**LABORatorio R. Revelli**
**Centre for Employment Studies**

# Generalizing Gibrat

*Reasonable multiplicative models of firm dynamics with entry and exit*

Matteo Richiardi
LABORatorio Riccardo Revelli

Working Papers Series
No. 21

Collegio "Carlo Alberto" via Real Collegio, 30 - 10024 Moncalieri (TO)
Tel. +39 011.640.26.59/26.60 - Fax +39.011.647.96.43 – www.labor-torino.it - labor@labor-torino.it
LABOR is an independent research centre within Coripe Piemonte

# Generalizing Gibrat

*Reasonable multiplicative models of firm dynamics with entry and exit*

Matteo Richiardi
LABORatorio Riccardo Revelli

**Version**: February 2003

***Abstract***

Multiplicative models of firm dynamics 'à la Gibrat' have become a standard reference in industrial organization. However, some unpleasant properties of their implied dynamics – namely, their explosive or implosive behaviour (firm size and number collapsing to zero or increasing indefinitely) - have been given only very little attention. In this paper I investigate which modifications to the standard multiplicative model of firm dynamics lead to stable (and reasonable) distributions of firm size.

An agent-based simulation study is performed, and a methodology is proposed to recover the (aggregate) laws governing the system by estimating the reduced form, i.e. the *local* data generating process, on the artificial data resulting from a number of artificial experiments.

I show that in order to obtain stable systems for a wide range of average growth rate, either heteroskedasticity in the growth rates has to be assumed, or entry and exit mechanisms included. While other particular, ad hoc, entry and exit mechanisms could be imagined, I show that combining the broad class of *threshold entry mechanisms* and the more restricted class of *threshold exit mechanisms* with overcapacity penalizing all firms (where entry and exit are determined with reference to an exogenously defined total capacity of the market), lead to stable distributions even in the case of growth rate homoskedasticity, given a non-zero minimum threshold for firm size.

**Keywords**: Firm growth, Gibrat's Law, Entry, Exit, Simulation

**JEL classification**: L11, C63

**Correspondence**: Matteo Richiardi, LABORatorio Riccardo Revelli Centre for employment studies, Collegio Carlo Alberto, via Real Collegio 30 - 10024 Moncalieri, Torino, Italy, email: m.richiardi@labor-torino.it

## Introduction

Multiplicative models of firm dynamics 'à la Gibrat' have become a standard starting point in the industrial organization and economic geography literature. However, some unpleasant properties of their implied dynamics – namely, their explosive or implosive behaviour (firm size and number collapsing to zero or increasing indefinitely) - have been given only very little attention. Overall, these models may be well suited to study particular urban population dynamics in non-mature economies, where 'explosive' dynamics may also be welcome, but they seem to offer very little to the study of firm growth.

It is thus surprising that such a big strand in the literature of applied industrial economics is devoted to trying to confirm or reject Gibrat assumptions of mean and variance homoskedasticity in the rate of firm growth[1]. Most papers don't pay attention at all to the fact that the growth rates they found in the data are incompatible with any definition of equilibrium in the industry, given the multiplicative model they assume. Here, two issues are at stake. First, we may actually face an out-of-equilibrium situation, but then it seems to make little sense to characterize the industry at that stage. Second, changes in the theoretical model could be more appropriate, but this could in turn imply the need to focus also on other things than the growth rate of existing firms. Entry and exit dynamics are first candidate, as it will be shown below.

Multiplicative models have nourished because they lead very easily to nice aggregate distributions of firm size. However, the focus of the literature on the exact *shape* of the resulting firm size distribution is surprising, compared with the little attention its *anchorage* has received. We should not pay too much attention to a model that predicts a Power Law distribution of firm size, as long as this distribution degenerates rapidly to zero, or to infinity.

In this paper, I want both less and more. I look at 'reasonable' distributions of firm size, where I define a Reasonable distribution (R-distribution) as:

- a left skewed and truncated distribution
- with fat right tail, in order to span empirically over some orders of magnitude
- with finite stationary mean and variance,
- of a finite stationary number of firms.

In the next sections, I will look at which conditions a multiplicative model of firm growth must satisfy in order to show an R-distribution of firm size. In particular, I will consider different entry and exit mechanisms. The paper is structured as follows. The first section is devoted to a brief survey of the use of multiplicative models in the literature. Section 2 describes a general multiplicative model with entry and exit mechanisms embedding most models described in section 1 as particular cases. The case for a simulation study is put forward in section 3. After discussing the strength and weaknesses of this approach, a methodology for overcoming some of its traditional limits is proposed. Section 4 deals with the simulation set-up, while in the following section I deal with the issue of recognizing the long-term equilibrium of each simulation run. Section 6 contains the results, both for models without (6.1) and for models with (6.2) entry and exit dynamics. Section 7 offers my conclusions.

[1] First empirical tests date back to the work of Hart and Prais (1956), Hart (1962), Mansfield (1962), Hymer and Pashigian (1962) and Samuels (1965). Other related work can be found in Singh and Whittington (1975), Chesher (1979), Kumar (1985), Leonard (1986), Evans (1987 a, b), Hall (1987), Boeri (1989), Contini and Revelli (1989), FitzRoy and Kraft (1991), Variyam and Kraybill (1992), Wagner (1992), Amirkhalkhali and Mukhopadhyay (1993), Bianco and Sestito (1993), Dunne and Hughes (1994), Tschoegl (1996), Amaral, Buldyrev, Havlin, Leschhorn, Maass, Salinger, Stanley and Stanley (1997), Harhoff, Stahl and Woywode (1998), Hardwick and Adams (1999), Hart and Oulton (1999), Fariñas and Moreno (2000), Geroski, Lazarova, Urga and Walters (2000), Machado and Mata (2000), Acs and Armington (2001), Vander Vennet (2001), Audretsch, Klomp, Santarelli and Thurik (2002), Delmar, Davidsson and Gartner (2002), Goddard, Wilson and Blandon (2002). For a survey of the main results see Audretsch, Klomp, Santarelli and Thurik (2002). Stylized facts are worked out in Caves (1998).

## 1. The literature

It is a well known fact that firm size distribution – as well as many other aggregate phenomena like city size, web graph and file size, word frequency, average weight of different species… - exhibits common features across time and space. In particular, it is recognized to be highly skewed to the left, and to span over several order of magnitude, i.e. to show 'fat' tails. Moreover, many studies have found that this distribution can be very well approximated by a Power Law, given by the probability distribution function:

[1] $$P[X = x] \sim x^{-(k+1)} = x^{-a}$$

The first appearance of this - now very fashionable - distribution dates back to Pareto in the late XIX century. He showed that income distribution follows what since then has been called a Pareto distribution, with a cumulative distribution function given by:

[2] $$P[X > x] \sim x^{-k}$$

More than 30 years later, George Kingsley Zipf, a Harvard linguistics professor, sought to determine the 'size' (or frequency of use in English text) $y$ of the *i*th most common word. He found that this frequency is inversely proportional to it's rank $r$. This regularity has of course been baptized after him and with the name of Zipf's Law

[3] $y_i \sim r_i^{-b}$, with $b$ close to unity

has found many applications, in particular to the study of city size distribution, where it is known to be surprisingly robust and stable.

Even if a long time had to pass before the connection was recognized, Zipf, Pareto and Power Law distributions are actually the same thing, Pareto being the c.d.f. of a Power Law and Zipf its expression in terms of rank, in the particular case of a Pareto coefficient equal to 1 (Adamic, 2000).

Once these regularities were noted, the challenge became finding simple statistical models to explain them[2]. Here, the benchmark is still Gibrat's 1930 Law of Proportionate Effect, stating that if growth rates of firms in a fixed population (i.e. abstracting from entry and exit dynamics) are independent of size and uncorrelated, the resulting distribution is lognormal. He thus introduced the first multiplicative model of firm dynamics

[4] $$S_{t+1} = \lambda_t S_t$$

Taking logs, this model reduces to

[5] $$\ln S_{t+1} = \sum \ln \lambda_t$$

The model is sometimes expressed in terms of instantaneous growth rate, $S_{t+1} = S_t\, exp(\lambda_t)$ , but it leads to the same conclusions. If the growth rates are independently distributed, by the law of great numbers each firm's logarithm of size at any time sufficiently far from the start is a

[2] see Sutton (1997) for a survey on the developments of these models

random extraction from a normal distribution. Thus, in this very simple model with no interaction between firms, firm size follows a lognormal distribution. Moreover, concentration in the industry will keep on increasing indefinitely, the more the higher the variance of the growth rate distribution.

Many variations of the Gibrat model have been developed in the literature, while remaining within a purely statistical description of firm dynamics. The main results of this strand of research was to show that even small variations from Gibrat Law lead to Power Law distributions. Indeed, Gibrat model itself lead to a Power Law distribution, although only in the limit and only if $\lambda$ is predominantly > 1.

Simon and Bonini (1958) introduced very simple entry dynamics, by assuming that:

- only a fixed number of independent opportunities arise in the market at each time,
- the probability of an existing firm taking up each opportunity is proportional to its size (Gibrat Law)
- the probability of a new firm taking up each opportunity is constant

With this model of 'preferential attachment' they showed that firm size distribution follows a Power Law, although only in the upper tail.

Kesten (1973) added to Gibrat model an additive term

[6]
$$S_{t+1} = \lambda_t S_t + \rho_t$$

showing that the model leads to a Power Law distribution if $\lambda$ predominantly > 1, while the distribution remains lognormal otherwise.

Levy and Solomon (1996) showed that a Power Law can be obtained also by adding a reflection condition to Gibrat model, i.e. by assuming that firm size is bounded from below to a threshold proportional to average firm size:

[7]
$$\begin{aligned} S_{t+1} &= \lambda_t S_t \\ S_{t+1} &= s_o \cdot \overline{S}_t \text{ if } S_{t+1} < s_o \cdot \overline{S} \end{aligned}$$

In another paper (Levy and Solomon, 1996b), they generalize this strand of literature by showing that power Law distributions arise very naturally from stochastic multiplicative dynamics[3].

Then, Blank and Solomon (2000) incorporate both entry and exit dynamics by assuming that firms disappear if they fall below a certain threshold $S_{min}$ (of magnitude 1), and that at each period $\Delta \mathrm{N} = \mathrm{k} \cdot (\mathrm{S}*_{\mathrm{t+1}} - S*_t)$ new firms enter the market , with size $S_{min}$ ($S*$ is the sum of all firms size, i.e. the total dimension of the industry). This model also leads to a Power Law distribution of firms size.

