Optimum Population Growth with CIES Preference in the Infinite-Horizon Ramsey Model

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June 13, 2005

Abstract

When the preference over intertemporal consumptions is of CIES type and human capital enters the production function in a specific way, we show that the optimum population growth rate is exactly the subjective discount rate in an infinite-horizon Ramsey Model.

Keywords: Population growth; CIES function; Ramsey Model

JEL Classification: D91; J24; O21

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1 Introduction

In the well-known classical theories of optimal growth (e.g., Cass, 1965; Koopmans, 1965), the population growth was taken to be exogenous neoclassically. When life-cycle consumptions was taken into account, however, Samuelson (1975) derived conditions for an optimum population growth rate in the steady state governed by *the goldenest golden rule*, with no regard to the growth path. If fertility was endogenized,[1] Palivos (1995) has shown that multiple steady states and growth paths in terms of per capita capital[2] may emerge, which confirms the convergence groups found in the empirical studies, whereas the population growth path has not been treated though it is a control variable in the model.

With linkage to the existing literature, the main aim of this paper is to analyze the determinancy of the optimum growth rate for population when the preference over intertemporal consumptions is of CIES (Constant Intertemporal Elasticity of Substitution) type, taking the conventional Ramsey's approach. Contrary to a recent paper developed by Lehmijoki (2004), where the population growth (or the net fertility level) was presumed to be first increasing and then decreasing in per capita capital and the corresponding demographic transition point was given exogenously due to country-specific features, this paper imposes no presumptions on the specific functional form of population growth. Human capital enters the production function and the fertility choice influences both the budget constraint and the household utility.

[1] For the contributive works, see, for example, Becker and Lewis (1973), Becker and Barro (1988), Barro and Becker (1989), Becker, Murphy, and Tamura (1990), Wang, Yip, and Scotese (1994).

[2] The capital can be viewed as a composite of both physical and human capital throughout his analysis. See Palivos (1995, p.1493).

2 Basic assumptions

2.1 Prefenrence

As mentioned above, the individual's preference over intertemporal consumptions is indicated by a utility function $u : \mathbb{R}_+ \to \mathbb{R}$ that is of CIES form. Specifically,

$$u(c(t)) = c(t)^{1-\theta}/(1-\theta), \quad \theta > 0 \text{ and } \theta \neq 1, \tag{1}$$

where $c(t)$ is the individual's consumption over time and θ is the elasticity of marginal utility with respect to $c(t)$ or the coefficient of relative risk aversion such that the intertemporal elasticity of substitution is $\sigma = 1/\theta$.[3] The size of the representative household is denoted by $N(t)$ and its exponential growth rate is $n(t)$. $N(0)$ is normalized to unity for simplicity, so $N(t) = e^{nt}$. The instantaneous utility of the household takes the familiar expression $u(c(t))\exp\{-\int_0^t [\rho - n(\tau)]\mathrm{d}\tau\}$ at time t, where ρ is the subjective discount rate. In the standard competitive setting where the households are identical and no externality exists, the competitive solution achieved in a decentralized manner coincides the planner's solution. Thus, we limit the analysis below to the household's optimization problem without loss of generality.

2.2 Technology

The technology used by the economy is represented by a C^2 production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ that has physical capital $K(t)$, human capital $H(t)$, and labor $L(t)$ as its arguments.[4] $F(K(t), H(t), L(t))$ is concave, monotonically increasing, and homogenous of degree one. The Inada conditions are satisfied in order to preclude corner solutions. For empirical robustness, it can be specified according to Mankiw, Romer, and Weil (1992) as

$$F(K(t), H(t), N(t)) = K(t)^{\alpha} H(t)^{\beta} L(t)^{1-(\alpha+\beta)}, \quad 0 < \alpha < 1, 0 < \beta < 1.$$

[3] The time indicator t is suppressed later without confusion when necessary.

[4] By C^2 we describe a function that is continuous over the domain of definition and has continuous partial, second-order derivatives at each interior point.

