Local network effects and network structure

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Abstract

This paper presents a model of local network effects in which agents connected in a social network each value adoption by a heterogeneous subset of others, and have incomplete information about the structure and strength of adoption complementarities between all other agents. I show that the symmetric Bayes-Nash equilibria of a general adoption game are in monotone strategies, can be strictly Pareto-ranked, and that the greatest equilibrium is uniquely coalition-proof. Each Bayes-Nash equilibrium has a corresponding fulfilled-expectations equilibrium under which agents form local-adoption expectations. Examples analyze three special cases including a standard model with completely connected agents, and characterize the distributions of equilibrium networks of adopters when the social network is an instance of a generalized random graph.

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1. Introduction

This paper studies network effects that are "local". Rather than valuing an increase in the size of a product's user base or network in general, each agent values adoption by a (typically small) subset of other agents, and this subset varies across agents.

The motivation for this paper can be explained by some examples. A typical user of communication software like AOL's Instant Messenger (IM) is generally interested communicating with a very small fraction of the potential set of users of the product (his or her friends and family, colleagues, or more generally, members of an immediate 'social network'). This user benefits when more members of their immediate social network adopt IM; they get no *direct* benefits from the adoption of IM by other users who they have no desire to communicate with (this observation is true of most person-to-person communication technologies). Similarly, firms often benefit when their business partners (suppliers, customers) adopt compatible information technologies; this set of partners (the business network local to the firm) is a small subset of the total set of potential corporate adopters of these technologies. Buyers benefit when more sellers join a common electronic market (and vice versa), though each buyer benefits directly only when those sellers whose products they want to buy join their market. This is typically a small fraction of the potential set of sellers.

Although local to individual users, these networks (social, business, trading) are not isolated from each other. Each individual agent (user, business, trader) values adoption by a distinct subset of other agents, and is therefore connected to a different "local network". However, each member of this local network is in turn connected to their own local network of other agents, and so forth. Returning to the examples above, each member of the immediate social network of a potential IM user has their own (distinct, though possibly overlapping) immediate social network. A firm's suppliers have their own suppliers and customers. The sellers a buyer is interested in purchasing from each have their own potential set of buyers, who in turn each may have a different potential set of sellers. The local networks are therefore interconnected into a larger network – the entire social network of potential AOL users, the entire network of businesses who transact with each other, and the entire network of potential trading partners.

The interconnection of these local networks implies that even if agent A is not directly connected to agent B (and does not benefit directly from agent B's adoption of a network good), the choices of agents A and B may affect each other in equilibrium. Additionally, different agents have information about the structure of a different portion of the entire network; each agent knows the structure of their own local network, but is likely to know less about the structure of their neighbors' local networks, and probably almost nothing about the exact structure of the rest of the network.

The goal of this paper is to study the adoption of a network good that displays network effects that are *local* in the way described above. In the model, potential adoption complementarities between agents are specified by a graph representing an underlying 'social network'. Each agent is a vertex in this graph, connected to a typically small subset of the other vertices (their neighbors), the subset of other agents whose adoption the agent values. The size of this subset (the number of neighbors, or *degree* of the agent) varies across agents. Each agent knows the local structure of the graph in their neighborhood (that is, they know who their neighbors are), but does not know the structure of the rest of the social network. This lack of exact information about the entire graph is modeled by treating it as instance of a random graph which drawn from a known distribution of graphs. Each agent who adopts the network good values the good more if more of their neighbors adopt the good. Additionally, agents are indexed by a heterogeneous valuation type parameter, and higher valuation type agents value adoption by a fixed number of their neighbors more than lower valuation type agents.

The adoption of the network good is modeled as a simultaneous-move game of incomplete information. This game is shown to have a greatest and least Bayes-Nash equilibria that are in monotone pure strategies. Under some assumptions about the independence and symmetry of the posteriors implied by the distribution over graphs, every symmetric Bayes-Nash equilibria of this game is shown to involve all agents playing a threshold strategy, which is defined by a vector of thresholds on valuation type, each component of which is associated with a different degree. When there are multiple equilibria, these threshold vectors can be strictly ordered, and the ordering is based on a common equilibrium probability of adoption by each neighbor of any agent. This ordering also determines a ranking of equilibria: outcomes under a higher-ranked equilibrium strictly Pareto dominate those under a lower-ranked equilibrium. The greatest equilibrium is shown to be the unique symmetric equilibrium that satisfies a refinement of being coalition-proof with respect to self-enforcing deviations in pure strategies.

Outcomes under each symmetric Bayes-Nash equilibrium are shown to be identical to those under a corresponding "fulfilled expectations" equilibrium (and vice versa), under which agents form expectations locally about the probability that each of their neighbors will adopt, and make unilateral adoption choices based on this expectation, which is then fulfilled. The greatest Bayes-

Nash equilibrium corresponds to the fulfilled expectations equilibrium that maximizes expected adoption. Coordination may be considerably simpler when each agent only needs to coordinate their choice of strategy with their neighbors, rather than with the entire set of agents. This is discussed briefly.

Three examples are presented, each of which corresponds to a different kind of randomness and structure in the graphs used to model the underlying social network. The latter two examples are also meant to show how the general analysis can be extended in a straightforward way to handles non-standard cases. In the first example, the social network is an instance of a Poisson random graph (Erdös and Rényi, 1959). The second example analyzes a complete graph which reduces the model to a "standard" model of network effects, under which each adopting agent benefits from the adoption of all other agents.

In the final example, the social network is an instance of a generalized random graph (Newman, Strogatz and Watts, 2001) with an exogenously specified degree distribution. This example is presented in a separate section. It illustrates how the set of Bayes-Nash equilibria can be equivalently characterized by a threshold function on degree, which is useful when all customers have the same valuation type. It also presents a result that relates the structure of the "adoption networks" that emerge as equilibrium outcomes of the adoption game to the structure of the underlying social network, and discusses how this result may be used to test specific instances of models of local network effects.

The results of this paper are related to those of many prior papers about network effects in which adoption is modeled as a game of complete information between "completely connected" agents. Rohlfs (1974), Dyvbig and Spatt (1983) and Katz and Shapiro (1986) each establish that specific games of this kind have a Pareto-dominant Nash equilibrium involving adoption by a maximal number of agents (often all). Milgrom and Roberts (1990) show that since the players' actions are strategic complements in adoption games with network effects, one would expect them to have a greatest and least pure strategy Nash equilibrium. A complete information version of the game in this paper has the same properties, despite its inherent asymmetries; the generalization of Van Zandt and Vives (2004) to Bayesian games of strategic complementarities applies to the model in this paper, and leads to Proposition 1.

An early paper by Farrell and Saloner (1985) analyze a game of incomplete information in which firms decide whether to switch to a new standard in the presence of positive adoption externalities; they establish that there is a unique symmetric Bayes-Nash equilibrium that is monotone in the firm's type. Nault and Dexter (2003) model an alliance formation game in which payoffs are supermodular in investment levels and participation, yet heterogeneity among participants leads to equilibria with "exclusivity" agreements that are most profitable. Recent papers by Segal (1999, 2003) have studied identity-based price discrimination in a general class of games with incomplete information and inter-agent externalities (both positive and negative). In each of these papers, the network effects are "global": all agents benefit from positive actions by each.

The adoption game in this paper bears a natural resemblance to the global game analyzed by Carlsson and Van Damme (1993) and Morris and Shin, (2002), and recently studied experimentally by Heinemann et al. (2004). Both are coordination games with a binary action space and adoption complementarities whose strength varies according to agent type. There are many reasons why the equilibrium monotonicity and uniqueness results of that literature do not immediately carry over to the adoption game with local network effects. First, agents in this paper have private values drawn from a general distribution with bounded support; this is in contrast with the two cases for which Morris and Shin (2002) establish a unique Bayes-Nash equilibrium in the global game: a model with "improper" uniformly distributed types, and a model with common values. Second, the adoption game of this paper is a more general version of the global game – one might think of the adoption game as a "local" game, with the special case of a complete network in Section 4.1 bearing the closest resemblance to the global game. Finally, the equilibrium uniqueness results of the global game are based on a model with a continuum (rather than a finite number) of players, though this restriction may not be critical.

The model of this paper is related quite closely to the analysis of communication in a social network by Chwe (2000), which also models strategic actions by agents in a game influenced by the existence of an underlying social network. However, like much of the literature highlighted in the previous paragraphs, the externalities in Chwe's coordination game are "global" – the agents' payoffs depend on the actions of all other agents – the similarity to this model arise from the fact that prior to choosing their (binary) action, each agent can exchanges information only with their 'neighbors' – and this neighborhood is determined by an exogenous social network, which specifies a subset of other agents for each agent.

Returning to the literature on network effects, the model in Rohlfs (1974) does admit the possibility of local network effects, since the marginal benefit of each agent from adoption by others can be zero for a subset of other agents; however, he does not explore this aspect of his model in any detail. An interesting example of an attempt to induce local network effects was MCI's Friends and Family pricing scheme; a model of the dynamic behavior such pricing induces in networks of agents has been analyzed by Linhart et al. (1994). In a related paper that examines how the local structure of an underlying social network affects economic outcomes, Kakade et al. (2004) model an economy as an undirected graph which specifies agents (nodes) and potential trading opportunities (edges), and provide conditions for the existence of Arrow-Debreu equilibria based on a condition that requires "local" markets to clear.

There is also a growing literature on endogenous strategic network formation. In these "network games", each agent, represented by a vertex, chooses the set of other agents they wish to share a direct edge with, and the payoffs of each agent are a function of the graphs they are connected to. Network effects in these models are also local in some sense, though the set of connections that define "local" are endogenously determined. Proposition 6 in this paper suggests a complementary approach to studying network formation, since it characterizes the structure of equilibrium adoption networks that emerge endogenously while also depending on an underlying social or business network. An excellent recent survey of this literature is available in Jackson (2003). Results on general structures of graphs one might expect as strict Nash equilibria are established by Bala and Goyal (2000). More recently, Jackson and Rogers (2004) analyze a dynamic model of network formation with costly search which explains when networks with low inter-vertex distances and a high degree of clustering ("small-world networks") and those with power-law degree distributions are likely to form. Bramoullé and Kranton (2004) study the effect of the structure of social networks on agents' incentives to experiment, and find that certain asymmetric network structures lead to a high level of agent specialization. Lippert and Spagnolo (2004) analyze the structure of networks of inter-agent relations, which could form the basis for an underlying social network. In work that preceded this literature, Hendricks, Piccone and Tan (1995) study the effect of scale economies on the structure of a specific economic network, an airline network designed by a monopolist, and examine the conditions under which simple structures (such as hub-and-spoke models) are optimal.

