

## **Preparing Relative Intensity Triangles**

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### **Abstract**

This paper provides the formulas required for the triangular representation of three dimensional primary input intensities. The first section provides an exposition of "endowment triangles"; and the second presents relative intensity triangles, i.e. those for representing relative primary input endowments of countries, or relative primary input intensity of products. The third section contains the formulas to calculate the coordinates, and the last is devoted to a numerical example.

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The analysis of international trade propositions in the framework of a three-primary-inputs model has led to the use of triangular two-dimensional representations of primary input endowments of countries or primary input contents of products.<sup>1</sup> The main objective of this paper is to provide the formulas required to create one such graphical representations – in this case for representing relative primary input intensities of products (endowments of countries). The resulting formulas may also be used for representing primary input contents (endowments). The first section provides an exposition of the triangular representation of primary input endowments; the second contains a presentation of relative intensity triangles, i.e. those for representing relative primary input endowments of countries, or relative primary input intensity of products; the third section provides the formulas to calculate the coordinates for this kind of graphical representation, while the last one is devoted to a numerical example.

### **"Endowment Triangles"**

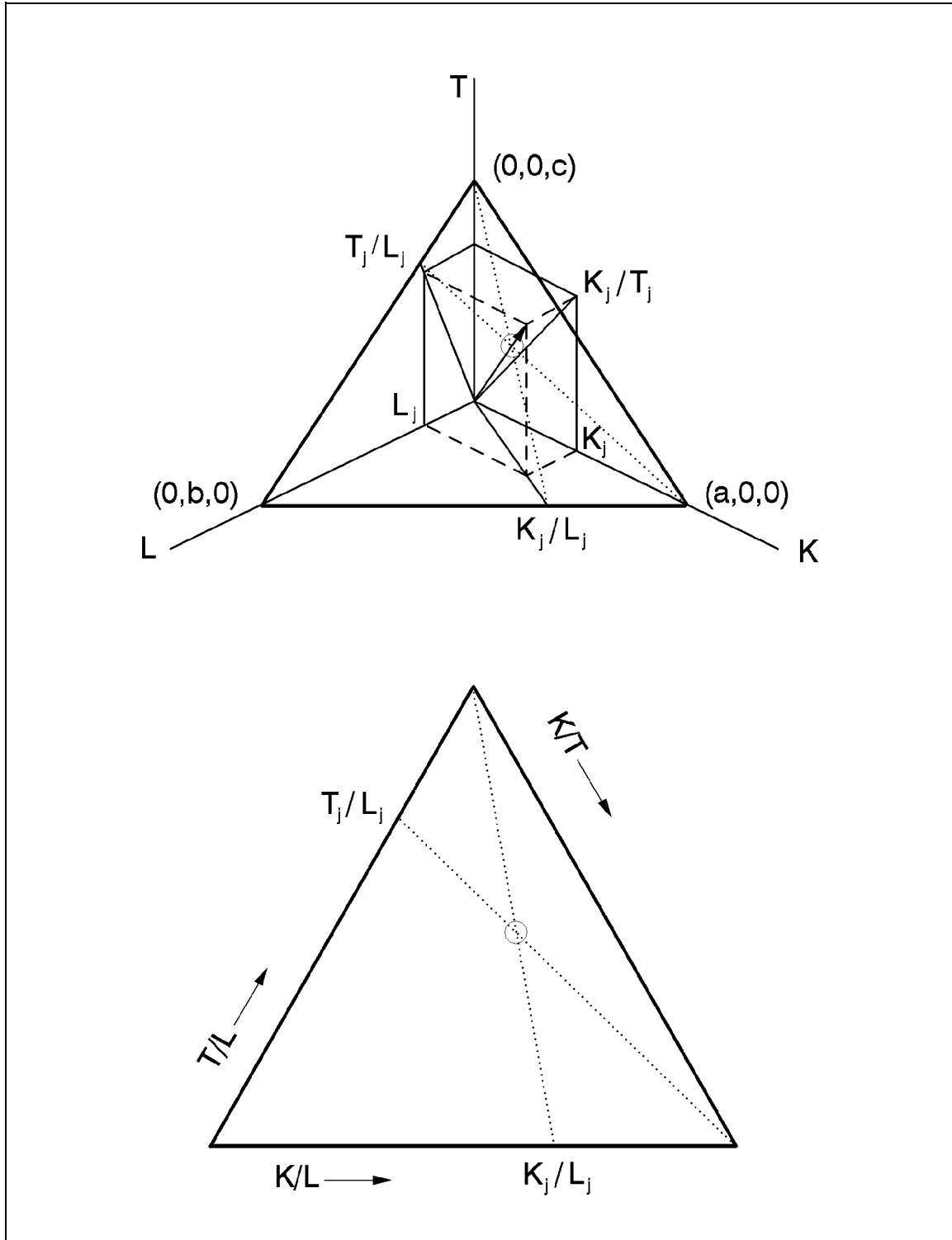
When there are three primary inputs ( $K$ ,  $L$ , and  $T$ ), primary input endowments of country  $j$  can be represented in subspace  $(K, L, T)$  by vector  $[v_j] = (K_j, L_j, T_j)$ . Given that primary input endowments are assumed to be non-negative, the analysis is restricted to the non-negative orthant of such subspace, as shown in Figure 1. When this non-negative orthant is intersected by a plane, the result is a triangle, the sides of which would depend on the points  $(a, 0, 0)$ ,  $(0, b, 0)$  and  $(0, 0, c)$  where the plane intersects axes  $K$ ,  $L$ , and  $T$ , respectively.

