Appendix

Motivation of Cost Function. Suppose that customer acquisition consists of draws with replacement from the general population of L potential consumers. Let υ_m be the yield of unique customers, and A_m be the cumulative number at draw *m*. By these definitions,

$$A_m = A_{m-1} + \upsilon_m . \tag{A1}$$

In draw *m*, the probability that the additional draw duplicates the consumers already drawn is

$$\frac{A_{m-1}}{L} .$$

Hence, the expected yield of unique customers from draw *m*

$$\upsilon_m = 1 - \frac{A_{m-1}}{L} . \tag{A2}$$

Substituting (A2) in (A1), the cumulative number at draw *m*

$$A_m = A_{m-1} + 1 - \left[\frac{1}{L}\right] A_{m-1} = \left[1 - \frac{1}{L}\right] A_{m-1} + 1 . \tag{A3}$$

Hence,

$$A_{m-1} = \left[1 - \frac{1}{L}\right] A_{m-2} + 1 .$$

Substituting in (A3),

$$A_m = \left[1 - \frac{1}{L}\right]^2 A_{m-2} + \left[1 - \frac{1}{L}\right] + 1 .$$

Likewise, by recursively substituting for $A_{m-2}, A_{m-3}, \ldots, A_2$, and noting that the first draw yields a unique customer with certainty, i.e., $A_1 = 1$, we derive

$$A_m = \sum_{i=1}^{m} \left[1 - \frac{1}{L}\right]^{i-1} = \sum_{i=0}^{m-1} \left[1 - \frac{1}{L}\right]^{i} . \tag{A4}$$

Summing all the *m* terms in (A4), the number of unique customers from *m* draws

$$A = \left[1 - \left[1 - \frac{1}{L}\right]^m\right] L .$$

which implies that

$$m = \ln\left(1 - \frac{A}{L}\right) \Big/ \ln\left(1 - \frac{1}{L}\right) . \tag{A5}$$

Let *c* be the unit cost of each draw. Then, the expected cost of acquiring *A* customers

$$C(A) = cm(A) = c\ln\left(1-\frac{A}{L}\right)\Big/\ln\left(1-\frac{1}{L}\right). \tag{1}$$

With an exogenously convex cost function (e.g., Chen et al. 2001a; Chen and Iyer 2002), consumers must be properly ordered in some way such as location or taste. Otherwise, the cost of acquiring a particular number of customers can be reduced by dividing the acquisition into separate batches. The cost function (1) does not suffer from this weakness – if a seller acquires customers in separate batches, its total cost and net expected yield (after removing duplicates) would be the same as with acquisition in a single batch.

Proof of Lemma 1.

(a) There is no pure strategy equilibrium.

Proof: Suppose otherwise, that each seller *i* sets price p_i with probability one. If some $p_i = p_j > 0$, then seller *i* should cut price to $p_i - \varepsilon$: it would incur a second-order loss in price, but gain a first-order quantity of switchers from the segment,

$$\left\{\frac{A_j}{L}\prod_{l\neq i,j}\left[1-\frac{A_l}{L}\right]\right\}A_i, \tag{A6}$$

acquired by both sellers *i* and *j*, but not acquired by other sellers. If there are no $p_i = p_j > 0$, suppose that the highest price below p_j is $p_i < p_j$. Then seller *i* should raise price to $p_j - \varepsilon$, in which case, seller *j* should just under-cut seller *i*, as in the previous case. If all $p_i = 0$, then every seller would earn zero revenue, hence every seller *i* should raise price to v, and earn positive revenue. [ ]

(b) The supports of the pricing strategies of at least two sellers have supremum at v, i.e., $\hat{p}_i = v$, at least two *i*.

Proof: We prove this result in two steps, first that the pricing strategies of at least two sellers have the same supremum, say $\hat{p}$, which is the highest among the suprema of all the pricing strategies. In the second step, we prove that $\hat{p} = v$.

Suppose that seller 1's pricing strategy has the highest supremum, $\hat{p}$, and seller 2's pricing strategy has the next highest supremum, $\tilde{p} < \hat{p}$. Then seller 1 should shift its pricing support in the interval $(\tilde{p}, \hat{p})$ to $\hat{p}$. It would gain a higher margin on all consumers that it alone acquires, while it would not lose any consumers since, for $p > \tilde{p}$, its price is higher than that of any other seller. Accordingly, seller 1's pricing support would have a gap on $(\tilde{p}, \hat{p})$.

Then, seller 2 should shift its pricing support from the neighborhood $(\tilde{p}-\varepsilon, \tilde{p})$ to $(\hat{p}-\varepsilon, \hat{p})$. It would

- Gain from a higher price on all consumers that it alone acquires;
- Incur no loss of revenue on consumers who are also acquired by seller 1;[1] and
- Incur no loss of revenue on consumers also acquired by some other seller $j > 2$, since $\tilde{p}$ is higher than the maximum price in the support of those other sellers.

Thus, we have proved that the pricing strategies of at least two sellers must have the same supremum, say $\hat{p}$, which is the highest among the suprema of all the pricing strategies. The same proof also shows that $\hat{p} = v$. [ ]

(c) The supports of the pricing strategies of at least two sellers have infimum at $\underline{p}$, i.e., $\underline{p}_i = \underline{p}$, at least two *i*.

Proof: Suppose that seller *i*'s pricing strategy has the lowest infimum, $\underline{p}$, and seller *j*'s pricing strategy has the next lowest infimum, $p' > \underline{p}$. Then seller *i* should shift its pricing support in the interval $[\underline{p}, p')$ to p'. It would gain from a higher price on all consumers that it alone acquires, while it would not lose any consumers since, for $p \le p'$, its price is lower than that of any other seller. Accordingly, the pricing strategies of sellers *i* and *j* have the same infimum $\underline{p}$. [ ]

(d) Equilibrium pricing strategies do not include any mass points in the interval, $[\underline{p}, v)$. At most one seller may have a mass point, which must be at v.

Proof: Suppose that seller *i* has a mass point at some $p_i \in [\underline{p}, v)$. If no seller's pricing strategy has support on an interval above p_i, $[p_i, p_i + \varepsilon)$, then seller *i* should shift its support from $[p_i, p_i + \varepsilon)$ to a mass point at $p_i + \varepsilon$: it would gain on price, and lose no customers (since no other seller sets price in $[p_i, p_i + \varepsilon)$).

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[1] In the scenario of Varian (1980), Baye et al. (1992), and Chioveanu (2003), all sellers acquire same segment of switchers, hence these consumers would buy from any seller whose supremum is less than $\tilde{p}$, and this loss would be zero.

Accordingly, suppose that the pricing strategy of some seller *j* has support on $[p_i, p_i + \varepsilon)$. Then seller *j* should shift its support from $[p_i, p_i + \varepsilon)$ to $(p_i - \varepsilon, p_i)$. It would incur a second-order loss in price, but gain a first-order quantity of switchers from the segment,

$$\left\{\frac{A_i}{L}\prod_{l \neq i,j}\left[1-\frac{A_l}{L}\right]\right\}A_j \tag{A6}$$

acquired by both sellers *i* and *j*, but not acquired by other sellers.