A specific model of a random graph used in this paper is due to Newman, Strogatz and Watts (2001). A number of interesting structural properties of graphs of this kind have been established over the course of the last few years, primarily in the context of studying the properties of different social and technological networks, especially the World Wide Web (Kleinberg et al., 1999). An excellent and especially accessible overview of this literature can be found in Newman (2003); a discussion in Ioannides (2004) examines different ways in which these models might apply to economic situations. Models of networks in which agents make strategic choices are conspicuously absent from this literature, and one hopes this paper will establish a first link.

2. Overview of model and a preliminary result

This section introduces the model, and specifies a sufficient condition under which the greatest and least Bayes-Nash equilibria of an adoption game are in monotone pure strategies where monotonicity is defined based on a partial ordering of set inclusion on types. Subsequent sections place more structure on the payoff functions and a different restriction on the set of permitted distributions, in order to describe the properties of equilibria in terms of the scalar degree of an agent.

In what follows, x, y and X are used as placeholder variables that have no specific meaning in the model. x and y are used to represent elements of Euclidean space, and X is used to represent sets. The variables x_{-i} and X_{-i} are used in the following (standard) way: if x is a vector, then $x = (x_i, x_{-i})$, and x_{-i} is also used to represent the vector of all components of x except x_i. X_{-i} is used in the same way when X is a vector of set-valued variables. Additionally, t is often used as a placeholder variable for valuation type (to be defined).

The underlying social network is modeled as a graph G with n vertices. The set of vertices of G is $N \equiv \{1, 2, 3, ..., n\}$. Each vertex represents an agent $i \in N$. This agent is directly associated with the agents in the set G_i, the *neighbor set*[1] of vertex i, where $G_i \in \Gamma_i \equiv 2^{N \backslash \{i\}}$. The fact that $j \in G_i$ is often referred to as "agent j is a neighbor of agent i".

The set of permitted social networks is represented by $\Gamma \subset [\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n]$, restricted appropriately to ensure that each element of Γ is an undirected graph. The vector of neighbor sets of all agents $j \neq i$ is denoted $G_{-i} \in \Gamma_{-i}$. The number of neighbors agent i has is referred to as agent i's *degree*, and is denoted $d_i \equiv |G_i|$. Additionally, each agent is indexed by a valuation type parameter $\theta_i \in \Theta \equiv [0, 1]$ which influences their payoffs as described below (in general, θ could be multidimensional, so long as Θ is compact).

Each agent i chooses an action $a_i \in A$. The payoff to agent i from an action vector $a = (a_1, a_2, ..., a_n)$ is:

$$\pi_i(a_i, a_{-i}, G_i, \theta_i) \equiv w_i(y, \theta_i), \tag{2.1}$$

where $y = (y_1, y_2, ..., y_n)$, $y_j = a_j$ if $j \in G_i$, and $y_j = 0$ otherwise. This means that the payoff to agent i is influenced by the actions of only those agents in their neighbor set G_i, and is also

[1] For instance, one might think of the members of G_i as friends or business associates of agent i.

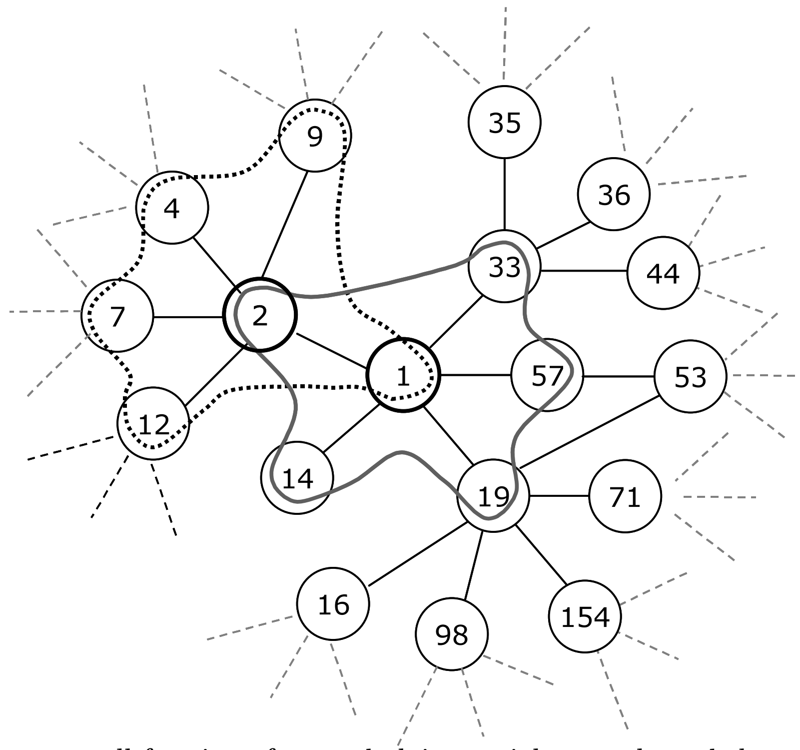

Figure 2.1: Depicts a small fraction of an underlying social network, and the neighbor sets G_i for two candidate agents. In the figure, $G_1 = \{2, 14, 19, 33, 57\}$ and $G_2 = \{1, 4, 7, 9, 12\}$. The solid grey line around agent 1 depicts the information agent 1 has about their neighborhood (the agent knows how many neighbors s/he has, and who they are, does not know anything else about the neighbor sets of these neighbors, and knows nothing else deterministic about the agents $j \notin G_1$). The dotted black line around agent 2 depicts the corresponding information for agent 2.

not influenced by θ_j for $j \neq i$. The function $w_i(y, \theta_i)$ is increasing in y_j for each $j \in N$, and is increasing in θ_i. Therefore, $\pi_i(a_i, a_{-i}, G_i, \theta_i)$ has increasing differences in each component of a, which implies that if w_i is supermodular in y_i (this is trivially true when a_i is one-dimensional), a complete information game in which agents choose actions simultaneously is a supermodular game. It follows from the results of Milgrom and Roberts (1990) that the game has a greatest and least pure strategy Nash equilibrium, independent of any asymmetries in payoffs or in the structure of the underlying social network.

The agent's *uncertainty* about the exact structure of the social network is modeled by drawing G from a known distribution ρ over Γ, and each θ_i independently from a common distribution F over Θ. Agent i observes G_i and θ_i, but does not know G_j or θ_j for $j \neq i$. Therefore, each agent has knowledge of the local structure of the social network; specifically, their own neighborhood. ρ is assumed to be *symmetric* with respect to the agents (that is, it does not change with a permutation of labels i). Each agent's posterior beliefs about (G_{-i}, θ_{-i}) are therefore identically distributed. In order to specify an "increasing posteriors" condition below, one needs a partial ordering on $\Gamma_i \times \Theta$. The result in Proposition 1 uses the natural ordering of set inclusion on Γ_i, though this ordering is not used in subsequent sections: based on further assumptions, monotone strategies are specified with respect to the *degree* of an agent.

The timeline of the adoption game is as follows:

1. Nature draws $\theta_i \in \Theta$ independently for each agent i according to F, and draws $G \in \Gamma$ according to ρ.
2. Each agent i observes θ_i and G_i.
3. Agents simultaneously choose their actions $a_i \in A$.
4. Each agent i realize their payoff $\pi_i(a_i, a_{-i}, G_i, \theta_i)$.

The distribution ρ satisfies the condition of *increasing posteriors* if the posterior distribution it implies is increasing (in the sense of first-order stochastic dominance) with respect to the partial order on Γ_i. That is, if $G_i' \subseteq G_i$ implies that $\Pr[G \subseteq G_j | G_i, \theta_i] \geq \Pr[G \subseteq G_j | G_i', \theta_i]$ for each $G \in \Gamma_j$.

The following result characterizes some properties of the Bayes-Nash equilibria of the adoption game, and follows from Theorem 1 of Van Zandt and Vives (2004).

Proposition 1. *(a) The adoption game has greatest and least Bayes-Nash equilibria that are in pure strategies.*

(b) If ρ satisfies the increasing posteriors condition, then the strategies in the greatest (and least) Bayes-Nash equilibria are monotone in G_i and θ_i. That is, if $s_i^*(G_i, \theta_i)$ is the equilibrium strategy of agent i, then for any $G_i' \subseteq G_i$ and $\theta_i' \leq \theta_i$, $s_i^*(G_i, \theta_i) \geq s_i^*(G_i', \theta_i)$ and $s_i^*(G_i, \theta_i) \geq s_i^*(G_i, \theta_i')$.

There are at least two scenarios of relevance under which the condition of increasing posteriors will hold. In the first, the fact that agent j is a neighbor of agent i increases agent i's posterior about how many neighbors agent j has (in the simplest case, trivially on account of the fact that $i \in G_j$), while not affecting agent i's posteriors about all $j \notin G_i$. For example, this condition is satisfied if G is an instance of a Poisson random graph with a known parameter p (the probability that any two vertices are connected to each other). Alternatively, the fact that agent i has more neighbors may lead her to believe that the entire social network is more densely connected, and this would increase her posteriors about all $j \neq i$. This is true, for instance, when G is an instance of a Poisson random graph with an unknown parameter p drawn from a distribution that is common knowledge.

Since ρ is symmetric, if payoffs are symmetric in the adoption game, the greatest and least Bayes-Nash equilibria are symmetric with respect to permutations in the labels of the agents. This suggests that the monotonicity of the equilibrium might be more conveniently characterized based on the degree of each agent $d_i \equiv |G_i|$ while ignoring the identity of each member of potential subsets of G_i. In subsequent sections, attention is restricted to all equilibria that can be characterized this way.

3. Symmetric equilibria, monotone strategies and fulfilled expectations

This section studies local network effects when the underlying social network is an instance of a specific class of random graphs that generate symmetric independent posteriors (which will be defined shortly). It shows that every symmetric Bayesian Nash equilibrium involves strategies that are monotonic in both valuation type and degree, provides a simple basis for ranking these equilibria and selecting one, relates the set of Bayesian Nash equilibria to a set of "fulfilled expectations" equilibria based on agents forming expectations locally about a scalar probability value, and discusses coordination issues.