Finally, an interesting and general link between multiplicative models and Markovian processes is found by Cordoba (2002). He refers to city growth, and shows that, if size follows a stationary Markov continuous diffusion process and total urban population keeps growing, in order to produce a Power Law distribution cities <<must exhibit (i) an expected growth rate

[3] (in particular, they showed that the appearance of the scaling power laws is as generic in multiplicative stochastic systems as the Boltzmann law is in additive stochastic systems).

that is independent of their size; and (ii) a growth variance that is proportional to size $^{\delta-1}$, where $\delta$ is the Pareto exponent found in the data>>. This means that Zipf's Law (the case $\delta$ = 1) *must* result from Gibrat Law, if the hypothesis about the diffusion process are correct. Cordoba's result also allows for the emergence of new city, as far as the emergency rate is lower than the growth rate of existing cities.

However, multiplicative models – and in particular most models proposed so far in the literature - have often very unpleasant properties, that more than compensate the nice distributions they produce.
Let's start with the father of all multiplicative models, i.e. 'pure' Gibrat. This model is either implosive or explosive, depending whether

[8] $$E(\ln(1+r)) = 0$$

The intuition is the following: suppose size can only increase or decrease by 10% each period, with the same probability. Starting from a size of 1, suppose we face first a decrease and then an increase. In the first period size shrinks to .90, and bounces back only to .99 in the second period. Consider on the other side we face first an increase and then a decrease. Size moves to 1.1 in the first period ad again to .99 in the second! This effect is of course stronger the bigger the variance, and can be contrasted only with a positive average growth rate.

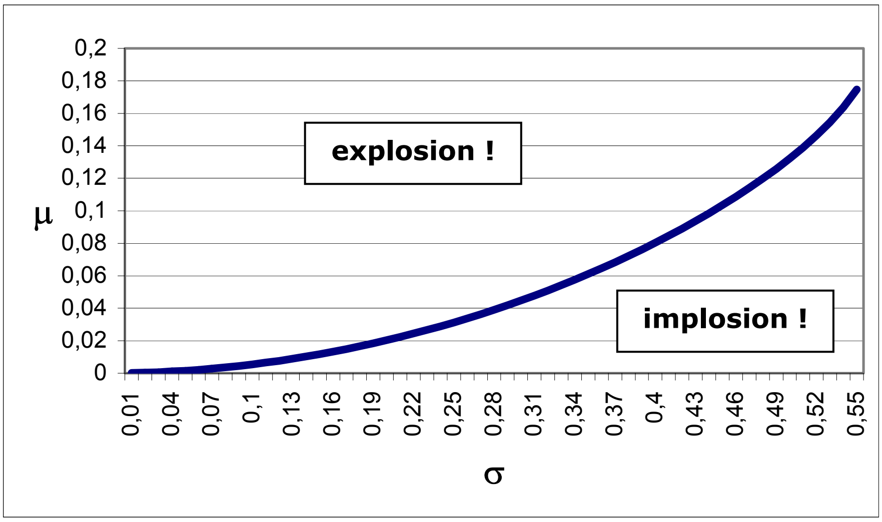
Solving equation [8] numerically, we find a very unstable equilibrium for which firm size remains on average constant, while departures from this equilibrium lead either to a degenerate distribution with zero mean and variance or to a degenerate distribution with infinite mean and variance.

In the case the growth rate is normally distributed

[9] $$\lambda_t = (1 + r_t)$$
$$r_t \sim N(\mu, \sigma)$$

we obtain the following combination of mean and standard deviation (that can be approximated to any degree by a polynomial of order n) satisfying the constraint, and thus leading to a non-degenerate distribution:

Figure 1: Stable equilibria in pure Gibrat model



Moreover, Gibrat model implies the concentration of the population (as - for instance – described by the Gini coefficient) always increases. Clearly, this is not a very reasonable model. Simon and Bonini variation does not help much: in their model, the number of firms in the market keeps rising; thus, existing firms face on average negative returns, since business opportunities are fixed. This implies the model implodes, no matter which value of the variance.

By adding a (stochastic) term $\rho$ Kesten preserves the model from the risk of implosion, but not from the risk of explosion. Moreover, for combinations of the parameters that guarantee an implosion in Gibrat model, firm size distribution in Kesten model is nothing else than that of $\rho$.

Levy and Solomon (1996 a) don't even solve the problem of implosion, since their floor threshold is parametrized to the average of the distribution, and thus can only slow down the process. Moreover, the existence of such a mobile threshold is difficult to justify from an economic point of view.

Blank and Solomon (2000) model faces the problem of implosion with the assumption that firms disappear if they fall below a minimum (constant) size. The model works fine only if the average growth rate is close to zero, i.e. in the implosion set. Implosion is avoided because the model provides an absolute anchorage to firm size, i.e. new firms size is independent of the existing firm size distribution. If the average growth rate is too high (where 'too high' means just slightly above 0) the number of firms keep rising indefinitely, and the convergence to a finite non-zero mean, equal to $1/k$, is guaranteed only by more and more new firms of minimum size contrasting the explosion of the existing firms. Again, not a reasonable dynamics, even if characterized by a Power Law distribution of firm size.

Cordoba's paper is the more general we have treated so far. However, it does not solve our problem of finding multiplicative models with 'stable' distributions, since total population in his model keeps also growing indefinitely, by assumption. Moreover, the characteristics of a continuous diffusion process do not fit properly an industrial dynamics perspective, since it is well known that firms can face drastic change in size, especially downwards. I won't investigate further relationships between other kind of Markovian processes and multiplicative models in this paper, although this could be of interest.

My paper follows a radically different approach. It investigates which kind of modifications to the pure Gibrat model lead to stable long-run equilibria. It is closer in spirit and in methods to McCloughan (1995). McCloughan considers a modified Gibrat model, which takes into account some of the main violations of Gibrat's Law (namely, the mean of the growth rate is allowed to be dependent on the size of the firm, and the growth rates are allowed to be serially correlated), together with a specific mechanism for entry and exit (entry is modelled as a *learning game* about the market opportunity between incumbents and potential entrants, which leads to a Poisson entry process; exit entails a distribution of critical sizes below which firms disappear). Calibration of the model yields thirteen empirical growth scenarios, six entry scenarios and three exit scenarios. The simulated data are then used to investigate the importance of growth, entry and exit in shaping concentration development.

My paper also simulates the effects of violations of Gibrat's Law, entry and exit on industry dynamics. However, it does not deal with real data. Rather, it is aimed at characterizing which processes lead to a distribution of firms size with some desired properties. A number of entry and exit mechanisms are considered, rather than just one. Which version of the general model better fits the data is left for future research.

## 2. The model

McCloughan discerns five types of violation of Gibrat's Law:

1) mean growth rate decreases with size of firm;
2) mean growth rate increases with size of firm;
3) growth variability decreases with size of firm;

4) growth variability decreases with age of firm;
5) growth rates exhibit first-order positive autocorrelation.

I'm concerned only with violations 1-3, i.e. with the issue of heteroskedasticity with respect to size. My starting point is a general multiplicative model, where growth rates are drawn from a Normal distribution, with mean and variance that are allowed to depend on the size of the firm (they can decrease to 0 with size at exponential speed, governed by two parameters, *m* and *s*; when *m* and *s* equal 0, mean and variance are homoskedastic). In order to reproduce the results of the literature, a minimum threshold is considered, so that firms cannot become smaller than a constant share $s_0$ of the average size. In the simplest case, where $s_0$ = 0, this guarantees no firm size becomes negative. Zero-dimension firms are cleaned up and exit the market, in order not to confound statistics.

[10]

$$
\begin{aligned}
&S_{t+1} = (1 + r_t) S_t \\
&r \sim \mathrm{N}(\mu(\mathrm{S_t}), \sigma(S_t)) \\
&\mu(\mathrm{S_t}) = \mu \cdot \exp(-m \cdot (S_t - 1)) \\
&\sigma(\mathrm{S_t}) = \sigma \cdot \exp(-s \cdot (S_t - 1)) \\
&S_{t+1} = s_0 \cdot \overline{S}_t \ \text{ if } S_{t+1} < s_0 \cdot \overline{S},\ s_0 \geq 0 \\
&\text{if } S_t = 0 \quad \text{firm exits}
\end{aligned}
$$

Different entry and exits dynamics are considered, representing two broad classes of mechanisms: threshold mechanisms and non-threshold ones.

**Non-threshold mechanisms** imply there is no reference to an exogenous threshold in determining entry and exit of firms, i.e. the model has no anchorage, except for the 'natural' floor at 0 for firms' size. Among this class, I consider:

- **Proportional Number entry**:
  - ➲ a constant share of total population of firms is added at each period
  - ➲ initial size is drawn from the lower half of the existing firms size distribution

[11]

$$
\begin{aligned}
&\Delta N_t = \alpha \cdot N_t \\
&\alpha \geq 0 \\
&S_{i,t} \sim \mathrm{D}_{1/2,\,\mathrm{t}}
\end{aligned}
$$

were $D_{1/2,\, t}$ is the lower half of the (empirical) firm size distribution at time t.

- **Proportional Number exit**:
  - ➲ a constant share of total population of firms exits the market at each period

- **Proportional Dimension entry** (the mechanism postulated by Blank & Solomon in their 2000 paper):
  - ➲ new firms are added in proportion of the increase in the total sector dimension
  - ➲ initial size is the exogenously determined minimum size allowed

[12]

$$
\begin{aligned}
&\Delta N_t = k \cdot (S_{tot,t} - S_{tot,t-1}) \\
&k \geq 0 \\
&S_{i,t} = S_{\min}
\end{aligned}
$$

- **Minimum Size exit** (the mechanism postulated by Blank & Solomon in their 2000 paper):
  - ➲ firms below minimum size $S_{min}$ exit the market

Among **threshold mechanisms**, where entry and exit are determined with reference to an exogenously defined demand (i.e. to a maximum capacity of the market, that could change exogenously from period to period), I consider:

- **Excess Demand entry**:
  - ➲ if total size of the market is less than the optimal dimension, (a part of) the gap is filled with new firms;
  - ➲ initial size is drawn from the lower half of the existing firms size distribution

[13]

$$\Delta N_t = \alpha \cdot (S_t^* - S_{t-1})$$
$$0 \le \alpha \le 1$$
$$S_{i,t} \sim \mathrm{D}_{1/2,\,\mathrm{t}}$$

were $S^*$ is the optimal size of the market, i.e. the dimension that keep supply and (exogenous) demand in equilibrium, given (exogenous) prices, and $D_{1/2,\, t}$ is the lower half of the (empirical) firm size distribution at time t.