The preceding proof also shows that at most one seller may have a mass point, and that it must be at v. [ ]

(e) The supports of the pricing strategies of all sellers have the same infimum, $\underline{p}$, i.e., $\underline{p}_i = \underline{p}$, all *i*, where

$$\underline{p} = v\prod_{i \neq m}\left[1-\frac{A_i}{L}\right]. \tag{5}$$

Proof. The proof follows that of McAfee's (1994) Lemma 1. Suppose that seller *m* has a mass point at v. Then, in randomized pricing equilibrium, seller *m*'s revenue at v must not be less than its revenue at any other price, or by (4),

$$\begin{aligned} vA_m\prod_{l \neq m}\left[1-\frac{A_l}{L}\right] &\geq pA_m\prod_{l \neq m}\left[1-\frac{A_l}{L}F_l(p)\right] \\ &= pA_i\frac{A_m}{A_i}\frac{1-\frac{A_i}{L}F_i(p)}{1-\frac{A_m}{L}F_m(p)}\prod_{l \neq i}\left[1-\frac{A_l}{L}F_l(p)\right]. \end{aligned} \tag{A7}$$

Consider any seller $i \neq m$. In equilibrium, its revenue at any price $p \in S_i$,

$$pA_i\prod_{l \neq i}\left[1-\frac{A_l}{L}F_l(p)\right] \geq vA_i\prod_{l \neq i}\left[1-\frac{A_l}{L}\right].$$

Substituting into the right-hand side of (A7),

$$vA_m\prod_{l \neq m}\left[1-\frac{A_l}{L}\right] \geq \frac{A_m}{A_i}\frac{1-\frac{A_i}{L}F_i(p)}{1-\frac{A_m}{L}F_m(p)}vA_i\prod_{l \neq i}\left[1-\frac{A_l}{L}\right],$$

which simplifies to

$$\left[1-\frac{A_i}{L}\right]\left[1-\frac{A_m}{L}F_m(p)\right] \geq \left[1-\frac{A_m}{L}\right]\left[1-\frac{A_i}{L}F_i(p)\right].$$

At $p = \underline{p}_i$, $F_i(\underline{p}_i) = 0$, hence the above simplifies to

$$\left[1-\frac{A_i}{L}\right]\left[1-\frac{A_m}{L}F_m(\underline{p}_i)\right] \geq \left[1-\frac{A_m}{L}\right]. \tag{A8}$$

Since $F_m(\underline{p}_i) \geq 0$,

$$\left[1-\frac{A_i}{L}\right] \geq \left[1-\frac{A_i}{L}\right]\left[1-\frac{A_m}{L}F_m(\underline{p}_i)\right]. \tag{A9}$$

By (A8) and (A9), $A_m \geq A_i$, for all $i \neq m$. Thus, only the seller with the largest number of customers may have a mass point at v.

Now, suppose that seller *k* has the lowest of the infima of the supports, i.e., $\underline{p}_k \leq \underline{p}_j$, all *j*, and suppose that seller *i*'s infimum $\underline{p}_i \geq \underline{p}_k$. Then, since $\underline{p}_k \in S_k$, seller *k*'s revenue at $\underline{p}_k$ must not be less than its revenue at $\underline{p}_i$,

$$\underline{p}_k A_k \geq \underline{p}_i A_k \prod_{l \neq k}\left[1-\frac{A_l}{L}F_l(\underline{p}_i)\right] = \underline{p}_i A_k \frac{1-\frac{A_i}{L}F_i(\underline{p}_i)}{1-\frac{A_k}{L}F_k(\underline{p}_i)} \prod_{l \neq i}\left[1-\frac{A_l}{L}F_l(\underline{p}_i)\right]. \tag{A10}$$

Similarly, for seller *i*, since $\underline{p}_i \in S_i$,

$$\underline{p}_i A_i \prod_{l \neq i}\left[1-\frac{A_l}{L}F_l(\underline{p}_i)\right] \geq \underline{p}_k A_i \prod_{l \neq i}\left[1-\frac{A_l}{L}F_l(\underline{p}_k)\right] = \underline{p}_k A_i,$$

since $\underline{p}_k \leq \underline{p}_j$, all *j*, which implies that $F_l(\underline{p}_k) = 0$, all *l*. Hence

$$\underline{p}_i \prod_{l \neq i}\left[1-\frac{A_l}{L}F_l(\underline{p}_i)\right] \geq \underline{p}_k. \tag{A11}$$

Substituting (A11) into the right-hand side of (A10), and using $F_i(\underline{p}_i) = 0$,

$$\underline{p}_k A_k \geq \frac{\underline{p}_k A_k}{1-\frac{A_k}{L}F_k(\underline{p}_i)},$$

which implies $F_k(\underline{p}_i) = 0$. Since $F_k(\underline{p}_k) = 0$, we conclude that $\underline{p}_i = \underline{p}_k = \underline{p}$, say.

Finally, in equilibrium, since $\underline{p} \in S_m$ and $v \in S_m$, seller m must derive equal revenue at the two prices,

$$\underline{p}A_m = vA_m \prod_{j \neq m}\left[1-\frac{A_j}{L}\right],$$

which yields (5). [ ]

(f) The supports of equilibrium pricing strategies are intervals (and so, do not have any gaps).

Proof. Suppose that there are just three sellers. By (b), the support of the pricing strategies of sellers 1 and 2 include v. By (e), the sellers have the same infimum in the pricing support. Hence seller 3's support must overlap with the supports of sellers 1 and 2.

Now suppose that seller 3's pricing strategy includes a gap, $(\tilde{p}, p')$, but seller 1's does not. By (4), seller 3's revenue

$$R_3(p) = \left[1 - \frac{A_1}{L} F_1(p)\right]\left[1 - \frac{A_2}{L} F_2(p)\right] A_3 p. \tag{A12}$$

Since $\hat{p}$ is in the support of seller 3's pricing strategy,

$$R_3(\hat{p}) = R_3^* = \left[1 - \frac{A_1}{L} F_1(\hat{p})\right]\left[1 - \frac{A_2}{L} F_2(\hat{p})\right] A_3 \hat{p}. \tag{A13}$$

Likewise, for seller 1,

$$R_1(p) = \left[1 - \frac{A_2}{L} F_2(p)\right]\left[1 - \frac{A_3}{L} F_3(p)\right] A_1 p, \tag{A14}$$

and, since $\hat{p}$ is in the support of seller 1' pricing strategy,

$$R_1(p) = \left[1 - \frac{A_2}{L} F_2(\hat{p})\right]\left[1 - \frac{A_3}{L} F_3(\hat{p})\right] A_1 \hat{p}. \tag{A15}$$