In what follows, θ_i is referred to as agent i's *valuation type*, and G_i as agent i's *neighbor set*[2].

Denote the degree of agent i as $d_i \equiv |G_i|$, and let $D \subset \{0, 1, 2, ..., n-1\}$ be the set of possible values that d_i can take. The symmetry of ρ implies that there is a common prior on the degree of each agent i, which is referred to as the *prior degree distribution*, and its density (mass) function is denoted $p(x)$. For each $x \in D$, denote by $\Gamma_j(x)$ the subset of Γ_j such that for each $X \in \Gamma_j(x)$, $|X| = x$. That is, $\Gamma_j(x)$ is the set of all elements of Γ_j with cardinality x, or equivalently, the set of all potential neighborhoods of j that result in j having degree x. The set of permissible distributions ρ over Γ is restricted by assuming that ρ generates posteriors that have marginal

[2] In the game of incomplete information analyzed, agent i's *type* is (θ_i, G_i). For clarity, references to each component (valuation type θ_i and neighbor set G_i) are made individually, and generic type is not referenced.

distributions with the following properties:

$$\text{For each } i \text{, for each } j \in G_i \text{ , } \Pr[G_j \in \Gamma_j(x) \,|G_i, \theta_i] = q(x) \tag{3.1}$$

$$\text{For each } i \text{, for each } j \notin G_i \text{ , } \Pr[G_j \in \Gamma_j(x) \,|G_i, \theta_i] = \widehat{q}(x) \tag{3.2}$$

$q(x)$ is referred to as the density (mass) function of the *posterior neighbor degree distribution*, and $\widehat{q}(x)$ as the density (mass) function of the *posterior non-neighbor degree distribution.* A distribution that satisfies (3.1) and (3.2) is said to satisfy the *symmetric independent posteriors* condition. Qualitatively, this condition implies that if the presence of agent j in the neighbor set of agent i changes agent i's priors, the change is symmetric for each neighbor $j \in G_i$ (which is a natural assumption, given the symmetry with respect to labeling), and is independent of agent i's degree (which is a restriction that, while still admitting a wide variety of distributions, precludes some social networks that have systematic clustering or display "small world" effects; this is discussed further in the concluding section). It is easily verified that if the posterior neighbor degree distribution first-order stochastically dominates the posterior non-neighbor degree distribution, that is, for each $x \in D$, if

$$\sum_{j=0}^{x} q(j) \leq \sum_{j=0}^{x} \widehat{q}(j), \tag{3.3}$$

then symmetric independent posteriors imply the increasing posteriors condition of Proposition 1.

Each agent makes a binary choice between adopting and not adopting a homogeneous network good, and therefore $A = \{0, 1\}$. An extension to variable quantity or to multiple compatible goods is straightforward. Payoffs take the following form for each agent i:

$$w_i(y, \theta_i) = y_i[u([\sum_{j \in N} y_j], \theta_i) - c)], \tag{3.4}$$

where w_i was defined in Section 2, and therefore,

$$\pi_i(a_i, a_{-i}, G_i, \theta_i) = a_i[u([\sum_{j \in G_i} a_j], \theta_i) - c)]. \tag{3.5}$$

(3.5) implies that the payoff to an agent who does not adopt is zero, and to an agent who adopts is according to a value function $u(x, \theta_i)$ which is a function of the number of the agent's neighbors who have adopted the good, and differs across agents only through differences in their valuation type θ_i. I assume that $u(x, \theta_i)$ is continuously differentiable in θ_i, and has the following properties:

1. $u(x+1,\theta_i) > u(x,\theta_i)$ for each $\theta_i \in [0,1]$ (the goods display positive network effects "locally")

2. $u_2(x,\theta_i) > 0$ for each $x \in D$ (the ordering of valuation types is such that higher valuation types value adoption by each of their neighbors more than lower valuation types)

c could be any cost associated with adoption, including a price paid for the good, a cost associated with finding and installing the good, or an expected cost of learning how to use the good. The distributions ρ and F, the utility function u and the adoption cost c are common knowledge. The *adoption game* has the same timeline as the game in Section 2:

1. Nature draws $\theta_i \in \Theta$ independently for each agent i according to F, and draws $G \in \Gamma$ according to ρ.

2. Each agent i observes their valuation type θ_i and their neighbor set G_i.

3. Agents simultaneously choose their actions $a_i \in A \equiv \{0,1\}$. A choice of $a_i = 1$ corresponds to agent i adopting the product.

4. Each agent i realizes their payoff $a_i[u([\sum\limits_{j\in G_i} a_j],\theta_i) - c)]$.

3.1. Monotonicity of all symmetric Bayes-Nash equilibria

The main result of this section is Proposition 2, which specifies that all symmetric Bayes-Nash equilibrium involve strategies that are monotone in both degree and valuation type, and can therefore be represented by a *threshold strategy* with a vector of thresholds θ^*, each component $\theta^*(x)$ of which is associated with a degree $x \in D$.

If the symmetric independent posteriors condition is satisfied, the posterior belief of agent i about degree is:

$$\Pr[d_{-i} = x_{-i}|d_i, G_i] = \left(\prod_{j\in G_i} q(x_j)\right)\left(\prod_{j\notin(G_i\cup\{i\})} \widehat{q}(x_j)\right), \tag{3.6}$$

for each $x_{-i} \in D^{(n-1)}$. Similarly, the posterior belief of agent i about valuation type is $\mu(t_{-i}|[n-1])$ for each $t_{-i} \in \Theta^{(n-1)}$, where $\mu(t|x)$ is the probability measure over $t \in \Theta^x$ defined as follows: for any $g(t)$,

$$\int\limits_{t\in\Theta^x} g(t)d\mu(t|x) = \int\limits_{t_1=0}^{1}\left(\int\limits_{t_2=0}^{1} \dots \left(\int\limits_{t_x=0}^{1} g(t)dF(t_x)\right)\dots dF(t_2)\right)dF(t_1) \tag{3.7}$$

From (3.5), the adoption game is symmetric, and the strategy of each agent i is simply a function of their valuation type θ_i and degree d_i. We look for symmetric equilibria in which all agents play the strategy $s : D \times \Theta \to A$. Suppose all agents $j \neq i$ play s. The expected payoff to agent i from a choice of action a_i is:

$$a_i \left(\left[\int_{t_{-i} \in \Theta^{(n-1)}} \left(\sum_{x_{-i} \in D^{(n-1)}} \left[u(\sum_{j \in G_i} s(x_j, t_j), \theta_i) \left(\prod_{j \in G_i} q(x_j) \right) \left(\prod_{j \notin (G_i \cup \{i\})} \widehat{q}(x_j) \right) \right] \right) d\mu(t_{-i} | [n-1]) \right] - c \right). \tag{3.8}$$

Given a fixed set of actions by each agent $j \in G_i$, the actions of agents $j \notin G_i$ do not affect agent i's payoffs. Symmetric independent posteriors imply that the marginal distributions of each x_j and t_j in (3.8) are independent. The expression (3.8) can therefore be rewritten as $a_i \left[\Pi(d_i, \theta_i) - c \right]$,where

$$\Pi(d_i, \theta_i) \equiv \int_{t \in \Theta^{(d_i)}} \left(\sum_{x \in D^{(d_i)}} \left[u(\sum_{j=1}^{d_i} s(x_j, t_j), \theta_i) \prod_{j=1}^{d_i} q(x_j) \right] \right) d\mu(t | d_i). \tag{3.9}$$

Assuming that indifferent agents adopt, a symmetric strategy s is therefore a Bayes-Nash equilibrium if it satisfies the following conditions for each i:

$$\text{If } s(d_i, \theta_i) = 1\text{: } \Pi(d_i, \theta_i) \geq c; \tag{3.10}$$

$$\text{If } s(d_i, \theta_i) = 0\text{: } \Pi(d_i, \theta_i) < c. \tag{3.11}$$

Proposition 2. *(a) In each symmetric Bayes-Nash equilibrium, the equilibrium strategy $s : D \times \Theta \to A$ is non-decreasing in both degree and valuation type. Therefore, in every symmetric Bayes-Nash equilibrium, the equilibrium strategy takes the form:*

$$s(d_i, \theta_i) = \begin{cases} 0, & \theta_i < \theta^*(d_i) \\ 1, & \theta_i \geq \theta^*(d_i) \end{cases} \tag{3.12}$$

where $\theta^* : D \to A$ is non-increasing.

(b) If $u(0, \theta) = 0$ for each $\theta \in \Theta$, then $s(x, t) = 0$ for each $x \in D, t \in \Theta$ is a symmetric Bayes-Nash equilibrium for any adoption cost $c > 0$.

A strategy of the form (3.12) is referred to as a *threshold strategy* with threshold vector $\theta^* \equiv (\theta^*(1), \theta^*(2), ..., \theta^*(n))$. To avoid introducing additional notation (such as 0^- or 1^+), we sometimes use $\theta^*(x) = 1$ as being equivalent to $s(x, t) = 0$ for all $t \in \Theta$. An implication of Proposition

2 is that there are likely to be multiple symmetric Bayes-Nash equilibria of the adoption game. The following section provides a ranking these equilibria, and a basis for the selection of a unique outcome.