Primary input endowments of country  $j$  can also be represented in two dimensions by the point (represented by a small circle in Figure 1) where vector  $[v_j]$  intersects an appropriately defined plane in the non-negative orthant. Note that each of the three primary input ratios characterizing country  $j$  may also be represented by a two dimensional vector starting at the origin and intersecting the sides of the

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<sup>1</sup> Leamer (1984, 1987) called them "endowment triangles". For an early use of the method, see McKenzie (1955); for a more recent one see Jones (1992).

Figure 1. A Plane Intersecting the Positive Orthant Defines an "Endowment Triangle"



triangle at a point uniquely associated with that particular primary input ratio. Consequently, all countries with the same  $K / L$  ratio as country  $j$  will lie on the line that passes through both the origin and point  $(K_j, L_j)$ , and each  $K / L$  ratio will be associated with one (and only one) point on the base of the triangle, with values increasing from left to right. The left side of the triangle measures (increasing from bottom to top) the corresponding  $T / L$  ratio, and the right side the  $K / T$  ratio (increasing from top to bottom).

At the same time, all three dimensional vectors associated with countries with the same  $K / L$  ratio, but with different  $K / T$  ratios, will intersect the plane at points lying on the straight line connecting the upper vertex (point  $(0, 0, c)$ ) and the point associated with that particular  $K / L$  ratio on the base of the triangle. Similarly, all countries with the same  $K / T$  ratio as product  $j$  will lie on the line that passes through the origin and through point  $(K_j, T_j)$ , and all products with the same  $K / T$  ratio, but with different  $K / L$  ratios, will lie on the line connecting the left vertex and the point associated with that particular  $K / T$  ratio on the right side of the triangle. Consequently, the primary input endowment of a country will be associated with one (and only one) point inside the triangle, and that point will be determined by the intersection of the two rays originating in the vertexes of the triangle, and intersecting the sides at the points corresponding to the respective primary input ratios.

### **Relative Intensity Triangles**

To represent *relative* primary input intensities of products, i.e. the primary input composition of product  $j$  relative to that of another product or basket of products  $i$ , we can define the corresponding relative primary input uses

$$k_j = (K_j / K_i)$$

$$l_j = (L_j / L_i)$$

$$t_j = (T_j / T_i)$$

Then, we can say that product  $j$  is relatively intensive in  $K$  with respect to  $L$  if

$$kl_j = (K_j / L_j) / (K_i / L_i) > 1$$

Following the same steps as in our previous example, we can now represent relative primary input intensities by a vector  $(k_j, l_j, t_j)$  in the non-negative orthant (Figure 2). When that vector (or the result of multiplying it by a scalar) intersects the previously defined plane, the resulting point represents the relative primary input intensity.

Note that when primary input composition of  $j$  is identical to that of  $i$ , relative primary input intensities  $kl_j$ ,  $tl_j$ , and  $kt_j$  are all equal to one. Also note that when any of the relative primary input ratios is equal to one, the corresponding two dimensional vector connecting the origin and the side of the triangle will determine a  $45^\circ$  angle. Finally, if the points defining the plane verify that  $a = b = c$ , the resulting triangle will be equilateral and relative primary input ratios equal to one will be associated with the midpoint of the corresponding side. Consequently, a product with the same primary input composition as the product/basket being used as the basis for comparison, will have all relative primary input ratios equal to one and will be represented by the point at the center of the equilateral triangle.

The triangle may be subdivided into six regions (Table 1 and Figure 2) according to the values taken by the relative primary input ratios (smaller or greater than one). For example, in region IV products are  $K$ -intensive with respect to labor ( $kl_j > 1$ ) and  $T$ -intensive with respect to  $K$  ( $kt_j < 1$ ).

**Table 1. Regions of a Relative Intensity Triangle**

	$kl < 1$	$kl > 1$
$tl < 1$ and $kt > 1$	I	VI
$tl < 1$ and $kt < 1$	II	
$tl > 1$ and $kt < 1$	II	IV
$tl > 1$ and $kt > 1$		V



**Formulas**

Given parameters  $a$ ,  $b$ , and  $c$  defining a plane that intersects the  $k, l, t$  space, it can be demonstrated<sup>2</sup> that the triangle can be represented on the  $(x, y)$  plane by the  $x$  axis and two straight lines defined by equations

$$y = x \tan \beta \quad [1]$$

$$y = -x \tan \alpha + \sqrt{(a^2 + b^2)} \tan \alpha \quad [2]$$

where

$$\tan \beta = \frac{\sqrt{(a^2 + c^2)(a^2 + b^2)} - b^4}{b^2} \quad [3]$$

$$\tan \alpha = \frac{\sqrt{(a^2 + c^2)(a^2 + b^2)} - a^4}{a^2} \quad [4]$$

Then, relative primary input composition  $(k_j, l_j, t_j)$  can be represented inside the triangle by coordinates  $(x_j, y_j)$  such that

$$x_j = \frac{\sqrt{(a^2 + b^2)} \tan \alpha_1}{\tan \beta_2 + \tan \alpha_1} \quad [5]$$

$$y_j = \tan \beta_2 \frac{\sqrt{(a^2 + b^2)} \tan \alpha_1}{\tan \beta_2 + \tan \alpha_1} \quad [6]$$

where

$$\tan \alpha_1 = \frac{b t_j \sqrt{(a^2 + c^2)(a^2 + b^2)} - b^4}{(a^2 + b^2)(b t_j + c l_j) - b^3 t_j} \quad [7]$$

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<sup>2</sup> A simple demonstration is provided in the Appendix.

$$\tan \beta_2 = \frac{a t_j \sqrt{(a^2 + c^2)(a^2 + b^2) - a^4}}{(a^2 + b^2)(a t_j + c k_j) - a^3 t_j} \quad [8]$$

The reader can verify that when the triangle is defined equilateral, i.e.  $a = b = c$ , equations [1] and [2] for defining the triangle become

$$y = x \sqrt{3} \quad [9]$$

$$y = \sqrt{3} (a \sqrt{2} - x) \quad [10]$$

and equations [5] and [6] for  $(x_j, y_j)$  simplify to

$$x_j = \frac{a \sqrt{2} [2 (t_j + k_j) - t_j]}{2 (t_j + k_j + l_j)} \quad [11]$$

$$y_j = \frac{a t_j \sqrt{6}}{2 (t_j + k_j + l_j)} \quad [12]$$

Note that both the preceding equations and those for  $(x_j, y_j)$  simplify even further if we define the plane with points  $a = b = c = \sqrt{2}$ .

