Network Competition with Reciprocal Proportional Access Charge Rules *

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Abstract

This paper presents a model of two competing local telecommunications networks, similar in spirit to the model of Laffont, Rey and Tirole (1996). The networks have different attributes which we assume are fixed and the consumers have idiosyncratic tastes for these attributes. The networks are mandated to interconnect and the access charges are determined cooperatively in the first stage. In the second stage, the two network companies are engaged in a price competition to attract consumers. In the third stage, each consumer selects a network and determines the consumption of phone calls.

Laffont, Rey and Tirole (1996) have shown that except for restrictive scenarios, the local price competition does not result in a pure strategy equilibrium. In this paper, we assume that the two companies choose access charge rules rather than simply access charges. These rules determine the access charges as a function of the future local prices. We show that the family of reciprocal proportional access charge rules generates a pure strategy equilibrium and we discuss its properties. (JEL D4, K21, L41,43, L51, L96)

Introduction

The telecommunications industry has been one of the fastest developing industries of the last half century. Traditionally, the telephone industry is considered to be a natural monopoly, since the cost structure consists of a large fixed cost component and a decreasing average operating cost (see Noam (1994)). For this reason, in most countries the telecommunications companies are owned by the governments. The inefficiencies generated by government ownership of a technologically very sophisticated industry has led to privatization in several countries and this trend is still in effect. In the US, while the telecommunications industry has always been privately owned, it has been subject to substantial regulation.

Over the last thirty years, with the emergence of entrants in several segments of the market, the question of regulating the telecommunications industry has become even more complex. The two well-known examples of such entrants are Microwave Communications Inc., MCI, in the US and Mercury Communications in the UK, which led to major policy changes in their respective countries. Both of these companies have provided long distance service for the consumers using local networks of the incumbent monopolists, AT&T in the US, and BT in the UK. For the history of the evolution of competition in the telecommunications industries in these two countries see Vogelsang and Mitchell (1994). In the US, the AT&T monopoly was broken down into a long distance company and seven Regional Bell Operating Companies (RBOCs) which were awarded monopolies for local service in their operating areas. The motivation behind this was that a vertically integrated monopoly would not have incentives to let a competitor enter in some segment of their business. The RBOCs were not allowed to provide long distance service, while MCI and AT&T were banned from local access markets. The long distance companies had to pay the local network companies access fees to interconnect with their networks and the fees were determined by the FCC. In the UK, the monopoly stayed intact and an access charge mechanism was designed for competitors to access BT's local networks. OFTEL, the regulatory agency in the UK, has set the rates for interconnection (see Laffont and Tirole (1994a)).

The access charge mechanism design has been a subject of immense discussion in the last decade, most of it in the context of one vertically integrated firm competing in one segment of the market. A competitor has to have access to the other segment and access charges need to be determined. Laffont and Tirole (1994b) proposed a mechanism which yields the welfare maximizing access charges. The Efficient Component Pricing Rule (ECPR) of Baumol and Sidak (1994), despite its ease, generated a lot of discussion since it is efficient only under very strict assumptions, and seems to favor the incumbent monopolies. Economides and White (1995) present a critique of ECPR, while Armstrong Doyle and Vickers (1995) reinterpret the ECPR in the light of Laffont and Tirole (1994b). Economides and Woroch (1992) examine the incentives of two long distance companies for interconnection when the bottleneck facility is owned only by one of them. They calculated the final prices for several different scenarios as well as the access charges.

In recent years, breakthrough technologies like the introduction of the fiber optic lines, mobile communication networks, the transformation of cable networks to carry phone calls and the amazing growth of Internet, have questioned the necessity of monopolies in the local access market (see Noam (1994)). The current trend is to open the whole telecommunications market to competition. Several countries are planning for a competitive telecommunications industry and passing legislation to prepare the legal grounds for competition. The latest example of such policies is the Telecommunications Act of 1996 in the US, which essentially allows entry into the telecommunications market. These developments bring new questions to mind concerning the new policies that will be required to accord with the existence of several networks and their mutual interconnection.

One of the most important articles in the 1996 act is the one which addresses interconnection between networks. It asserts that interconnection should be provided on a nondiscriminatory manner to everyone who wishes; the access to networks should be at a just and fair price; the access charges should be negotiated between interacting firms and binding agreements should be signed. These agreements are subject to the approval of FCC and Public Utility Commissions.[1] Like most laws, the 1996 bill uses vague language and it is subject to interpretation. Once the access fees are set by mutual consent, the networks act competitively, i.e. they compete in local pricing schemes, service quality, etc.