3.2. Equilibrium ranking and selection

Consider any threshold strategy of the form derived in Proposition 2:

$$s(d_i, \theta_i) = \begin{cases} 0, & \theta_i < \theta^*(d_i) \\ 1, & \theta_i \geq \theta^*(d_i) \end{cases} \tag{3.13}$$

When s is played by all n agents, each of their expected payoffs can be characterized in the following way. For any agent i, the realized payoff under s is

$$u(\sum_{j \in G_i} s(d_j, \theta_j), \theta_i) - c \tag{3.14}$$

Now, for each $j \in G_i$, according to (3.13),

$$s(d_j, \theta_j) = 1 \Leftrightarrow \theta_j \geq \theta^*(d_j). \tag{3.15}$$

Therefore, conditional on d_i, ex-ante (that is, after the agents has observed her own degree and type, but before she make her adoption choices):

$$\Pr[s(d_j, \theta_j) = 1 | d_i] = 1 - F(\theta^*(d_i)). \tag{3.16}$$

Since the posterior probability that an arbitrary neighbor of i has degree x is $q(x)$, it follows that

$$\Pr[s(d_j, \theta_j) = 1] = \sum_{x=1}^{m} q(x) \left[1 - F(\theta^*(x)\right]. \tag{3.17}$$

Note that this probability does not depend on j, and, given player i's information, is the *same* ex-ante (that is, after agents has observed their degree and type, but before they make their adoption choices) for each neighbor $j \in G_i$. Denote this common probability as $\lambda(\theta^*)$, which is termed the *neighbor adoption probability* under the symmetric strategy with threshold θ^*. From (3.17),

$$\lambda(\theta^*) = \sum_{x=1}^{m} q(x) \left[1 - F(\theta^*(x)\right]. \tag{3.18}$$

Moreover, the payoff to agent i only depends on the *number* of their neighbors who adopt the product. Let Y be the random variable which is binomially distributed with parameters x and $\lambda(\theta^*)$, and whose probability mass (frequency) function is:

$$B(y|x,\theta^*) \equiv \Pr[Y=y] = \binom{x}{y}[\lambda(\theta^*)]^y[1-\lambda(\theta^*)]^{(x-y)}, \tag{3.19}$$

If all agents $j \neq i$ play the symmetric strategy (3.13), the expected payoff to agent i is

$$\left(\sum_{y=1}^{d_i} u(y,\theta_i)B(y|d_i,\theta^*)\right) - c. \tag{3.20}$$

We have therefore established that under a threshold strategy with threshold vector θ^*,

$$\Pi(d_i,\theta_i) = \sum_{y=1}^{d_i} u(y,\theta_i)B(y|d_i,\theta^*), \tag{3.21}$$

where Π was defined in (3.9).

The following is a well-known result about the binomial distribution:

Lemma 1. *Let X be a random variable distributed according to a binomial distribution with parameters n and p. If $g(x)$ is any strictly increasing function, then $E[g(X)]$ is strictly increasing in p.*

The lemma is a consequence of the fact that a binomial distribution with a higher p strictly dominates one with a lower p in the sense of first-order stochastic dominance.

Define the set of threshold vectors associated with symmetric Bayes-Nash equilibria as $\Theta^* \subset \Theta^{m+1}$. The next lemma shows that $\lambda(\theta^*)$ provides a basis on which one can rank the different Bayes-Nash equilibria of the agent adoption game:

Lemma 2. *For any two threshold vectors θ^A and $\theta^B \in \Theta^*$*

(a) If $\lambda(\theta^A) > \lambda(\theta^B)$, then, for each $x \in D$, either

$$\theta^A(x) < \theta^B(x),$$

or

$$\theta^A(x) = \theta^B(x) = 1.$$

(b) If $\lambda(\theta^A) = \lambda(\theta^B)$, then $\theta^A(x) = \theta^B(x)$ for each $x \in D$.

Lemma 2 establishes that if multiple Bayes-Nash equilibria exist, their threshold vectors can be strictly ordered, this ordering is determined completely by the neighbor adoption probability $\lambda(\theta^*)$, and two different equilibria cannot have the same neighbor adoption probability. An implication of the lemma is that if there are multiple Bayes-Nash equilibria, then these equilibria can be strictly Pareto-ranked:

Proposition 3. *For any two threshold vectors $\theta^A, \theta^B \in \Theta^*$, the equilibrium with threshold vector θ^A strictly Pareto-dominates*[3] *the equilibrium with threshold vector θ^B if and only if $\lambda(\theta^A) > \lambda(\theta^B)$.*

Together, Lemma 2(b) and Proposition 3 establish that symmetric independent posteriors are sufficient to rank the set of *all* Bayes-Nash equilibria of the adoption game; Proposition 2 confirms that the strategies of each are monotonic in degree and valuation type. An immediate corollary is the existence of a unique greatest Bayes-Nash equilibrium. Denote the threshold vector associated with this equilibrium as θ^*_{gr}. From Lemma 2,

$$\theta^*_{gr} = \arg\max_{\theta \in \Theta^*} \lambda(\theta).$$

The greatest equilibrium of the adoption game is similar to the "maximum user set" equilibrium characterized by Rohlfs (1974), the "maximal Nash" equilibrium characterized by Dyvbig and Spatt (1983), and the Pareto-dominant outcome in Katz and Shapiro (1986), since it is the outcome that maximizes (expected) adoption. Proposition 4 also establishes that the greatest equilibrium is also the only symmetric Bayes-Nash equilibrium that satisfies a refinement of coalition-proofness.

Proposition 4. *The symmetric Bayes-Nash equilibrium with threshold vector θ^*_{gr} is coalition-proof with respect to self-enforcing deviations in pure strategies by anonymous coalitions.*

Coalition-proofness is a refinement which seems especially appealing in the context of an adoption game with direct network effects. Given that the equilibria in this paper are symmetric and in pure strategies, restricting attention to coalitions that are anonymous ensures that players in a coalition do not condition their deviation on the identity of their coalition members. Note, however, that this refinement is not as strong as that of coalition-proof correlated equilibrium (Moreno and Wooders, 1996). The restriction of anonymity would be attractive to relax, as it would make the

[3] *Strict Pareto-dominance: If equilibrium A strictly Pareto-dominates equilibrium B, it means that all types have at least as high an expected payoff under A as they do under B, and there exists $x \in D, t \in \Theta$ such that a player with degree x and valuation type t is strictly better off under A than under B.*

refinement more robust, ensuring that the equilibrium is robust to deviations that follow private communication between agents who are members of the same local network.

Clearly, since a deviation by the grand coalition (all N agents) to the greatest Bayes-Nash equilibrium will be a Pareto-improving self-enforcing deviation for any other symmetric Bayes-Nash equilibrium, none of the other symmetric equilibria satisfy the condition of Proposition 4, and the greatest Bayes-Nash equilibrium is uniquely coalition proof in the way described in the proposition. Furthermore, it is the only candidate equilibrium for stronger refinements based on coalition-proofness.

The result of Proposition 4 is somewhat related to a "sequential choice" argument given by Farrell and Saloner (1985) for the selection of the maximal Nash equilibrium as the outcome in a complete graph (though not characterized that way) with K players and complete information.

3.3. Relationship to fulfilled expectations equilibria

This section describes an adoption process under which each agent locally forms an identical expectation λ, the probability that each of their neighbors will adopt, and makes their adoption choice unilaterally based on this expectation. It defines a condition which specifies what expectations λ are "fulfilled", and shows that for each fulfilled expectations equilibrium, there is a corresponding Bayes-Nash equilibrium, and vice versa. Since some notion of rational or fulfilled expectations is widely used to define outcomes in models with network effects, this connection seems important.

Suppose that each agent i forms the same expectation about the behavior of other agents in their neighborhood – a probability λ that any arbirtrary neighbor of theirs will adopt. Based on this expectation λ, the probability that y of their d_i neighbors will adopt is according to the binomial distribution $b(y|d_i, \lambda)$, where

$$b(y|x,\lambda) \equiv \binom{x}{y} \lambda^y [1-\lambda]^{(x-y)}, \tag{3.22}$$

and agent i's expected value from adoption is $[v(d_i, \theta_i, \lambda) - c]$, where:

$$v(x,t,\lambda) \equiv \sum_{y=1}^{x} b(y|x,\lambda) u(y,t). \tag{3.23}$$

Therefore, agent i adopts the product if $[v(d_i, \theta_i, \lambda) - c] \geq 0$. For a fixed λ, define the adoption threshold $\underline{\theta}(x, \lambda)$ as follows:

$$\underline{\theta}(x, \lambda) = \begin{cases} 1 \text{ if } v(x, 1, \lambda) < c; \\ t : v(x, t, \lambda) = c \text{ otherwise.} \end{cases} \tag{3.24}$$

Since $u_2(x, t) > 0$, it is easily verified that $v_2(x, t, \lambda) > 0$, and therefore, $\underline{\theta}(x, \lambda)$ is well defined. Additionally, an agent of valuation type θ_i and degree d_i adopts if and only if $\theta_i \geq \underline{\theta}(d_i, \lambda)$. Therefore, ex-ante, the probability that a neighbor of agent i who has degree x will adopt is $[1 - F(\underline{\theta}(x, \lambda)]$. Since all agents share a common expectation λ, the *actual* probability $\Lambda(\lambda)$ that an arbitrary neighbor of any agent adopts the product, given the posterior neighbor degree distribution $q(x)$ is

$$\Lambda(\lambda) = \sum_{x=1}^{m} q(x)[1 - F(\underline{\theta}(x, \lambda)]. \tag{3.25}$$

Therefore, λ is fulfilled as an expectation of the probability of neighbor adoption only if it is a fixed point of $\Lambda(\lambda)$. Each outcomes associated with a fulfilled expectation λ is a *fulfilled expectations equilibrium.* Since $b(y|x, 0) = 0$, it follows that $v(x, t, 0) = 0$ for each $x \in D, t \in \Theta$, and consequently, $\Lambda(0) = 0$. The expectation $\lambda = 0$ is therefore fulfilled. Define L as the set of all fixed points of $\Lambda(\lambda)$.

$$L \equiv \{\lambda : \Lambda(\lambda) = \lambda\}. \tag{3.26}$$

Consider any Bayes-Nash equilibrium with threshold vector θ^*. From (3.18), the neighbor adoption probability associated with θ^* is

$$\lambda(\theta^*) = \sum_{x=1}^{m} q(x)[1 - F(\theta^*(x)]. \tag{3.27}$$

Now, examine the possibility that $\lambda(\theta^*)$ is a rational expectation. Since $b(y|x, \lambda(\theta^*))$ is equal to $B(y|x, \theta^*)$, the adoption thresholds associated with $\lambda(\theta^*)$ are

$$\underline{\theta}(x, \lambda(\theta^*)) = \theta^*(x), \tag{3.28}$$

and therefore, from (3.25) and (3.27), $\lambda^*(\theta)$ is a fixed point of $\Gamma(\lambda)$, and therefore, a rational expectation. Conversely, consider any $\lambda \in L$, and define a candidate Bayes-Nash equilibrium with threshold vector

$$\theta^*(x) = \underline{\theta}(x, \lambda). \tag{3.29}$$

The neighbor adoption probability associated with the threshold θ^* is

$$\lambda(\theta^*) = \sum_{x=1}^{m} q(x)[1 - F(\theta^*(n)], \tag{3.30}$$

and since λ is a fixed point of $\Lambda(\lambda)$, it follows from (3.25) that $\lambda = \lambda^*(\theta)$, and consequently, $\theta^* \in \Theta^*$. We have therefore proved:

Proposition 5. *(a) For each Bayes-Nash equilibrium of the adoption game with threshold θ^*, the expectation $\lambda^*(\theta)$ defines a fulfilled expectations equilibrium.*

(b) For each fulfilled expectation λ, the threshold strategy with threshold vector defined by $\theta^*(x) = \underline{\theta}(x, \lambda)$ is a Bayes-Nash equilibrium of the adoption game.