### Numerical Example

Let us suppose that we want to graphically represent the primary input compositions of six products,  $P^1$  to  $P^6$ , relative to that of another ( $P^0$ ), i.e. the primary input intensities of the six products will be compared to that of  $P^0$  in order to obtain relative primary input intensities. (Access to a spreadsheet software with graphics capabilities is assumed.)

The spreadsheet is reproduced in Table 2 and the resulting triangle is presented in Figure 3. The spreadsheet contains the primary input compositions (rows  $P^0$  to  $P^6$ , columns labeled  $K_j$ ,  $L_j$ , and  $T_j$ ) and the values of parameters  $a$ ,  $b$ , and  $c$  needed to define the triangle.

This will be all the data required to calculate the coordinates  $(x_j, y_j)$ , as well as to draw the triangle. The values for  $\tan \alpha$  and  $\tan \beta$ , represented in the spreadsheet by  $\tan A$  and  $\tan B$ , are calculated according to formulas [4] and [3]. Then we calculate  $\tan \alpha_1$  and  $\tan \beta_2$ , represented in the spreadsheet by  $\tan a1$  and  $\tan b2$ , using formulas [7] and [8]. To finish this stage, we calculate coordinates  $(x_j, y_j)$  according to formulas [5] and [6] (columns labeled  $x_j$  and  $y_j$  in the spreadsheet).

The next stage is to calculate the coordinates for the points on the sides of the triangle that will represent the scale. Those are the twenty one rows (seven labels on each side for each ratio:  $kl_j$ ,  $tl_j$  and  $kt_j$ ) following the primary input requirements and labeled from 0.125 to 8.00 (see Figure 3). For that purpose we simulate that they are also primary input intensities, but we create vectors  $(K_j, L_j, T_j)$  such that their relation to  $P^0$  will generate coordinates  $(x_j, y_j)$  located in the appropriate places. For example, the label for  $kl_j = 0.5$  will correspond to  $T_j = 0$  and to values for  $K_j$  and  $L_j$  such that

$$\left\{ \begin{array}{l} K(0.5) / L(0.5) = [K(P^0) / L(P^0)]0.5 \\ K(0.5) + L(0.5) = 1 \end{array} \right. \quad [13]$$

where  $K(0.5)$  and  $L(0.5)$  are the capital and labor requirements we are looking for, and  $K(P^0)$  and  $L(P^0)$  are those of the comparator. As some of the formulas do not allow for  $T_j$  to take a zero value, we make it sufficiently small (e.g., cell E14). Then we use the first column to define  $K_j$  as the solution of system [13], i.e.