Solving (A12) and (A13) for $\hat{p}/p$, and likewise with (A14) and (A15), and equating,

$$\frac{[L - A_1 F_1(p)][L - A_2 F_2(p)]}{[L - A_1 F_1(\hat{p})][L - A_2 F_2(\hat{p})]} = \frac{\hat{p}}{p} = \frac{[L - A_2 F_2(p)][L - A_3 F_3(p)]}{[L - A_2 F_2(\hat{p})][L - A_3 F_3(\hat{p})]},$$

and hence,

$$\frac{1}{[L - A_1 F_1(\hat{p})]}\left[1 - \frac{A_1}{L} F_1(p)\right] = \frac{1}{[L - A_3 F_3(\hat{p})]}\left[1 - \frac{A_3}{L} F_3(p)\right]. \tag{A16}$$

Since seller 3's pricing strategy has a gap, $(\tilde{p}, p')$, the distribution, $F_3(\tilde{p}) = F_3(p')$. Applying (A16) to $\tilde{p}$ and p', and using $F_3(\tilde{p}) = F_3(p')$, we infer that

$$F_1(\tilde{p}) = F_1(p'),$$

which is a contradiction.

Essentially, the proof shows that, if there is an overlap in the support of any two sellers' pricing strategies, then if there is any gap in the strategy of the seller whose support has a lower maximum, then the support of the other seller must have the same gap. [ ]

(g) For sellers 1 and 2, $\hat{p}_1 = \hat{p}_2 = v$, while among all other sellers, a seller which acquires more consumers will have a support with a higher supremum, i.e., if $A_i > A_j$, then $\hat{p}_i > \hat{p}_j$.

Proof: Suppose that $A_i > A_k$, but seller *i*'s support has a lower supremum, $\hat{p}_i \le \hat{p}_k$. By (e), all the supports of all the pricing strategies have the same infimum, $\underline{p}$, and by (f), all the supports are intervals. Since both $\underline{p}$ and $\hat{p}_i$ belong to seller *i*'s support, seller *i*'s revenue from the two prices must be equal, or, by (4),

$$\underline{p}A_i\prod_{j\neq i}\left[1-\frac{A_j}{L}F_j(\underline{p})\right]=\hat{p}_iA_i\prod_{j\neq i}\left[1-\frac{A_j}{L}F_j(\hat{p}_i)\right]. \tag{A17}$$

Similarly for seller *k*,

$$\underline{p}A_k\prod_{j\neq k}\left[1-\frac{A_j}{L}F_j(\underline{p})\right]=\hat{p}_iA_k\prod_{j\neq k}\left[1-\frac{A_j}{L}F_j(\hat{p}_i)\right]. \tag{A18}$$

Solving each of (A17) and (A18) for $\underline{p}/\hat{p}_i$ and equating,

$$\frac{\prod_{j\neq i}\left[1-\frac{A_j}{L}F_j(\underline{p})\right]}{\prod_{j\neq i}\left[1-\frac{A_j}{L}F_j(\hat{p}_i)\right]}=\frac{\hat{p}_i}{\underline{p}}=\frac{\prod_{j\neq k}\left[1-\frac{A_j}{L}F_j(\underline{p})\right]}{\prod_{j\neq k}\left[1-\frac{A_j}{L}F_j(\hat{p}_i)\right]},$$

which simplifies to

$$\frac{1-\frac{A_k}{L}F_k(\underline{p})}{1-\frac{A_k}{L}F_k(\hat{p}_i)}=\frac{1-\frac{A_i}{L}F_i(\underline{p})}{1-\frac{A_i}{L}F_i(\hat{p}_i)}. \tag{A19}$$

Using $F_i(\underline{p})=F_k(\underline{p})=0$ and $F_i(\hat{p}_i)=1$, (A19) reduces to

$$1-\frac{A_i}{L}=1-\frac{A_k}{L}F_k(\hat{p}_i),$$

which implies that $A_i=A_kF_k(\hat{p}_i)$, which contradicts the assumption that $A_i>A_k$. [ ]

(h) For every seller *i*, revenue

$$R_i=\underline{p}A_i=vA_i\prod_{j\neq m}\left[1-\frac{A_j}{L}\right]. \tag{6}$$

Proof: By (e), since $\underline{p}$ belongs to the support of every seller *i* and is the minimum of all supports, $R_i=R_i(\underline{p})=\underline{p}A_i$. The result then follows from (5). [ ]

Proof of Proposition 1.

We claim that any equilibria that involve randomized acquisitions must have at least two sellers having the highest supremum, hence, $\hat{A}_1=\hat{A}_2\geq\ldots\geq\hat{A}_n$. Suppose otherwise, that $\hat{A}_1>\hat{A}_2$. Then, for $A_1\in[\hat{A}_2,\hat{A}_1]$,

$$R_1(A_1 \mid A_2,\ldots,A_n) = vA_1 \cdot E\left\{\prod_{j=2}^{n}\left[1-\frac{A_j}{L}\right]\right\},$$

and hence,

$$\Pi_1(A_1) = vA_1 \cdot E\left\{\prod_{j=2}^{n}\left[1-\frac{A_j}{L}\right]\right\} - C(A_1)\,. \tag{A20}$$

Consider the right-hand side of (A20): the first term (expected revenue) is linear in A_1, while the second term (cost) is strictly convex in A_1, hence the sum is strictly concave on $[\hat{A}_2,\hat{A}_1]$, and so seller 1 will not randomize on $[\hat{A}_2,\hat{A}_1]$. A similar argument shows that if seller 1 acquires customers deterministically at some $\hat{A}_1 > \hat{A}_2$, then the other $n-1$ sellers would not randomize acquisitions. Therefore, in equilibrium, if sellers were to randomize acquisitions, at least two sellers must have the highest supremum.

Define $A_{m(i)} = \max_{j\neq i}(A_j)$ to be the highest realized acquisition among the sellers other than *i*. Let $\Phi_{m(i)}(\cdot)$ be its distribution. If $A_{m(i)} \geq A_i$, then by (5),

$$\underline{p} = v\left[1-\frac{A_i}{L}\right]\prod_{j\neq i,m(i)}\left[1-\frac{A_j}{L}\right],$$

while, if $A_{m(i)} < A_i$, then

$$\underline{p} = v\prod_{j\neq i}\left[1-\frac{A_j}{L}\right].$$

Accordingly, seller *i*'s expected profit would be

$$\begin{aligned}\Pi_i(A_i) = {} & vA_i\left[1-\frac{A_i}{L}\right]\int_{A_{m(i)}\geq A_i} E\left\{\prod_{j\neq i,m(i)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}d\Phi_{m(i)}(A_{m(i)}) \\ & + vA_i\int_{A_{m(i)}<A_i} E\left\{\prod_{j\neq i}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}d\Phi_{m(i)}(A_{m(i)}) - \frac{c\ln(1-A_i/L)}{\ln(1-1/L)},\end{aligned} \tag{A21}$$

where

$$E\left\{\prod_{j\neq i,m(i)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}$$

represents the expectation of the multiple product

$$\left[1-\frac{A_1}{L}\right]\ldots\left[1-\frac{A_{i-1}}{L}\right]\left[1-\frac{A_{i+1}}{L}\right]\ldots\left[1-\frac{A_{m(i)-1}}{L}\right]\left[1-\frac{A_{m(i)+1}}{L}\right]\ldots\left[1-\frac{A_n}{L}\right]$$

over the joint distribution of the A_j s, $j \neq i, m(i)$, conditional on the value of $A_{m(i)}$, and similarly for

$$E\left\{\prod_{j\neq i}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}.$$

We separate the remainder of the proof into two cases.