Laffont, Rey and Tirole (1996) (hereafter LRT) have analyzed a model of two local network companies that possess different attributes for consumers.[2] In their model, the two companies, given access charges, set the local prices competitively. The customers of a network are charged the same price independent of the network which completes their call. The networks compete only in prices since the other attributes are assumed to be fixed. Then consumers select their preferred networks à la Hotelling. It is shown in LRT that equilibrium in local prices may not exist except for restrictive values of access charges.

[1] See Telecommunications Act of 1996 .

[2] Carter and Wright (1996) build on LRT and examine the effects of brand loyalty.

This paper deals with a similar model. It differs from LRT in two aspects: the determination of access fees and the determination of consumer demands. In the first stage, the two companies negotiate access charge rules rather than access charges. The access charges are functions of the local prices and they are determined only after the second stage, when the local prices are determined. Thus, any change in demands and therefore in local prices will automatically result in new access charges. The other difference between our model and that of LRT is in the consumer preferences. The utilities of consumers in both models consist of a deterministic part and a random part. While the deterministic part in LRT generates demands with constant price elasticity, we use quadratic utilities which generate linear demands. This avoids the unboundedness of the consumption for small prices and provides a satiation point, which is natural for this type of services.[3] For the random component we use the Weibull distribution, which resembles the Normal distribution and provides analytical convenience. Furthermore, it is easy to extend this model to deal with more than two network companies.

It is shown here that if the network companies restrict their first stage negotiation to reciprocal[4] proportional access charge rules (RPACR), which determine the access charges as certain proportions of the future local prices, then a pure strategy equilibrium always exists and the local prices are explicitly computed. If the companies are allowed to set the proportion factor to maximize their joint profits then they will set access charges smaller than local prices when the networks are not close substitutes. However, in this case the resulting local price coincides with the monopoly price. In other words, the companies charge the customers the monopoly price but charge each other a lower price.

If the two networks are close substitutes, they act competitively and the result is low local prices even under joint profit maximization. In this case, the access charges will coincide with the local prices. The adoption of RPACR can be viewed as a mild and simple regulatory policy. It does not require information about the determinants of the industry. If the services of the networks are close substitutes then the use of RPACR results in competitive prices. In addition, as noted above, RPACR generates access charges which are responsive to future changes like demand shocks and technological innovations. However, if one believes that the competing companies will find a way to increase differentiation then further government intervention may be necessary.

The paper is organized as follows. In Section 1, we introduce the general

[3] There is a limit to the time individuals spend on phone calls.

[4] Reciprocal here means that both networks employ the same rule.

model for the industry. In Section 2, we develop a model for consumer choice and derive their demands. In Section 3, we analyze the competition between the two network companies. Concluding remarks appear in Section 4.

1 The Industry Model

The telecommunications industry is a complicated industry to model. We deal with a simple case of two network companies which only provide local telecommunications services. We refer to them as Network 1 and Network 2. The networks incur zero marginal cost for each call.

There is a fixed cost associated with building a network. Operating and maintenance costs are assumed to be independent of the amount of service provided. Usually, operating and maintenance costs do depend on the number of customers of a network. But to simplify the analysis, we assume that these costs are included in the fixed cost component.[5] The fixed cost of each network is denoted by F. Each company faces two demand functions. The demand function X_{11}, for calls initiated and completed at Network 1, and the demand function X_{12}, for the calls initiated at Network 1 and completed at Network 2. The demand functions for Network 2, X_{22} and X_{21}, are defined similarly. The demand functions X_{ij} will be derived from utility maximization. This is done in the next section.

The two companies first negotiate access charge rules, $\hat{a}_1$ and $\hat{a}_2$, where $\hat{a}_1$ is the per unit price that Network 2 will pay Network 1 for each unit call that is completed in Network 1. The term $\hat{a}_2$ is similarly defined. Then they are engaged in a price competition which determines the local prices p_k, ($k = 1,2$). The price p_k is the per unit charge of company k to each of its customers whether their calls are completed locally or in the other network. In contrast to LRT, the access fees may depend on the local prices p_1 and p_2. Thus, it is assumed that $\hat{a}_k = a_k(p_1, p_2)$ for $k = 1,2$. We further restrict our attention to the simple case of the proportional rule

$$\hat{a}_k = a_k p_k,$$

where $0 \leq a_k \leq 1$, for $k = 1,2$. If $a_1 = a_2 = a$, then the access charge rule will be reciprocal, and it is called the Reciprocal Proportional Access Charge Rule (RPACR). Thus, we treat each network as a regular customer who only buys partial service (just completion of a call) and therefore pays only a proportion of the