$$K_j = \frac{[K(P^0) / L(P^0)]^\ell}{1 + [K(P^0) / L(P^0)]^\ell}$$

**Table 2. Spreadsheet to Represent Relative Primary Input Intensities in a Relative Intensity Triangle**

Product	Description	Kj	Lj	Tj	tan a1	tan b2	xj	yj	Labels	Sides	Ray	Ray	Ray
		a = 1.4142		tan B = 1.732									
		b = 1.4142		tan A = 1.732									
		c = 1.4142											
P0	Name 0	0.400	0.350	0.250	0.5773	0.5773	1.0000	0.5773					
P1	Name 1	0.300	0.600	0.100	0.1809	0.3646	0.6633	0.2418					
P2	Name 2	0.150	0.650	0.200	0.3069	0.8939	0.5112	0.4569					
P3	Name 3	0.100	0.450	0.450	0.7131	1.3555	0.6895	0.9346					
P4	Name 4	0.450	0.100	0.450	1.3146	0.7698	1.2614	0.9710					
P5	Name 5	0.650	0.150	0.200	0.8361	0.3421	1.4193	0.4855					
P6	Name 6	0.600	0.300	0.100	0.3276	0.2037	1.2332	0.2512					
0.125	KL label	0.1250	0.8750	1E-40	1E-40	1E-39	0.2222	2E-40	5E-41				
0.25	KL label	0.2222	0.7777	1E-40	2E-40	6E-40	0.4000	2E-40	1E-40				
0.50	KL label	0.3636	0.6363	1E-40	2E-40	4E-40	0.6667	3E-40	2E-40				
1.00	KL label	0.5333	0.4666	1E-40	3E-40	3E-40	1.0000	3E-40	3E-40				
2.00	KL label	0.6956	0.3043	1E-40	4E-40	2E-40	1.3333	3E-40	4E-40				
4.00	KL label	0.8205	0.1794	1E-40	7E-40	2E-40	1.6000	3E-40	4E-40				
8.00	KL label	0.9014	0.0985	1E-40	1E-39	2E-40	1.7778	3E-40	5E-40				
0.125	TL label	1E-40	0.9180	0.0819	0.1018	1.7320	0.1111	0.1924	0.1924				
0.25	TL label	1E-40	0.8484	0.1515	0.1924	1.7320	0.2000	0.3464	0.3464				
0.50	TL label	1E-40	0.7368	0.2631	0.3464	1.7320	0.3333	0.5773	0.5773				
1.00	TL label	1E-40	0.5833	0.4166	0.5773	1.7320	0.5000	0.8660	0.8660				
2.00	TL label	1E-40	0.4117	0.5882	0.8660	1.7320	0.6667	1.1547	1.1547				
4.00	TL label	1E-40	0.2592	0.7407	1.1547	1.7320	0.8000	1.3856	1.3856				
8.00	TL label	1E-40	0.1489	0.8510	1.3856	1.7320	0.8889	1.5396	1.5396				
0.125	KT label	0.1666	1E-40	0.8333	1.7320	1.3856	1.1111	1.5396	1.5396				
0.25	KT label	0.2857	1E-40	0.7142	1.7320	1.1547	1.2000	1.3856	1.3856				
0.50	KT label	0.4444	1E-40	0.5555	1.7320	0.8660	1.3333	1.1547	1.1547				
1.00	KT label	0.6153	1E-40	0.3846	1.7320	0.5773	1.5000	0.8660	0.8660				
2.00	KT label	0.7619	1E-40	0.2380	1.7320	0.3464	1.6667	0.5773	0.5773				
4.00	KT label	0.8648	1E-40	0.1351	1.7320	0.1924	1.8000	0.3464	0.3464				
8.00	KT label	0.9275	1E-40	0.0724	1.7320	0.1018	1.8889	0.1924	0.1924				
	Upper vertex	1E-40	1E-40	1.0000	1.7320	1.7320	1.0000	1.7320		1.7320			
	Right vertex						2.0000	0.0000		0			
	Left vertex						0.0000	0.0000		0			
	Upper vertex	1E-40	1E-40	1.0000	1.7320	1.7320	1.0000	1.7320		1.7320			
1.0	Ray from kw=1	0.5333	0.4666	1E-40	3E-40	3E-40	1.0000	3E-40			3E-40		
	to (0,0,1)	1E-40	1E-40	1.0000	1.7320	1.7320	1.0000	1.7320		1.7320			
1.0	Ray from kt=1	1E-40	0.5833	0.4166	0.5773	1.7320	0.5000	0.8660			0.8660		
	to (1,0,0)	1.0000	1E-40	1E-40	0.7131	1E-40	2.0000	3E-40			3E-40		
1.0	Ray from kt=1	0.6153	1E-40	0.3846	1.7320	0.5773	1.5000	0.8660				0.8660	
	to (0,1,0)	1E-40	1.0000	1E-40	1E-40	0.7698	3E-40	2E-40				2E-40	

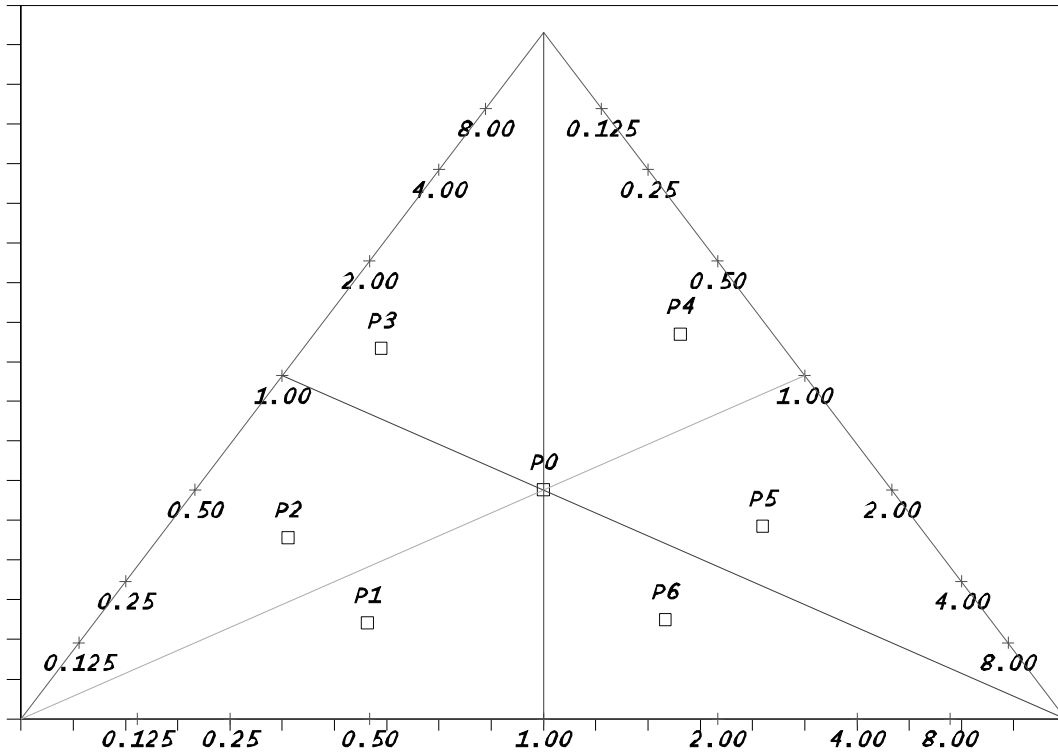
Note: For the resulting triangle, see Figure 3.

(e.g., cell C14) where  $\ell$  is the value of the label we want to place on the base of the triangle, i.e., when  $T_j = 0$ . Then,  $L_j$  simply is

$$L_j = 1 - K_j$$

(e.g., cell D14). In order to differentiate these labels by using a different marker and by placing the number below the marker, we have defined the corresponding numerical values as a separate series (column designated as *Labels* in Table 2).