Case (i): The strategy, $G_i(\cdot)$, of at least one seller is continuous at $\hat{A}$.

Without loss of generality, suppose seller 2's strategy is continuous at $\hat{A}$. Differentiating (A21) with respect to A_i and substituting from (3),

$$\begin{aligned}\frac{d}{dA_i}\Pi_i(A_i) &= v\left[1-\frac{2A_i}{L}\right]\int_{A_{m(i)}\geq A_i} E\left\{\prod_{j\neq i,m(i)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}d\Phi_{m(i)}(A_{m(i)}) \\ &+v\int_{A_{m(i)}<A_i} E\left\{\prod_{j\neq i}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}d\Phi_{m(i)}(A_{m(i)}) - vX\left[1-\frac{A_i}{L}\right]^{-1}.\end{aligned} \tag{A22}$$

Further differentiating,

$$\begin{aligned}\frac{d^2}{dA_i^2}\Pi_i(A_i) &= \frac{vA_i}{L}E\left\{\prod_{j\neq i,m(i)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}=A_i\right\}\frac{d}{dA_{m(i)}}\Phi_{m(i)}(A_i) \\ &-\frac{2v}{L}\int_{A_{m(i)}\geq A_i} E\left\{\prod_{j\neq i,m(i)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(i)}\right\}d\Phi_{m(i)}(A_{m(i)}) - \frac{vX}{L}\left[1-\frac{A_i}{L}\right]^{-2}.\end{aligned} \tag{A23}$$

Now, for seller 1, at $A_1=\hat{A}$, it is necessary that $d\Pi_1(\hat{A})/dA_1=0$ and $d^2\Pi_1(\hat{A})/dA_1^2\leq 0$. Otherwise, there would exist A_1 that yields higher profit than $A_1=\hat{A}$, which would contradict the randomized-strategy equilibrium. By (A22), the first-order condition resolves to

$$\begin{aligned}&\left[1-\frac{2\hat{A}}{L}\right]\int_{A_{m(1)}\geq \hat{A}} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}(A_{m(1)}) \\ &+\int_{A_{m(1)}<\hat{A}} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}(A_{m(1)}) = X\left[1-\frac{\hat{A}}{L}\right]^{-1}.\end{aligned} \tag{A24}$$

Note that $\hat{A}$ is the highest of the suprema of the various supports. Hence we have

$$\begin{aligned}\left[1-\frac{2\hat{A}}{L}\right]^{n-1} &< \left[1-\frac{2\hat{A}}{L}\right]\int_{A_{m(1)}\geq \hat{A}} E\left\{\prod_{j\neq i,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}(A_{m(1)}) \\ &+\int_{A_{m(1)}<\hat{A}} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}(A_{m(1)}).\end{aligned}$$

Substituting in (A24) and rearranging the terms,

$$\left[1-\frac{2\hat{A}}{L}\right]^{n-1}\left[1-\frac{\hat{A}}{L}\right] < X \ ,$$

which implies

$$\left[1-\frac{2\hat{A}}{L}\right]^{n} < X$$

and hence

$$\frac{\hat{A}}{L} \geq \frac{1-\sqrt[n]{X}}{2}. \qquad \text{(A25)}$$

By (A23), the second-order condition, $d^2\Pi_1(\hat{A})/dA_1^2 \leq 0$, resolves to

$$\begin{aligned}\frac{d^2}{dA_1^2}\Pi_1(\hat{A}) &= \frac{v\hat{A}}{L}E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}=\hat{A}\right\}\frac{d}{dA_{m(1)}}\Phi_{m(1)}(\hat{A}) \\ &\quad -\frac{2v}{L}\int_{A_{m(1)}>\hat{A}} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}(A_{m(1)}) \ -\frac{vX}{L}\left[1-\frac{\hat{A}}{L}\right]^{-2} \leq 0.\end{aligned} \qquad \text{(A26)}$$

Consider the right-hand side of (A26). Since seller 2's acquisition strategy is left continuous up to $A_2 = \hat{A}$, the probability density, $d\Phi_{m(1)}(\hat{A})/dA_{m(1)} > 0$. Together with (A25), this implies that the first term is positive and finite for all values of L. The second term vanishes as $L \to \infty$. As for the third term on the right-hand side of (A26), by (A24),

$$\left[1-\frac{\hat{A}}{L}\right] > X \ ,$$

and hence

$$vX\left[1-\frac{\hat{A}}{L}\right]^{-2} < \ vX\,\frac{1}{X^2} = \frac{v}{X},$$

which is finite, hence the third term vanishes as $L \to \infty$. Thus, if L is sufficiently large, $d^2\Pi_1(\hat{A})/dA_1^2 > 0$, which is a contradiction.

<u>Case (ii)</u>: The strategies, $G_i(\cdot)$, of all sellers with support in the neighborhood $(\hat{A}-\varepsilon, \hat{A}]$ comprise discrete mass points in that neighborhood.

Without loss of generality, suppose that the strategies of sellers 1 and 2 have mass points at $\hat{A}$. Consider seller 1's profit. By (A22), a variation dA_1 would change seller 1's profit by

$$d\Pi_1 = v\left[1-\frac{2A_1}{L}\right]\int_{A_{m(1)}\geq A_1} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}dA_1$$
$$+v\int_{A_{m(1)}<A_1} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}dA_1 - vX\left[1-\frac{A_1}{L}\right]^{-1}dA_1. \quad \text{(A27)}$$

Applying (A27) to $A_1 = \hat{A}-\varepsilon, \hat{A}+\varepsilon$,

$$d\Pi_1(\hat{A}+\varepsilon) - d\Pi_1(\hat{A}-\varepsilon)$$
$$\to v\left[1-\frac{2\hat{A}}{L}\right]\left\{\int_{A_{m(1)}\geq \hat{A}+\varepsilon} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}\right.$$
$$\left. - \int_{A_{m(1)}\geq \hat{A}-\varepsilon} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}\right\}dA_1 \quad \text{(A28)}$$
$$+v\left\{\int_{A_{m(1)}<\hat{A}+\varepsilon} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}\right.$$
$$\left. - \int_{A_{m(1)}<\hat{A}-\varepsilon} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}\right\}dA_1$$

as $\varepsilon \to 0$.