[5] Computer simulations suggest that our results will remain true without this assumption, at least when these costs are not too large relative to the fixed cost. However the methods we use to prove our result does not apply to this case.

full service price. This is consistent with the nondiscrimination requirement of the Telecommunications Act of 1996. The profit functions of the two networks are given by

$$\Pi_1 = p_1X_{11} + (p_1 - ap_2)X_{12} + ap_1X_{21} - F, \tag{1}$$

$$\Pi_2 = p_2X_{22} + (p_2 - ap_1)X_{21} + ap_2X_{12} - F. \tag{2}$$

Consider the following sequence of events

Stage 1. The network companies select an RPACR by mutual consent.

Stage 2. The two companies choose their prices p_1 and p_2 simultaneously and independently and announce them publicly.

Stage 3. After observing the prices p_1 and p_2, every consumer selects a network to subscribe to and chooses the amount of calls.

The Telecommunications Act of 1996 requires companies to negotiate and then to sign binding agreements concerning their access charges before they engage in the competition for local prices, services, etc. In our model, the parties negotiate rules, not charges. The rules then determine access prices as functions of their local prices. That is, the access charges are determined only after local prices p_1 and p_2 are determined. Any change in local prices will have an immediate impact on the access prices. A crucial point is to determine how to select access charge rules in the bargaining stage. In our model it amounts to the selection of the proportion $a \in [0,1]$.

2 Consumer Demand For Local Telecommunications Services

In this section, we will specify the consumers' utilities and model the process by which they select their networks. Suppose that there are N potential consumers. Each consumer is assumed to have idiosyncratic tastes which depend on the various attributes of the networks (like specific services offered by the companies, the intensity of advertising, accounting methods, etc.). This allows us to explain the existence of several networks with similar products but different prices. Consumer i who subscribes to network k has the following utility function,

$$u_i(k,x,y) = ((r - sx)x + y)e^{\sigma\varepsilon_{ik}}, \tag{3}$$

where x is the total consumption of telephone calls in minutes and y is the amount of numeraiare (money) consumed. The term ε_{ik} measures the idiosyncratic taste variable of the consumer. It is assumed that ε_{ik}'s are Weibull distributed ((Domencich and McFadden 1975)), statistically independent for all i and k, and that they are private information of the consumers. The cdf of the Weibull distribution closely resembles that of the Normal distribution. The term σ is a measure of the dispersion of tastes , that is, σ measures the substitutability between the services of the two networks. As $\sigma \to 0$ the networks become perfect substitutes, while as $\sigma \to \infty$ the networks become perfect complements. One important feature of this utility function is that the deterministic part will result in a linear demand function. This implies a bounded amount of calls demanded at prices close to zero. Another feature of the quadratic utility function is that it provides a satiation point, which is natural for such services as there is a limit to the time an individual will spend on the phone.

Random utility models of this kind have been extensively employed in the literature, starting with Domencich and McFadden (1975); see Anderson, de Palma and Thisse (1992) for a wide variety of examples.

Let $V_{ik}(x)$ be the deterministic part of the surplus of consumer i who subscribes to Network k and consumes x units. Then

$$V_{ik}(x) = (r - sx)x - p_k x. \tag{4}$$

This is maximized for

$$x_k = \frac{1}{2s}(r - p_k), \tag{5}$$

and the maximum is given by

$$V_{ik} = \frac{1}{4s}(r - p_k)^2. \tag{6}$$

Observe that (as in LRT) the demand of each consumer, x_k does not depend on i; thus we have dropped the index i.

By (3) and (6) consumer i prefers Network k to Network $\bar{k}$ if and only if

$$V_{ik} e^{\sigma\varepsilon_{ik}} \geq V_{i\bar{k}} e^{\sigma\varepsilon_{i\bar{k}}}.$$

Therefore, the network companies assign probability P_{ik} that i will choose Network k over $\bar{k}$, where

$$P_{ik} = Probability(V_{ik} e^{\sigma\varepsilon_{ik}} > V_{i\bar{k}} e^{\sigma\varepsilon_{i\bar{k}}}). \tag{7}$$

Since the ε_{ik}'s are independent and Weibull distributed, this probability is given by (Anderson, de Palma and Thisse 1992)

$$P_{ik} = \frac{1}{1 + (\frac{V_{i\bar{k}}}{V_{ik}})^{\frac{1}{\sigma}}}. \tag{8}$$

For the derivation of (8) see Anderson, de Palma and Thisse (1992).