- **Excess Supply Affects All exit**:
  - ➲ if total size of the market is less than the optimal dimension, first all firms are reduced proportionally. Then, firms smaller than a minimum size exit the market;
- **Excess Supply Affects Small exit**:
  - ➲ if total size of the market is less than the optimal dimension, smaller ones exit the market;
- **Excess Supply Affects Large exit**:
  - ➲ if total size of the market is less than the optimal dimension, larger ones exit the market;

Table 1: Entry and exit mechanisms

| | **ENTRY** | **EXIT** |
|---|---|---|
| **NON-THRESHOLD** | • Proportional Number<br>• Proportional Dimension | • Minimum Size<br>• Proportional Number |
| **THRESHOLD** | • Excess Demand | • Excess Supply Affects All<br>• Excess Supply Affects Small<br>• Excess Supply Affects Large |

Most of these mechanisms seem plausible, and altogether they characterize a fairly general class of entry and exit dynamics. Of course, the list could (and should) be extended, but it can work as an initial set of analysis.

It must be noted that I have not specified how size is measured: it could be both an input variable (employment) or an output variable (turnover). Of course, in case size employment is used as the target variable, the additional hypothesis of constant returns to scale must be made in order to justify threshold mechanisms.

## 3. Methodology

One reason why the literature has focused so far only on more restrictive models, with single entry and/or exit mechanisms, is that it becomes quite hard to deal analytically with more general models. Since my goal is studying more generally the interaction between multiplicative models and entry and exit mechanisms, I have to give something, and abandon the purity of analytical models. I will *simulate* my models.

Simulation is often thought to be less general than analytical models. This is because analytical results are conditional on the specific hypothesis made about the *model* only, while simulation results are conditional both on the specific hypothesis of the model and the specific values of the *parameters* used in the simulation runs.

This is partly reversed by the fact that simulation allows fairly less restrictive hypothesis about the model, since the results are computed and need not to be solved analytically. However, the problem to state general propositions about the dynamics of the model starting only from point observations remains.
Another way of stating more or less the same thing is the follow. Both an analytical model and a simulation model are expressed in their structural form, although in the simulation model more flexible *rules* may be specified instead of equations (one example is the following, hypothetical, rule for determining firm growth: "first look at the particular firm which is closest in size; if it exits the market, do the same with a probability *p*, otherwise grow accordingly to a function of some other parameters" – try to express this with a formula!). By solving an analytical model, we find the only one reduced form corresponding to the structural form of the model. This is impossible to do in a simulation model. The reduced form (the data generating process) remains unknown. But we may estimate the reduced form (better: the *local* data generating process) on the artificial data resulting from a number of (somehow designed) artificial experiments. This is the way I proceed. Of course, it is always possible that as soon as we move to other values of the parameters, the local data generating process will change dramatically, for example exhibiting singularities. But if the design of the experiments is sufficiently accurate, this problem becomes marginal, since we're not really interested in what happens only with an infinitesimal probability. Moreover, critical values of the parameters can often be guessed, and thus included in the experiments.

Before moving on, one last issue has to be addressed. In estimating the local data generating process, one functional form must be chosen. Having specified the micro-rules of the artificial world, the researcher generally knows which variables affect the outcome variable of interest, even if sometimes, in complicated models, the causal link between inputs and outputs may be quite indirect, and thus remain at first unnoticed. However, there are methodologies to reconstruct the causal structure from statistical data, as well as software applications that do it automatically (see, for instance, the Tetrad project at Carnegie Mellon University - http://www.phil.cmu.edu/projects/tetrad/). I plan to further investigate this approach in future publications, since I'm confident that the causal structure of the models developed here is simple enough to be known in advance. However, the issue of the choice of a functional form remains to be addressed. This is always an arbitrary choice, and may lead to very different specifications of the aggregate laws of the system.

However, insofar two different specifications provide the same description of the dynamics of the model in the relevant range of the parameters, we should not bother too much about which one is closest to the 'true' data generating process. Differently from estimation on real world data, the problem of a mispecified model unable to make good predictions in out-of-sample data is not important here, since we're not constrained to particular ranges of the parameters in the design of the artificial experiments.

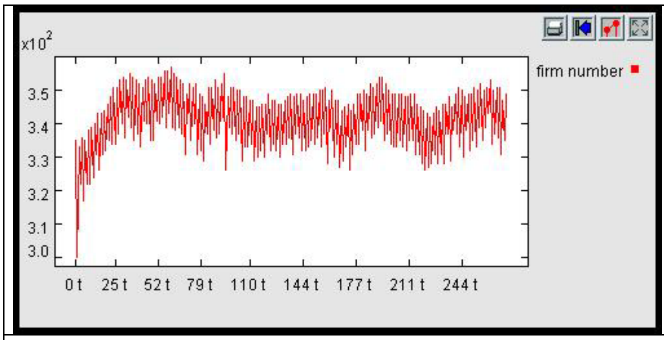
My approach to (meta-)model selection is to adopt the general-to-specific methodology (Hendry et al., 2002 ): a very general specification is first laid down, where all input variables are included among the regressors, as well as their powers, interaction terms and so on. Then, through a multi-test procedure, the specification is reduced to relevant terms only, by progressive elimination of non-significant variables.

## 4. The simulation

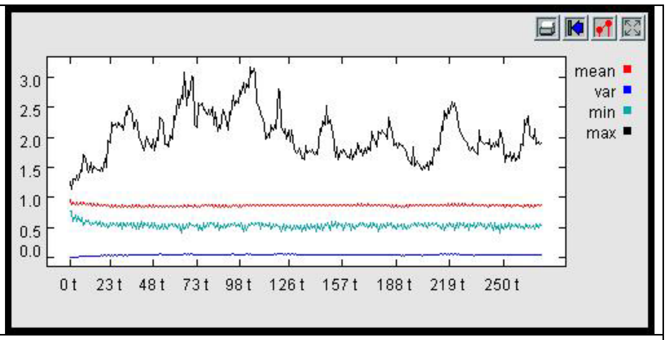
The simulation model is written in Java code, using JAS libraries, developed by Michele Sonnessa at the University of Torino (http://sourceforge.net/projects/jaslibrary/). Simulations can be monitored graphically. Here is a typical simulation output with parameters:

| | | |
|---|---|---|
| Entry mechanism: | Excess Demand | |
| Exit mechanism: | Excess Supply Affects All | |
| Initial number of firms: | 100 | |
| Maximum sector dimension: | 300 | |
| $\alpha$: | 1.0 | (Take-up rate) |
| $\mu$: | 0.1 | |
| $\sigma$: | 0.1 | |
| m: | 1.0 | (mean heteroskedasticity) |
| s: | 0.0 | (variance homoskedasticity) |

