mathcal{H}}{\partial C} = \alpha \cdot h \cdot C^{\alpha - 1} \cdot N^{\beta - \alpha} - q = 0 \qquad \langle A.2.1 \rangle$$

$$\frac{\partial \mathcal{H}}{\partial N} = (\beta - \alpha) \cdot h \cdot C^{\alpha} \cdot N^{\beta - \alpha - 1} + q \cdot \frac{\partial F}{\partial N} = 0 \qquad \langle A.2.2 \rangle$$

$$\dot{q} = \frac{dq}{dt} = -\left(q \cdot \frac{\partial F}{\partial K} - \mu q\right). \qquad \langle A.3.1 \rangle$$

The last of these equations give the percentage change in the costate variable over the entire range.

$$\boldsymbol{\frac{\dot{q}}{q} = \mu - \frac{\partial F}{\partial K}} \qquad \langle A.4 \rangle$$

In order to get the time derivatives of population and consumption, we take time derivatives of equations <A.2.1> and <A.2.2> and get:

$$\alpha h[\,(\alpha-1)C^{\alpha-2}N^{\beta-\alpha}\dot{C}+(\beta-\alpha)C^{\alpha-1}N^{\beta-\alpha-1}\dot{N}\,]-\dot{q}=0, \qquad \text{<A.5>}$$

$$(\beta-\alpha)h[\,\alpha C^{\alpha-1}N^{\beta-\alpha-1}\dot{C}+(\beta-\alpha-1)C^{\alpha}N^{\beta-\alpha-2}\dot{N}]+\dot{q}\frac{\partial F}{\partial N}+q\frac{d\left(\frac{\partial F}{\partial N}\right)}{dt}=0 \qquad \text{<A.6>}$$

respectively. Putting $\dot{q}$ in equation <A.5> on one side of the equal sign, we divide both sides of the equation by q lifted from equation <A.2.1> to obtain:

$$\frac{\dot{q}}{q}=\frac{\alpha h[\,(\alpha-1)C^{\alpha-2}N^{\beta-\alpha}\cdot\dot{C}+(\beta-\alpha)C^{\alpha-1}N^{\beta-\alpha-1}\dot{N}\,]}{\alpha hC^{\alpha-1}N^{\beta-\alpha}}$$

which upon cancellations gives:

$$\frac{\dot{\mathbf{q}}}{\mathbf{q}}=\frac{(\alpha-1)}{\mathbf{C}}\dot{\mathbf{C}}+\frac{(\beta-\alpha)}{\mathbf{N}}\dot{\mathbf{N}} \qquad \text{<A.7>}$$

Similarly in equation <A.6> we put $\dot{q}\cdot\frac{\partial F}{\partial N}+q\frac{d\left(\frac{\partial F}{\partial N}\right)}{dt}$ on one side of the equation and divide both sides by $q\cdot\frac{\partial F}{\partial N}$ lifted from <A.2.2> to get:

$$\frac{\dot{q}\cdot\frac{\partial F}{\partial N}+q\frac{d\left(\frac{\partial F}{\partial N}\right)}{dt}}{q\cdot\frac{\partial F}{\partial N}}=\frac{-(\beta-\alpha)\cdot h\cdot[\,\alpha\cdot C^{\alpha-1}\cdot N^{\beta-\alpha-1}\cdot\dot{C}+(\beta-\alpha-1)\cdot C^{\alpha}\cdot N^{\beta-\alpha-2}\cdot\dot{N}\,]}{-(\beta-\alpha)\cdot h\cdot C^{\alpha}\cdot N^{\beta-\alpha-1}}$$

which simplifies to:

$$\frac{\dot{\mathbf{q}}}{\mathbf{q}}=\frac{\alpha}{\mathbf{C}}\dot{\mathbf{C}}+\frac{(\beta-\alpha-1)}{\mathbf{N}}\dot{\mathbf{N}}-\frac{\dot{\mathbf{MP}}_N}{\mathbf{MP}_N}, \qquad \text{<A.8>}$$

where $\frac{\dot{MP}_N}{MP_N}=\left(\frac{d\left(\frac{\partial F}{\partial N}\right)}{dt}\right)\Big/\frac{\partial F}{\partial N}$ = percentage change in the marginal product of labor.

Equations <A.4>, <A.7> and <A.8> are the key to find optimum control trajectories, the $\dot{C}$ and the $\dot{N}$ equations that we seek. These may be extracted by the help of linear algebra. To economize on long equations we introduce four new symbols momentarily.

Let $\bar{Q}=\frac{\dot{q}}{q}$, $\bar{C}=\frac{\dot{C}}{C}$, $\bar{N}=\frac{\dot{N}}{N}$ and $M\bar{P}_N=\frac{\dot{MP}_N}{MP_N}$. By equating equations <A.7> and <A.8> we get:

$\bar{\mathrm{Q}} \;=\; \alpha\bar{\mathrm{C}} + (\beta - \alpha \;-1)\bar{\mathrm{N}} \;- \mathrm{M}\bar{\mathrm{P}}_N \;=\; (\alpha - 1)\bar{\mathrm{C}} \;+ (\beta - \alpha)\bar{\mathrm{N}},$ which simplifies to:

$$\bar{\mathrm{C}} \;=\; \bar{\mathrm{N}} \;+\; \mathrm{M}\bar{\mathrm{P}}_N \qquad \text{<A.9>}$$