Applying (6) and (8) to the case k=1, we have

$$P_{i1} = \frac{(r-p_1)^\tau}{(r-p_1)^\tau + (r-p_2)^\tau}, \tag{9}$$

where $\tau = \frac{2}{\sigma}$. Thus, this probability does not depend on i, and we can drop the index i from P_{i1} and write P_1. From the point of view of the two companies, every consumer will select Network 1 with probability P_1. The expected number of consumers who will subscribe to Network 1 is therefore NP_1. Notice that P_k can be viewed as the market share of Network k. Consequently by (9), the expected market share, $m(p_1, p_2)$, of Network 1 is given by

$$m(p_1, p_2) = \frac{(r-p_1)^\tau}{(r-p_1)^\tau + (r-p_2)^\tau}. \tag{10}$$

The expected market share of Network 2 is obviously $P_2 = 1 - m(p_1, p_2)$. Observe that the aggregate subscriber demand faced by Network k is given by,

$$X_k = X_k(p_1, p_2) = \frac{N}{2s}(r - p_k)P_k, \qquad k = 1, 2. \tag{11}$$

Next let us find the expected number of calls initiated in Network k, and completed in Network j where $k, j \in \{1, 2\}$. From the companies' point of view, the probability that a customer of Network k, $k \in \{1, 2\}$, calling a customer of Network j is $m(p_1, p_2)$ if $j = 1$ and $1 - m(p_1, p_2)$ if $j = 2$. Therefore by (10) and (11), we conclude that,

$$X_{11} = \frac{N}{2s}(r-p_1)(m(p_1,p_2))^2, \tag{12}$$

$$X_{12} = \frac{N}{2s}(r-p_1)m(p_1,p_2)(1-m(p_1,p_2)), \tag{13}$$

$$X_{22} = \frac{N}{2s}(r-p_2)(m(p_1,p_2))^2, \tag{14}$$

$$X_{21} = \frac{N}{2s}(r-p_2)m(p_1,p_2)(1-m(p_1,p_2)). \tag{15}$$

We use these demand functions in the next section to analyze the competition between the two companies.

3 Network Competition with RPACR

In this section, we analyze the price competition when the access charge rules are reciprocal proportional rules. This means that both networks use the same rule, $\hat{a}_k = ap_k, \;\; 0 \leq a \leq 1, \;\; k \in 1,2$, to calculate access charges. The number a is determined in the first stage of the game. To simplify notation, we use $p_1 = p$ and $p_2 = q$. Using this rule we obtain, by (1), (2), and (12)-(15)

$$\begin{aligned} \Pi_1 &= \frac{N}{2s}\{p(r-p)m(p,q) + (p-ap)(r-p)m(p,q)(1-m(p,q)) + \\ &\quad +ap(r-q)m(p,q)(1-m(p,q))\} - F, \qquad (16) \\ \Pi_2 &= \frac{N}{2s}\{q(r-q)(1-m(p,q)) + (q-aq)(r-q)m(p,q)(1-m(p,q) + \\ &\quad +aq(r-p)m(p,q)(1-m(p,q))\} - F. \qquad (17) \end{aligned}$$

Given $a, \; 0 \leq a \leq 1$, the two network companies are engaged in a price competition. We find that when N is sufficiently large to cover the fixed costs, a symmetric equilibrium exists[6] and can be computed explicitly.

Theorem 1: For every $0 \leq a \leq 1$, $\tau > 0$ and $r > 0$, there exists one and only one symmetric equilibrium. It is given by:

$$p* = q* = \frac{(2+a)r}{\tau + 4}$$

Proof. See Appendix.

The main contribution of Theorem 1 is that pure strategy equilibrium in local prices always exists with RPACR. The equilibrium prices have intuitive properties. They are decreasing in substitutability rate, τ. The higher the substitutability rate τ, the stronger is the competition between the networks, and, therefore, the lower are the prices. Second, the prices are increasing in the demand intensity, r, in a linear way. Finally they are increasing in the access charge proportion factor a. The proportion a can be viewed as the marginal cost of a call that is completed in the other network. The result asserts that the higher this cost is, the higher is the price that the company charges to its customers.

The value of a is determined in the first stage by negotiation. The bargaining between the two parties may result in essentially any number in $[0,1]$ with the restriction that the revenue of a company covers the fixed cost. Let us first

[6] Although we did not succeed in proving the uniqueness of this equilibrium, computer simulations suggest that it is indeed unique.

examine the case where the companies set a to maximize total profit, Π_T. Since $m(p*, p*) = 1/2$, by Theorem 1,

$$\Pi_T = \Pi_1 + \Pi_2 = \frac{N}{2s} \frac{(\tau + 2 - a)(2 + a)r^2}{(\tau + 4)^2} - 2F. \tag{18}$$

This function is concave in a and it obtains its unconstrained maximum at $a = \frac{\tau}{2}$. To guarantee that $a \leq 1$, the companies should select $a = min(\frac{\tau}{2}, 1)$ provided that they cover fixed costs. We summarize this in the next theorem.