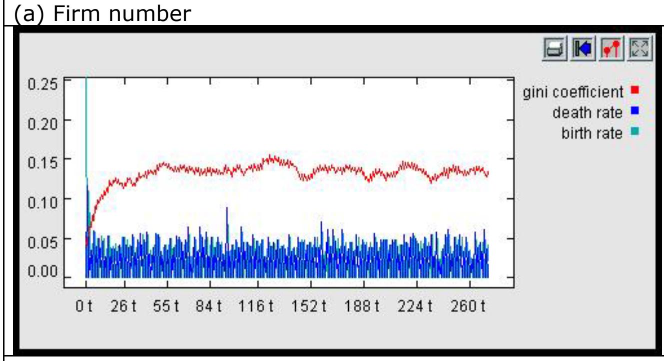
Figure 2: Simulation output



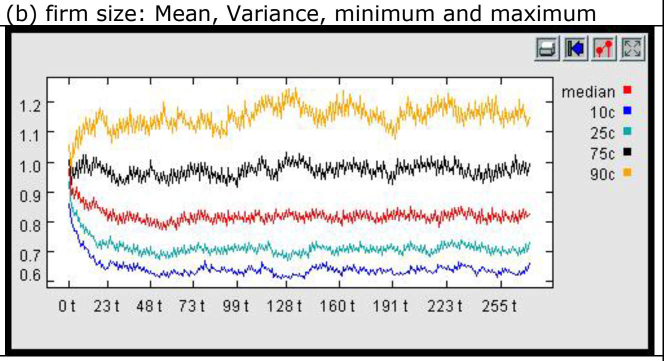

(a) Firm number



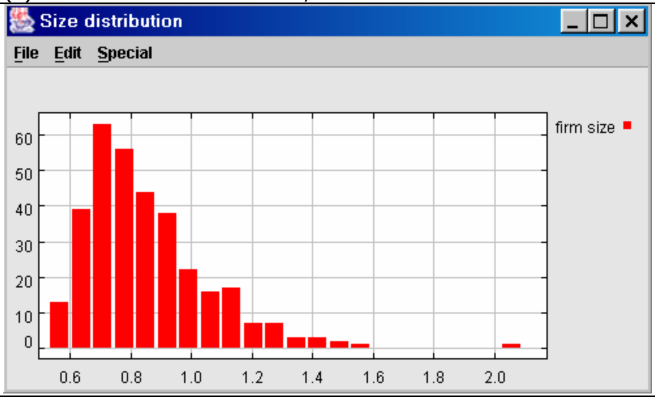

(b) firm size: Mean, Variance, minimum and maximum



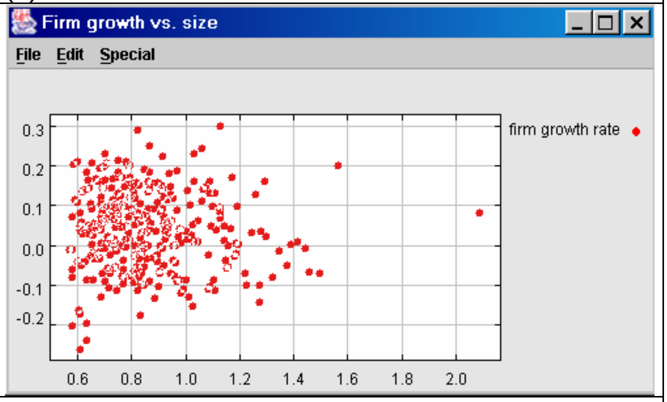

(c) Firm size Gini coefficient, Birth and Death rate



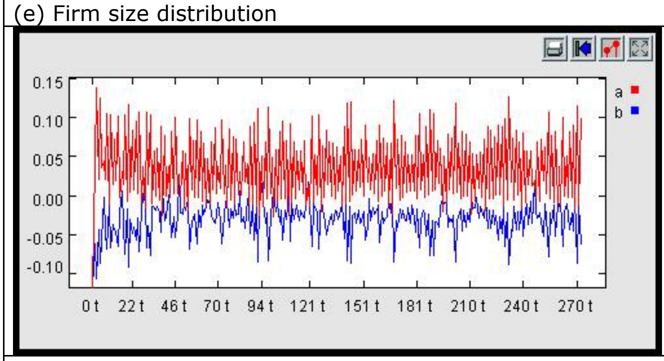

(d) Firm size: Percentiles



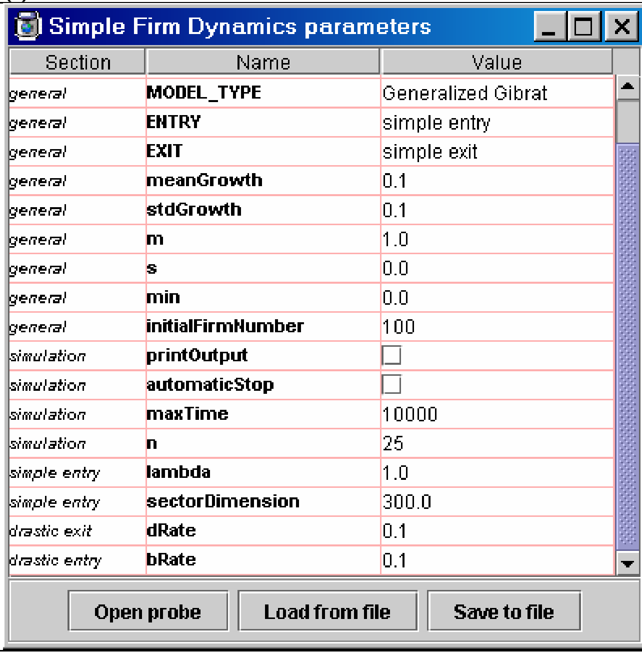

(e) Firm size distribution




(f) Growth rate vs. size




(g) Growth rate vs. size linear regression coefficients:
$R(t) = a + b\,S(t)$


Simple Firm Dynamics parameters

| Section | Name | Value |
|---|---|---|
| general | MODEL_TYPE | Generalized Gibrat |
| general | ENTRY | simple entry |
| general | EXIT | simple exit |
| general | meanGrowth | 0.1 |
| general | stdGrowth | 0.1 |
| general | m | 1.0 |
| general | s | 0.0 |
| general | min | 0.0 |
| general | initialFirmNumber | 100 |
| simulation | printOutput | ☐ |
| simulation | automaticStop | ☐ |
| simulation | maxTime | 10000 |
| simulation | n | 25 |
| simple entry | lambda | 1.0 |
| simple entry | sectorDimension | 300.0 |
| drastic exit | dRate | 0.1 |
| drastic entry | bRate | 0.1 |

Open probe | Load from file | Save to file

Here, parameters have to be edited by hand at each run. Clearly, this reduces the feasibility of a big number of experiments, and compromises the possibility of replication. JAS Multi-Run feature has thus been used, which allows batch runs of the simulation with parameters changing according to a pre-defined algorithm at each run[4].

## 5. Long run equilibrium

The R-distribution we're interested in is obviously an equilibrium distribution. Thus, the issue of when the equilibrium is reached becomes relevant. In other words, we need to decide when to stop each simulation run. Moreover, since simulations are performed in a batch mode, the 'human eye' criterion cannot be applied. The fact that we want to focus only on equilibrium relationships between the variables doesn't mean we're not interested in the out-of equilibrium dynamics. After all, one of the strengths of simulations is exactly the possibility of studying the disequilibrium dynamics of a system: even when a system can be characterized analytically in equilibrium, its out-of-the-equilibrium behaviour can often be investigated only through numerical simulations. Moreover, it is quite relevant to know how long it takes to reach the equilibrium: a model could be empirically relevant also when it explodes or implodes, as long as it takes long enough.

Thus, the wait-for-long-enough criterion is also not appropriate. We cannot wait – say – 10.000 periods before stopping a simulation and analysing its results because we want to know whether the system reached approximately the same state much earlier! Moreover, waiting too much reduces the number of experiments that can be done.

So, I developed a few algorithms to determine when the system becomes approximately stable. The first algorithm looks at whether a moving average of the median size and a moving average of the firm number remain constant, i.e. the first differences of the moving average remain around zero ($0 \pm 0.1$ times the moving standard error) for a sufficiently long period of time (if *n* is the moving average windows, the first differences must remain in the range in 95% of the last *n* periods of the simulation). The median is considered, rather than the mean, for rubustness concerns.

The second algorithm looks directly at the mean of the size distribution and at the firm number, and checks whether these series remain in a given range for long enough. Every *5n*, periods, where *n* is the moving average window used in the first algorithm, the two ranges are computed again. The first one, centred on the average dimension at that moment in time, is 0.2 times the standard deviation of firm size wide, while the second one goes from $1/2$ to $3/2$ of the firm number at that moment in time. The two series must remain bounded within these ranges for *5n* periods.

Altogether, these two algorithms work pretty well (which one is invoked first depending on the values of the parameters), and save on average around 75% of the time compared to an appropriate constant length criterion.

Simulation runs are also stopped when the firm number becomes too high (of course, entry dynamics are needed), in order to prevent the simulation to go slower and slower. Firm size in this case implodes, and the system is considered to exhibit 'non-reasonable' behaviour.

If no other stop mechanisms have already become binding, the wait-for-long-enough criterion is invoked, and the simulation is stopped after 2,000 periods. If the outcome looks 'reasonable', the simulation is re-run in the graphical mode, in order to investigate more in depth the dynamics of this slowly- or non-converging system.

---

[4] this feature also allows model calibration, if needed.

## 6. Results

As expected, the homoskedastic system (mean and variance of the growth rate independent of size) with no entry and exit exhibits very fast diverging dynamics, as soon as we move away from the line of figure 1.

### *6.1 Reasonable dynamics without entry and exit mechanisms*

#### *6.1.1 Mean heteroskedasticity*

The only way of obtaining stable systems within a 'pure' multiplicative model is to give away with the homoskedasticity assumption. In particular, heteroskedasticity of the mean of the growth rate is needed. However, as it will be shown, this heteroskedasticity doesn't have to be assumed all the times, since it can be found in the data when variance heteroskedasticity instead is assumed.

Figure 3 presents the outcomes of many simulation runs, where all parameters are kept constant except for the mean of the growth rate.

Figure 3 shows the case of mean heteroskedasticity: the mean of the growth rate keeps declining with size with an exponential speed, governed by the parameter *m*, which remains fixed at 1.0 (see equation 10). It plots the different values of long-run (equilibrium) mean firm size, for different values of the mean of the growth rate (the latter are referred to a size 1 firm). Standard deviation of the growth rate remains fixed at 0.1. This value of the standard deviation is low enough to guarantee that the number of firms remains constant, since it is very unlikely that any firm would pick up a growth rate lower than −1. However, in an infinite amount of time all firms will eventually experience such a casualty, and thus disappear.

The reason why the system reaches a 'reasonable' steady state, for values of $\mu(1)$ big enough, is the balance between the explosive tendency of smaller firms and the implosive tendency of bigger firms. Thus, all we need to have R-distributions is that the average of the growth rate at initial size is in the explosive region of figure 1. One typical resulting firms size distribution is reported in figure 4.

Figure 3: Long-run dynamics, mean heteroskedasticity (multiple simulation runs)

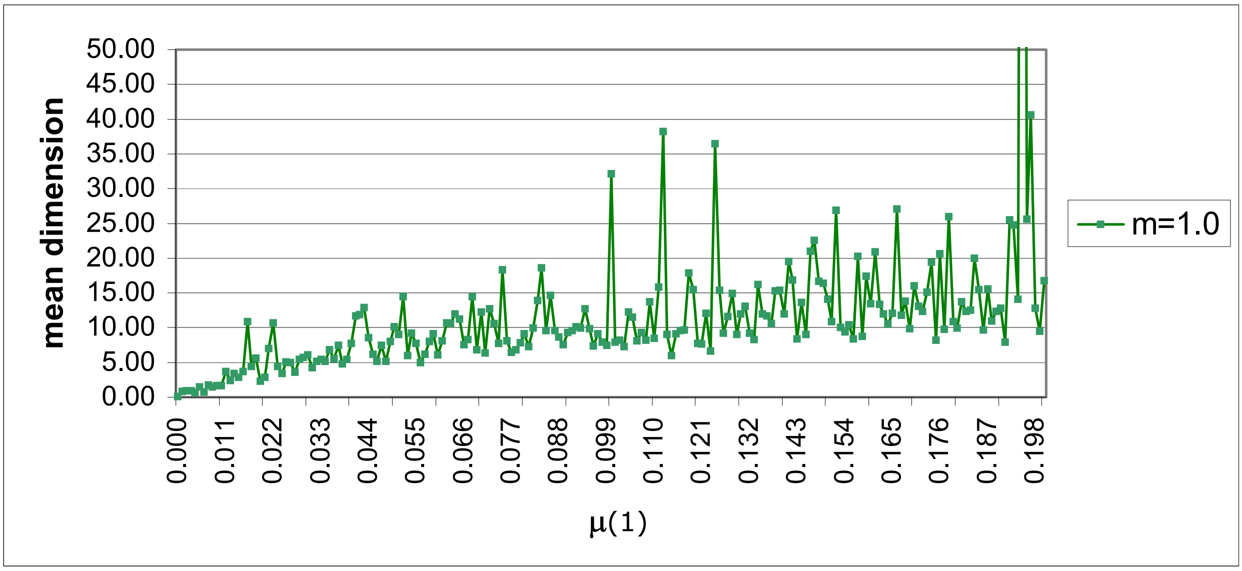
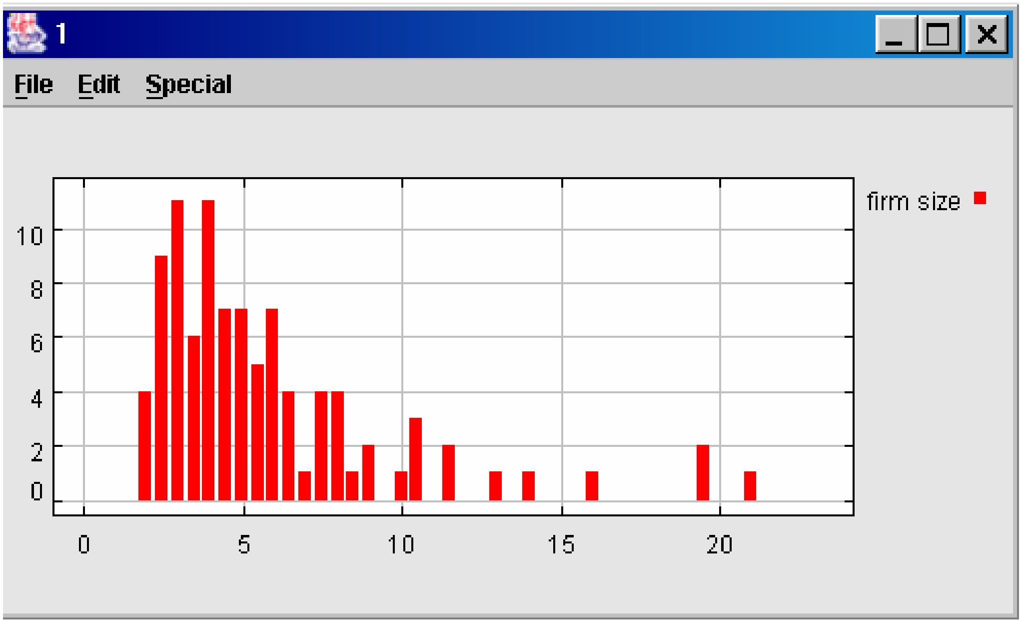