Figure 3. Example of a Relative Intensity Triangle



The next four rows, designated by *Upper vertex*, *Right vertex*, *Left vertex* and *Upper vertex*,<sup>3</sup> are used to define the points that, connected by lines, will be used to draw the sides of the triangle. The base of the triangle is simply defined by the line going from  $(x, y) = (0,0)$  to

$$(x_j, y_j) = (\sqrt{(a^2 + b^2)}, 0)$$

and the other two sides by lines connecting the points

$$(K_j, L_j, T_j) = (0,0,1)$$

<sup>3</sup> The upper vertex is included twice for the spreadsheet program to be able to complete the triangle.

$$(K_j, L_j, T_j) = (0,1,0)$$

$$(K_j, L_j, T_j) = (1,0,0)$$

where zeroes were approximated by very small numbers to avoid errors. Note that in order to be able to connect these points with straight lines, they were defined as a separate series (column designated by *Sides* in Table 2).

Finally, in order to represent the rays going from each vertex to the midpoint of the opposite side (three pairs of rows labeled *Ray from ... to ...* in Table 2), where the relative primary input ratios are equal to one, we defined the midpoints on each side using the same equations used to place the "1.0" labels. Then we connected each one of these points with the corresponding vertex, and for that we defined each pair of points as a separate series (columns designated by *Ray* in Table 2).

A spreadsheet file containing the numerical example presented here is available from the author upon request. Such file may be used to create other graphical representations.

## Appendix

### Formulas for the Coordinates inside the Triangle

Given the relative primary input use of product  $j(k_j, l_j, t_j)$ , we use simple (but tedious) geometry to deduce formulas for the coordinates  $(x_j, y_j)$  that allow us to represent vector  $(k_j, l_j, t_j)$  by the point where it intersects the triangle formed by the intersection of the plane determined by coordinates  $[(a, 0, 0);(0, b, 0);(0, 0, c)]$ , and the  $k, l$ , and  $t$  axes. In order to draw the triangle in the  $(x, y)$  plane, with its left vertex coinciding with the origin, we start with the angles of the triangle, labeled  $\alpha, \beta$ , and  $\phi$  in Figure A.1. If we designate the sides of the triangle opposite to those angles by  $A, B$ , and  $C$ , from basic trigonometry we know that

$$B^2 = A^2 + C^2 - 2 A C \cos \beta \quad [\text{A.0}]$$

We also know that a side of the triangle, say  $A$ , can be represented in the  $(t, l)$  plane as shown in the upper part of Figure A.1. Consequently, the length of each side will depend on the corresponding points defining the plane. According to the Pythagorean theorem, the length of side  $A$  will be