Theorem 2. If a maximizes joint profits then

$$a = \begin{cases} 1 & \tau \geq 2, \\ \frac{\tau}{2} & \tau \leq 2, \end{cases} \tag{19}$$

and

$$p^* = q^* = \begin{cases} \frac{3r}{\tau+4} & \tau \geq 2, \\ \frac{r}{2} & \tau \leq 2. \end{cases} \tag{20}$$

Consequently, when $\tau \geq 2$ then $a = 1$ and the access fee coincides with the local price. If the rate of substitutability is large, consumers will be charged a small price (reflecting strong competition) and each company will be treated as any other customer by the other company. If $\tau \leq 2$ (reflecting low substitutability), the access fee is $\frac{\tau p}{2}$ and this is smaller than the local price. Therefore, in equilibrium the local price is $\frac{r}{2}$, which is the monopoly price.

In this paper the attributes of the networks (other than the prices) are exogenously given. These attributes determine the value of σ and therefore τ. One could add another stage to the game, for instance before a is determined. In this stage the companies compete in attributes and their selections determine the value of τ. Then Theorem 1 determines the equilibrium prices in terms of a and these attributes. The equilibrium choice of attributes will be a function of their costs. If the costs of different attributes are quite similar and a is selected to maximize joint profits, then they will choose their attributes so that τ will be sufficiently small to guarantee monopoly local prices (See Theorem 2). This will eliminate the effectiveness of the price competition and will justify government intervention. An extreme case is when government sets a to maximize total social surplus, SS, subject to the constraint that the companies cover fixed costs. The social surplus is the sum of the total industry profits and the total consumer surplus, CS, where

$$CS = \frac{N}{4s}(r - p*)^2.$$

It is easy to verify that in equilibrium

$$SS = \frac{Nr^2}{4s(\tau+4)^2}(\tau+2-a)(\tau+6+a) - 2F.$$

This function is a decreasing function of a for $0 \leq a \leq 1$. Hence the social surplus is maximized for the lowest level of a which still covers fixed costs. Observe that if $a = 0$ then $p^* = q^* = \frac{2r}{\tau+4}$ and

$$N \geq \frac{(\tau+4)^2 s}{(\tau+2)r^2} F$$

should hold to cover fixed costs. Hence fixed costs may not be covered for large τ. This suggests that for a competitive industry the ”Bill and Keep” policy may jeopardize the viability of the industry.

4 Conclusion

We have analyzed the competition between two network companies which choose access charges using RPACR. The most important result is that an equilibrium in local prices always exists. The equilibrium prices exibit desirable properties when the services of the networks are close substitutes and we believe that this will be the case in a competitive telecommunications industry. The imposition of RPACR can be viewed as a mild regulatory policy. One important advantage is that implementation of RPACR does not require information about the industry parameters. Since RPACR is responsive to future changes of the determinants of the market, it provides flexibility for the companies to react to changes in the environement and in strategies. For low differentiation between the networks, the equilibirium local prices are low, provided that both companies cover their costs. If one believes that this industry will not be as competitive and the companies will be able to differentiate themselves considerably, then RPACR may not be useful. It may serve as a collusion device for the companies to induce monopoly prices. Further regulations should then be imposed to prevent such a case.

References

Anderson, S., A. de Palma, J.F. Thisse(1992): “ Discrete Choice Theory of Product Differentiation”, MIT Press.

Armstrong, M., C. Doyle, J. Vickers(1995):“ The Access Pricing Problem: A Synthesis”, *mimeo*.

Baumol, W. J., G. Sidak(1994): “The Pricing of Inputs Sold to Competitors”, *Yale Journal of Regulation*, vol.11, no.1, pp.171-202.

Carter, M., J. Wright(1996): “ Interconnection in Network Industries”, *mimeo*.

Domencich,T., D. McFadden(1975): “Urban Travel Demand”, *North-Holland, Amsterdam*.

Economides, N. S., L.J. White(1995): “ Access and Interconnection Pricing: How Efficient is the ”Efficient Component Pricing Rule”?”, *Antitrust Bulletin*, forthcoming1 .

