A R&D Race with Dual Uncertainties: Technical (In)Feasibility and Timing of Innovation

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This paper develops a simple model of a R&D race with two sources of uncertainty about the innovation. The first is about its technical feasibility, and, the second, if the innovation is technically feasible, about its timing. Firms have prior beliefs that the innovation is technically feasible. As the race continues without success, firms become increasingly pessimistic that the innovation is technically feasible, which impacts investment, modeled as feedback strategies. Closed form solutions are developed, which include several earlier models as special cases. It is shown that the equilibrium path of research intensities under technical feasibility uncertainty: (a) lies below that of technical feasibility, (b) may rise or fall over time, and (c) will converge to a 'steady state' level in the long run. Increased rivalry raises innovative activity in the short run, but dampens it in the long run, implying that the equilibrium paths of research intensities for various number of rivals will cross. Allowing for an endogenous market structure: (a) under free entry, a competitive market structure emerges but the race lasts for an instant, and (b) under fixed cost of entry, a finite number of firms participate, where this number is a function of how optimistic firms are that the innovation is technically feasible.

1 Introduction

Stochastic R&D race models are arguably the fulcrum of the economics of innovation[1]. In such models, firms vie with each other to be the first to innovate. The first firm to do so is declared the 'winner' and is awarded a 'prize'[2], which is often interpreted as a patent[3]. A key assumption of these models is that the timing of innovation is uncertain, which is modeled through the probability

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[1]Reinganum (1989) is an excellent overview of the literature.

[2]An example is the X-prize, awarded to the first private space flight (http://www.xprizefoundation.com).

[3]The loser(s) may receive a 'consolation' prize, which may reflect the notion that the winner's patent is imperfect, thereby allowing the loser(s) to innovate around it.

of innovating before a certain date being generated by an exponential distribution with an *a priori* known hazard rate[4]. Choi (1991) points out that this is equivalent to the assumption that the innovation is technically feasible. As such, the obvious question is: how are results from these models impacted by uncertainties regarding *both* technical feasibility *and* timing of innovation?

An example of a R&D race consistent with earlier models of stochastic R&D races was the one organized by Eli Lilly in the 1970s to develop synthetic human insulin based on recombinant DNA technology. It involved four stages, each of which was thought to be 'technically feasible' but with uncertain time of completion (Barese (1991)). As Brandenburger and Krishna (1991) point out, the dynamics of this race is consistent with Harris and Vickers (1987).

On the other hand, consider the race to develop energy through cold fusion. In 1989 two University of Utah scientists, Stanley Pons and Martin Fleischman, announced the results of a single experiment, in which energy had apparently been produced through fusion at room temperature. The announcement was made without verifying the initial experiment because the Utah team was competing with a rival team in Italy, who were rumored to be on the verge of discovering cold fusion. This prompted the Utah team to preempt the Italians by announcing the 'discovery' of cold fusion energy. Following the announcement, a barrage of attempts to reproduce the Pons/Fleischman experiment followed. But despite some claims of success, the majority of attempts to replicate the finding ended in failure. The Pons/Fleischman experiment was discredited and research into cold fusion tapered off.

But despite widespread ridicule of cold fusion, there were some scientists, like Peter Hagelstein (at MIT), who continued to believe that there had been a phenomenon in the Pons/Fleischman experiment[5]. These renegade scientists have maintained steady research into cold fusion and their efforts seem to have paid off: there is tentative evidence suggesting that cold fusion is possible[6] and both the Departments of Defense and Energy are funding cold fusion R&D[7].

The pattern of research in cold fusion– rising at first, falling thereafter and tapering off to a steady level– is inconsistent with existing models of R&D races which have, almost without exception[8], assumed technical feasibility of the innovation. In fact, the only formal treatment of R&D investment in possibly technical feasible races (Malueg & Tsutsui (1997)) does not derive closed form solutions, thereby rendering impossible a comparison with earlier models; it also generates dynamics which are inconsistent with scenarios like cold fusion.

Given the importance of stochastic R&D race models in the economics of innovation[9], the impact from modeling technical infeasibility is surely of con-

[4]As shown in Appendix A and argued below, this is equivalent to the assumption that the instantaneous probability of innovation is governed by a homogeneous Poisson process.

[5]*The Washington Post*, November 21, 2004 and *Wired* magazine, Issue 6.11, November 1998

[6]*The Economist*, April 28th, 2005

[7]http://physicsweb.org/articles/world/12/3/8 and the *New York Times*, March 25, 2004.

[8]These being: Choi (1991) and Malueg & Tsutsui (1996)

[9]For example, stochastic R&D race models are used in the economics of growth- see Aghion & Howitt (1992).

siderable interest. For example, what is the difference in levels and dynamics of research intensity under technical feasibility versus technical infeasibility? Could uncertainty about technical feasibility generate non-monotonic patterns for research intensity? What are the implications for econometric studies? How will increased rivalry affect research intensity when there is uncertainty about technical feasibility? In turn, what implications does this have for antitrust policy?

That stochastic R&D race models implicitly assume technical feasibility was first raised by Choi (1991)[10], who models leader-follower dynamics under innovation uncertainty and does not, by construction, analyze research intensities levels. Instead, the focus in Choi (1991) is on reversing the $\epsilon-$preemption result in Fudenberg, Gilbert, Stiglitz, and Tirole (1983), which refers to the idea that once the leader in a deterministic R&D race pulls 'ahead' by an ϵ 'distance'[11], the follower drops out of the race; there are also temperate versions of $\epsilon-$ preemption[12] in which the follower slows down as it lags behind. Choi (1991) shows that with innovation uncertainty, if a firm moves from one stage into the next, it signals that a heretofore thought of impossible innovation is in fact possible. Thus, instead of dropping out or slowing down as $\epsilon-$ preemption predicts, the follower *speeds* up instead[13].

Malueg & Tsutsui (1997) extend Choi (1991) by developing a model of research intensities with hazard rate parameter uncertainty. At the outset, firms have identical prior beliefs on the probability that the innovation is technically possible. As the race continues without success, firms become increasingly pessimistic that the innovation exists and accordingly adjust research intensities, which are modeled as feedback strategies. Their model relies on numerical simulations and therefore does not generate closed form solutions[14]. Moreover, their model employs control and state variables which differ from earlier models. As such, comparing Malueg & Tsutsui (1997) with earlier models of stochastic R&D races is impossible. The question still remains: how are results from earlier models of stochastic R&D races impacted by uncertainty surrounding technical feasibility and the timing of innovation?

This model seeks to fill these gaps in the literature. As in Malueg & Tsutsui (1997), the model incorporates two sources of uncertainty. The first is whether the innovation is technically feasible. The second is that if the innovation is technically feasible, there is uncertainty about the timing of innovation. Closed

[10] Earlier models have recognized innovation uncertainty. For example, Reinganum (1982) does this by imposing a terminal date on the R&D race which, "may be regarded as a time at which the firms abandon the project entirely if they haven't yet perfected the innovation. That is, if a firm fails to succeed by [the terminal date], it becomes convinced that the innovation is infeasible" (N2, p. 673 in Reinganum (1982).

[11] Where the 'distance' is a function of various parameters– see Fudenberg, Gilbert, Stiglitz, and Tirole (1983).

[12] See for example: Grossman and Shapiro (1987), Harris and Vickers (1987), and Lippman and McCardle (1987).

[13] Doraszleski (2003) develops a model with knowledge accumulation in which the follower may speed up as the leader pulls ahead.

[14] Closed form solutions are derived in Malueg and Tsutsui (1997) only in the special case of no discounting.

form results are developed, which includes Loury (1979), Lee & Wilde (1980), and Reinganum (1982) amongst others, as special cases. As such, besides developing new results, the model permits a comparison of results from technical infeasibility (hereafter 'innovation uncertainty') with earlier models of technical feasibility (hereafter 'innovation certainty').