Each individual owns one unit of nonleisure time per period, which can be divided between working and education. We use $\omega(t)$ to denote the fraction of time spent on working by the individual, such that the labor supply $L(t) = \omega(t)N(t)$. Because of homogeneity of degree one, the per capita production function can be written as

$$f(k(t), h(t)) \equiv F(K(t), H(t), L(t))/N(t) = k(t)^{\alpha} h(t)^{\beta} \omega(t)^{1-(\alpha+\beta)}, \tag{2}$$

where $k(t)$ and $h(t)$ represent per capita physical capital and per capita human capital respectively. The technology for per capita human capital growth we postulate is as in Lucas (1988):

$$\dot{h}(t) = \delta(1 - \omega(t))h(t), \quad \delta > 0.$$

Solving this differential equation gives $h(t) = me^{\phi(t)}$, where $\phi(t) = \delta \int (1 - \omega(t))\mathrm{d}t$ and the constant m can be determined from the initial condition. Substitute the expression of $h(t)$ into Eq. (2), one obtains

$$f(k(t)) = \gamma(t)k(t)^{\alpha}, \quad \text{where } \gamma(t) \equiv m^{\beta} e^{\beta\phi(t)} [\omega(t)]^{1-(\alpha+\beta)}. \tag{3}$$

3 Equilibrium

Without loss of generality, I assume there is no capital depreciation and the households are infinitely lived. Thus the essential problem is to

$$\max \int_0^{\infty} u(c(t))\exp\{-\int_0^t [\rho - n(\tau)]\mathrm{d}\tau\}\mathrm{d}t \tag{4}$$

$$\text{subject to} \quad \dot{k}(t) = f(k(t)) - c(t) - n(t)k(t). \tag{5}$$

It can be seen from above that the effective discount rate is variable. In order to solve the problem, an equivalent formulation can be made, following Uzawa (1968), by constructing virtual time $\Delta(t) = \int_0^t [\rho - n(\tau)]\mathrm{d}\tau$, or equivalently $\mathrm{d}\Delta(t)/\mathrm{d}t = \rho - n(t)$. The problem can thus be rewritten in terms of virtual time as

$$\max \int_0^{\infty} \{u(c(t))/[\rho - n(t)]\} e^{-\Delta(t)} \mathrm{d}\Delta(t) \tag{6}$$

$$\text{subject to} \quad \mathrm{d}k(t)/\mathrm{d}\Delta(t) = [f(k(t)) - c(t) - n(t)k(t)]/[\rho - n(t)]. \tag{7}$$

The current-value (in terms of virtual time) Hamiltonian is

$$\mathscr{H}(c, \omega, n, k, \lambda) = [1/(\rho - n)]\{u(c) + \lambda[f(k) - c - nk]\}.$$

Applying the Pontryagin Maximum Principle gives

$$u'(c) = \lambda, \tag{8}$$

$$\omega = \{se^{\alpha\beta t/[1-(\alpha+\beta)]} + 1\}^{-1}, \quad \text{where } s \text{ is a constant}, \tag{9}$$

$$u + \lambda[f(k) - c - \rho k] = 0, \tag{10}$$

$$\dot{\lambda} = (\mathrm{d}\lambda/\mathrm{d}\Delta)(\mathrm{d}\Delta/\mathrm{d}t) = \lambda[\rho - f'(k)]. \tag{11}$$

The transversality conditions are assumed to be fulfilled. Combining these conditions with Eq. (1), we find the evolutions of consumption and capital over time are governed by

$$\dot{c}/c = \sigma[f'(k) - \rho] \tag{12}$$

and

$$\dot{k} = \sigma[f(k) - \rho k] \tag{13}$$

respectively. In addition, Eq. (1), (8), and (10) generate the optimum consumption as a function of per capita physical capital k, i.e.,

$$c = (1 - \sigma)[f(k) - \rho k]. \tag{14}$$

Finally, substituting Eq. (13) and (14) into the state quation in natural time (Eq. (5)), the optimum growth rate for population, $[n(t)]^*$, is thus solved:

$$[n(t)]^* = \rho, \tag{15}$$

which is constant over time and exactly the same as the subjective discount rate!

4 Concluding remarks

The specification of CIES preference plays a key role in this highly simplified model. In this context, the consumption grows (or decays) linearly in the intertemporal elasticity of substitution, which ensures that the optimum magnitude of resources allocated to capital widening would just be proportionate to the existing per capita stock (i.e., $n^*k = f(k) - c - \dot{k} = \rho k$). Thus the optimum population growth rate is constant—exactly the discount rate—and can be sustained by appropriate savings decision even under *laissez faire*, similar to Samuelson (1975)'s results. The interpretation of the equality between n^* and ρ is intuitive: the more impatient the individual is, the higher the population growth rate or fertility should be in order to secure discounted future income because per capita human capital is increasing over time (Eq. (9)). Approximately, if we take the real interest rate as the proxy for the subjective discount rate, the relationship obtained can as well be explained in terms of the opportunity costs of physical capital investment or consuming more today. Although one may reasonably argue that the quantitative equality would not hold in a more general environment, the logic will remain qualitatively unchanged to a large extent. Admittedly, however, further analyses in a more relaxed setting regarding the other aspects of the model as extensions of this paper will surely enrich the results.

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