Economides, N. S., G. Woroch(1992):“ Benefits and Pitfalls of Network Interconnection”, *Stern School of Bussiness, Working Paper*.

Laffont, J-J., J.Tirole (1996):“Network Competition: I. Overview and Nondiscriminatory Pricing ”, *mimeo*, 38.

Laffont, J-J., J. Tirole(1994a) :“Creating Competition Through Interconnection: Theory and Practice”, *mimeo*.

Laffont, J-J., J. Tirole(1994b) :“Access Pricing and Competition”, *European Economic Review*, vol. 38, pp.1673-1710.

Muench, T(1988): “Quantum Agglomoration Formation During Growth in a Combined Economic/Gravity Model”, *Journal of Urban Economics*, vol. 23, pp. 199-214.

Noam, E.M.(1994):“ Interconnecting the Network of Networks”, *AEI Working Paper*.

Vogelsang, I., B.M. Mitchell(1994):“ Telecommunications Competition: The Last 10 Miles”, *AEI Working Paper*.

Telecommunications Act of 1996, *US Congress*.

Appendix

Proof of Theorem 1. Let $\alpha = r - p$ and $\beta = r - q$. Then the profit functions of the two companies are

$$\Pi_1(\alpha,\beta) = \frac{N}{2s}\frac{\alpha^x}{(\alpha^x+\beta^x)}\left[\alpha(r-\alpha)+a(\beta-\alpha)\frac{\beta^x}{(\alpha^x+\beta^x)}\right]-F,$$

and

$$\Pi_2(\alpha,\beta) = \frac{N}{2s}\frac{\beta^x}{(\alpha^x+\beta^x)}\left[\beta(r-\beta)+a(\alpha-\beta)\frac{\alpha^x}{(\alpha^x+\beta^x)}\right]-F.$$

Let

$$p^* = q^* = \frac{(2+a)r}{\tau+4}.$$

Then

$$\alpha^* = \frac{(\tau+2-a)r}{\tau+4} = \beta^*.$$

We have to prove that $\Pi_1(\alpha,\beta^*)$ is maximized for $\alpha = \alpha^*$ and that $\Pi_2(\alpha^*,\beta)$ is maximized for $\beta = \beta^*$. By symmetry, it is enough to show this for $\Pi_1(\alpha,\beta^*)$. Since $\Pi_1(\alpha,\beta)$ is continuous in the parameter τ, it is sufficient to prove our claim for positive rational values of τ. Let $\tau = \frac{m}{n}$, where $m > 0$, $n > 0$ are integers. Let

$$t \equiv \alpha^{\frac{1}{n}} \quad \mathit{and} \quad s \equiv \beta^{\frac{1}{n}}.$$

Then $t^* = (\alpha^*)^{\frac{1}{n}} = s^*$, where

$$s^* = (\alpha^*)^{\frac{1}{n}} = \left(\frac{m+2n-an}{m+4}r\right)^{\frac{1}{n}}, \tag{21}$$

and

$$\Pi_1(t,s) \equiv \frac{N}{2s}\frac{t^m}{(t^m+s^m)}\left[t^n(r-t^n)+ar(s^n-t^n)\frac{s^m}{(t^m+s^m)}\right]-F.$$

Since t^n is an increasing function of t, it is sufficient to prove that $\Pi_1(t,s^*)$ is maximized for $t^* = (\alpha^*)^{\frac{1}{n}}$ when $s^* = (\beta^*)^{\frac{1}{n}}$.

It can be easily verified that

$$\frac{\partial \Pi_1}{\partial t} = \frac{t^m}{t(t^m+s^m)^3}P(t,s),$$

where the polynomial $P(t,s)$ is given by

$$\begin{aligned} P(t,s) &= ms^m t^n (r-t^n)(t^m+s^m) + arms^m(s^n-t^n)(s^m-t^m) \\ &\quad +(rnt^n - 2nt^{2n})(t^m+s^m)^2 - arnt^n s^m (t^m+s^m). \end{aligned} \tag{22}$$

After rearranging the terms we have

$$\begin{aligned} P(t,s) &= s^{2m}\left[-(m+2n)t^{2n} + (rm-arm+rn-arn)t^n + arms^n\right] \\ &\quad s^m\left[-(m+4n)t^{m+2n} + (rm+arm+2rn-arn)t^{m+n} - armt^n s^n\right] \\ &\quad -2nt^{2m+2n} + rnt^{2m+n}. \end{aligned} \tag{23}$$