Figure 4: Firm size distribution, mean heteroskedasticity (single simulation run)




Mainly with the purpose of testing my (general-to-specific) interpretation procedure, we run over 1,500 simulations with varying values of the input parameters, namely:

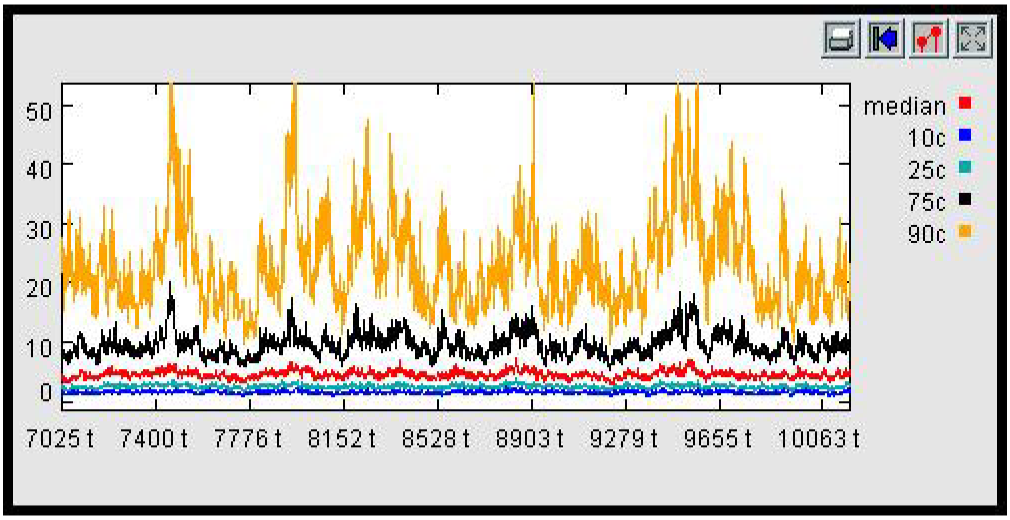
$$\mu(1) \in [0.0, 0.40]$$
$$\sigma(1) \in [0.1, 0.25]$$
$$m \in [0.1, 1.5]$$

Most simulations ended because one of the two algorithms hit (codes 1 and 2 in the table below). 59 runs ended because no firms were active (i.e. had positive size) anymore (code 3), and 72 runs ended because maximum time (10,000 periods) was reached (code 4).

| stop | Freq. | Percent | Cum. |
|---|---|---|---|
| 1 | 603 | 37.95 | 37.95 |
| 2 | 855 | 53.81 | 91.76 |
| 3 | 59 | 3.71 | 95.47 |
| 4 | 72 | 4.53 | 100.00 |
| Total | 1589 | 100.00 | |

Code 4 does not necessarily imply the system did not reach yet its 'equilibrium' behaviour. More often (especially maximum time was set so high), it means that its long run dynamics are simply more volatile than allowed by the stop algorithms. This is shown for example in figure 5, which represents one specific code-4 run outcome.

Figure 5: Firm size distribution when long-run stopping mechanisms don't work, mean heteroskedasticity (single simulation run)




I tested a number of different specifications in order to recover the functional (reduced) form of the average firm size as a function of the input parameters, namely $\mu(1)$, $\sigma(1)$ and m.

Following the argument stated above, it could be argued that code-4 runs should not be excluded from the econometric analysis: the probability of any particular value being observed at maximum time being its (stable) distribution density. However, since they show more outcome variability, they lower the explanatory power of the (mean) regression. In order to choose an appropriate functional form we excluded these runs, but in the appendix we report also the coefficients for the whole sample.

My final choice for model specification is the following:

[14] $$\log(\bar{S}) = \alpha + \beta_1 \log \mu(1) + \beta_2 \log \sigma(1) + \beta_3 \log m + \beta_4 m\mu(1) + \beta_5 m\sigma(1)$$

which corresponds to the following expression for the mean firms size:

[15] $$\bar{S} = k \frac{\mu(1)}{m\sigma(1)} \exp[m(\sigma(1) - \mu(1)]$$

Regression results are reported below (mG is $\mu(1)$, sG is $\sigma(1)$, mG_m stands for m*$\mu(1)$, sG_m for m*$\sigma(1)$):

Table 2: regression results for long-run average firm size, mean heteroskedasticity

| Source | SS | df | MS |
|---|---|---|---|
| Model | 817.530929 | 5 | 163.506186 |
| Residual | 130.69403 | 1449 | .090196018 |
| Total | 948.224959 | 1454 | .652149215 |

| | |
|---|---|
| Number of obs | = 1455 |
| F( 5, 1449) | = 1812.79 |
| Prob > F | = 0.0000 |
| R-squared | = 0.8622 |
| Adj R-squared | = 0.8617 |
| Root MSE | = .30033 |

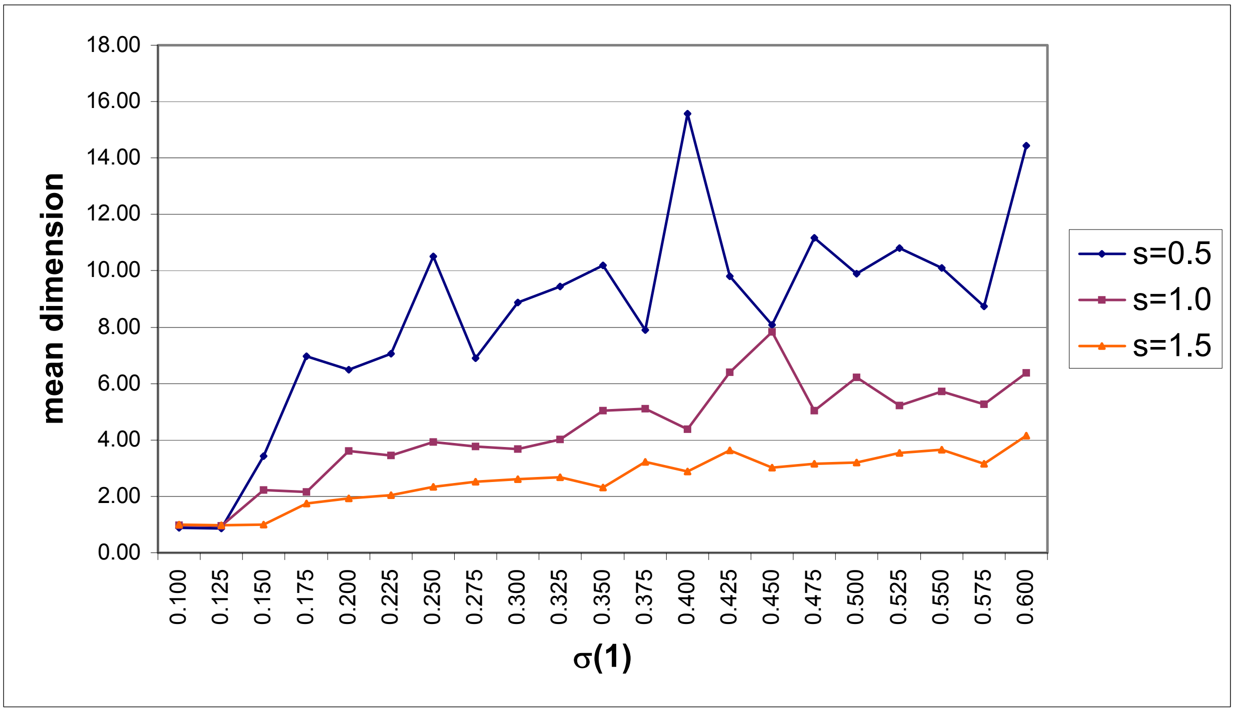
| log_mS | Coef. | Std. Err. | t | P>\|t\| | [95% Conf. Interval] | |
|---|---|---|---|---|---|---|
| log_mG | .7277577 | .017495 | 41.598 | 0.000 | .6934394 | .762076 |
| log_sG | -1.073993 | .0607705 | -17.673 | 0.000 | -1.1932 | -.9547855 |
| log_m | -.9772666 | .0471145 | -20.742 | 0.000 | -1.069686 | -.8848467 |
| mG_m | -1.868199 | .2107114 | -8.866 | 0.000 | -2.281531 | -1.454867 |
| m_sG | 2.438006 | .6188691 | 3.939 | 0.000 | 1.224031 | 3.651981 |
| _cons | 1.58892 | .1888207 | 8.415 | 0.000 | 1.218529 | 1.959311 |

Of course, I don't expect this aggregate law to be the *true* data generatine process (DGP). However, this turns out to be a good approximation of the local DGP. Such a law could for example be used to test whether a market satisfying the assumptions of this particular model has reached a sort of 'steady state'.