$$A = \sqrt{(b^2 + c^2)} \quad [\text{A.1}]$$

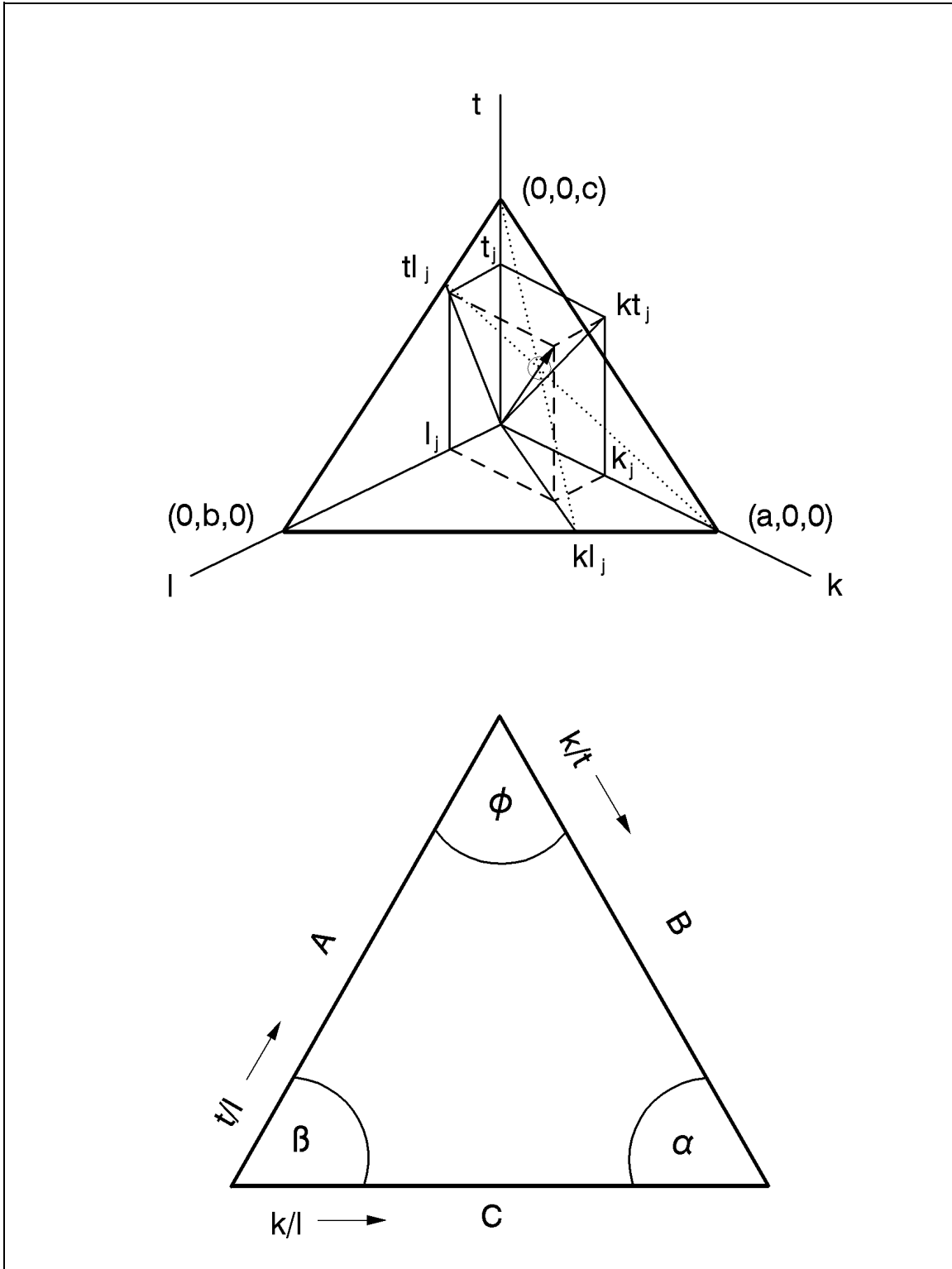
and those of the other sides will be

$$B = \sqrt{(a^2 + c^2)} \quad [\text{A.2}]$$

$$C = \sqrt{(a^2 + b^2)} \quad [\text{A.3}]$$

Replacing  $A, B$ , and  $C$  in [A.0] by [A.1], [A.2], and [A.3], we obtain

Figure A.1. Relative Intensity Triangle



$$\cos \beta = \frac{b^2}{\sqrt{(b^2 + c^2)} \sqrt{(a^2 + b^2)}} \quad [\text{A.4}]$$

and, considering that  $\sin \beta = \sqrt{(1 - \cos^2 \beta)}$ , we obtain

$$\sin \beta = \frac{\sqrt{(a^2 + c^2)(a^2 + b^2) - b^4}}{\sqrt{(b^2 + c^2)} \sqrt{(a^2 + b^2)}} \quad [\text{A.5}]$$

$$\tan \beta = \frac{\sqrt{(a^2 + c^2)(a^2 + b^2) - b^4}}{b^2} \quad [3]$$

Note that when  $a = b = c$ , i.e. when the triangle is equilateral,  $\tan \beta = \sqrt{3}$ .

Following the same approach, in the case of  $\alpha$  we have that

$$\cos \alpha = \frac{a^2}{\sqrt{(a^2 + c^2)} \sqrt{(a^2 + b^2)}} \quad [\text{A.6}]$$

$$\sin \alpha = \frac{\sqrt{(a^2 + c^2)(a^2 + b^2) - a^4}}{\sqrt{(a^2 + c^2)} \sqrt{(a^2 + b^2)}} \quad [\text{A.7}]$$

$$\tan \alpha = \frac{\sqrt{(a^2 + c^2)(a^2 + b^2) - a^4}}{a^2} \quad [4]$$

These two angles allow us to represent the triangle as the intersection of three straight lines.

The first one is the  $x$  axis, and the other two are those represented by the equations

$$y = x \tan \beta$$

$$y = x \tan (180^\circ - \alpha) + n$$

where  $n$  can be deduced from the second equation as

$$x = 0 \Rightarrow y = n$$

$$y = 0 \Rightarrow x = \sqrt{(a^2 + b^2)} = n / \tan (180 - \alpha)$$

$$n = -\sqrt{(a^2 + b^2)} \tan (180 - \alpha)$$

Recalling that  $\tan (180^\circ - \alpha) = -\tan \alpha$ , the two equations defining the triangle will be

$$y = x \tan \beta \tag{1}$$

$$y = -x \tan \alpha + \sqrt{(a^2 + b^2)} \tan \alpha \tag{2}$$

A point inside the triangle will be determined by two rays starting at the vertexes and intersecting the sides at points uniquely associated with the relative primary input intensities. In order to find the equations for those rays, we need to calculate the distances between vertexes of the triangle and those intersections.<sup>4</sup> Let us start in the  $(t, l)$  plane depicted in Figure A.2. The point corresponding to  $(t_j / l_j)$  can be expressed as the solution of the linear system