Note that the probability of $A_{m(1)} \geq \hat{A}+\varepsilon$ is zero, and $A_{m(1)} < \hat{A}+\varepsilon$ occurs with certainty. Further, because no strategies have support in $(\hat{A}-\varepsilon, \hat{A})$, the condition $A_{m(1)} \geq \hat{A}-\varepsilon$ is equivalent to $A_{m(1)} = \hat{A}$, and the condition $A_{m(1)} < \hat{A}-\varepsilon$ is equivalent to $A_{m(1)} < \hat{A}$.

Therefore, (A28) simplifies to

$$d\Pi_1(\hat{A}+\varepsilon) - d\Pi_1(\hat{A}-\varepsilon)$$
$$\to \; - \; v\left[1-\frac{2\hat{A}}{L}\right]\int_{A_{m(1)}=\hat{A}} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}dA_1$$
$$+v\left\{\int_{A_{m(1)}} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)} - \int_{A_{m(1)}<\hat{A}} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}\right\}dA_1$$
$$= \; - \; v\left[1-\frac{2\hat{A}}{L}\right]\int_{A_{m(1)}=\hat{A}} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}dA_1$$
$$+v\int_{A_{m(1)}=\hat{A}} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\}d\Phi_{m(1)}dA_1, \quad \text{(A29)}$$

since the complement of $A_{m(1)} < \hat{A}$ is $A_{m(1)} = \hat{A}$.

Now,

$$v\left[1-\frac{2\hat{A}}{L}\right]\int_{A_{m(1)}=\hat{A}} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\} d\Phi_{m(1)} dA_1$$

$$< v\left[1-\frac{\hat{A}}{L}\right]\int_{A_{m(1)}=\hat{A}} E\left\{\prod_{j\neq 1,m(1)}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\} d\Phi_{m(1)} dA_1 \qquad \text{(A30)}$$

$$\leq v\int_{A_{m(1)}=\hat{A}} E\left\{\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]\middle| A_{m(1)}\right\} d\Phi_{m(1)} dA_1,$$

where the first inequality is strict because seller 2 has a mass point at $\hat{A}$, which implies that $d\Phi_{m(1)}(\hat{A}) > 0$.

Therefore, by (A29) and (A30),

$$d\Pi_1(\hat{A}+\varepsilon) > d\Pi_1(\hat{A}-\varepsilon),$$

which implies that seller 1 should deviate by shifting its mass point above $\hat{A}$, which is a contradiction. [ ]

Proof of Proposition 2.

The first part of this proof essentially follows the proof of McAfee's (1994) Theorem 3. Substituting (5) in Lemma 1(h), the revenue of seller $i = 2,...,n$,

$$R_i = vA_i\prod_{j=2}^{n}\left[1-\frac{A_j}{L}\right] = vA_i\left[1-\frac{A_i}{L}\right]\prod_{j\neq 1,i}\left[1-\frac{A_j}{L}\right].$$

Looking forward from the first stage, the seller's profit

$$\Pi_i(A_i) = vA_i\left[1-\frac{A_i}{L}\right]\prod_{j\neq 1,i}\left[1-\frac{A_j}{L}\right] - c\frac{\ln(1-A_i/L)}{\ln(1-1/L)}. \qquad \text{(A31)}$$

Seller *i* maximizes profit (A31) subject to the constraint that its maximum be on the concave part of its revenue function,

$$A_i^* < \max_{j\neq i}(A_j). \qquad \text{(A32)}$$

By contrast with the other sellers, seller 1 maximizes profit

$$\Pi_1(A_1) = vA_1\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right] - c\frac{\ln(1-A_1/L)}{\ln(1-1/L)} \qquad \text{(A33)}$$

subject to the constraint that its maximum be on the *linear* part of its revenue function,

$$A_1^* \geq \max_{j \neq 1}(A_j). \tag{A34}$$

Below, we show that, in equilibrium, the constraints (A32) and (A34) do not bind. Accordingly, seller *i*'s choice of customer acquisitions is characterized by the first order condition,

$$\left[1-\frac{A_i^*}{L}\right]\left[1-\frac{2A_i^*}{L}\right]\prod_{j\neq 1,i}\left[1-\frac{A_j}{L}\right]=X \ , \tag{A35}$$

after substituting from (3) and simplifying, for all $i=2,...,n$. In particular, for seller l,

$$\left[1-\frac{A_l^*}{L}\right]\left[1-\frac{2A_l^*}{L}\right]\prod_{j\neq 1,l}\left[1-\frac{A_j}{L}\right]=X \ . \tag{A36}$$

Equating (A35) with (A36) and simplifying, we have $A_i^* = A_l^* = A^*$, say, for all $i, l > 1$, which proves (7).

Further, seller 1's choice of customer acquisitions is characterized by the first order condition,

$$\left[1-\frac{A_1^*}{L}\right]\prod_{j\neq 1}\left[1-\frac{A_j}{L}\right]=X \ , \tag{A37}$$

after substituting from (3) and simplifying. Substituting $A_j = A^*$ in (A37) and comparing with (7), we have $A_1^* = 2A^*$.

By Lemma 1, the sellers $i=2,..,n$ will have an identical pricing strategy, say $F(p)$, with support, $S=[\underline{p},v)$, and no mass point. Consider seller 1. By Lemma 1(h) and (4), with $A_j = A^*$ and $F_j = F$, for $i=2,..,n$,

$$R_1 = p\left[1-\frac{A^*}{L}F(p)\right]^{n-1}A_1 = v\left[1-\frac{A^*}{L}\right]^{n-1}A_1,$$

which simplifies to (8). Similarly, for any seller $i=2,..,n$, (4) and (6) imply

$$R_i = pA^*\left[1-\frac{A_1}{L}F_1(p)\right]\left[1-\frac{A^*}{L}F(p)\right]^{n-2} = vA^*\left[1-\frac{A^*}{L}\right]^{n-1}.$$

Substituting from (8), this simplifies to

$$F_1(p)=\frac{L}{A_1}\left\{1-\left[1-\frac{A^*}{L}\right]\left[\frac{v}{p}\right]^{\frac{1}{n-1}}\right\}=\frac{1}{2}F(p). \tag{A38}$$

Finally, substituting $p=v$ in (A38), seller 1's mass point at v has weight,

$$1-F_1(v)=1-\frac{A^*}{A_1}=\frac{1}{2}, \tag{A39}$$

which completes the derivations of (7) to (9).