The main results are as follows. Research intensity under innovation uncertainty is the product of the posterior belief that the innovation is possible, and, the research intensity under innovation certainty. This implies that the equilibrium path of research intensities under innovation uncertainty lies below the equilibrium path of research intensities under innovation certainty. In contrast to the innovation certainty 'memoryless' R&D races, where research intensities do not depend on past stock of knowledge, research intensities under innovation uncertainty are a function of knowledge stock[15].

Whereas the equilibrium path of research intensities under innovation certainty rises[16] (or remains constant[17]), over time, the equilibrium path of research intensities under innovation uncertainty may rise *or* fall. It also shown that, regardless of the degree of initial pessimism, if the race continues long enough without success, the equilibrium path of research intensities under innovation uncertainty, converges to a "steady state" level which depends inversely with increased rivalry. Put another way, increased rivalry dampens innovative activity in the long run. This inverse relationship between rivalry and innovative activity is at variance with earlier models (such as Lee & Wilde (1980) and Reinganum (1982)), in which increased rivalry spurs innovative activity.

In the short run, however, the link between increased rivalry and innovative activity is complex. This is because research intensity under innovation uncertainty is equal to research intensity under innovation certainty times the posterior belief that the innovation is impossible. With increased rivalry, research intensity under innovation certainty rises, but at the same time, firms become pessimistic at a faster rate. Thus, whether research intensity under innovation uncertainty rises or falls due to increased rivalry depends on which effect dominates. With increased rivalry, it is shown that the equilibrium paths will cross, confirming and formalizing the same result in Malueg & Tsutsui (1997).

The closed form solutions permit an endogenous determination of market structure. Allowing for a free entry condition results in a perfectly competitive market structure. But with perfect competition, innovative activity lasts only for an instant. This is because the rate at which firms become pessimistic is a function of the number of rivals: with an infinite number of rivals, if there is no discovery at the very outset, firms become convinced in an instant that the innovation is technically infeasible. The race therefore is a flash of innovative activity.

Allowing for a fixed cost of entry results in a finite number of firms. This

[15] Doraszleski (2003) also has a model where research intensities depend upon knowledge stocks.

[16] As in Reinganum (1982)

[17] As in Lee & Wilde (1980)

number, however, depends on the prior belief that the innovation is technically possible. It is shown that feasible entry requires that firms be sufficiently confident about the technical feasibility of the innovation. It is also shown that for a given set of parameters, the number of rivals in an innovation uncertainty R&D race is fewer than the number of rivals in an innovation certain R&D race.

This article is organized as follows: section 2 contains the model, section 3 contains the equilibrium analysis, section 4 analyzes the link between increased rivalry and innovative activity, while section 5 discusses endogenous market structure. The paper concludes in section 6.

2 Model

In many respects, the model parallels Reinganum (1982) and Malueg & Tsutsui (1997). Let $n \geq 2$ be an exogenously given finite number of firms, competing in continuous time over a single stage R&D race. The race ends at date T; for now, this date is exogenous, but will be determined endogenously below. The winner of the race receives a perfect patent with constant value W, while losers receive nothing.

□ **Investment Strategies.** Firms choose R&D investment as a function of the date and past observable cumulative R&D expenditure by all firms. Investment therefore takes the form of closed loop feedback strategies[18]. Vectors are denoted in bold. Let $x_i(t, \mathbf{X}(t))$ denote firm i's R&D investment flow at time t, where $\mathbf{X}(t) = (X_1(t), ..., X_n(t))$ is the observable vector of cumulative expenditure by all n firms till date t. For simplicity denote $x_i(t, \mathbf{X}(t))$ by $x_i(t)$.

The investment flows will be derived by solving the dynamic programming problem facing each firm and are chosen for arbitrary pairs of date, t, and history of cumulative R&D investments by all firms till date t, $\mathbf{X}(t)$. Firms take into account the impact of this investment flow on future actions until the terminal date T.

□ **Innovation: Discovery Process and Beliefs.** Let t_i denote the time at which firm i discovers the innovation (if, of course, it is technically feasible). Let the instantaneous probability that firm i innovates over a time interval $[t, t+s]$ be governed by a non-homogeneous Poisson process[19]:

$$P(t_i \in [t, t+s)) = \Psi[2x_i(t)]^{1/2} \ s + o(s) \tag{1}$$

where a function is $o(s)$ if $\lim_{s\to 0} f(s)/(s) = 0$. Note that the time dependent arrival rate of this process is $\Psi[2x_i(t)]^{1/2}$.

Following Choi (1991) and Malueg & Tsutsui (1997), innovation uncertainty is modeled by assuming that Ψ can take either of two values[20]:

$$\Psi = \{0, \lambda > 0\}$$

[18] Fudenberg & Tirole (1989) contains a discussion of open vs. closed loop strategies.

[19] See appendix A for a discussion of homogeneous vs. non-homogeneous Poisson processes.

[20] Malueg and Tsutsui (1997) have a version in which Ψ takes a range of values.

If $\Psi = 0$, then regardless of R&D expenditure, the probability of discovery is zero, which is labeled innovation uncertainty. If $\Psi = \lambda > 0$, the innovation exists, which is labeled innovation certainty.

At the outset of the race, firms have uniform prior beliefs that with probability p_0 there is innovation uncertainty ($\Psi = 0$), and with probability $(1 - p_0)$ there is innovation certainty ($\Psi = \lambda > 0$). As the race continues without success, firms use Bayesian updating to attach a greater probability that the innovation is impossible.

□ **Payoff.** Denote the vector of R&D investment flows of all firms as $\mathbf{x(t)}$, where $\mathbf{x(t)} = (x_1(t), .., x_n(t))$. Firm i's payoff at time 0 is:

$$J_i(\mathbf{x(t)}) = \int_0^T [P(t_i = t, t_j > t, j \neq i)\ W - P(t_1 > t, ..., t_n > t)e^{-rt}x_i(t)]dt$$

The first term in the payoff represents the prize W if firm i wins the race at time $t_i = t$, which happens with probability $P(t_i = t, , t_j > t, j \neq i)$. The second term represents the investment expenditure incurred through time t, so long as no firm has succeeded until then, which happens with probability $P(t_1 > t, ..., t_n > t)$. Appendix A derives the expressions for $P(t_i = t, t_j > t, j \neq i)$ and $P(t_1 > t, ..., t_n > t)$ from the non-homogeneous Poisson process governing innovation in this race– with these expressions and denoting $\sum_{i=1}^n$ by $\sum_k$, the payoff to firm i is:

$$\begin{aligned} J_i(\mathbf{x(t)}) &= \int_0^T [(1 - p_0)\lambda[2x_i(t)]^{1/2} e^{-\lambda \sum_k \int_0^t [2x_i(s)]^{1/2} ds}] W dt \\ &- \int_0^T [p_0 + (1 - p_0) \prod_{i=1}^n e^{-\lambda \int_0^t [2x_i(s)]^{1/2} ds}] e^{-rt} x_i(t) dt \end{aligned}$$

□ **Research and Knowledge Intensities.** The model becomes tractable if instead of choosing R&D investment flows as a function of date and cumulative expenditure, firms choose "knowledge intensities" as a function of date t and "knowledge stock" at t. Define the knowledge intensity of firm i at time t as:

$$u_i(t, \mathbf{z(t)}) \equiv [2x_i(t)]^{1/2}$$

where the knowledge stock vector, $\mathbf{z(t)} = (z_1(t), .., z_n(t))$ is the knowledge stock of all firms at date t. Denoting $u_i(t, \mathbf{z(t)})$ with $u_i(t)$, the knowledge stock is defined as the cumulative knowledge until time t:

$$z_i(t) \equiv \int_0^t u_i(s) ds$$

Note that in Equation 1, $u_i(t, \mathbf{z})$ is the time dependent component of the non-homogeneous Poisson process[21]. The payoff to firm i in terms of knowledge intensities and stock is:

[21] That is: $P(t_i \in [t, t + s)) = \Psi u_i(t, \mathbf{z})s + o(s)$.

$$J_i(\mathbf{u}) = \int_0^T W(1-p_0)\lambda e^{-\lambda \sum_k z_k(t)} u_i(t)\ dt \qquad (2)$$
$$- \int_0^T e^{-rt}[p_0 + (1-p_0)e^{-\lambda \sum_k z_k(t)}] \frac{u_i^2(t,\mathbf{z})}{2}\ dt$$

where $\mathbf{u} = (u_1(t), u_2(t), .., u_n(t))$. As in Reinganum (1982), integrate-by-parts the first term of the payoff function and impose the initial condition $z(0) = 0$ to obtain:

$$J_i(\mathbf{u}) = \int_0^T W(1-p_0)(1-e^{-\lambda z_i(t)})\lambda e^{-\lambda \sum_{k\neq i} z_k(t)} \sum_{k\neq i} u_k(t)\ dt$$
$$- \int_0^T e^{-rt}(p_0 + (1-p_0)e^{-\lambda \sum_k z_k(t)}) \frac{u_i^2(t,\mathbf{z})}{2}\ dt$$
$$+ W(1-p_0)(1-e^{-\lambda z_i(T)})e^{-\lambda \sum_{k\neq i} z_k(T)}$$

□ **Dynamic Program.** The value to firm i ($\forall i = 1, .., n$) of behaving optimally from any arbitrary date t and knowledge stock z pair is:

$$V_i(t,\mathbf{z}) = \int_t^T W(1-p_0)(1-e^{-\lambda z_i(s)})\lambda e^{-\lambda \sum_{k\neq i} z_k(s)} \sum_{k\neq i} u_k(s)\ ds$$
$$- \int_t^T e^{-rs}(p_0 + (1-p_0)e^{-\lambda \sum_k z_k(s)}) \frac{u_i^2(s,\mathbf{z})}{2}\ ds \qquad (3)$$
$$+ W(1-p_0)(1-e^{-\lambda z_i(T)})e^{-\lambda \sum_{k\neq i} z_k(T)}$$
$$\ni \dot{z}_i(t,\mathbf{z}) = u_i(t,\mathbf{z}(\mathbf{t})),\ \mathbf{z}(\mathbf{t}) = \mathbf{z},\ p_0 \in [0,1]$$

where firm i chooses its knowledge intensity at each instant and the exit date, T.
The dynamic programming problem is handled in two steps. First, the terminal date is taken as given and the value function is derived, which leads to knowledge intensities as a function of date and knowledge stocks. Second, the exit date T is chosen to maximize the value function[22].

3 Equilibrium Analysis

□ **Value Function.** The first step entails obtaining the Hamiltonian Jacobi Bellman (HJB) equations, which yield the first order condition for $u_i(t,\mathbf{z})$, which are inserted back in the HJB equations. A guess is made for the value function,

[22]This two step technique for dynamic programming problems with endogenous exit dates is described in Dixit (1990) and has been used in Grossman & Shapiro (1986).

which is then used in the HJB equations to it, which allows the derivation of the expected payoff and knowledge intensities[23].

Denote $V_i(t, \mathbf{z(t)})$ in equation 3 by $V_i(t)$ to save notation. The HJB equations are for firms $i = 1, \ldots, n$:

$$\begin{aligned} -(V_i)_t(t) &= \max_{u_i(t,\mathbf{z})} [(V_i)_{z_i} u_i(t,\mathbf{z}) + \sum_{k \neq i} (V_i)_{z_j} u_j^*(t,\mathbf{z}) \\ & +W(1-p_0)(1-e^{-\lambda z_i})\lambda e^{-\lambda \sum_{k\neq i} z_k} \sum_{k \neq i} u_k(t,\mathbf{z}) + \\ & -e^{-rt}(p_0 + (1-p_0)e^{-\lambda \sum_k z_k(t)}) \frac{u_i^2(t,\mathbf{z})}{2} \end{aligned} \tag{4}$$

where the subscript to (V_i) denotes its partial derivative with respect to that argument. By the conditions in Reinganum (1982), $u_1^*, \ldots, u_n^*$ must satisfy the system of equations above for $i = 1, \ldots, n$. The first order condition for Equation 4 yields[24]:

$$(p_0 + (1-p_0)e^{-\lambda \sum_k z_k})e^{-rt} u_i(t,\mathbf{z}) = (V_i)_{z_i}$$

Define $B \equiv (p_0 + (1-p_0))e^{-\lambda \sum_k z_k}$. The expression above becomes: $u_i(t,\mathbf{z}) = e^{rt} B^{-1} (V_i)_{z_i}$ which implies that:

$$u_j(t,\mathbf{z}) = e^{rt} B^{-1} (V_i)_{z_j} \tag{5}$$

which substituting into equation 4 yields:

$$\begin{aligned} BV_t &+ e^{rt}/2((V_i)_{z_i})^2 + e^{rt} \sum_{j \neq i} (V_i)_{z_j} V_{z_j}^j \\ &+ e^{rt} W(1-p_0)(e^{-\lambda \sum_{k\neq i} z_k} - e^{-\lambda \sum_k z_k}) \sum_{k \neq i} \lambda V_{z_j}^j = 0 \end{aligned} \tag{6}$$

The value function from Equation 3 evaluated at T is:

$$V_i(T) = W(1-p_0)e^{-\lambda \sum_{k\neq i} z_k(T)} - W(1-p_0)e^{-\lambda \sum_k z_k(T)}$$

which suggests the following guess for the value function $V_i(t, z)$:

$$V_i(t) = a(t)e^{-\lambda \sum_{k\neq i} z_k} + b(t)e^{-\lambda \sum_k z_k} \tag{7}$$

for $i = 1, \ldots, n$, and where $a(t), b(t)$ are time varying functions to be determined. The value function at the terminal date T is:

$$a(T) = W(1-p_0), b(T) = -W(1-p_0) \tag{8}$$

[23] This is the same technique in Reinganum (1982).

[24] Assuming an interior solution.

Equation 7 then implies:

$$
\begin{aligned}
(V_i)_{z_i}(t) &= -\lambda b(t) e^{-\lambda \sum_k z_k} \\
(V_i)_{z_j}(t) &= -\lambda b(t) e^{-\lambda \sum_k z_k} - a(t) e^{-\lambda \sum_{k \neq i} z_k} \\
V^j(t) &= b(t) e^{-\lambda \sum_k z_k} + a(t) e^{-\lambda \sum_{k \neq j} z_k} \\
V^j_{z_j}(t) &= -\lambda b(t) e^{-\lambda \sum_k z_k} \\
V_t(t) &= \dot{b}(t) e^{-\lambda \sum_k z_k} + \dot{a}(t) e^{-\lambda \sum_{k \neq i} z_k}
\end{aligned}
$$

Inserting the expressions above into Equation 6 yields the following differential equations and terminal conditions:

$$
\begin{aligned}
\dot{b}(t) &+ (b(t))^2 \lambda^2 e^{rt} \frac{2n-1}{2(1-p_0)} + b(t)(n-1) W \lambda^2 e^{rt} = 0, \\
&\ni b(T) = -W(1-p_0) \qquad (9) \\
\dot{a}(t) &+ a(t) b(t)(n-1) \lambda^2 e^{rt} - \lambda^2 b(t) W (n-1) e^{rt} (1-p_0) = 0, \\
&\ni a(T) = W(1-p_0) \qquad (10)
\end{aligned}
$$