The connection established by Proposition 5 seems important, because many earlier papers have derived their results based on some idea of expectations that are self-fulfilling, and this idea is still used to make predictions in models of network effects. Establishing that there is an underlying inter-agent adoption game which has a Bayes-Nash equilibrium that leads to identical outcomes may make this usage more robust. Clearly, in every game of incomplete information, if an "expectation" of an agent comprises a vector of strategies for all other agents, then each vector of Bayes-Nash equilibrium strategies is (trivially) a fulfilled expectations equilibrium. What makes the proposition more interesting is that a scalar-valued expectation that is intuitively natural (how likely are my neighbors to adopt this product), that the agent only needs to make locally, and that has a natural connection to realized demand, is sufficient to establish the correspondence.

Together, Propositions 4 and 5 indicate that the rational expectations equilibrium corresponding to the unique coalition-proof Bayes-Nash equilibrium is the one that maximizes expected adoption. This is the equilibrium customarily chosen in models of demand with network effects that are based on "fulfilled expectations" (for instance, in Katz and Shapiro, 1985, Economides and Himmelberg, 1995; also see Economides, 1996). An argument provided for the stability of this equilibrium is typically based on tattonement, rather than it being an equilibrium of an underlying adoption game. For pure network goods, the non-adoption equilibrium is stable under the former procedure, but not under the refinement of Proposition 4.

3.4. Equilibrium determination and coordination

Propositions 3 and 5 suggest a simple method for determining the set of all Bayes-Nash equilibria of the adoption game. Proposition 3 establishes that each equilibrium is parametrized by a unique probability $\lambda(\theta^*) \in [0,1]$. Proposition 5 establishes that each of these values is a fixed point of the function $\Lambda(\lambda)$ defined in (3.25). Therefore, to determine the set of all symmetric Bayes-Nash equilibria, all one needs are the set of fixed points λ of $\Lambda(\lambda)$, after which one can use (3.24) to specify the equilibrium associated with each $\lambda \in L$. Finding the fixed points of $\Lambda(\lambda)$ is likely to be a substantially simpler exercise than finding each vector θ^* that is a fixed point of the associated equation for the game, as illustrated by the example in Section 4.1.

As discussed briefly in Section 2.4, a natural choice for the outcome of the adoption games appears to be the threshold equilibrium with threshold θ^*_{gr}, since it Pareto-dominates all the others, and is also the unique coalition-proof equilibrium. While this presents a strong case for choosing this as the outcome, it does not resolve how agents might coordinate on choosing this equilibrium. Note, however, that the coordination problem with local network effects is (loosely speaking) considerably less complicated, since each player i *only* needs to coordinate on their choice of strategy with their *neighbors* $j \in G_i$, rather than with every other player. Of course, this is does not guarantee a unique equilibrium in an appropriately defined sequential coordination game, but merely makes it more likely (again, loosely speaking). A more realistic mechanism that determines the actual outcome is likely to be an adjustment mechanism of the kind described by Rohlfs (1974).

4. Two simple examples

This section presents two examples of adoption games in which specific assumptions are made about the distribution over the set of graphs from which the social network is drawn, and/or the distribution of valuation types. In the first example, the social network is an instance of a Poisson random graph and valuation types are drawn from a general distribution F. In the second example, the social network is a complete graph and each adopting agent benefits from the adoption of all other agents. This shows how the model of this paper encompasses a "standard" model of network effects as a special case.

4.1. Poisson random graph

This section analyzes an example in which the social network is an instance of a Poisson random graph (Erdös and Rényi, 1959), and for which the value of adoption is linear in both valuation type and in the number of complementary adoptions, that is, $u(x,t) = xt$. Poisson random graphs are constructed as follows: take n vertices and connect each pair (or not) with probability r (or $1-r$). It is well-known that the prior degree distribution of these random graphs has the density (mass) function:

$$p(x) = \binom{n-1}{x} r^x (1-r)^{(n-1-x)}. \tag{4.1}$$

Excluding agent i, the distribution of the number of neighbors that an arbitrary agent j has (the so called *excess degree* of agent j from agent i's perspective) is simply according to the prior degree distribution of a Poisson random graph with $(n-1)$ nodes:

$$p_{ex}(x) = \binom{n-2}{x} r^x (1-r)^{(n-2-x)} \tag{4.2}$$

Therefore, the posterior *neighbor* degree distribution is according to

$$q(x) = \begin{cases} 0, \ x = 0 \\ p_{ex}(x-1), \ 1 \leq x \leq n-1, \end{cases} \tag{4.3}$$

since any agent i knows that for each of their neighbors j, $i \in G_j$, but has no additional information about j's neighbors on account of j being a neighbor of i. Similarly, the posterior *non-neighbor* degree distribution is according to

$$\widehat{q}(x) = \begin{cases} p_{ex}(x), \ 0 \leq x \leq n-2 \\ 0, \ x = n-1, \end{cases} \tag{4.4}$$

since any agent i knows that for each of their non-neighbors j, $i \notin G_j$.

The Poisson random graph therefore satisfies the symmetric independent posteriors condition. Based on Proposition 5, the set of Bayes-Nash equilibria can be determined by constructing the set of neighbor adoption probabilities that define a fulfilled expectation equilibrium. For any candidate adoption probability λ, since $u(x,t) = xt$, from (3.23),

$$v(x,t,\lambda) = \sum_{y=1}^{x} b(y|x,\lambda) yt = txr \tag{4.5}$$

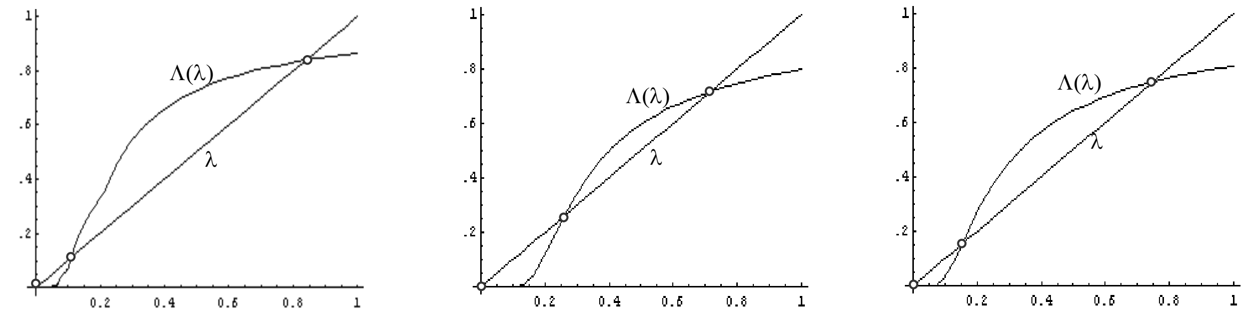

Figure 4.1: Illustrates the fixed points of $\Lambda(\lambda)$ for three numerical examples. The circled points correspond to the neighbor adoption probabilities that determine each Bayes-Nash equilibrium. In the first figure, $n = 10, r = 0.1, c = 0.2$ and $F(\theta) = \theta$. In the second figure, $n = 100, r = 0.1, c = 2$ and $F(\theta) = \theta$. In the final figure, $n = 100, r = 0.1, c = 1$ and $F(\theta) = 1 - (1 - \theta)^2$. In each of the figures (though not clearly visible), $\Lambda(\lambda)$ has a portion between 0 and a positive value where $\Lambda(\lambda) = 0$.

Based on (3.24) and (4.5), for any cost of adoption $c > 0$, the adoption thresholds $\underline{\theta}(x, \lambda)$ are

$$\underline{\theta}(x, \lambda) = \begin{cases} 1, \; x \leq \frac{c}{\lambda}; \\ \frac{c}{x\lambda}, \; x > \frac{c}{\lambda}. \end{cases} \tag{4.6}$$

Consequently, from (3.25) and (4.6),

$$\Lambda(\lambda) = \sum_{x=\ulcorner c/\lambda \urcorner}^{n-1} \left[\binom{n-2}{x-1} \lambda^{(x-1)} (1-\lambda)^{(n-1-x)} \right] [1 - F(\frac{c}{x\lambda})], \tag{4.7}$$

where $\ulcorner c/\lambda \urcorner$ denotes the smallest integer greater than or equal to c/λ. Note that $\Lambda(\lambda)$ is continuous in λ, though there are discontinuous changes in its slope at each λ for which c/λ is an integer. For each $\lambda \in L$, the set of all fixed points of $\Lambda(\lambda)$, there is a Bayes-Nash equilibria with threshold vector:

$$\theta^*(x) = \begin{cases} 1, \; x \leq \frac{c}{\lambda}; \\ \frac{c}{x\lambda}, \; x > \frac{c}{\lambda}. \end{cases} \tag{4.8}$$

Three numerical examples are depicted in Figure 4.1. In the first two, F is the uniform distribution. Figure 4.1(a) is for a low n, and the structure of $\Lambda(\lambda)$ is visible; in Figure 4.1(b), n is relatively high. In Figure 4.1(c) $F(\theta) = [1 - (1 - \theta)^2]$, which is the beta distribution with parameters $a = 1$ and $b = 2$. In each case, c is chosen low enough to ensure that $\Lambda(\lambda)$ has fixed points in addition to $\lambda = 0$.

4.2. Complete social network

In the next example, each agent is connected to all $(n-1)$ others, and the social network is therefore a complete graph. This special case of the model resembles many standard models of network effects, in which the payoffs to agents are directly influenced by the actions of all other agents (see, for example, Dyvbig and Spatt, 1983, Farrell and Saloner, 1985, Segal, 1999, 2003).

The degree distribution takes the following form:

$$p(x) = \begin{cases} 1, \ x = n-1 \\ 0, x < n-1 \end{cases}, \tag{4.9}$$

and it is easy to see that this is identical to the neighbor degree distribution:

$$q(x) = \begin{cases} 1, \ x = n-1 \\ 0, x < n-1 \end{cases}. \tag{4.10}$$

Clearly, the condition of symmetric independent posteriors is trivially true. It follows from Proposition 2 that any symmetric Bayes-Nash equilibrium is defined by a single threshold $\theta^*(n-1)$, which (for brevity, and only in this section) we refer to as θ^*.