It remains to prove that the constraints (A32) and (A34) do not bind, and hence $A_i = A^*$ is the global optimum for seller $i = 2,...,n$, and $A_1 = 2A^*$ is the global optimum for seller 1. By Lemma 1(h), the revenue for any seller *k* is:

$$R_k(A_k)=\begin{cases} vA_k\left[1-\frac{A_k}{L}\right]\prod_{j\neq 1,k}\left[1-\frac{A_j}{L}\right] & \text{if } A_k<\max(A_j) \\ vA_k\prod_{j\neq k}\left[1-\frac{A_j}{L}\right] & \text{if } A_k=\max(A_j) \end{cases} \tag{A40}$$

Consider seller $i = 2,...,n$. For $A_i < 2A^*$, its profit is

$$vA_i\left[1-\frac{A_i}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-A_i/L)}{\ln(1-1/L)}, \tag{A41}$$

which, by (A31) with $A_j = A^*$, $j \neq 1, i$, is maximized at $A_i = A^*$. By (A40), if seller *i* deviates to $A_i \geq 2A^*$, its profit would be

$$vA_i\left[1-\frac{2A^*}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-A_i/L)}{\ln(1-1/L)}, \tag{A42}$$

which, referring to Figure A1, is represented graphically by the vertical distance between the broken straight line and the solid cost curve.

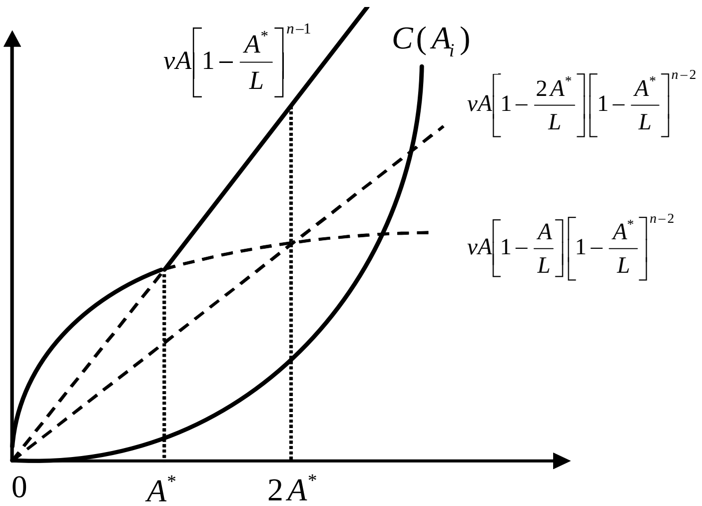

Figure A1.

By (A37), with $A_j = A^*$, $j \neq 1$, for $A_1 \geq A^*$, seller 1's profit is

$$vA_1\left[1-\frac{A^*}{L}\right]^{n-1}-\frac{c\ln(1-A_1/L)}{\ln(1-1/L)}, \tag{A43}$$

which, in Figure A1, is represented graphically by the vertical distance between the solid straight line and the solid cost curve. Profit functions (A42) and (A43) differ in the average revenue: specifically, since

$$\left[1-\frac{A^*}{L}\right]>\left[1-\frac{2A^*}{L}\right],$$

the average revenue in (A43) is higher than that in (A42). Graphically, the solid straight line is steeper than the broken straight line. Now, seller 1 maximizes its profit at $A_1=2A^*$, which means that the vertical distance between the solid straight line and the solid cost curve is maximized at $A_1=2A^*$. Since the broken straight line is gentler, the vertical distance between the broken straight line and the solid cost curve must be maximized at some $A_i<2A^*$. Hence, for $A_i\geq 2A^*$,

$$\begin{aligned} &vA_i\left[1-\frac{2A^*}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-A_i/L)}{\ln(1-1/L)} \\ &\qquad\leq v[2A^*]\left[1-\frac{2A^*}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-2A^*/L)}{\ln(1-1/L)}. \end{aligned} \tag{A44}$$

In turn, referring to Figure A1, for $A_i<2A^*$,

$$\begin{aligned} &v[2A^*]\left[1-\frac{2A^*}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-2A^*/L)}{\ln(1-1/L)} \\ &\qquad\leq vA_i\left[1-\frac{A_i}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-A_i/L)}{\ln(1-1/L)}. \end{aligned} \tag{A45}$$

Accordingly, by (A44) and (A45), if seller *i* deviates to $A_i\geq 2A^*$, its profit would be less than that with $A_i<2A^*$. Thus, seller *i* will not deviate to $A_i\geq 2A^*$.

Next, consider seller 1. By (A43), for $A_1\geq A^*$, seller 1 maximizes its profit at $A_1=2A^*$. By (A40), if seller 1 deviates to $A_1<A^*$, its profit would be

$$vA_i\left[1-\frac{A_i}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}-\frac{c\ln(1-A_i/L)}{\ln(1-1/L)},$$

which is identical to (A41), and hence is maximized at $A_1=A^*$. But, from Figure A1, seller 1's revenue (the solid straight line) is continuous up to $A_1=A^*$, but its optimal profit is attained at $A_1=2A^*$. Hence, seller 1's profit with $A_1=A^*$ must be less than its profit with $A_1=2A^*$. Accordingly, seller 1 will not deviate to any

$A_1 < A^*$, and the (asymmetric) equilibrium characterized in Proposition 2 is both unique and stable. [ ]

Proof of Proposition 3

For any price p within the support of its pricing strategy, seller i's expected profit is

$$\Pi_i(p, A_i) = E\{R_i(p, A_i)\} - C(A_i)$$
$$= E\left\{\prod_{j \neq i}\left[1 - \frac{A_j}{L} F_j(p)\right]\right\} pA_i - c\ln\left(1 - \frac{A_i}{L}\right) \Big/ \ln\left(1 - \frac{1}{L}\right). \tag{A46}$$

Consider the right-hand side of (A46): given the choices of other sellers, the first term (expected revenue) is linear in A_i, whereas the second term (cost) is strictly convex in A_i. Hence, the sum is strictly concave in A_i, which means that seller i will not randomize acquisitions.[2] [ ]

Proof of Proposition 4

By (4), the profit of seller $i = 1,...,n$, at any price p within the support of its pricing strategy, is

$$\Pi_i(A_i, F_i) = pA_i \prod_{j \neq i}\left[1 - \frac{A_j}{L} F_j(p)\right] - \frac{c\ln(1 - A_i/L)}{\ln(1 - 1/L)}. \tag{A47}$$

Since sellers acquire customers and set prices simultaneously, the other sellers' pricing strategies, $F_j(\cdot)$, do not depend on seller i's acquisition, A_i. Hence, the profit maximization problem of seller i is globally concave.