Equation 9 along with the terminal conditions yields:

$$
b(t) = \frac{-2(n-1)W(1-p_0)}{(2n-1) - e^{-(n-1)m(t)}} \qquad (11)
$$

where $m(t) \equiv W\lambda^2(e^{rT} - e^{rt})/r$. Substituting Equation 11 in Equation 10:

$$
a(t) = W(1-p_0) \qquad (12)
$$

Inserting equations 11 and 12 into equation 7 yields the value function for a firm facing innovation uncertainty with exogenous exit date T:

$$
V_i(t, \mathbf{z}) = (1-p_0) W \{ e^{-\lambda \sum_{k \neq i} z_k} - \frac{2(n-1) e^{-\lambda \sum_k z_k}}{(2n-1) - e^{-(n-1)m(t)}} \} \qquad (13)
$$

where $m(t) \equiv W\lambda^2(e^{rT} - e^{rt})/r$. Note $V_i^*(t, \mathbf{z})$ is increasing in T which implies that the "optimal" exit time is at infinity[25]. With $\lim_{T \to \infty} V_i(t, \mathbf{z})$, Equation 13 leads to:

Proposition 1 *(a) The equilibrium value function for a firm engaged in innovation uncertain R&D race for arbitrary date t and knowledge stocks z is:*

$$
V_i^*(t, \mathbf{z}) = (1-p_0) W [e^{-\lambda \sum_{k \neq i} z_k} - \frac{2(n-1) e^{-\lambda \sum_k z_k}}{(2n-1)}] \qquad (14)
$$

[25] This results appears to be at odds with Malueg & Tsutsui (1997) who endogenously obtain a finite exit date. In that model, however, firms incur a fixed flow cost at each instant. If the fixed flow cost in their model is zero, as has been assumed in this model, then an infinite exit date also emerges endogenously in Malueg & Tsutsui (1997).

(b) $V_i^*(t,\mathbf{z})$ is increasing in W, decreasing in p_0, λ and, decreasing in n so long as $e^{\lambda z} < 2(n-1)/(2n-1)$.
Proof: Comparative statics of $V_i^*(t,\mathbf{z})$ with respect to W, λ, p_0 follow from straightforward derivatives. For comparative statics with respect to n, assume symmetry, and rearrange $V_i^*(t,\mathbf{z})$ in equation 14:

$$V_i^*(t,\mathbf{z}) = (1-p_0)We^{-\lambda nz}[e^{\lambda z} - \frac{2(n-1)}{(2n-1)}]$$

Taking partial derivatives:

$$\partial V_i^*(t,\mathbf{z})/\partial n = -(1-p_0)We^{-\lambda nz}[e^{\lambda z} - \frac{2(n-1)}{(2n-1)}] - (1-p_0)We^{-\lambda nz}\frac{2}{(2n-1)^2}$$

which will be negative as long as $e^{\lambda z} - \frac{2(n-1)}{(2n-1)} > 0$.

The equilibrium value function for a firm engaged in innovation certainty R&D is derived by substituting p_0 in equation 14:

$$V_i^*(t,\mathbf{z}: p_0 = 0) = W[e^{-\lambda\sum_{k\neq i} z_k} - \frac{2(n-1)e^{-\lambda\sum_k z_k}}{(2n-1)}]$$

which is equal to the value function in Reinganum (1982)[26]. Notice that for all $p \in [0,1), V_i^*(t,\mathbf{z}: p_0 = 0) \geq V_i^*(t,\mathbf{z})$, or that, for a given set of parameters and arbitrary date, state variable pairs, the value of innovation certainty R&D is greater than the value of innovation uncertainty R&D.
□ **Payoff Function.** Evaluating $V_i^*(t,\mathbf{z})$ at $t=0$, and recognizing that $z(0) = 0$, leads to:

Proposition 2 *(a) The expected payoff under innovation uncertainty is:*

$$J^*(\mathbf{u}) = (1-p_0)W - (1-p_0)W\frac{2(n-1)}{(2n-1)}$$

which is interpreted as the expected benefit minus expected cost.
(b) $J^*(\mathbf{u})$ is increasing in W, decreasing in p_0, λ and n.
Proof: Take derivatives.

The expected payoff under innovation certainty is obtained by setting $p_0 = 0$ above:

$$J^*(\mathbf{u}: p_0 = 0) = W - W\frac{2(n-1)}{(2n-1)}$$

which is equal to the payoff function in Reinganum (1982)[27]. As with the value function, $J^*(\mathbf{u}: p_0 = 0) \geq J^*(\mathbf{u}), \forall p_0 \in [0,1]$.

[26] Reinganum (1982), p. 678, for the case $P_L = W, P_F = 0$.
[27] Reinganum (1982), Proposition 1(b), for the case $P_L = W, P_F = 0$.

□ **Knowledge Intensities.** For the derivation of the equilibrium knowledge intensities, it is convenient to first define the posterior belief that the innovation is impossible:

$$p(t) \equiv P(U|t_1 > t, \ldots, t_n > t) = \frac{p_0}{p_0 + (1-p_0)e^{-\lambda \sum_k z_k}}$$

where "U" denotes innovation "uncertainty". From $p(t)$, the expression for the posterior belief that the innovation is possible, is:

$$1 - p(t) = P(C|t_1 > t, \ldots, t_n > t) = [\frac{(1-p_0)}{p_0 + (1-p_0)e^{-\lambda \sum_k z_k}}]e^{-\lambda \sum_k z_k}$$

where "C" indicates innovation "certainty". Recall from equation 5 that $u_i(t, \mathbf{z}) = e^{rt}B^{-1}(V_i)_{z_i}$ which combined with the expression for $V_i(t, z)$ in Equation 14 leads to:

Proposition 3 *(a) The equilibrium knowledge intensity for a firm engaged in innovation uncertainty R&D is:*

$$u_i^*(t, \mathbf{z}) = \dot{z}_i^*(t, \mathbf{z}) = \frac{(1-p_0)e^{-\lambda \sum_k z_k}}{p_0 + (1 - p_0 e^{-\lambda \sum_k z_k}} \frac{e^{rt} 2\lambda(n-1)W}{(2n-1)}$$

which, using the expression for $1 - p(t)$, *is:*

$$u_i^*(t, \mathbf{z}) = (1 - p(t)) \frac{e^{rt} 2\lambda(n-1)W}{(2n-1)} \tag{15}$$

(b) $u_i^*(t, \mathbf{z})$ *is increasing in* W, λ, r. *See discussion on link between increased rivalry and innovative activity for comparative statics with respect to* n.
Proof: Take partial derivatives.

Equation 15 yields a closed-form solution for $u_i^*(t, \mathbf{z})$ and extends Malueg & Tsutsui (1997) which contains a closed form solution only when there is no discounting and, also assumed in this model, zero fixed flow costs[28].
Setting $p_0 = 0$ in equation 15 yields the equilibrium knowledge intensities under innovation certainty:

$$u_i^*(t, \mathbf{z} : p_0 = 0) = \frac{e^{rt} 2\lambda(n-1)W}{(2n-1)} \tag{16}$$

which is identical to the expression for knowledge intensities in Reinganum (1982) (Proposition 1(a), for the case $P_L = W, P_F = 0$)[29].

[28] Equation 14, p. 763 in Malueg & Tsutsui (1997).