*6.1.2 Variance heteroskedasticity*

Figure 6 shows the case of variance heteroskedasticity: the standard deviation of the growth rate keeps declining with size with an exponential speed, governed by the parameter *s*, which takes the three different values of 0.5, 1.0 and 1.5 (see again equation 10). It plots the different values of long-run (equilibrium) mean firm size, for different values of the standard deviation of the growth rate (the latter are referred to a size 1 firm). Average growth rate remains fixed at 0.

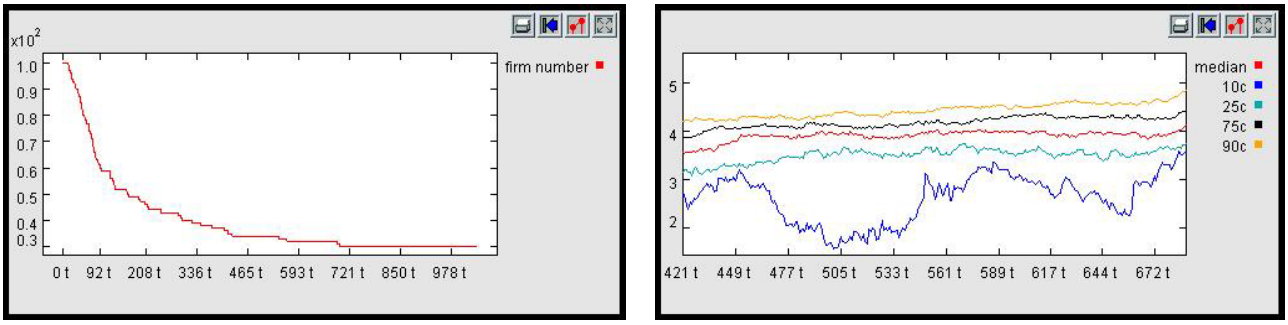
Figure 6: Long-run dynamics, variance heteroskedasticity (multiple simulation runs)



The system works as follows. Bigger variance for small firms imply some firms will grow, while others will shrink considerably. When a firm shrinks to zero, it exits the market. This implies the average size does not decline steadily to zero, as in the standard Gibrat version. The asymmetry due to the threshold at 0 induce negative correlation between the mean of the observed growth rate and size, as in the mean heteroskedasticity case. Of course, the number of active firms becomes smaller and smaller. However, some firms will pick up high growth rate at their initial stage. As they grow bigger, the variance of their growth rate becomes smaller. This induces a sort of lock-in effect: once a firm has become big, it can't move backwards as easily. However the tendency to implode due to the zero growth rate mean assumption is still at work: slowly, big firms keep shrinking. But the smaller they become, the bigger the variance of the growth rate: again, some will disappear and others will grow. Empirically, after a short time the number of active firms becomes quite stable, i.e. 'unlucky' firms who got negative returns in their initial stage disappear, while 'lucky' firms become big enough to avoid this risk. Only running the simulation for a very long time the system will eventually reach its final, empty, state. To give an idea, with $s$ = 1.0 and a standard deviation for a size-1 firm of 0.5 (we chose such a high but not unrealistic value in order to magnify the effects of this parameter), starting from a population of 100 firms the system adjusts to around 30 firms in 700 periods, and then becomes quite stable. After 20.000 periods there are still more than 20 firms in the system.

No matter what the very-long-run equilibrium is, the system exhibits a 'reasonable' behaviour, for reasonable values of the parameters and for a reasonable time horizon (see Figure 7 for one specific simulation outcome). In particular, the mean growth rate $\mu$ must be included in the 'implosion set' of figure 1, because the system cannot contrast explosive firm dynamics.

Figure 7: Total population and firms size distribution, variance heteroskedasticity (individual simulation run)




## *6.2 Reasonable dynamics with entry and exit mechanisms*

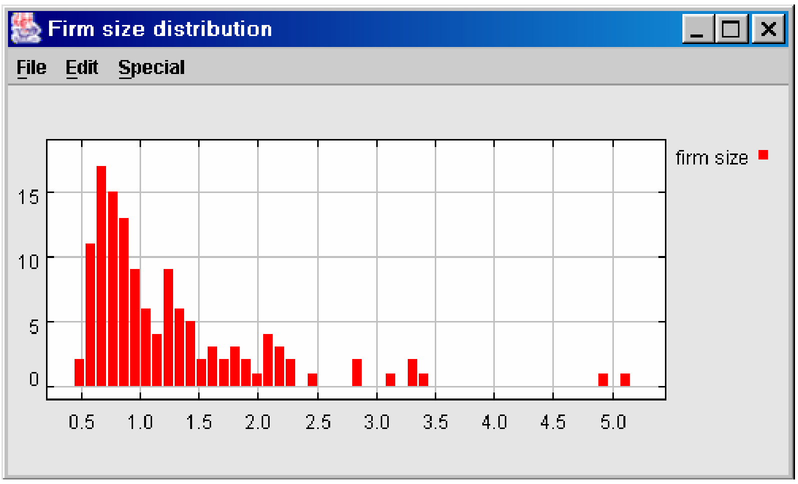
So far, I have showed that in order to get 'reasonable' dynamics in a multiplicative model, at least some form of mean or variance heteroskedasticity in the growth rate of firms must be assumed. However, models without entry or exit dynamics are not very interesting in themselves, but for benchmarking purposes. I now turn to investigating extensions of the basic multiplicative model which take into consideration that the market could be characterized by a turnover of firms. As it will be shown, weaker conditions are necessary in order to get 'reasonable' dynamics in such a market.

When entry or exit mechanisms are specified within the context of mean or variance heteroskedasticity, R-distributions are again obtained. Moreover, the risk of the market vanishing because all firms dry out is avoided either by specifying a threshold entry (like Excess Demand entry) or by specifying a non-threshold entry guaranteeing a sufficiently high number of new firms, like Proportional Number or Proportional Dimension entry.

R-distributions can also be obtained in the case of homoskedasticity. Here, however, a threshold mechanism must be specified both for entry and for exit.

This is an explanation of the underlying dynamics.
We have already seen that multiplicative models suffer from two contrasting tendencies, either to implosive or to explosive behaviour. To reach R-distributions, these two tendencies have to be balanced. To contrast implosion, a natural threshold arises from the fact that no firm can shrink below size 0. Removing 0-sized firms from the market (or any firm below a minimum size) induces, as I showed above, a negative correlation between size and growth rate. All we need then is that enough new firms are created, in order to preserve the system from extinction. Of course, if the growth rate variance is too small, it becomes too rare for any firm to actually fall below the minimum size. Firm dimension would then slowly deflate towards zero. Moreover, if new firm dimension is drawn from the (lower part of the) existing firm size distribution, the resulting distribution will be centered very close to the removal threshold (see figure 8 for one typical simulation outcome). However, by choosing for example a minimum size of 0.5 the resulting distribution (for an average growth rate of 0 and a standard deviation of 0.1) is obtained:

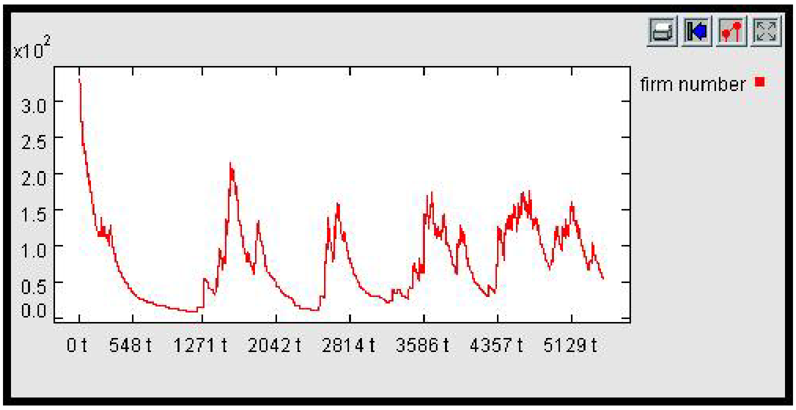
Figure 8: Firm size distribution with positive minimum size, zero average growth and no entry (single simulation run)




The next problem is to specify a mechanism that lets an appropriate number of new firms to enter the market, i.e. a number 'in the long run' approximately equal to that of disappearing old firms (falling below the minimum size threshold). **Threshold entry mechanisms** guarantee this condition is always met. Non-threshold mechanisms may not work. For instance, Proportional Dimension (Blank & Solomon) mechanism is able to contrast implosion, as we have already seen, but Proportional Number is not.

By specifying an Excess Demand entry mechanism, interesting dynamics as the one depicted in figure 9 for the total number of firms in the market can be obtained:

Figure 9: Total firm number with positive minimum size, zero average growth and Excess Demand entry (single simulation run)



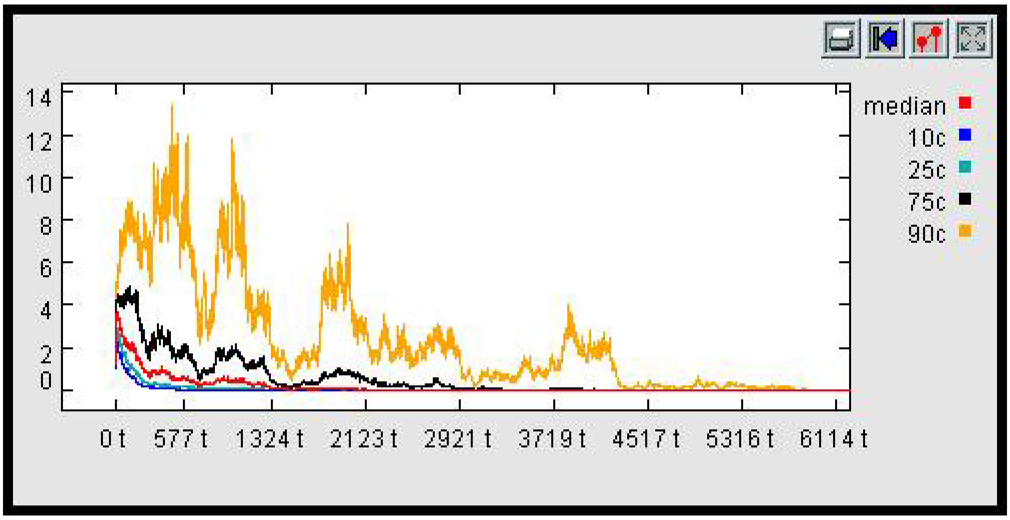

One final note. Threshold mechanisms, as I have defined them, work by comparing total dimension (capacity) with an exogenously given demand. New firms are added if supply falls short of demand, and new firm size is drawn from the lower part of the existing firm size distributions. If the lower threshold is fixed at 0, these mechanisms keep introducing new firms as the existing ones shrink, since the effect of new capacity on total capacity gets smaller and smaller (adding a 0-sized firm doesn't help in increasing supply!).