Rearranging equation <A.7> and using the new notation we get:

$$\bar{\mathrm{C}} = \frac{\bar{\mathrm{Q}} - (\beta-\alpha)\bar{\mathrm{N}}}{(\alpha-1)} \qquad \text{<A.10>}$$

and inserting this equation in <A.8> gives:

$$\bar{\mathrm{Q}} \;=\; \alpha\left(\frac{\bar{\mathrm{Q}} - (\beta-\alpha)\bar{\mathrm{N}}}{(\alpha-1)}\right) \;+\; (\beta - \alpha - 1)\,\bar{\mathrm{N}} \;\;-\;\; \mathrm{M}\bar{\mathrm{P}}_N$$

This equation can be solved for $\bar{\mathrm{N}}$ (and hence for $\dot{\mathrm{N}}$) in terms of $\bar{\mathrm{Q}}$ and $\mathrm{M}\bar{\mathrm{P}}_N$. The equation above can be rewritten as:

$$\bar{\mathrm{Q}} \;-\; \frac{\alpha\,\bar{\mathrm{Q}}}{(\alpha-1)} \;-\mathrm{M}\bar{\mathrm{P}}_N \;\;=\;\; (\beta - \alpha - 1)\,\bar{\mathrm{N}} \;\;-\;\; \frac{\alpha(\beta-\alpha)}{(\alpha-1)}\,\bar{\mathrm{N}},$$

which after some algebraic dexterity yields:

$$\bar{\mathrm{N}} = \;\frac{\bar{\mathrm{Q}} \;-\mathrm{M}\bar{\mathrm{P}}_N\cdot(\alpha-1)}{(\beta-1)} \quad \text{(See Endnote)} \qquad \text{<A.11>}$$

Substituting for $\bar{\mathrm{N}}$ and $\bar{\mathrm{Q}}$ from the symbols introduced earlier and from equation <A.4>, we get the optimum population growth trajectory:

$$\dot{\mathbf{N}} \;=\; \frac{\left(\mu \;-\; \frac{\partial\mathbf{F}}{\partial\mathbf{K}} \;\;-\mathbf{M}\bar{\mathbf{P}}_N\cdot(\alpha-1)\right)}{(\beta-1)}\cdot\mathbf{N} \qquad \text{<A.11>}$$

By substituting for $\bar{\mathrm{N}}$ from equation <A.11> in equation <A.9> we get $\bar{\mathrm{C}}$ in terms of $\bar{\mathrm{Q}}$ and $\mathrm{M}\bar{\mathrm{P}}_N$.

$$\bar{\mathrm{C}} = \left(\frac{\bar{\mathrm{Q}} - \mathrm{M}\bar{\mathrm{P}}_N \cdot (\alpha - 1)}{(\beta - 1)} \right) + \mathrm{M}\bar{\mathrm{P}}_N \qquad \text{<A.12>}$$

and once again by substitution from equation <A.4> we get the optimum trajectory for consumption:

$$\dot{\mathbf{C}} = \left(\frac{\left(\mu - \frac{\partial \mathbf{F}}{\partial \mathbf{K}} - \mathbf{M}\bar{\mathbf{P}}_N \cdot (\alpha - 1)\right)}{(\beta - 1)} + \mathbf{M}\bar{\mathbf{P}}_N \right) \cdot \mathbf{C} \qquad \text{<A.13>}$$

Equations <A.11> and <A.13> are the optimum control trajectories implying that in order to maximize social welfare, population and consumption growth rate exponential. If the population growth is exogenous and is, say, changing at a constant exponential rate η, then η must be equal to:

$$\eta = \frac{\left(\mu - \frac{\partial \mathrm{F}}{\partial \mathrm{K}} - \mathrm{M}\bar{\mathrm{P}}_N \cdot (\alpha - 1)\right)}{(\beta - 1)} , \qquad \text{<A.14>}$$

thus:

$$\frac{\partial \mathbf{F}}{\partial \mathbf{K}} = \mathbf{MP}_K = \mu - \eta \cdot (\beta - 1) - \mathbf{M}\bar{\mathbf{P}}_N \cdot (\alpha - 1) \qquad \text{<A.15>}$$

which when substituted into <13>, simplifies to:

$$\dot{\mathrm{C}} = (\eta + \mathrm{M}\bar{\mathrm{P}}_N) \cdot \mathrm{C} \qquad \text{<A.16>}$$

Endnote: If we had provisioned for neutral technology in the production function, that is Y(t) = A(t) · F(K,N) and had not assumed $d(\partial \mathrm{F}/\partial \mathrm{N})/dt = 0$, equations <A.10>, <A.12> and <A.4> would be:

$$\bar{\mathrm{N}} = \frac{\bar{\mathrm{Q}} - \bar{\mathrm{M}}\mathrm{P}_N\ (\alpha - 1) - \bar{\mathrm{A}}(\alpha - 1)}{(\beta - 1)};$$

$$\bar{\mathrm{C}} = \left(\frac{\bar{\mathrm{Q}} - \bar{\mathrm{M}}\mathrm{P}_N\ (\alpha - 1) - \bar{\mathrm{A}}(\alpha - 1)}{(\beta - 1)} \right) + \bar{\mathrm{M}}\mathrm{P}_N + \bar{\mathrm{A}}; \text{ and,}$$

$$\bar{Q} = \mu - A \cdot \frac{\partial F}{\partial K}, \text{ respectively,}$$

$$\text{where: } \bar{MP}_N = \left(\frac{d(\partial F/\partial N)}{dt}\right) \Big/ \partial F/\partial N \quad \text{and} \quad \bar{A} = \dot{A}/A\,.$$

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