[29] Setting $r = 0$ in $u_i^*(t, \mathbf{z} : p_0 = 0)$ yields Lee & Wilde (1980) knowledge intensities. This result derives from Malueg & Tsutsui (1995) who show that Lee & Wilde's (1980) knowledge intensities in their constant flow costs model can also be derived through a feedback strategies differential game ala Reinganum (1982). However, knowledge intensities with feedback equilibrium requires setting $r = 0$ in Malueg & Tsutsui (1997). Then: $u_i^*(t, \mathbf{z} : p_0 = r = 0) = 2(n-1)\lambda W/(2n-1)$ is identical to Malueg & Tsutsui (1995), equation 21, for $r = 0$

The equilibrium path of knowledge intensities under innovation uncertainty lies at or below the equilibrium path of knowledge intensities under innovation certainty, since for all $p_0 \in [0,1], u_i^*(t,\mathbf{z}) \leq u_i^*(t,\mathbf{z}: p_0 = 0)$. In turn, since, $u_i(t,\mathbf{z}(\mathbf{t})) = [2x_i(t)]^{1/2}$, this implies that R&D investment under innovation uncertainty is less than R&D investment under innovation certainty.
Recognizing that if $p_0 = 1$, firms will not engage in R&D (i.e. $u_i^*(t,\mathbf{z}: p_0 = 1) = 0$), $u_i^*(t,\mathbf{z})$ can be re-expressed as:

$$\begin{aligned} u_i^*(t,\mathbf{z}) &= (1-p(t))u_i^*(t,\mathbf{z}: p_0 = 0) + p(t)u_i^*(t,\mathbf{z}: p_0 = 1) \\ u_i^*(t,\mathbf{z}) &= (1-p(t))u_i^*(t,\mathbf{z}: p_0 = 0) \end{aligned}$$

which lends $u_i^*(t,\mathbf{z})$ an "expected value" interpretation. Notice also that knowledge intensities under innovation certainty does not depend on the knowledge stock. This has led to criticism (Doraszleski (2003)) that the 'memoryless' R&D race models (such as Loury (1979), Lee & Wilde (1980) and Reinganum (1982)) are incapable of exhibiting strategic interaction between firms[30]. But examining equation 15 indicates that $u_i^*(t,\mathbf{z})$ *is* a function the knowledge stock z, since $1-p(t)$ is a function of z.
Differentiating $u_i^*(t,\mathbf{z}: p_0 = 0)$ with respect to time t characterizes the dynamics of the innovation certainty equilibrium knowledge intensities path:

$$\dot{u}_i^*(t,\mathbf{z}: p_0 = 0) = e^{rt} 2\lambda r(n-1)W/(2n-1) > 0$$

That is, as the race continues without success and firms believe the innovation is technically feasible, then firms will invest in R&D at a faster rate over time. While the equilibrium path for $u_i^*(t,\mathbf{z}: p_0 = 0)$ rises over time, the same is not true for $u_i^*(t,\mathbf{z})$:

$$\dot{u}_i^*(t,\mathbf{z}) = \theta[(p_0 e^{n\lambda z} + (1-p_0))r - p_0 n\lambda u_i^*(t,\mathbf{z})e^{n\lambda z}] \tag{17}$$

where:

$$\theta = \frac{2\lambda(n-1)We^{rt}}{2n-1}\frac{p(t)(1-p(t))}{p_0} = u_i^*(t,\mathbf{z}: p_0 = 0)\frac{p(t)(1-p(t))}{p_0}$$

which shows that as long as there is innovation uncertainty, the sign of $\dot{u}_i^*(t,\mathbf{z})$ depends on the sign of the terms in the square brackets in equation 17.
Define: $\bar{u} \equiv r/n\lambda$. Then $u_i^*(t,\mathbf{z})$ will rise (or fall with the opposite inequality) if: $\dot{u}_i^*(t,\mathbf{z}) > 0$ if $(p_0 e^{n\lambda z} + (1-p_0))r \geq p_0 n\lambda u_i^*(t,\mathbf{z})e^{n\lambda z}$, which occurs when:

$$\bar{u} \geq p(t)u_i^*(t,\mathbf{z}) \tag{18}$$

The date t^* at which, for any given set of parameters, the equilibrium path of knowledge intensities under innovation uncertainty reaches its peak, is therefore implicitly given by: $\bar{u} = p(t^*)u_i^*(t^*,\mathbf{z})$. $\bar{u}$ will be shown to have a "steady state" interpretation below.

[30] Doraszleski (2003) develops a model in which knowledge intensities depend upon knowledge stocks.

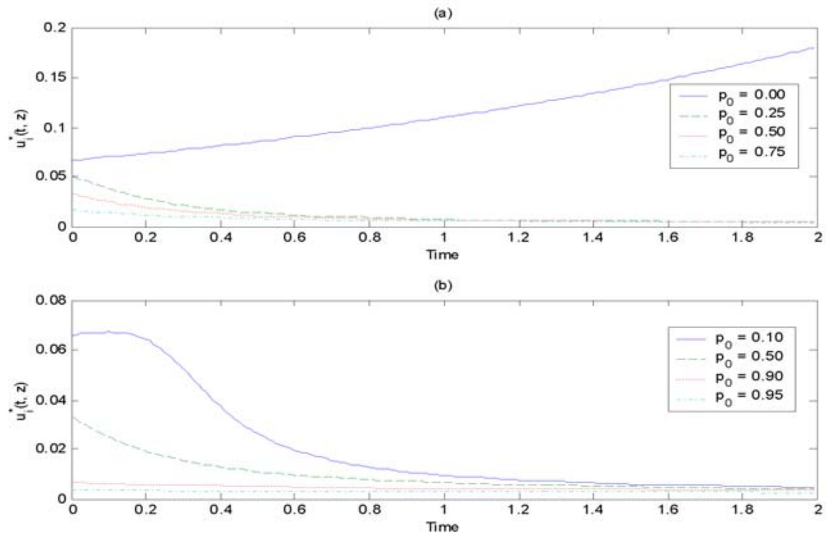

Figure 1: Equilibrium Knowledge Intensities for Various Prior Beliefs the Innovation is Technically Infeasible. ($W = 0.1, r = 0.5, \lambda = 1$)

Figure 1 (a) depicts knowledge intensities equilibrium paths for $p_0 = 0, 0.25, 0.50$ and 0.75 (for parameter values: $W = 0.1, r = 0.5, \lambda = 1$). Note how $u_i^*(t, \mathbf{z} : p_0 = 0)$ rises over time, while the equilibrium paths for $p_0 = 0.25, 0.5, 0.75$ begin to tapers off from the start of the race[31]. Figure 1 (b) depicts knowledge intensities equilibrium paths for $p_0 = 0.1, 0.5, 0.9, 0.95$, (again, for parameter values are $W = 0.1, r = 0.5, \lambda = 1$). Note how $u_i^*(t, \mathbf{z} : p_0 = 0.1)$ at first rises, peaks and then tapers off.

These equilibrium time paths are consistent with a variety of scenarios. For example, the R&D race to develop synthetic human insulin is characterized by the equilibrium path for $p_0 = 0$, while the cold fusion scenario, with a flurry of innovative activity followed by persistent effort, is consistent with the equilibrium path for $p_0 = 0.1$. Figure 1 (a) and (b) also depicts an especially interesting result: regardless of the value for p_0, the equilibrium path of knowledge intensities converges to a "steady state" value. The persistent effort in cold fusion research is consistent with this result.