Thus, my conclusion is that **in order to contrast implosion and get R-distributions, threshold entry mechanisms are enough, given a non-zero minimum size**.
It is important to note that – as long as the danger remains on the implosion side – non-threshold exit mechanisms could well be assumed, in conjunction with threshold entry mechanisms.

Let's now turn to explosive behaviour, where things get tougher.
Clearly, non-threshold mechanisms cannot work, in general, because they do not pose any limit to firm growth.
But even if a **threshold exit mechanism** is specified, the exogenously defined total capacity available is generally monopolized by a single firm. In Excess Supply Affects Small total overcapacity hits only small firms, while in Excess Supply Affects Large it hits only large firms. However, both mechanisms lead to monopoly. This is obvious if no entry mechanism is specified, because no new firms enter the market, while old firms keep exiting. For Excess Supply Affects Small it remains true with any entry mechanisms. For Excess Supply Affects Large it remains true with most entry mechanisms. The only way out would be specifying an entry mechanism where new firms enter the market at every period, with size independent of the existing firm size distribution and small. The number of firms to enter the market should be either independent of the number of existing firms, or negatively dependent. Not an appealing solution.

There is only one class of threshold exit mechanisms which preserve the system from moving towards a monopoly: exit mechanisms where overcapacity affects all firms in the market, reducing their growth rate, like in the Excess Supply Affects All one. This ultimately provides a solution to the explosive behaviour caused by a high average growth rate, by reducing it! However, even when such a mechanism is specified, if the minimum size is set to 0 the system moves towards a monopolistic situation, with all very small firms but one.

Figure 10: Firm size distribution with Excess Supply Affects All exit mechanism, 0 minimum size and $\mu$ = 0.2;



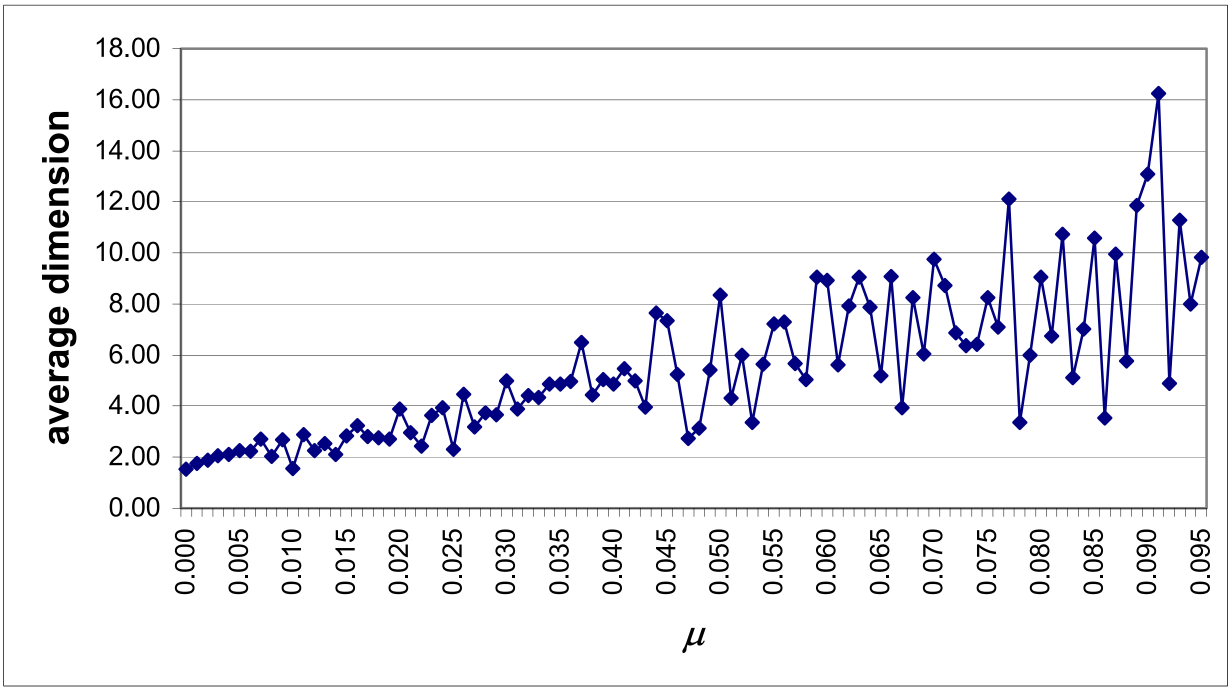

With positive minimum size this risk is avoided.

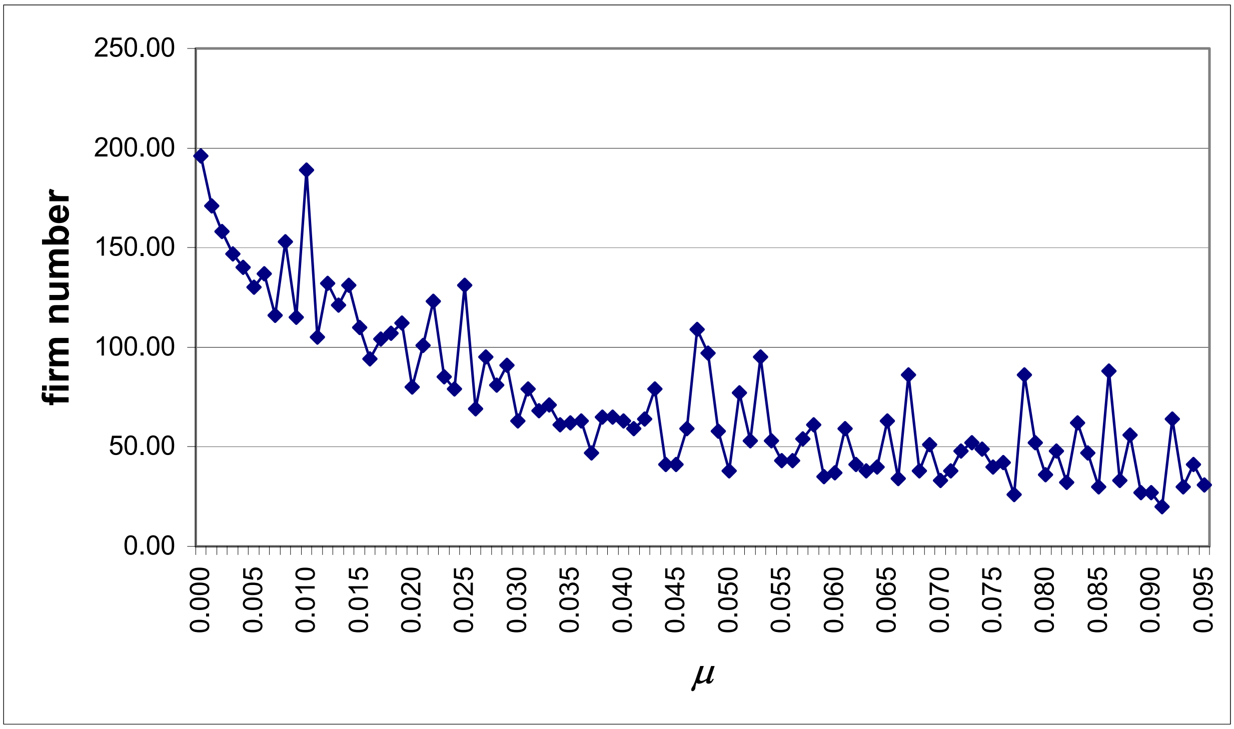
In conclusion, I have shown that **in order to contrast explosion and get R-distributions, threshold exit mechanisms that penalize all firms are enough, given a non-zero minimum size**.

At this point, it can be noted that the two problems of implosion and explosion are com disjoint. This is a general characteristic of multiplicative models. To avoid implosion, one should care about entry, and to avoid explosion one should care about exit. Thus, in order to obtain a stable system for a wide range of (homoskedastic) mean growth rates, we can combine the two results presented above.

Figure 11 shows the long-run relation between mean firm size and mean growth rate when Excess Demand entry and Excess Supply Affects All exit mechanisms are included, with a minimum size of 0.5 and a standard deviation of the growth rate of 0.1.

Figure 11: Long run dynamics, Excess Demand entry and Excess Supply Affects All exit




## 8. Concluding remarks

In this paper I have shown that – in order for a multiplicative model of firm growth to exhibit reasonable dynamics for a wide range of average growth rate, we have either to assume heteroskedasticity in the growth rates, or to include entry and exit mechanisms. While other particular, ad hoc, entry and exit mechanisms could be imagined, I have shown that combining the broad class of threshold entry mechanisms and the more restricted class of threshold exit mechanisms where overcapacity penalizes all firms, lead to R-distributions even in the case of growth rate homoskedasticity, given a non-zero minimum threshold for firm size. Threshold mechanisms are defined as rules where entry and exit are determined with reference to an exogenously defined total capacity of the market.

It is interesting to note that real data seem to confirm the existence of many of the above examined features leading to R-distributions at work at the same time. Caves (1998) in his enumeration of stylized facts on the turnover and mobility of firms, indicates that «1. The variance of firms' proportional growth rates is not independent of their size but diminishes with it […] 2. Mean growth rates of surviving firms are not independent of their sizes but tend to decline with size and also with the unit's age (given size) […] 3. Entry and exit are intimately involved in growth-size relations. Entry is more likely to occur into smaller size classes, and the likelihood of a unit's exit declines with its size». A positive minimum size is obviously at work, since it is difficult to imagine a firm asking for no more than a very small fraction of the entrepreneur's time.