It is easy to verify that $t = t^* = s^*$ is the root of the polynomial $P(t,s^*)$. Therefore

$$P(t,s^*) = (t-s^*)g(t,s^*), \tag{24}$$

for a certain polynomial $g(t,s^*)$. It can be verified that

$$\begin{aligned} g(t,s) &= -2n\sum_{j=0}^{n-1} s^j t^{2m+2n-j-1} + n(r-2s^n)\sum_{j=0}^{2m-1} s^j t^{2m+n-j-1} \\ &\quad -ws^m \sum_{j=0}^{n-1} s^j t^{2n-j-1} + [y - ws^n + n(r-2s^n)]s^{2m}\sum_{j=0}^{n-1} s^j t^{n-j-1} \\ &\quad -vs^m \sum_{j=0}^{n-1} s^j t^{m+2n-j-1} + (z-vs^n)\sum_{j=0}^{n-1} s^j t^{m+n-j-1} \\ &\quad +(z-vs^n-arm)s^{m+n}\sum_{j=0}^{n-1} s^j t^{m-j-1}, \end{aligned} \tag{25}$$

where $s = s^* = \left(\frac{m+2n-an}{m+4n} r\right)^{\frac{1}{n}}$ and where

$$\begin{aligned} w &= m+2n, \\ v &= m+4n, \\ y &= rm-arm+rn-arn, \\ z &= rm+arm+2rn-arn. \end{aligned} \tag{26}$$

It can be easily checked that

$$z - vs^n - arm = 0, \tag{27}$$

and

$$y - ws^n - n(r - 2s^n) = -arm. \tag{28}$$

Lemma 1. let $0 \le t \le r^{\frac{1}{n}}$. Then, for $0 \le t \le r^{1/n}$, $g(t,s^*) < 0$.
Proof. Let us sum up the geometric series appearing in (25) and apply (26), (27), (28). We obtain

$$\begin{aligned} g(t,s) &= -2nt^{2m+2n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} + n(r-2s^n)t^{2m+n-1}\frac{1-(\frac{s}{t})^{2m}}{1-\frac{s}{t}} \\ &\quad -(m+2n)s^m t^{2n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} arms^{2m}t^{n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} \\ &\quad -(m+4n)s^m t^{m+2n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} + arms^m t^{m+n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}}, \end{aligned} \tag{29}$$

where $s = s^*$. Let us first show that the sum of the last three terms of the right-hand side of (29) is negative. Consider the case where $t < s^*$. Observe that

$$arms^{2m}t^{n-1} > arms^m t^{m+n-1},$$

for all $s > t$. Therefore the sum of the last three terms of (29) is negative. Consider next the case where $t > s^*$. It is sufficient to show that

$$(m+4n)s^m t^{m+2n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} > arms^m t^{m+n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}}.$$

Therefore, it is sufficient to show that

$$(m+4n)s^n - arm < 0. \tag{30}$$

It is easy to check that (30) holds for $s = s^*$. Consequently, the sum of the last three terms of (29) is negative for every $0 \le t \le r^{1/n}$, and $s = s^*$. Returning to (29), we are left to show that when $s = s^*$

$$\begin{aligned} g(t,s^*) &< -2nt^{2m+2n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} + n(r-2s^n)t^{2m+n-1}\frac{(1-\frac{s}{t})^{2m}}{1-\frac{s}{t}} \\ &\quad -(m+2n)s^m t^{2n-1}\frac{1-(\frac{s}{t})^n}{1-\frac{s}{t}} < 0. \end{aligned}$$

Let

$$h(t,s) = -2t^{2m} - 2s^{2m} + (r-2s^n)\frac{t^{2m}-s^{2m}}{t^n - s^n}. \tag{31}$$

It is easy to check that

$$g(t,s) \leq \frac{n(t^n - s^n)}{(t-s)} h(t,s).$$

Therefore, it is sufficient to show that $h(t,s) < 0$ for all $0 \leq t \leq r^{1/n}$ and for $s = s^*$. To this end, we need the following lemma.

Lemma 2. Let

$$k(t,s) = \frac{t^{2m} - s^{2m}}{t^n - s^n}.$$

Then, $\frac{\partial k}{\partial t}(t,s) > 0$ if and only if $(2m-n)(t-s) > 0$.

Proof. It can be easily verified that

$$\frac{\partial k}{\partial t}(t,s) = -\frac{2mt^{2m-1}}{(t^n - s^n)} \left[\frac{n}{2mt^{2m-n}} \frac{t^{2m} - s^{2m}}{t^n - s^n} - 1 \right].$$

By the mean value theorem

$$\frac{t^{2m} - s^{2m}}{t^n - s^n} = \frac{2m}{n} c^{2m-n}, \tag{32}$$

where $min(t,s) \leq c \leq max(t,s)$. Therefore

$$\frac{\partial k}{\partial t}(t,s) = -\frac{2mt^{2m-1}}{(t^n - s^n)} \left[(\frac{c}{t})^{2m-n} - 1 \right],$$

and it is now easy to verify that the condition of Lemma 2 holds.