Rather than computing the associated adoption probabilities λ and using Proposition 5, it is straightforward in this case to compute θ^* directly, since θ^* is a scalar. If all agents play the symmetric strategy $s : \Theta \to A$ with threshold θ^* on valuation type, then from (3.19) and (3.20), the expected value to an agent of valuation type t is

$$w(t,\theta^*) \equiv \left(\sum_{y=0}^{n-1} u(y,t) \binom{n-1}{y} (1-F(\theta^*)]^y [F(\theta^*)]^{(n-1-y)} \right) - c, \tag{4.11}$$

and therefore, the set Θ^* of all thresholds corresponding to symmetric Bayes-Nash equilibria is defined by:

$$\Theta^* = \{t : w(t,t) = 0\} \tag{4.12}$$

Correspondingly, from (4.10) and (3.18), the neighbor adoption probability associated with each threshold $\theta^* \in \Theta^*$ is:

$$\lambda(\theta^*) = 1 - F(\theta^*), \tag{4.13}$$

and from Proposition 5, each $\lambda(\theta^*)$ defines a fulfilled-expectations equilibrium in which agents form homogeneous expectations about the probability that each other agent will adopt.

5. A third example and the structure of adoption networks

In this third example, the social network is an instance of a generalized random graph. Agents have the same valuation type $\theta_i = 1$. Therefore, this example also illustrates how the model applies to situations where all the uncertainty is in the structure of the underlying social network. The set of Bayes-Nash equilibria can be equivalently characterized by a threshold function on degree (rather than a threshold vector on valuation type), which is necessary for models with homogeneous adoption complementarities across agents. This characterization leads to a result about the structure of the network of agents who adopt the product, and some empirical implications of this result are discussed.

Generalized random graphs (Newman, Strogatz and Watts, 2001) have been used widely to model a number of different kinds of complex networks (for an overview, see section IV.B of Newman, 2003). They are specified by a number of vertices n, and an exogenously specified degree distribution with probability mass (frequency) function $p(x)$ defined for each $x \in D$. For each vertex i , the degree d_i is realized as an independent random draw from this distribution. Once each of the values of n_i have been drawn, the instance of the graph is constructed by first assigning d_i 'stubs' to each vertex i, and then randomly pairing these stubs[4].

Recall that m is the largest element of D. Given this procedure for drawing G from Γ, the neighbor degree distribution is described by:

$$q(x) = \frac{xp(x)}{\sum_{j=0}^{m} jp(j)}. \tag{5.1}$$

The reason why the degree of an arbitrary neighbor of a vertex has the distribution $q(x)$ is as follows. Given the 'algorithm' by which each instance of the random graph is generated, since there are n other vertices connected to a vertex of degree n, it is n times more likely to be a neighbor of an arbitrarily chosen vertex than a vertex of degree 1. The neighbor degree distribution is essentially

[4] The process described above has some shortcomings in generating representative elements of Γ; for instance, it may create a graph with multiple edges between a pair of vertices. Two algorithms that are used to account for this while preserving uniform sampling are the switching algorithm (Rao et al., 1996, Roberts, 2000) and the matching algorithm (Molloy and Reed, 1995). Recent studies have contrasted the performance of these algorithms with a third procedure called "go with the winners"; for details, see Milo et al. (2003).

identical to the *excess degree distribution* discussed in Newman (2003). The *non-neighbor* degree distribution is somewhat more complicated; for large enough n, it is approximately the same as the prior degree distribution, that is, $\widehat{q}(x) \cong p(x)$.

It is straightforward to see that the characterization based on threshold types in Section 3 is "invertible" in the following sense: for each vector θ^*, one can define a corresponding function:

$$\delta^*(t) = \min\{x : \theta^*(x) = t\}, \tag{5.2}$$

and a symmetric Bayes-Nash equilibria of the game are completely defined by the functions $\delta^*(t)$. The strategy that corresponds to $\delta^*(t)$ is

$$s(d_i, \theta_i) = \begin{cases} 0, & d_i < \delta^*(\theta_i) \\ 1, & d_i \geq \delta^*(\theta_i)) \end{cases}. \tag{5.3}$$

If $\theta_i = 1$ for all agents, then

$$F(t) = \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases}. \tag{5.4}$$

Therefore, in this example, each Bayes-Nash equilibrium is completely determined by its value of $\delta^*(1)$, which we refer to as δ^* for brevity. Define

$$\overline{Q}(x) \equiv \Pr[d_j \geq x | j \in G_i] = \sum_{j=x}^{m} q(i),$$

and with a slight abuse of notation, denote the neighbor adoption probability defined in (3.18) as $\lambda(\delta^*)$, which is correspondingly

$$\lambda(\delta^*) = \overline{Q}(\delta^*). \tag{5.5}$$

As in Section 4.2, given that the thresholds defining the Bayes-Nash equilibria are scalar values, we can compute them directly. If all agents play the symmetric strategy $s : D \to A$ with threshold δ^* on degree, following (3.19) and (3.20), the expected payoff to an agent of degree $x \in D$ is

$$w(x, \delta^*) \equiv \left(\sum_{y=0}^{x} u(y, 1) \binom{x}{y} (1 - \overline{Q}(\delta^*)]^y [\overline{Q}(\delta^*)]^{(x-y)} \right) - c, \tag{5.6}$$

and therefore, the set Δ of all thresholds on degree corresponding to symmetric Bayes-Nash equilibria is defined by:

$$\Delta = \{x : w(x-1, x) < 0, w(x, x) \geq 0\} \tag{5.7}$$

Two points are specifically worth highlighting about this example. First, while (4.11) and (5.6) are quite similar, the latter is based on the posterior neighbor degree distribution. Therefore, if one were to try and approximate away the structure of the underlying social network into a continuous type variable of some kind, the results would tend to systematically underestimate adoption unless the type distribution was based on the posterior degree distribution.

More importantly, explicitly modeling the structure of the social network allows one to study the structure of the adoption network G_α, which is the graph whose vertices are agents who have adopted, and whose edges are those edges in G connecting vertices corresponding to adopting agents. Denote the degree distribution of the adoption network as $\alpha(y)$. Now, the probability that a agent has y neighbors in the adoption network, conditional on the agent's degree being $x < \delta^*$ is zero, since no agents of degree less than δ^* adopt the product. For $x \geq \delta^*$,

$$\Pr[\alpha(y) = y | n_k = x] = \begin{cases} \binom{x}{y}[\lambda(\delta^*)]^y[1-\lambda(\delta^*)]^{(x-y)}, & y \leq x \\ 0, & y > x. \end{cases} \tag{5.8}$$

Summing over all $n \in D$, weighted by the degree distribution $p(n)$, one gets

$$\alpha(y) = \begin{cases} A \sum\limits_{x=\delta^*}^{m} \left[\binom{x}{y}[\overline{Q}(\delta^*)]^y[1-\overline{Q}(\delta^*)]^{(x-y)}p(x)\right] & \text{for } y \leq \delta^* \\ A \sum\limits_{x=y}^{m} \left[\binom{x}{y}[\overline{Q}(\delta^*)]^y[1-\overline{Q}(\delta^*)]^{(x-y)}p(x)\right] & \text{for } y > \delta^* \end{cases} \tag{5.9}$$

where A is a constant that ensures that the probabilities sum to 1. The following proposition relates the structure of the underlying social network to that of the adoption network in a more general way:

Proposition 6. *Let $\Phi_p(w) \equiv \sum\limits_{x=0}^{\infty} p(x)w^x$ denote the moment generating function of the degree distribution of the social network G, and correspondingly, let $\Phi_\alpha(w) \equiv \sum\limits_{x=0}^{\infty} \alpha(x)w^x$ denote the moment generating function of the degree distribution of the adoption network G_α, If agents play the symmetric Bayes-Nash equilibrium with threshold δ^*, then for δ^* sufficiently smaller than m,*

$$\Phi_\alpha(w) \cong \Phi_p(1 - \overline{Q}(\delta^*) + w\overline{Q}(\delta^*)).$$

The result in Proposition 6 may be important for at least three reasons. First, the "adoption networks" of many products can form the underlying social network on which the adoption of complementary products is based. For example, compatible applications may only be adopted by existing adopters of a specific platform. Second, if inverted appropriately, it could possibly provide important information to sellers of network goods who only observe the structure of their adoption networks (or a sample of this structure), and who may be interested in understanding the structure of the underlying social network of their potential customers towards increasing product adoption.

Third, the result is a first step towards developing techniques that may test the predictions of this theory. An adoption network is likely to be the richest empirical object that an interested researcher can observe. Proposition 6 establishes that given a distribution over the set of possible underlying social networks, and a parameter associated with a specific equilibrium, one can describe the distribution over the set of possible adoption networks. This presents at least two empirical possibilities. Assuming a distribution over social networks and given an empirical distribution for the structure of the adoption network, one might infer which equilibrium is being played by estimating its associated threshold, and assess whether it is in fact the greatest Bayes-Nash equilibrium. Alternatively, assuming that the best equilibrium is being played, the degree distribution of the underlying social networks and potentially the strength of the local adoption complementarities can be estimated from empirically observed adoption networks. Each of these represents an interesting direction for future research.

Further analysis of the structure of adoption networks is available in Sundararajan (2004).

6. Summary and directions for future work

This paper has presented a new model of local network effects. It allows one to model local structures that determine adoption complementarities between agents. These structures are discrete and can be specified quite generally, while still incorporating a (standard) continuous variation in the strength of network effects across customers. It admits standard models of network effects and models based on widely used generalized random graphs as special cases. It provides a simple basis – the neighbor adoption probability – for determining and ranking all symmetric Bayes-Nash equilibria. It establishes a simple one-to-one correspondence between each Bayes-Nash equilibrium and a corresponding fulfilled-expectations equilibrium based on agents forming expectations locally. A number of economic situations involve network effects that are localized, and one hopes that this paper forms a first step in providing a general basis for modeling them more precisely.