Now, in a randomized-strategy equilibrium, seller i must earn the same profit for all $p \in [\underline{p}, v]$. In particular, at $p = \underline{p}$, by Lemma 1, $F_j(\underline{p}) = 0$, all $j = 1,...,n$. Substituting into (A47), seller i's profit simplifies to

$$\Pi_i(A_i, F_i) = \underline{p}A_i - \frac{c\ln(1 - A_i/L)}{\ln(1 - 1/L)}. \tag{A48}$$

The solution is characterized by the first-order condition

[2] Note that this proof does not apply to the case of sequential action, because there the prices depend on the customer acquisitions.

$$\left[1-\frac{A_i}{L}\right]\underline{p} = vX \ , \tag{A49}$$

after substituting from (3). In particular, for any other seller *l*,

$$\left[1-\frac{A_l}{L}\right]\underline{p} = vX \ . \tag{A50}$$

Equating (A49) and (A50) and simplifying, we have $A_i = A_l = A_s$, say. Hence, the acquisitions are symmetric. By Lemma 1, when acquisitions are symmetric, the pricing strategies will not include any mass points. Hence, at $p = v$, $F_j(v) = 1$, all $j = 1,...,n$. Since the revenue of seller *i* at $p = \underline{p}$ and $p = v$ must be the same in a randomized-strategy equilibrium, and the cost of acquisition does not depend on p, seller *i*'s revenue

$$pA_i \prod_{j \neq i}\left[1-\frac{A_j}{L}F_j(p)\right] = vA_i\left[1-\frac{A_s}{L}\right]^{n-1} = \underline{p}A_i = A_i\frac{vX}{1-\dfrac{A_s}{L}}, \tag{A51}$$

after substituting from (A49). This proves (11).

Now, by Lemma 1, since acquisitions are symmetric, the pricing strategies are also symmetric, with $F_j = F_s$, say, for all $j = 1,...,n$. Substituting $A_i = A_s$ and $F_j = F_s$ in the first equality of (A51), we have

$$\left[1-\frac{A_s}{L}F_s(p)\right]^{n-1} = \frac{v}{p}\left[1-\frac{A_s}{L}\right]^{n-1}. \tag{A52}$$

After simplifying and using (11), this yields (12). Finally, at $p = \underline{p}$, $F_s(\underline{p}) = 0$. Substituting in (12) yields (13).

Next, we show that sellers would not deviate from this equilibrium. In equilibrium, by (A47), (A49), and (A51), for every seller $i = 1,...,n$,

$$\Pi_i(A_s, F_s) = vA_s\left[1-\frac{A_s}{L}\right]^{n-1} - \frac{c\ln(1-A_s/L)}{\ln(1-1/L)} \geq vA_i\left[1-\frac{A_s}{L}\right]^{n-1} - \frac{c\ln(1-A_i/L)}{\ln(1-1/L)}, \tag{A53}$$

for all A_i. Using (5), this simplifies to

$$\Pi_i(A_s, F_s) = \underline{p}A_s - \frac{c\ln(1-A_s/L)}{\ln(1-1/L)} \geq \underline{p}A_i - \frac{c\ln(1-A_i/L)}{\ln(1-1/L)}. \tag{A54}$$

We first consider a deterministic deviation to some price p and acquisition A_i', and show that such a deviation will not raise profit. Specifically, suppose seller *i* deviates to a strategy (A_i', p), where $A_i' \neq A_s$.

(i) $p < \underline{p}$. In this case, seller i's price is lower than that of every other seller. So, it would sell to every customer that it acquires; hence its expected profit from the deviation would be

$$\Pi(A_i', p) = pA_i' - \frac{c\ln(1-A_i'/L)}{\ln(1-1/L)} < \underline{p}A_i' - \frac{c\ln(1-A_i'/L)}{\ln(1-1/L)} \le \underline{p}A_s - \frac{c\ln(1-A_s/L)}{\ln(1-1/L)} = \Pi_i(A_s, F_s),$$

where the second inequality is implied by (A54). Hence, the seller would prefer not to deviate.

(ii) $\underline{p} \le p \le v$. In this case, seller i sells to those customers who are not acquired by other sellers, and those customers who are acquired by other seller(s) if seller i's price is lowest. By (A47) and (A52), seller i's expected profit from the deviation would be

$$\begin{aligned}\Pi(A_i', p) &= pA_i' \cdot \frac{v}{p}\left[1 - \frac{A_s}{L}\right]^{n-1} - \frac{c\ln(1-A_i'/L)}{\ln(1-1/L)} \\ &= vA_i'\left[1 - \frac{A_s}{L}\right]^{n-1} - \frac{c\ln(1-A_i'/L)}{\ln(1-1/L)} \\ &\le vA_s\left[1 - \frac{A_s}{L}\right]^{n-1} - \frac{c\ln(1-A_s/L)}{\ln(1-1/L)} = \Pi_i(A_s, F_s),\end{aligned}$$

where the inequality is implied by (A53). Hence, the seller would prefer not to deviate.

Finally, the profit from any deviation, (A_i', F_i), where the pricing strategy is randomized, cannot exceed the expectation over the profits from the various (A_i', p) with deterministic prices. Since each of the various deviations (A_i', p) does not increase profit, the deviation with a randomized pricing strategy would also not increase profit. [ ]

Proof of Table 1

(a) We prove the results with respect to c and v through X.

Proof: Differentiating (7) with respect to X and evaluating at $A = A^*$, we have

$$[n-1]\left[1 - \frac{2A^*}{L}\right]\left[1 - \frac{A^*}{L}\right]^{n-2}\left[-\frac{1}{L}\frac{dA}{dX}\right] + \left[1 - \frac{A^*}{L}\right]^{n-1}\left[-\frac{2}{L}\frac{dA}{dX}\right] = 1,$$

or

$$\left\{[n-1]\left[1 - \frac{2A^*}{L}\right] + 2\left[1 - \frac{A^*}{L}\right]\right\}\left[1 - \frac{A^*}{L}\right]^{n-2}\frac{dA}{dX} = -L.$$

The coefficient on the left-hand side is positive, hence we conclude that

$$\frac{dA^*}{dX} < 0. \tag{A55}$$

By (7) and (9),

$$\underline{p} = \left[1 - \frac{A^*}{L}\right]^{n-1} v = \left[1 - \frac{2A^*}{L}\right]^{-1} Xv. \tag{A56}$$

Accordingly,

$$\frac{d}{dX}(\underline{p}) > 0.$$

Further, by (8),

$$F(p) = -\frac{L}{A^*}\left\{\left[\frac{v}{p}\right]^{\frac{1}{n-1}} - 1\right\} + \left[\frac{v}{p}\right]^{\frac{1}{n-1}}, \tag{A57}$$

hence by (A39), $dF/dX < 0$ and, by Proposition 2,

$$\frac{dF_1}{dX} = \frac{1}{2}\frac{dF}{dX} < 0.$$

The results with respect to c and v follow from (3), by which X is increasing in c and decreasing in v. The proof for the symmetric equilibrium is similar.

(b) Results with respect to increase in *L*.