$$t = (t_j / l_j) l$$

$$t = -(c / b) l + c$$

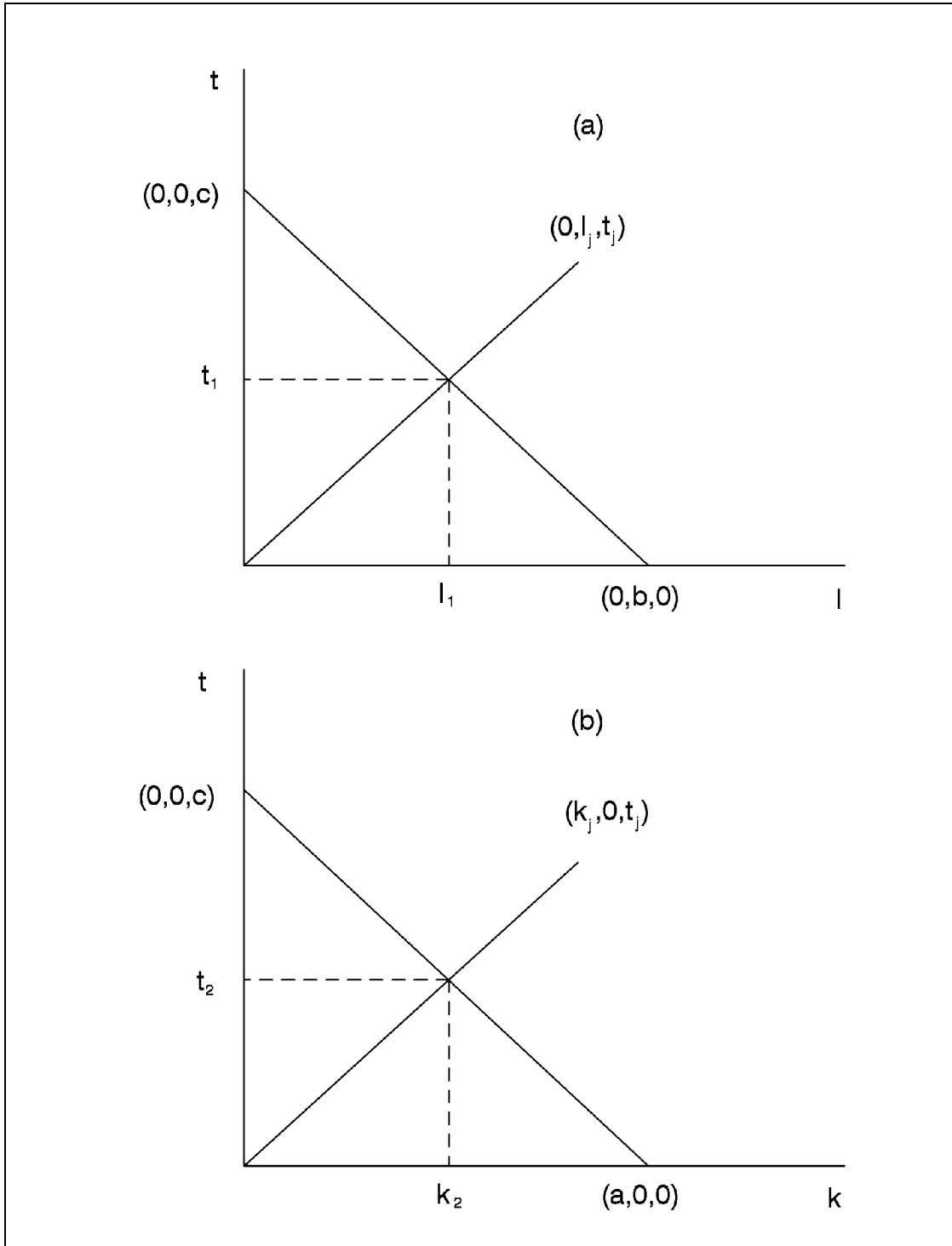
from where

$$l_l = \frac{c}{(t_j / l_j) + (c / b)} = \frac{b c l_j}{b t_j + c l_j}$$

$$t_l = \frac{(t_j / l_j) c}{(t_j / l_j) + (c / b)} = \frac{b c t_j}{b t_j + c l_j}$$

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<sup>4</sup> Glenn Westley provided valuable help at this stage.

Figure A.2. Sides of the Triangle Represented on Planes of the  $(k, l, t)$  Space

Then, the distance  $h_1$  from  $(0, b, 0)$  to  $(0, l_1, t_1)$  will be

$$h_1 = \sqrt{(t_1)^2 + (b - l_1)^2} = b t_j \frac{\sqrt{(b^2 + c^2)}}{(b t_j + c l_j)} \quad [\text{A.8}]$$

that can also be written as

$$h_1 = \frac{\sqrt{(b^2 + c^2)}}{1 + (c / b)(l_j / t_j)}$$

According to the Pythagorean theorem, the hypotenuse will measure  $\sqrt{(b^2 + c^2)}$  , and,

obviously,

$$\lim_{(l_j / t_j) \rightarrow 0} h_1 = \sqrt{(b^2 + c^2)}$$

Note that if  $a = b = c$ , i.e. if the condition needed for obtaining an equilateral triangle is satisfied, then

$$h_1 = \frac{b t_j \sqrt{2}}{t_j + l_j}$$

and the length of the hypotenuse will be  $\sqrt{2}$ .

Following the same reasoning, the intersection on the  $(t, k)$  plane (Figure A.2) will be the point determined by

$$t = (t_j / k_j) k$$

$$t = -(c / a) k + c$$

from where

$$k_2 = \frac{c}{(t_j / k_j) + (c / a)} = \frac{a c k_j}{a t_j + c k_j}$$

$$t_2 = \frac{(t_j / k_j) c}{(t_j / k_j) + (c / a)} = \frac{a c t_j}{a t_j + c k_j}$$

Then, distance  $h_2$  from  $(a, 0, 0)$  to intersection  $(k_2, 0, t_2)$  will be

$$h_2 = \sqrt{(t_2)^2 + (a - k_2)^2} = a t_j \frac{\sqrt{(a^2 + c^2)}}{(a t_j + c k_j)} \quad [\text{A.9}]$$