The focus of this paper is on local network effects arising out of direct adoption complementarities between small heterogeneous groups of agents. Additionally, it is well known that many goods display indirect network effects, under which the benefits to each adopter are through the development of higher quality complementary goods (for instance). Many economic situations involve both direct and indirect network effects, and developing a model that also admits indirect network effects is an interesting direction for future work (this would involve payoff functions that were a combination of the general form in Section 3 and the special case of Section 4.2). A related extension of the model might study two-sided local network effects which arise in many marketplaces (Rysman, 2004), and it appears that each of the results in Propositions 1 through 5 would continue to hold when the set of underlying social networks is restricted to containing only bipartite graphs. Another natural extension would involve agents adopting one of many incompatible network goods, perhaps dynamically and using an evolutionary adjustment process based on the state of adoption of one's neighbors (Sandholm, 2002). Some ideas towards developing ways of testing the predictions of theories based on local network effects are discussed immediately following Proposition 6, and these represent yet another promising direction of future work.

A contrast between the equilibria of the adoption game in this paper and those obtained when agents have progressively "more" information about the structure of the underlying social network would be interesting, since it would improve our understanding of whether better informed agents adopt in a manner that leads to more efficient outcomes. It would also indicate how robust the predictions of models that assume that agents know the structure of these graphs are, if in fact these agents do not.

While the assumption of symmetric independent posteriors models uncertainty about the social network for a wide range of cases, as illustrated by the examples in Section 4, it may preclude distributions over social networks that display "small world" effects (Milgram, 1967, Watts, 1999). Models of these networks have a specific kind of clustering that lead to posteriors that, while independent across neighbors for a given agent, are conditional on the agent's degree. A natural next step is to extend the analysis of this paper to admit symmetric *conditionally* independent posteriors of this kind, and then to explore how more elaborate local clustering of agents may affect equilibrium. This may be of particular interest in a model of competing incompatible network goods.

7. References

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A. Appendix: Proofs

Proposition 1 is a direct application of the results in Theorem 1 of Van Zandt and Vives (2004). Proposition 5 is proved in the text of the paper preceding the result. The proofs of all the other results in the paper are in this appendix.

Proof of Proposition 2

(a) The definition of $\Pi(d_i, \theta_i)$ is reproduced below from (3.9):

$$\Pi(d_i, \theta_i) \equiv \int\limits_{t \in \Theta^{(d_i)}} \left(\sum_{x \in D^{(d_i)}} \left[u(\sum_{j=1}^{d_i} s(x_j, t_j), \theta_i) \prod_{j=1}^{d_i} q(x_j) \right] \right) d\mu(t|d_i). \tag{A.1}$$

If there exists $d_i \in D$, and $\theta_i, \theta_i' \in \Theta$ such that $s(d_i, \theta_i) = 1$ and $s(d_i, \theta_i') = 0$, it follows from (3.10) and (3.11) that $\Pi(d_i, \theta_i) \geq c$ and $\Pi(d_i, \theta_i') < c$, which implies that

$$\Pi(d_i, \theta_i) > \Pi(d_i, \theta_i'). \tag{A.2}$$

Since $u_2(x, t) > 0$ for all $x \in D, t \in \Theta$, it follows that for a fixed strategy s, $\Pi_2(x, t) \geq 0$. Therefore, (A.2) implies that $\theta_i > \theta_i'$, which establishes that $s(x, t)$ is non-decreasing in t.

From (A.1), for any symmetric Bayesian Nash equilibrium strategy s,

$$\Pi(d_i + 1, \theta_i) = \int\limits_{t \in \Theta^{(d_i+1)}} \left(\sum_{x \in D^{(d_i+1)}} \left(u(\sum_{j=1}^{d_i+1} s(x_j, t_j), \theta_i) \prod_{j=1}^{n+1} q(x_j) \right) \right) d\mu(t|d_i + 1), \tag{A.3}$$

which can be rewritten as

$$\Pi(d_i{+}1, \theta_i) = \int\limits_{t'=0}^{1} \left[\int\limits_{t \in \Theta^{(d_i)}} \left(\sum_{x' \in D} \left(\sum_{x \in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j, t_j) + s(x', t'), \theta_i) \prod_{j=1}^{d_i} q(x_j) \right) \right) q(x') \right) d\mu(t|d_i) \right] f(t')dt', \tag{A.4}$$

Since $s(x, t) \geq 0$, and $u_1(x, t) > 0$, it follows that

$$u(\sum_{j=1}^{d_i} s(x_j, t_j) + s(x', t'), \theta_i) \geq u(\sum_{j=1}^{d_i} s(x_j, t_j), \theta_i), \tag{A.5}$$

which in turn implies that

$$\sum_{x\in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j,t_j)+s(z',t'),\theta_i) \prod_{j=1}^{d_i} q(x_j) \right) \geq \sum_{x\in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j,t_j),\theta_i) \prod_{j=1}^{d_i} q(x_j) \right). \quad \text{(A.6)}$$

Since $\sum_{x\in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j,t_j)+s(x',t'),\theta_i) \prod_{j=1}^{d_i} q(x_j) \right)$ is independent of x', and $\sum_{x'\in D} q(x')=1$,

$$\sum_{x'\in D} \left(\sum_{x\in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j,t_j),\theta_i) \prod_{j=1}^{d_i} q(x_i) \right) \right) q(x') = \sum_{x\in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j,t_j),\theta_i) \prod_{j=1}^{d_i} q(x_j) \right), \quad \text{(A.7)}$$

which in turn implies that $\Pi(d_i+1,\theta_i)$ can be written as:

$$\Pi(d_i+1,\theta_i) = \int_{t\in\Theta^{(d_i)}} \left(\sum_{x\in D^{(d_i)}} \left(u(\sum_{j=1}^{d_i} s(x_j,t_j)+s(x',t'),\theta_i) \prod_{j=1}^{d_i} q(x_j) \right) \right) d\mu(t|d_i), \quad \text{(A.8)}$$

and since this expression is independent of t', (A.1), (A.5) and (A.8), verify that

$$\Pi(d_i+1,\theta_i) \geq \Pi(d_i,\theta_i). \quad \text{(A.9)}$$

Based on (A.9), it follows from (3.10) and (3.11) that $s(x,t)=1 \Rightarrow s(x+1,t)=1$, and therefore, $s(x,t)$ is non-decreasing in x. We have now established that any symmetric Bayesian Nash equilibrium strategy $s(x,t)$ is non-decreasing in both x and t for each $x\in D, t\in\Theta$. For a given $s(x,t)$, define

$$\theta^*(x) = \max\{t : s(x,t)=0\} \quad \text{(A.10)}$$

Clearly, $s(x,t)=1 \Leftrightarrow t>\theta^*(x)$. Moreover, since $s(x,t)$ is non-decreasing in x, it follows that $\theta^*(x)$ is non-increasing, which completes the proof.

(b) Follows directly from the fact that $u(0,t)=0$ for all $t\in\Theta$.

Proof of Lemma 2

(a) Assume the converse: that there are threshold vectors θ^A and θ^B such that $\lambda(\theta^A)>\lambda(\theta^B)$, and that $1\geq\theta^A(x)>\theta^B(x)$ for some $x\in D$. Therefore, there exists $t\in\Theta, \theta^B(x)<t<\theta^A(x)$ such that $s^A(x,t) = 0$ and $s^B(x,t) = 1$. From (3.10) and (3.11), and (3.21), this in turn implies that

$$\sum_{y=1}^{x} u(x,t)B(y|x,\theta^A) < \sum_{y=1}^{x} u(x,t)B(y|x,\theta^B). \tag{A.11}$$

Since $\lambda(\theta^A) > \lambda(\theta^B)$, this contradicts Lemma 1, and the result follows.

(b) If $\lambda(\theta^A) = \lambda(\theta^B)$, then from (3.21), the payoff to agent i from adoption is identical under A and B, for any $d_i \in D, \theta_i \in \Theta$. The result follows immediately from (3.10) and (3.11).

Proof of Proposition 3

Denote the payoff from adoption under strategy s^I for an agent of type (d_i, θ_i) as $\Pi^I(d_i, \theta_i)$.

(i) $\lambda(\theta^A) > \lambda(\theta^B) \Rightarrow s^A$ *strictly Pareto-dominates* s^B*:* If $\lambda(\theta^A) > \lambda(\theta^B)$, then from Lemma 2:

$$s^B(d_i, \theta_i) = 1 \Rightarrow s^A(d_i, \theta_i) = 1,$$

which ensures that $s^A(d_i, \theta_i) \geq s^B(d_i, \theta_i)$ for each $(d_i, \theta_i) \in D \times \Theta$. Also, if $\lambda(\theta^A) > \lambda(\theta^B)$, Lemma 1 and (3.21) imply that $\Pi^A(d_i, \theta_i) > \Pi^B(d_i, \theta_i)$. Therefore, for each $(d_i, \theta_i) \in D \times \Theta$ under which $s^A(d_i, \theta_i) = s^B(d_i, \theta_i) = 1$, the payoff to each agent from the symmetric strategy s^A is strictly higher, which implies that so long as the set of (d_i, θ_i) under which $s^A(d_i, \theta_i) = 1$ is non-empty, s^A strictly Pareto-dominates s^B.

(ii) s^A *strictly Pareto-dominates* $s^B \Rightarrow \lambda(\theta^A) > \lambda(\theta^B)$*:* Suppose s^A strictly Pareto-dominates s^B, and assume that $\lambda(\theta^A) \leq \lambda(\theta^B)$. From Lemma 1 and (3.21), this implies that $\Pi^A(d_i, \theta_i) \leq \Pi^B(d_i, \theta_i)$, which in turn implies that for $(d_i, \theta_i) \in D \times \Theta$ under which $s^A(d_i, \theta_i) = s^B(d_i, \theta_i) = 1$, the payoff to each agent from the symmetric strategy s^A is (weakly) lower than the payoff to each agent from the symmetric strategy s^B. Therefore, for s^A to strictly Pareto-dominate s^B, there must be $(d_i, \theta_i) \in D \times \Theta$ such that $s^A(d_i, \theta_i) = 1$ and $s^B(d_i, \theta_i) = 0$, which, given that $\lambda(\theta^A) \leq \lambda(\theta^B)$, contradicts Lemma 2.

Proof of Proposition 4

The notion of coalition-proofness used here involves the following ideas:

(i) Anonymous coalitions: Each player in the coalition knows how many other agents there are in the coalition, but does not know the identity of these agents (or they know the identity of these agents but do not base their strategies on this information). Specifically, a player i does not use the information that one or more members of the coalition might be members of G_i.

(ii) Deviations in pure strategies: Under the deviation, each member i of the coalition plays a pure strategy that depends on her type (d_i, θ_i).