□ **Steady State Knowledge Intensities.** The steady state level is derived by exploiting symmetry and re-expressing equation 15 as:

$$u_i^*(t, \mathbf{z}) = \dot{z}_i(t) = \frac{\beta}{(1 - p_0) + p_0 e^{n\lambda z(t)}} e^{rt}$$

where $\beta = (1 - p_0)2\lambda(n - 1)W/(2n - 1)$. This differential equation in $z(t)$ can

[31] These plots are robust to grid size.

be solved, assuming $p_0 \in (0,1)$ and the boundary condition $z(0) = 0$, to obtain:

$$(1-p_0)z_i(t) + \frac{p_0}{n\lambda}e^{n\lambda z_i(t)} = \frac{\beta}{r}e^{rt} + \frac{p_0}{n\lambda} - \frac{(1-p_0)2\lambda(n-1)W}{r(2n-1)}$$

The left hand side of this expression has a linear and exponential term in $z_i(t)$ while the right hand side has an exponential term in t and a constant. Therefore, for very large values of t, examine the behavior of the term containing $e^{n\lambda z_i(t)}$ and the term containing e^{rt}. For $t \to \infty$, the behavior of $z_i(t)$ is approximated by examining:

$$\frac{p_0}{n\lambda}e^{n\lambda z_i(t)} \approx \frac{\beta}{r}e^{rt}$$

which implies that:

$$z_i(t) = \frac{1}{n\lambda}\{\ln(\frac{\beta}{r}) - \ln(\frac{p_0}{n\lambda})\} + \frac{r}{n\lambda}t$$

That is, as $t \to \infty$, $z(t)$ is a linear function of t and grows at the rate $r/n\lambda$. But this means that, since $\dot{u}_i(t, \mathbf{z}) = z_i(t)$:

Proposition 4 *(a) As $t \to \infty$, the equilibrium knowledge intensities path converges to $\bar{u}$, where*[32]*:*

$$\bar{u} = \lim_{t\to\infty} u_i^*(t, \mathbf{z}) = \frac{r}{n\lambda} \tag{19}$$

(b) $\bar{u}$ is decreasing in n and λ and increasing in r.

□ **Knowledge Stock.** The expression for $u_i^*(t, \mathbf{z})$ yields an implicit equation for the knowledge stock $z(t)$. Recall that: $u_i^*(t, \mathbf{z}) = \dot{z}(t)$. Making this substitution in equation 15 yields a differential equation in z:

$$\dot{z}(t) = \frac{\alpha}{e^{n\lambda z}p_0 + (1-p_0)}e^{rt}$$

where $\alpha = (1-p_0)2\lambda(n-1)W/(2n-1)$. Imposing the boundary value condition $z(0) = 0$ yields an implicit equation for z:

$$\frac{p_0}{n\lambda}e^{n\lambda z} + (1-p_0)z = \frac{\alpha}{r}e^{rt} + \frac{p_0}{n\lambda} - \frac{\alpha}{r} \tag{20}$$

4 Increased Rivalry and Innovative Activity

The literature contains conflicting results on the link between the number of participants and knowledge intensities level. In Loury (1979), increased rivalry (i.e. raising n) lowers the equilibrium path of innovation certainty knowledge intensities, while in Lee & Wilde (1980) and Reinganum (1982) the opposite result occurs. In innovation uncertainty, however, increased rivalry results in a crossing of knowledge intensities equilibrium path.

[32] Recall from the finite t discussion that $\dot{u}_i^*(t, \mathbf{z}) > 0$ if $\bar{u} \geq p(t)u_i^*(t, \mathbf{z})$.

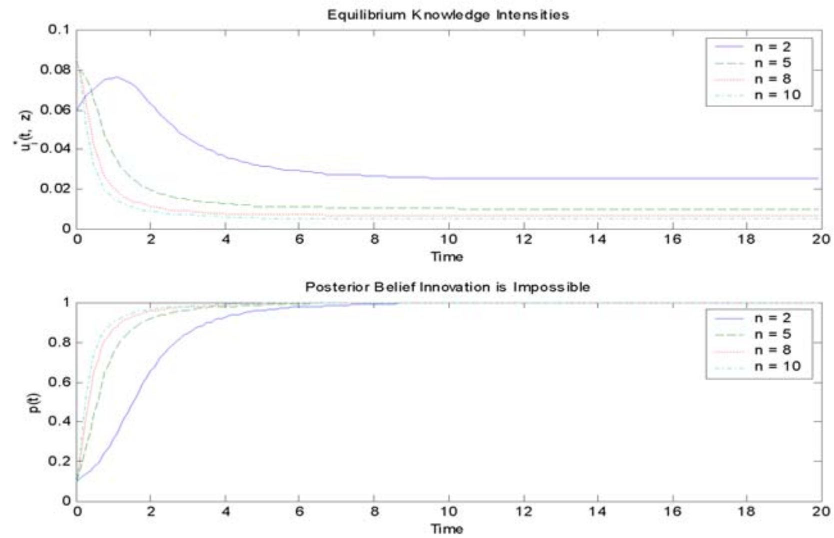

Figure 2: Equilibrium Knowledge Intensities and Posterior Beliefs that Innovation is Technically Infeasible for Various Number of Rivals. ($W = 0.1, r = 0.5, \lambda = 1, p_0 = 0.1$)

To see this, recall that the steady state knowledge intensities level is $\bar{u} = r/n\lambda$, which is inversely proportional to n, implying that increased rivalry lowers the equilibrium path in steady state. Contrast this with what happens at the outset of the race. Recall the "expected value" result: $u_i^*(t, \mathbf{z}) = (1 - p(t))u_i^*(t, \mathbf{z} : p_0 = 0)$. Differentiating with respect to n, and recognizing that $p(t)$ and z are functions of n, yields:

$$du_i^*(t, \mathbf{z})/dn = (1 - p(t))\frac{2\lambda W}{(2n-1)^2} + u_i^*(t, \mathbf{z} : p_0 = 0)\ d(1 - p(t))/dn \qquad (21)$$

Evaluating this at $t = 0$ yields:

$$du_i^*(0, \mathbf{0})/dn = (1 - p_0)\frac{2\lambda W}{(2n-1)^2} > 0$$

Thus, at the outset of the race, increase rivalry raises equilibrium knowledge intensities. When juxtaposed with the result that increased rivalry lowers steady state knowledge intensities, it means that the equilibrium paths for various n must cross, as depicted in figure 2.

To characterize the impact of increased rivalry upon the equilibrium path of knowledge intensities, examine the sign of $du_i^*(t, \mathbf{z})/dn$. This depends on the overall sign of the right hand side of equation 21: the first term is always positive while the sign of the second term depends on the sign of $d(1 - p(t))/dn$. This requires an equation linking $p(t)$ with n. To evaluate this, recall $u_i(t, \mathbf{z}) = (1-p(t))u_i(t, \mathbf{z} : p_0 = 0)$ which when substituted in $\dot{p}(t) = p(t)(1-p(t))\lambda n u_i(t, \mathbf{z})$ yields:

$$\dot{p}(t) = p(t)(1 - p(t))^2 \lambda n u_i^*(t, \mathbf{z} : p_0 = 0)$$

which yields the following differential equation:

$$\frac{dp(t)}{p(t)(1-p(t))^2} = \frac{2n\lambda^2(n-1)W}{(2n-1)} e^{rt} dt, \ni p(0) = 0$$

The solution to which is[33]:

$$\frac{1}{1-p(t)} + \ln \frac{p(t)}{1-p(t)} = \frac{2n\lambda^2(n-1)W}{r(2n-1)} e^{rt} + \frac{1}{1-p_0} + \ln \frac{p_0}{1-p_0} - \frac{2n\lambda^2(n-1)W}{r(2n-1)}$$

Differentiating with respect to $p(t)$ and n and collecting terms yields:

$$\frac{dp(t)}{dn} = p(t)(1-p(t))(e^{rt}-1)\frac{2\lambda^2 W}{r}\frac{2n^2-2n+1}{(2n-1)^2} \tag{22}$$

As long as $n \geq 2$, this implies that increased rivalry results in firms becoming pessimistic at a faster rate[34] as illustrated in figure 2. From equation 22, an expression for $d(1-p(t))/dn$, is substituted in equation 21 and:

Proposition 5 *Increased rivalry raises the equilibrium path of knowledge intensities path if:*

$$\frac{r(2n-1)}{(n-1)(2\lambda^2 W)(2n^2-2n+1)} > p(t)(e^{rt}-1)$$

At $t = 0$ the right hand side of this expression equals 0, and the inequality holds, implying that at the outset of the race, increased rivalry raises the equilibrium knowledge intensities path- which confirms the earlier result. As t increases, the right hand increases, which implies that eventually the inequality will be violated, implying that over time increased rivalry dampens the equilibrium path of knowledge intensities.