However, in order to preserve the skewness of the firm size distribution, the amount of variance heteroskedasticity (the rate of decline of the standard deviation of the growth rate, $s$) must be small, in order to maintain some variance in the growth rates when firm size approaches its long-run distribution. Thus, the smaller the mean growth rate (or the higher the amount of mean heteroskedasticity), the more shifted to the left will be the long-run distribution of firm size, and the smaller the amount of variance heteroskedasticity that can be supported.

Finally, to my judgement future research on these issue could go in three directions. First, generalized Gibrat models could be calibrated on real data, by choosing the appropriate entry and exit mechanisms and the values of the relevant parameters in order to characterize different industries. Second, the limit distribution of firm size could be computed using these calibrated models, thus providing a measure whether any particular market is in equilibrium (close to its implied long-run distribution). Third, economic models of firms or employees behaviour could be developed, with the aim to reconstruct 'from the inside out' or 'from the bottom up' the aggregate behaviour implied by these 'reasonable' multiplicative models.

## References


Acs Z.J. and Armington C. (2001), "Gibrat's Law Reconsidered: The Relationship Between Firm Growth, Establishment Age, Establishment Size and Firm Size", Working Paper of the Regional Entrepreneurship Catalyst Kauffman Center for Entrepreneurial Leadership, Kansas City, MO.

Adamic L.A. (2000) "Zipf, Power-laws, and Pareto - a ranking tutorial", http://ginger.hpl.hp.com/shl/papers/ranking/ranking.html

Amaral L.A.N, Buldyrev S.W., Havlin S., Leschhorn H., Maass P., Salinger M.A., Stanley H. E. and Stanley M.H.R. (1997), "Scaling Behavior in Economics: Empirical Results for Company Growth", *Journal de Physique I*, 7, 621-633

Amirkhalkhali S. and Mukhopadhyay A.K. (1993), "The Influence of Size and R&D on the Growth of Firms in the U.S.", *Eastern Economic Journal*, 19, 223-233.

Audretsch D. B., Klomp L., Santarelli E., and Thurik A.R. (2002), "*Gibrat's Law*: Are the Services Different?", mimeo

Blank A. and Solomon S. (2000), "Power laws in cities population, financial markets and internet sites (scaling in systems with a variable number of components)", Physica A, 287, 279-288

Boeri T. (1989), "Does Firm Size Matter?" Giornale degli Economisti e Annali di Economia, 48(9-lo), 477-95.

Bianco M. and Sestito P. (1993), "Entry and Growth of Firms: Evidence for the Italian Case", Unpublished manuscript, Banca d'Italia, Rome.

Caves R.E. (1998), "Industrial Organization and New Findings on the Turnover and Mobility of Firms", *Journal of Economic Literature*, 36, 1947-1983

Chesher A. (1979), "Testing the Law of Proportionate Effect", *Journal of Industrial Economics*, 27, 403-411.

Contini B. and Revelli R. (1989), "The Relationship between Firm Growth and Labor Demand", *Small Business Economics*, 1, 309-314.

Cordoba J.C. (2002), "Balanced City Growth and Zipf's Law", RICE Economics Working Papers 2002-03

Delmar F., Davidsson P. and Gartner W.B. (2002), "Arriving at the High-growth Firm", *Journal of Business Venturing*, 17, forthcoming.

Dunne P. and Hughes A. (1994), "Age, Size, Growth and Survival: UK Companies in the 1980s", *Journal of Industrial Economics*, 42, 115-140.

Evans D.S. (1987a), "The Relationship between Firm Growth, Size, and Age: Estimates for 100 Manufacturing Industries", *Journal of Industrial Economics*, 35, 567-581.

Evans D.S. (1987b), "Tests of Alternative Theories of Firm Growth", *Journal of Political Economy*, 95, 657-674.

Fariñas J.C. and Moreno L. (2000), "Firms' Growth, Size and Age: A Nonparametric Approach", *Review of Industrial Organization*, 17, 249-265.

FitzRoy F.R. and Kraft K. (1991), "Firm Size, Growth and Innovation: Some Evidence from West Germany", in Z.J. Acs and D.B. Audretsch (eds.), *Innovation and Technological Change: An International Comparison*, Harvester Wheatsheaf, London, 152-159.

Gibrat R. (1930), "Une loi des réparations économiques: l'effet proportionnel", *Bull. Statist. gén Fr.* 19, 469.

Gibrat R. (1931), "Les Inégalités Économiques", Librairie du Recueil Sirey, Paris.

Geroski P.A., Lazarova S., Urga G. and Walters C.F. (2000), "Are Differences in Firm Size Transitory or Permanent?", London Business School, Department of Economics, mimeo, August, forthcoming in *Journal of Applied Econometrics*.

Goddard J., Wilson J. and Blandon P. (2002), "Panel Tests of Gibrat's Law for Japanese Manufacturing", *International Journal of Industrial Organization*, 20, 415-433.

Hall B.H. (1987), "The Relationship between Firm Size and Firm Growth in the US Manufacturing Sector", *Journal of Industrial Economics*, 35, 583-606.

Hardwick P. and Adams M. (1999), "Firm Size and Growth in the United Kingdom Life Insurance Industry", University of Bournemouth, School of Finance and Law, Working Paper Series, No. 16.

Harhoff D., Stahl K. and Woywode M. (1998), "Legal form, Growth and Exit of West German Firms- Empirical Results for Manufacturing, Construction, Trade and Service Industries", *Journal of Industrial Economics*, 46, 453-488.

Hart P.E. (1962), "The Size and Growth of Firms", *Economica*, 29, 29-39.

Hart P.E. and Prais S.J. (1956), "The Analysis of Business Concentration: A Statistical Approach", *Journal of the Royal Statistical Society*, 119 (part 2, series A), 150-191.

Hart P.E. and Oulton N. (1999), "Gibrat, Galton and Job Generation", *International Journal of the Economics of Business*, 6, 149-164.

Hendry D.F and Krolzig H-M. (2002), "New Developments in Automatic General-to-specific Modelling", in Stigum B.P. (ed), *Econometrics and the Philosophy of Economics*, MIT Press, Cambridge, MA, forthcoming.

Hymer S. and Pashigian P. (1962), "Firm Size and Rate of Growth", *Journal of Political Economy*, 70, 556- 569.

Kesten H. (1973), "Random Difference Equations and Renewal Theory for Products of Random Matrices", *Acta Mathematica*, 131, 207-248.

Kumar M.S. (1985), "Growth, Acquisition Activity and Firm Size: Evidence from the United Kingdom", *Journal of Industrial Economics*, 33, 327-338.

Leonard J.S. (1986), "On the Size Distribution of Employment and Establishments," Working Paper No. 1951, National Bureau of Economic Research.

Levy M. and Solomon S. (1996 a), "Spontaneous Scaling Emergence in Generic Stochastic Systems", *International Journal of Modern Physics C*, 7, 745

Levy M. and Solomon S. (1996 b), "Power Laws are Logarithmic Boltzmann Laws", *International Journal of Modern Physics C*, 7, 595

Machado J.A.F. and Mata J. (2000), "Box-Cox Quantile Regression and the Distribution of Firm Sizes", *Journal of Applied Econometrics*, 15, 253-274.

Mansfield E. (1962), "Entry, Gibrat's Law, Innovation, and the Growth of Firms", *American Economic Review*, 52, 1023-1051.

McCloughan P. (1995), "Simulation of Concentration Development from Modified Gibrat Growth-Entry-Exit Processes", *Journal of Industrial Economics*, 43 (4), 405-432

Pareto V. (1896), *Cours d'Economie Politique*, Droz, Geneva.

Samuels J.M. (1965), "Size and Growth of Firms", *Review of Economic Studies*, 32, 105-112.

Simon H. and Bonini C. (1958), "The Size Distribution of BusinessFirms" *American Economic Review*, 48, 607-617.

Singh A. and Whittington G. (1975), "The Size and Growth of Firms", *Review of Economic Studies*, 42, 15-26.

Sutton J. (1997), "Gibrat's Legacy", *Journal of Economic Literature*, 35, 40-59.

Tschoegl A. (1996), "Managerial Dis(economies) of Scale: The Case of Regional Banks in Japan", Reginald H. Jones Center for Management and Policy, Strategy and Organization, The Wharton School of the University of Pennsylvania, Working Paper No. 96-04.

Vander Vennet R. (2001), "The Law of Proportionate Effect and OECD Bank Sectors", *Applied Economics*, 33, 539-546.

Variyam J.N. and Kraybill D.S. (1992), "Empirical Evidence on Determinants of Firm Growth", *Economics Letters*, 38, 31-36.

Wagner J. (1992), "Firm Size, Firm Growth, and Persistence of Chance: Testing Gibrat's Law with Establishment Data from Lower Saxony, 1978-1989", *Small Business Economics*, 4, 125-131.

Zipf G. (1932), *Selective Studies and the Principle of Relative Frequency in Language*, Harvard University Press, Cambridge, MA.

# Appendix

Regression results for Mean Heteroskedasticity model, full sample

| Source | SS | df | MS |
|---|---|---|---|
| Model | 1026.89056 | 5 | 205.378113 |
| Residual | 368.400658 | 1521 | .242209506 |
| Total | 1395.29122 | 1526 | .914345493 |

| | |
|---|---|
| Number of obs | 1527 |
| F( 5, 1521) | 847.94 |
| Prob > F | 0.0000 |
| R-squared | 0.7360 |
| Adj R-squared | 0.7351 |
| Root MSE | .49215 |

| log_mS | Coef. | Std. Err. | t | P>\|t\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| log_mG | .9226223 | .0263594 | 35.002 | 0.000 | .8709176 | .9743269 |
| log_sG | -1.312518 | .0958111 | -13.699 | 0.000 | -1.500453 | -1.124582 |
| log_m | -.9095693 | .0744303 | -12.220 | 0.000 | -1.055566 | -.7635725 |
| mG_m | -3.194521 | .3311262 | -9.647 | 0.000 | -3.844033 | -2.545008 |
| m_sG | 3.339967 | .9916963 | 3.368 | 0.001 | 1.39473 | 5.285204 |
| _cons | 1.58423 | .3006232 | 5.270 | 0.000 | .9945503 | 2.17391 |