The length of the corresponding hypotenuse will be  $\sqrt{(a^2 + c^2)}$ , and

$$\lim_{(k_j / t_j) \rightarrow 0} h_2 = \sqrt{(a^2 + c^2)}$$

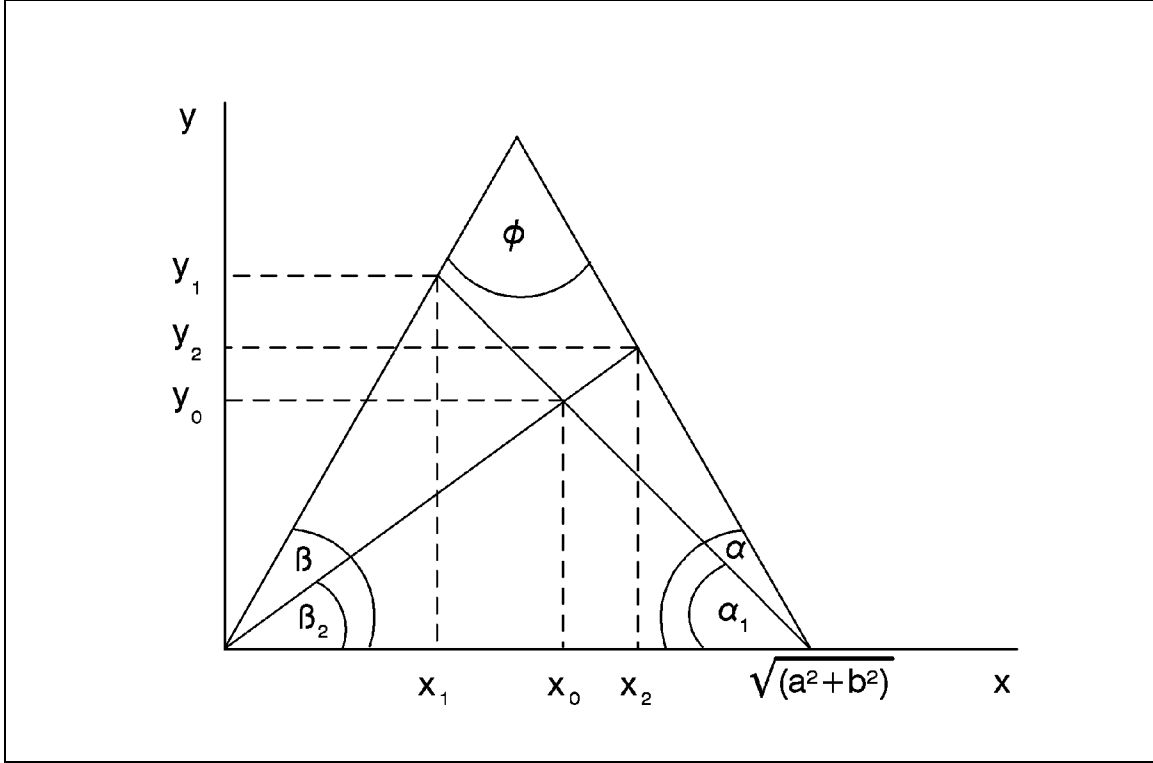
Also, when  $a = b = c$  the previous expression simply becomes

$$h_2 = \frac{a t_j \sqrt{2}}{t_j + k_j}$$

and the length of the hypotenuse will also be  $\sqrt{2}$ .

We can now turn our attention to the angles formed by the rays going from the vertexes on the base of the triangle to the opposite sides, as depicted in Figure A.3. These are the angles that will allow us to write equations for those rays and eventually find coordinates for the intersection of two rays on the  $(x, y)$  plane. In order to find the coordinates on the  $(x, y)$  plane of points determined by the intersection of one of the rays and the sides of the triangle, i.e. points previously referred to as  $(t_1, l_1)$  and  $(k_2, t_2)$ , we currently know angles  $\alpha$  and  $\beta$ , distances  $h_1$  and  $h_2$  from vertexes to intersections of rays

Figure A.3. Coordinates inside a Triangle



with the corresponding opposite sides of the triangle. Based on that information we can write

$$x_1 = h_1 \cos \beta$$

$$y_1 = h_1 \sin \beta$$

Then,

$$\tan \alpha_1 = y_1 / [\sqrt{(a^2 + b^2)} - x_1]$$

[A.10]

$$\tan \alpha_1 = \frac{h_1 \sin \beta}{\sqrt{(a^2 + b^2)} - h_1 \cos \beta}$$

Following the same approach, in the case of the ray starting at the right vertex we have

$$\sqrt{(a^2 + b^2)} - x^2 = h_2 \cos \alpha$$

$$x^2 = \sqrt{(a^2 + b^2)} - h_2 \cos \alpha$$

$$y_2 = h_2 \sin \alpha$$

$$\tan \beta_2 = \frac{h_2 \sin \alpha}{\sqrt{(a^2 + b^2)} - h_2 \cos \alpha} \quad [\text{A.11}]$$

Then, recalling that  $\tan (180^\circ - \alpha_1) = -\tan \alpha_1$ , equations for the rays starting at the left and right vertexes will be

$$y = x \tan \beta_2$$

$$y = x \tan (180^\circ - \alpha_1) + n = -x \tan \alpha_1 + n$$

where  $n$  can be deduced from the second equation as follows

$$x = 0 \Rightarrow y = n$$

$$y = 0 \Rightarrow x = \sqrt{(a^2 + b^2)} = n / \tan \alpha_1$$

$$n = \sqrt{(a^2 + b^2)} \tan \alpha_1$$

Consequently, coordinates  $(x_j, y_j)$  for the intersection of the two rays

$$y = x \tan \beta_2$$

$$y = x \tan \alpha_1 + \sqrt{(a^2 + b^2)} \tan \alpha_1$$

will be

$$x_j = \frac{\sqrt{(a^2 + b^2)} \tan \alpha_1}{\tan \beta_2 + \tan \alpha_1} \quad [5]$$

$$y_j = \tan \beta_2 \frac{\sqrt{(a^2 + b^2)} \tan \alpha_1}{\tan \beta_2 + \tan \alpha_1} \quad [6]$$

that can be evaluated using equations [A.10] and [A.11] for the two tangents. By replacing [A.4], [A.5] and [A.8] into [A.10], and [A.6], [A.7] and [A.9] into [A.11], the equations for the two tangents become

$$\tan \alpha_1 = \frac{b t_j \sqrt{(a^2 + c^2)(a^2 + b^2)} - b^4}{(a^2 + b^2)(b t_j + c l_j) - b^3 t_j} \quad [7]$$

$$\tan \beta_2 = \frac{a t_j \sqrt{(a^2 + c^2)(a^2 + b^2)} - a^4}{(a^2 + b^2)(a t_j + c k_j) - a^3 t_j} \quad [8]$$

\* \* \*

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