5 Endogenous Market Structure

□ **Free Entry.** This section relaxes the assumption that the number of rivals, n, is exogenous. Previous models (Dasgupta & Stiglitz (1980), Reinganum (1980), and Malueg & Tsutsui (1997)) have determined n endogenously through a free

[33] This solution differs from Malueg & Tsutsui (1997) since they obtain an implicit equation for $p(t)$ only for the case of no discounting.

[34] This formalizes the result in Malueg & Tsutsui (1997), which was obtained through numerical simulations in that paper.

entry equilibrium whereby entry occurs until the expected payoff to each firm is zero. From equation 14, the expected payoff is:

$$J^*(\mathbf{u}) = V_i(0, \mathbf{0}) = W(1 - p_0)/(2n - 1)$$

which approaches zero as $n \to \infty$. From this, the following result emerges for innovation certainty:

Proposition 6 *(a) Allowing for an endogenous market condition via a free entry condition, yields a 'competitive' market structure: $n \to \infty$*
(b) With a competitive market structure, the innovation certainty knowledge intensities are:

$$\lim_{n\to\infty} u_i(t, \mathbf{z} : p_0 = 0) = e^{rt} W\lambda, \forall t \geq 0$$

These results are consistent with Dasgupta & Stiglitz (1980), Reinganum (1982), and Malueg & Tsutsui (1997). A
For innovation uncertainty ($p_0 \in [0, 1)$) knowledge intensities at $t = 0$ are $\lim_{n\to\infty} u_i(0, \mathbf{z}(\mathbf{0})) = (1 - p_0)W\lambda > 0$ for . As $t \to \infty$, $\lim_{n\to\infty} \bar{u} = 0$. For finite $t > 0$, consider:

$$\begin{aligned} \lim_{n\to\infty} u_i(t, \mathbf{z}) &= \lim_{n\to\infty} \{(1 - p(t))u_i(t, \mathbf{z} : p_0 = 0)\} \\ &= e^{rt} W\lambda \lim_{n\to\infty} (1 - p_0)/(p_0 e^{\lambda nz} + (1 - p_0)) \end{aligned}$$

to evaluate this limit, note from equation 20 that $\lim_{n\to\infty} e^{\lambda nz} \to \infty$, which implies:

Proposition 7 *(a) Allowing for an endogenous market condition via a free entry condition, yields an atomistic or 'competitive' market structure:*
(b) With a competitive market structure, the innovation uncertainty knowledge intensities are:

$$\lim_{n\to\infty} u_i(0, \mathbf{z}(\mathbf{0})) = (1 - p_0)W\lambda, t = 0$$

$$\lim_{n\to\infty} u_i(t, \mathbf{z}) = 0, t > 0$$

With endogenous free entry, a startling conclusion emerges: if firms are engaged in innovation certainty R&D, the market will be competitive and the equilibrium path of knowledge intensities will be positive and increasing throughout the race. Thus, so long as there is no success, there is always a positive probability of discovery. But for innovation uncertainty R&D, while free entry also yields a fully competitive market, the equilibrium path of knowledge intensity is positive only at $t = 0$, plummeting to zero thereafter. This implies that there is a positive chance of discovery only at $t = 0$. The intuition behind this result is that with increased rivalry, firms become pessimistic at a faster rate. In perfect competition, firms become instantaneously convinced that the innovation does not exist. Unless there is an innovation at $t = 0$, the R&D race ends at the start– the R&D race is but a flash of innovative activity.

□ **Fixed Cost of Entry.** Suppose now firms incur a fixed cost of entry F. Firms enter until the expected payoff $J^*(\mathbf{u}) = F$. Then:

Proposition 8 *Allowing for an endogenous market condition via a fixed cost of entry, F, condition:*
(a) Let $n_c(F)$ denote the maximum integer such that the expected payoff under innovation certainty is greater than or equal to F, and assuming an integer solution for $n_c(F) \ni J^*(\mathbf{u}) = W/(2n_c(F) - 1) \geq F$, implies:

$$n_c(F) = \frac{W + F}{2F}$$

(b) Let $n_u(F)$ denote the maximum integer such that the expected payoff under innovation uncertainty is greater than or equal to F, or: $n_u(F) \ni J^*(\mathbf{u}) = W(1 - p_0)/(2n_u(F) - 1) \geq F$, implies:

$$n_u(F) = \frac{(1 - p_0)W + F}{2F}$$

(c) For $n \geq 2$ to participate in an innovation uncertainty R&D race, firms must be sufficiently optimistic that the innovation is possible:

$$n_u(F) \geq 2, \ \textit{if}\ , p_0^* \leq \frac{W - 3F}{W}$$

The proposition implies that for a given set of parameters, fewer firms will be engaged in innovation uncertainty than in innovation certainty R&D (i.e. $n_u(F) < n_c(F), \forall p_0 \in (0, 1]$). More interestingly, part (c) shows that in order to participate in an innovation uncertainty R&D race, a certain 'threshold' level of confidence about technical feasibility must prevail. As such, measures to raise the upper bound of the threshold pessimism ($(W - 3F)/W$), through a bigger prize or lower fixed cost of entry, or measures to lower p_0 through, say, conferences (as in Eli Lilly's conference on synthetic human insulin) or publications and dissemination of information[35], will lead to greater participation in innovation uncertainty R&D.

6 Conclusion

This paper has developed a simple model of a R&D race with two sources of uncertainty: the first is about the technical (in)feasibility of the innovation, while the second, provided the innovation is technically feasible, is about the timing of innovation. At the outset of the race, the identical firms have (uniform) prior beliefs that the innovation is technically feasible. As the race continues without success, firms update these beliefs and become increasingly pessimistic

[35] This has implications for models of economic growth, especially technology spillovers.

that the innovation is technically infeasible, which in turn impacts knowledge intensities. As in Reinganum (1982) and Malueg & Tsutsui (1997), the model is set up as a differential game, in which R&D knowledge intensities are modeled as feedback strategies. The model develops closed form solutions and includes Lee & Wilde (1980), Reinganum (1982) and Malueg & Tsutsui (1997) as special cases, thereby permitting direct comparison of results.

The major findings are as follows. The equilibrium path of knowledge intensities under innovation uncertainty lies below the equilibrium path of knowledge intensities under innovation certainty. In contrast to earlier models of innovation certainty in which the equilibrium path of knowledge intensities rises over time (Reinganum (1982)), or, remains steady (Lee & Wilde (1980)), the equilibrium path of knowledge intensities under innovation uncertainty may rise or fall. In the long run, the equilibrium path converges to a steady state level, which is independent of prior beliefs. Increased rivalry dampens this steady state level, but raises initial knowledge intensities, implying that the equilibrium paths for various number of rivals will cross. This result is in contrast to earlier models where increased rivalry has an unambiguous effect upon innovative activity. Econometric studies of the link between innovative activity and market structure should therefore be mindful of the fact that with innovation uncertainty, there is a positive link between increased rivalry and innovative activity early on in an R&D race, but that if the innovation is not discovered early on, there is a negative relationship between increased rivalry and innovative activity. Allowing for an endogenous market structure through free entry results in a perfectly competitive market structure– knowledge intensities are positive only at the beginning of the R&D race, falling to nil thereafter. With perfect competition, the race is a 'flash' of innovative activity for an instant. With a fixed cost of entry, there is positive innovative activity over the race. However, for participation to be feasible, firms must be sufficiently optimistic about the technical feasibility of the innovation. Hussain (2002) also extends this model to research joint ventures and to scenarios where the social value of the innovation differs from its private value.