(iii) Strictly Pareto-improving deviations: For a deviation to be valid, it should be strictly Pareto-improving for all agent in the coalition: that is, for each agent i in the coalition, and for each $(d_i, \theta_i) \in D \times \Theta$, the payoff under the deviation should be at least as high as the corresponding payoff under the symmetric strategy with threshold θ^*_{gr}, and strictly higher for some $(d_i, \theta_i) \in D \times \Theta$.

(iv) Self-enforcing deviations: For a deviation to be valid, there should be no strictly Pareto-improving deviations (of the kind described above – anonymous and in pure strategies) by any subset of players in the coalition. This is based on the standard idea of self-enforcing deviations in Bernheim, Peleg and Whinston (1987).

Define the following subsets of $D \times \Theta$:

$$\begin{aligned} H &= \{(x,t) \in D \times \Theta \text{ such that } t > \theta^*_{gr}(x)\} \\ M &= \{(x,t) \in D \times \Theta \text{ such that } t = \theta^*_{gr}(x)\} \\ L &= \{(x,t) \in D \times \Theta \text{ such that } t < \theta^*_{gr}(x)\} \end{aligned} \tag{A.12}$$

Denote the symmetric strategy with threshold θ^*_{gr} as s^*. Suppose there is a coalition $S \in N$ and a corresponding strategy $\sigma_i : D \times \Theta \to A$ for each $i \in S$ such that the deviation according to σ_i is strictly Pareto-improving for each i, and is self-enforcing. Since the payoff to player i when $\sigma_i(d_i, \theta_i) = 0$ is zero, a deviation involving σ_i is not strictly Pareto-improving unless $\sigma_i(x,t) = 1$ for each $(x,t) \in H$.

Consequently, for each $i \in S$, there must be $(x,t) \in L$ such than $\sigma_i(x,t) = 1$. This is because $s^*(x,t) = 1$ for each $(x,t) \notin L$, and if $\sigma_i(x,t) = 0$ for all $(x,t) \in L$, then σ_i yields identical (or weakly lower) payoffs as s^*, and cannot be strictly Pareto-improving.

Next, proceeding as in the proof of Proposition 2, it is straightforward to establish that for each $i \in S$, $\sigma_i(x,t)$ is non-decreasing in $x \in D$ and $t \in \Theta$; otherwise, there is a unilateral deviation by i that is strictly Pareto-improving for i, and the deviation by the coalition is not self-enforcing. As a consequence, the deviation by the coalition is of the form:

$$\sigma_i(x,t) = \begin{cases} 0, & t < \theta_i(x) \\ 1, & t \geq \theta_i(x) \end{cases} \tag{A.13}$$

for each $i \in S$, where $\theta_i(x) \leq \theta^*_{gr}(x)$ for each $i \in S, x \in D$; for each i, the inequality is strict for some $x \in D$.

Next, consider any two strategies s^A and s^B such that $s^A(x,t) \geq s^B(x,t)$ for all $(x,t) \in D \times \Theta$, and for some $y \in D$, $s^A(y,t) > s^B(y,t)$ for each $t \in T \subset \Theta$. Holding everything else constant, for each $j \in S$, the expected payoff from any strategy σ_j is strictly higher for each $(x,t) \in D \times \Theta$ when $i \in S$ plays s^A than when $i \in S$ plays s^B. As a consequence, any self-enforcing deviation must be *symmetric*. This is because if $\theta_i(y) < \theta_j(y)$ for some $y \in D$ and $i, j \in S$, and if both players i and j deviates to a strategy under which they play

$$\sigma_{ij}(x,t) = \begin{cases} 0, & t < \min[\theta_i(x), \theta_j(x)] \\ 1, & t \geq \min[\theta_i(x), \theta_j(x)] \end{cases} \tag{A.14}$$

then this strictly improves both of their expected payoffs, and constitutes a strictly Pareto-improving deviation by $\{i,j\}$ from the proposed deviation.

Therefore, to be self-enforcing, the deviation must be of the form

$$\sigma(x,t) = \begin{cases} 0, & t < \theta(x) \\ 1, & t \geq \theta(x) \end{cases} \tag{A.15}$$

for each $i \in S$, where where $\theta(x) \leq \theta^*_{gr}(x)$ for each $x \in D$, and $\theta(y) < \theta^*_{gr}(y)$ for some $y^* \in D$. Now, suppose all agents $i \in N$ play according to the strategy $\sigma(x,t)$. The switch by agents $i \in N \backslash S$ from playing s^* to playing σ increases the expected payoffs to all agents $i \in S$, since $\sigma(x,t) \geq s^*(x,t)$ with the inequality being strict for $y^* \in D$, $t \in [\theta(y), \theta^*_{gr}(y)]$. Since $\sigma(x,t)$ was a strictly Pareto-improving deviation to begin with, the symmetric strategy $\sigma(x,t)$ played by all agents strictly Pareto-dominates the symmetric strategy $s^*(x,t)$ played by all agents. Consequently, since each θ_i takes continuous values in Θ, and the action space $A = \{0,1\}$ is binary, one can now start with the threshold vectors of σ and construct a symmetric Bayes-Nash equilibrium that strictly Pareto-dominates s^*. This leads to a contradiction, since s^* is by definition, the greatest symmetric Bayes-Nash equilibrium, and completes the proof.

Proof of Proposition 6

Recall that $m = \max\{x : x \in D\}$, the maximum possible degree for any of the n agents. The expression for $\alpha(y)$, the degree distribution of the adoption network, is reproduced from (5.9) below:

$$\alpha(y) = \begin{cases} A \sum_{x=\delta^*}^{m} \left[\binom{x}{y} [\overline{Q}(\delta^*)]^y [1-\overline{Q}(\delta^*)]^{(x-y)} p(x)\right] & \text{for } y \leq \delta^* \\ A \sum_{x=y}^{m} \left[\binom{x}{y} [\overline{Q}(\delta^*)]^y [1-\overline{Q}(\delta^*)]^{(x-y)} p(x)\right] & \text{for } y > \delta^* \end{cases}, \tag{A.16}$$

which can be rearranged as:

$$\alpha(y) = \begin{cases} A \sum_{x=\delta^*}^{m} \left[\binom{x}{y} \left[\frac{\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^y [1-\overline{Q}(\delta^*)]^x p(x)\right] & \text{for } y \leq \delta^* \\ A \sum_{x=y}^{m} \left[\binom{x}{y} \left[\frac{\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^y [1-\overline{Q}(\delta^*)]^x p(x))\right] & \text{for } y > \delta^* \end{cases}, \tag{A.17}$$

By definition, the generating functions of the degree distributions of the social network $\Phi_p(w)$ and the adoption network $\Phi_\alpha(w)$ are:

$$\Phi_p(w) \equiv \sum_{k=0}^{\infty} p(k) w^k; \tag{A.18}$$

$$\Phi_\alpha(w) \equiv \sum_{k=0}^{\infty} \alpha(k) w^k. \tag{A.19}$$

From (A.17) and (A.19),

$$\begin{aligned} \Phi_\alpha(w) &= A \sum_{k=0}^{\delta^*-1} \left[\sum_{x=\delta^*}^{m} \left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k \binom{x}{k} [1-\overline{Q}(\delta^*)]^x p(x) \right)\right] \\ &\quad + A \sum_{k=\delta^*}^{m} \left[\sum_{x=k}^{m} \left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k \binom{x}{k} [1-\overline{Q}(\delta^*)]^x p(x) \right)\right]. \end{aligned} \tag{A.20}$$

One can interchange the order of summation for the first part of (A.20) with no changes in expressions, to:

$$A \sum_{x=\delta^*}^{m} \left[\sum_{k=0}^{\delta^*-1} \left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k \binom{x}{k} [1-\overline{Q}(\delta^*)]^x p(x) \right)\right]. \tag{A.21}$$

Interchanging the order of summation of the second part of (A.20), one gets:

$$A \sum_{x=\delta^*}^{m} \left[\sum_{k=\delta^*}^{x} \left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k \binom{x}{k} [1-\overline{Q}(\delta^*)]^x p(x) \right)\right]. \tag{A.22}$$

From (A.20-A.22), it follows that

$$\Phi_\alpha(w) = A\sum_{x=\delta^*}^{m}\left[\sum_{k=0}^{\delta^*-1}\left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k\binom{x}{k}[1-\overline{Q}(\delta^*)]^x p(x)\right)\right] + A\sum_{x=\delta^*}^{m}\left[\sum_{k=\delta^*}^{x}\left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k\binom{x}{k}[1-\overline{Q}(\delta^*)]^x p(x)\right)\right], \tag{A.23}$$

which reduces to

$$\Phi_\alpha(w) = A\sum_{x=\delta^*}^{m}\left[\sum_{k=0}^{x}\left(\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k\binom{x}{k}[1-\overline{Q}(\delta^*)]^x p(x)\right)\right]. \tag{A.24}$$

Grouping terms not involving k on the right hand side of (A.24), one gets:

$$\Phi_\alpha(w) = A\sum_{x=\delta^*}^{m}\left[[1-\overline{Q}(\delta^*)]^x p(x)\left(\sum_{k=0}^{x}\left(\binom{x}{k}\left[\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right]^k\right)\right)\right]. \tag{A.25}$$

Using the identity

$$(1+x)^n = \sum_{i=0}^{n}\left[\binom{n}{i}x^i\right], \tag{A.26}$$

(A.25) reduces to:

$$\Phi_\alpha(w) = A\sum_{x=\delta^*}^{m}\left[[1-\overline{Q}(\delta^*)]^x p(x)\left(1+\frac{w\overline{Q}(\delta^*)}{1-\overline{Q}(\delta^*)}\right)^x\right], \tag{A.27}$$

which simplifies to:

$$\Phi_\alpha(w) = A\sum_{x=\delta^*}^{m}\left(p(x)[1-\overline{Q}(\delta^*)+w\overline{Q}(\delta^*)]^x\right) \tag{A.28}$$

From (A.19), since $p(x) = 0$ for $x > m$,

$$\Phi_p([1-\overline{Q}(\delta^*)+w\overline{Q}(\delta^*)]) = \sum_{x=0}^{m}\left(p(x)[1-\overline{Q}(\delta^*)+w\overline{Q}(\delta^*)]^x\right), \tag{A.29}$$

and comparing (A.28) and (A.29), the result follows.