We will use Lemma 2 to prove that $h(t,s) < 0$ for every $0 \leq t \leq r^{1/n}$ and $s = s^*$. First observe that by (31)

$$h(t,s) \leq \left[(r - 2s^n) \frac{t^{2m} - s^{2m}}{t^n - s^n} - 2s^{2m} \right]. \tag{33}$$

Consider the following four cases.

Case 1. $2m - n < 0$ and $t < s^*$.

By Lemma 2, $\frac{\partial k}{\partial t}(t,s) > 0$. Hence, it is sufficient to show that the right-hand side of (33) is negative at $t = s^*$. Indeed for $s = s^*$

$$h(s,s) \leq \frac{2m}{n}(r - 2s^n)s^{2m-n} - 2s^{2m} = s^{2m} \left[\frac{2m}{n} \frac{(r - 2s^n)}{s^n} - 2 \right].$$

Using (21) , we just need to show that

$$\frac{2m}{n}\frac{(2an-m)}{(m+2n-an)}-2<0, \tag{34}$$

holds for all m and n such that $2m-n<0$. It is sufficient to show that this is true for $a=1$ since the left-hand side of (34) is increasing in a. The last inequality holds if and only if

$$m^2+n^2>mn.$$

Clearly this is true for all $m,n>0$; hence $h(t,s^*)$ is negative for $t<s^*$.
Case 2. $2m-n>0$ and $t<s^*$.
Again by Lemma 2, $\frac{\partial k}{\partial t}(t,s)<0$. Thus, we need to show that the right-hand side of (33) is negative at $t=0$. Indeed for $s=s^*$

$$h(0,s)\leq(r-2s^n)s^{2m-n}-2s^{2m}<s^{2m}\left[\frac{(r-2s^n)}{s^n}-2\right]. \tag{35}$$

The right hand side of (35) is negative at $s=s^*$ if and only if

$$\frac{2an-m}{m+2n-an}-2<0,$$

for all m and n such that $2m-n>0$. This is certainly true for $a=1$ and for all $m,n>0$.
Case 3. $2m-n<0$ and $t>s^*$.
Since $\frac{\partial k}{\partial t}(t,s)<0$, we need to show that the right-hand side (33) is negative at $t=s^*$. For $s=s^*$

$$h(s,s)\leq\frac{2m}{n}(r-2s^n)s^{2m-n}-2s^{2m}=s^{2m}\left[\frac{2m}{n}\frac{(r-2s^n)}{s^n}-2\right],$$

and this was shown to be negative in Case 1.
Case 4. $2m-n>0$ and $t>s^*$. In this case, by (31)

$$h(t,s)\leq\left[(r-2s^n)\frac{t^{2m}-s^{2m}}{t^n-s^n}2t^{2m}\right]. \tag{36}$$

By (32), for all $s<t$ there exists c, $s\leq c\leq t$, such that

$$\frac{t^{2m}-s^{2m}}{t^n-s^n}=\frac{2m}{n}c^{2m-n}. \tag{37}$$

If $2m-n>0$, c^{2m-n} increases in c. Therefore, whenever $s<t$,

$$c^{2m-n}<t^{2m-n}<\frac{t^{2m}}{s^n}.$$

Then by (36) and (37) we have

$$h(t,s)\leq(r-2s^n)\frac{2m}{n}t^{2m}s^{-n}-2s^{2m}=t^{2m}\left[\frac{2m}{n}\frac{(r-2s^n)}{s^n}-2\right]. \tag{38}$$

The right-hand side of (38) is negative if and only if

$$\frac{2m}{n}\frac{(r-2s^n)}{s^n}-2<0,$$

holds for $s=s^*$. But in Case 1, it was shown that this condition is satisfied. This completes the proof of Lemma 1.

By Lemma 1 and (24), $P(t,s^*)>0$ whenever $0\leq t\leq s^*$ and $P(t,s^*)<0$ whenever $s^*\leq t\leq r^{1/n}$. Therefore, $t=s^*$ is the unique maximizer of $\Pi_1(t,s^*)$ and hence α^* is the unique maximizer of $\Pi_1(\alpha,\beta^*)$. This completes the proof of Theorem 1.