This model can be extended in several ways which are interesting avenues for future research. Choi (1991) develops a multi stage R&D race with innovation uncertainty, but does not analyze investment. Both Malueg & Tsutsui (1997) and this paper analyze investment, but do so in a single stage model. As such, a natural extension of this model is to analyze investment in a multi-stage R&D race, where some or all stages may be characterized by innovation uncertainty. In this model, failure to innovate leads firms to become increasingly pessimistic that the innovation does not exist– as such, this is a "one-armed bandit" uncertainty approach. But, there may well be uncertainty about technical feasibility in multiple dimensions (Moscarini & Smith (2001)), and it is conceivable that firms make could make progress in one dimension, but fail to do so in others. As such, pessimism would depend on overall success. A model of a R&D race which incorporates multiple uncertainties will come closer to capturing innovation uncertainty, especially in pharmaceuticals. The model has assumed that prior beliefs and 'effectiveness' of R&D[36] is uniform across firms– a natural extension is to introduce heterogeneity amongst firms. Finally, another extension to this model is to allow for the possibility that the loser(s) may receive a consolation prize.

Appendix A: Homogeneous and Non-Homogeneous Poisson Process and Derivation of Probabilities

Denote by $N(t)$ the number of events that have occured up to time t. Then any stochastic process that satisfies the following is a homogeneous Poisson process (see Ross (1995)):

1. $N(0) \geq 0$
2. $N(t)$ is integer valued
3. $P(N(h) = 1) = \lambda h + o(h)$
4. $P(N(h) \geq 2) = o(h)$

where the function f is said to be $o(h)$ if $\lim_{h\to 0} f(h)/h = 0$. Notice that λ is independent of time.

From these assumptions, one can show that:

$$P(N(t+s) - N(s) = n) = e^{-\lambda t}(\lambda t)^n/n!, n = 0, 1, 2, ...$$

and in particular:

$$P(N(t) = 1) = e^{-\lambda t}\lambda t$$

Furthermore, $E[N(t)] = \lambda t$, where λ is the arrival rate.

Denote by t_i as the length of time firm i first innovates. Then the distribution function of t_i is:

$$P(t_i < t) = 1 - e^{-\lambda t}$$

from which, the density of t_i is:

$$f(t_i = t) = \lambda e^{-\lambda t}$$

The homogeneous Poisson process is attractive because of its tractability. But it also assumes that the arrival rate is always constant regardless of history. The non-homogeneous Poisson process assumes that the arrival rate is a function of the date. Any stochastic process that satisfies the following is a non-homogeneous Poisson process (see Ross (1995)):

1. $N(0) \geq 0$
2. $N(t)$ is integer valued
3. $P(N(h) = 1) = \lambda(t)h + o(h)$
4. $P(N(h) \geq 2) = o(h)$

Note that $\lambda(t)$ is a function of time: thus probability of an event taking place is a function of the date.

Define the "integrated hazard rate":

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

Then:

$$P(N(t+s) - N(s) = n) = e^{-[\Lambda(s+t)-\Lambda(s)]}[\Lambda(s+t) - \Lambda(s)]^n/n!, n = 0, 1, 2, ...$$

[36] Represented by λ in the innovation certainty arrival rate of the non-homogeneous Poisson process.

which implies that:

$$P(N(t) = 1) = e^{-\Lambda(t)}\Lambda(t)$$

Furthermore, $E[N(t)] = \Lambda(t)$. In contrast to the homogeneous Poisson process, the non-homogeneous process allows for a time dependent arrival rate. One can show that the distribution function of t_i is:

$$P(t_i < t) = 1 - e^{-\Lambda(t)}$$

$P(t_1 > t, ..., t_n > t)$ is derived as follows:

$$P(t_1 > t, ..., t_n > t) = p_0 P(t_1 > t, ..., t_n > t|U) + (1 - p_0)P(t_1 > t, ..., t_n > t|C)$$

To obtain the conditional probabilities, assume that the probabilities are distributed independently across all n firms:

$$P(t_1 > t, ..., t_n > t|U) = \prod_{i=1}^{n} P(t_i > t|U)$$

To derive the term $P(t_i > t|U)$ note that:

$$P(t_i \leq t|U) = 1 - e^{-[\Lambda(t)|U]} = 1 - e^{-(\Psi=0)\int_0^t [2x_i(s)]^{1/2}ds} = 0$$

which implies that: $P(t_i > t|U) = 1$, in turn implying that: $P(t_1 > t, ..., t_n > t|U) = 1$. To derive $P(t_1 > t, ..., t_n > t|C)$ assume independence again, and:

$$P(t_1 > t, ..., t_n > t|C) = \prod_{i=1}^{n} P(t_i > t|C)$$

Note:

$$P(t_i \leq t|C) = 1 - e^{-[\Lambda(t)|C]} = 1 - e^{-(\Psi=\lambda>0)\int_0^t [2x_i(s)]^{1/2}ds}$$

implies:

$$P(t_i \leq t|C) = 1 - e^{-\lambda \int_0^t [2x_i(s)]^{1/2}ds}$$

Therefore:

$$P(t_i > t|C) = e^{-\lambda \int_0^t [2x_i(s)]^{1/2}ds}$$

This implies that:

$$P(t_1 > t, ..., t_n > t|C) = \prod_{i=1}^{n} e^{-\lambda \int_0^t [2x_i(s)]^{1/2}ds}$$

Hence:

$$P(t_1 > t, ..., t_n > t) = p_0 + (1 - p_0) \prod_{i=1}^{n} e^{-\lambda \int_0^t [2x_i(s)]^{1/2}ds} = p_0 + (1 - p_0)e^{-\lambda \sum_{i=1}^{n} z_i(t)}$$

To derive the instantaneous probability $P(t_i = t, t_j > t, j \neq i)$, note that:

$$\begin{aligned} P(t_i = t, t_j > t, j \neq i) &= p_0 P(t_i = t, t_j > t, j \neq i|U) \\ &\quad + (1 - p_0)P(t_i = t, t_j > t, j \neq i|C) \end{aligned}$$

Now, if the task is impossible (state "U"), no firm will innovate, implying that $P(t_i = t, t_j > t, j \neq i|U)) = 0$. Next consider $P(t_i = t, t_j > t, j \neq i|C)$. By independence:

$$P(t_i = t, t_j > t, j \neq i|C) = P(t_i = t|C)P(t_j > t, j \neq i|C)$$

The calculation of the instantaneous probability $P(t_i = t|C)$ requires the calculation of the density $f(t_i = t|C)$, which by definition is: $f(t_i = t|C) = dF(t_i = t|C)/dt$, where, $F(\cdot)$ is the cumulative density function for firm i:

$$F(t_i = t|C) = P(t_i \leq t|C) = 1 - e^{-\lambda z_i(t)}$$

Then:

$$f(t_i = t|C) = e^{-\lambda \hat{z_i}(t)} d\lambda z_i(t)/dt$$

Using Leibniz's rule:

$$d\lambda z_i(t)/dt = \lambda[2x_i(t)]^{1/2} = \lambda u_i(t, \mathbf{z})$$

which in turn implies that:

$$f(t_i = t|C) = \lambda u_i(t, \mathbf{z}) e^{-\lambda z_i(t)}$$

The expression for $P(t_i = t, t_j > t, j \neq i|C)$ requires $P(t_j > t, j \neq i|C)$, which from previous calculations, assuming independence is:

$$P(t_j > t, j \neq i|C) = \prod_{j \neq i} e^{-\lambda z_j(t)} = e^{-\lambda \sum_{j \neq i} z_j(t)}$$

Combining the terms for $f(t_i = t|C)$ and $P(t_j > t, j \neq i|C)$:

$$P(t_i = t, t_j > t, j \neq i|C) = \lambda u_i(t, \mathbf{z}) e^{-\lambda \sum_k z_k(t)}$$

Therefore:

$$P(t_i = t, t_j > t, j \neq i) = (1 - p_0) \lambda u_i(t, \mathbf{z}) e^{-\lambda \sum_k z_k(t)}$$

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