Proof: Differentiating (7) with respect to L and evaluating at $A = A^*$, we have

$$-2\left[1 - \frac{A^*}{L}\right]^{n-1}\left[-\frac{A^*}{L^2} + \frac{1}{L}\frac{dA^*}{dL}\right] - [n-1]\left[1 - \frac{2A^*}{L}\right]\left[1 - \frac{A^*}{L}\right]^{n-2}\left[-\frac{A^*}{L^2} + \frac{1}{L}\frac{dA^*}{dL}\right]$$

$$= \frac{dX}{dL} = -\frac{X}{L}\left\{1 + \frac{1}{[L-1]\ln(1 - 1/L)}\right\}. \tag{A58}$$

Simplifying, dA^*/dL has the sign of

$$\frac{A^*}{L}\left\{2\left[1 - \frac{A^*}{L}\right]^{n-1} + [n-1]\left[1 - \frac{2A^*}{L}\right]\left[1 - \frac{A^*}{L}\right]^{n-2}\right\} + X\left\{1 + \frac{1}{[L-1]\ln(1 - 1/L)}\right\}. \tag{A59}$$

Using (7), we can rewrite (A59) as

$$\frac{2A^*}{L}\left[1 - \frac{2A^*}{L}\right]^{-1} X + [n-1]\frac{A^*}{L}\left[1 - \frac{A^*}{L}\right]^{-1} X + X\left\{1 + \frac{1}{[L-1]\ln(1 - 1/L)}\right\}$$

$$> [n+1]\frac{A^*}{L}\left[1 - \frac{A^*}{L}\right]^{-1} X + X\left\{1 + \frac{1}{[L-1]\ln(1 - 1/L)}\right\} = \Omega, \text{ say.} \tag{A60}$$

Note that

$$[n+1]\frac{A^*}{L} > 1 - X.$$

Substituting into (A60), we have

$$\Omega > \left\{ [1-X]\left[1-\frac{A^*}{L}\right]^{-1} + 1 + \frac{1}{[L-1]\ln(1-1/L)} \right\} X$$

$$> \left\{ 2 - X + \frac{1}{[L-1]\ln(1-1/L)} \right\} X$$

$$\approx [1-X]X > 0$$

for sufficiently large L. Hence we conclude that

$$\frac{dA^*}{dL} > 0. \qquad \text{(A61)}$$

Now differentiate (9),

$$\frac{d}{dL}(\underline{p}) = [n-1]\frac{1}{L}\left[1-\frac{A^*}{L}\right]^{n-2}\left[\frac{A^*}{L} - \frac{dA^*}{dL}\right]v > 0$$

since by (A42),

$$\frac{dA^*}{dL} < \frac{A^*}{L}.$$

Similarly, differentiate (8),

$$\frac{dF}{dL} = \left\{ \frac{1}{A^*} - \frac{1}{A^*}\left[\frac{v}{p}\right]^{\frac{1}{n-1}} \right\}\left[1 - \frac{L}{A^*}\frac{dA^*}{dL}\right] < 0$$

and

$$\frac{dF_1}{dL} = \frac{1}{2}\frac{dF}{dL} < 0.$$

The proofs for the symmetric equilibrium is similar.

(c) Results with respect to increase in *n*.

Proof: Totally differentiating (7) and evaluating at $A = A^*$, we have

$$\left\{ \left[1-\frac{2A^*}{L}\right]\ln\left(1-\frac{A^*}{L}\right)\left[1-\frac{A^*}{L}\right]^{n-1} \right\} dn$$
$$+ \left\{ [n-1]\left[1-\frac{2A^*}{L}\right]\left[1-\frac{A^*}{L}\right]^{n-2}\left[-\frac{1}{L}\right] - \frac{2}{L}\left[1-\frac{A^*}{L}\right]^{n-1} \right\} dA^* = 0,$$

or

$$\frac{dA^*}{dn} = \frac{\left[1-\frac{A^*}{L}\right]\left[1-\frac{2A^*}{L}\right]\ln\left(1-\frac{A^*}{L}\right)}{\frac{n-1}{L}\left[1-\frac{2A^*}{L}\right] + \frac{2}{L}\left[1-\frac{A^*}{L}\right]} = \frac{\left[1-\frac{A^*}{L}\right]\left[1-\frac{2A^*}{L}\right]\ln\left(1-\frac{A^*}{L}\right)}{\frac{1}{L}\left\{1 + n\left[1-\frac{2A^*}{L}\right]\right\}} < 0, \qquad \text{(A62)}$$

since the numerator is negative and the denominator is positive.

Regarding the impact on the pricing strategy, by (A56),

$$\frac{d}{dn}(\underline{p}) = \frac{d}{dn}\left\{vX\left[1-\frac{2A^*}{L}\right]^{-1}\right\} = \frac{2}{L}Xv\left[1-\frac{2A^*}{L}\right]^{-2}\frac{dA^*}{dn} < 0 \tag{A63}$$

since, by (A62), $dA^*/dn < 0$. Now, by (8) and (9),

$$F(p) = \frac{L}{A^*} - \left[\frac{L}{A^*}-1\right]\left[\frac{v}{p}\right]^{\frac{1}{n-1}} = \frac{L}{A^*}\left\{1-\left[\frac{\underline{p}}{p}\right]^{\frac{1}{n-1}}\right\}. \tag{A64}$$

Denote

$$\Phi(p) = 1-\left[\frac{\underline{p}}{p}\right]^{\frac{1}{n-1}}. \tag{A65}$$

By total differentiation of (A65), we have

$$\begin{aligned}\frac{d\Phi}{dn} &= \frac{\partial\Phi}{\partial n} + \frac{\partial\Phi}{\partial\underline{p}}\frac{d\underline{p}}{dn} = \frac{1}{[n-1]^2}\left[\frac{\underline{p}}{p}\right]^{\frac{1}{n-1}}\ln\left(\frac{\underline{p}}{p}\right) - \frac{1}{n-1}\left[\frac{\underline{p}}{p}\right]^{\frac{1}{n-1}}\frac{1}{\underline{p}}\frac{d\underline{p}}{dn} \\ &= \frac{1}{n-1}\left[\frac{\underline{p}}{p}\right]^{\frac{1}{n-1}}\left\{\frac{1}{n-1}\ln\left(\frac{\underline{p}}{p}\right) - \frac{1}{\underline{p}}\frac{d\underline{p}}{dn}\right\}.\end{aligned} \tag{A66}$$

By (A63), at $p = \underline{p}$, $d\Phi/dn > 0$. When p increases, $d\Phi/dn$ decreases. When $p = v$, (A66) becomes

$$\frac{d\Phi}{dn} = \frac{1}{n-1}\left[1-\frac{A^*}{L}\right]\left\{\ln\left(1-\frac{A^*}{L}\right) - \frac{2}{L}\left[1-\frac{A^*}{L}\right]^{-1}\frac{dA^*}{dn}\right\} < 0.$$

Since L/A^* is always positive, dF/dn (and hence $dF_1/dn = 1/2\ dF/dn$) also changes from positive to negative as p increases from $\underline{p}$ to v. Hence when the number of sellers increases, price becomes more extreme – the sellers have a higher tendency to set either low or high prices. The weight on the mass point at v does not change with n.

Finally, from (8) and (12), it is clear that as $n \to \infty$, both $F(p)$ and $F_s(p)$ equal one, and $F_1(p)$ equals 0.5 at every p. Hence, the price distributions degenerate into a mass point at $\underline{p}$. By (7), (9) and (12), when $n \to \infty$, $\underline{p} = Xv$. [ ]