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Multiproduct firms in the representative consumer model of product differentiation

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This note describes a worked-out example of the Dixit–Stiglitz model of product differentiation, generalized in two directions; career choice is endogenous and firms can produce many products.

1. Introduction

This note studies a version of the Dixit and Stiglitz (1977) model of product differentiation, generalized in two respects. Agents are allowed to choose their occupations, namely to be either workers or employers (firms); and firms are allowed to produce more than one commodity, and to choose the technique of production of each commodity. To see why this exercise might be of some interest, recall that in the Dixit–Stiglitz model variety can be over or undersupplied in equilibrium. Firms cannot capture all the consumer surplus generated by the introduction of new products, because they cannot perfectly price discriminate; as a result, variety tends to be undersupplied and the number of firms to be too small (there is a one-to-one correspondence between products and firms). At the same time, each firm ignores the negative externality it imposes on other firms when it decides to enter and supply a new product; variety, then, tends to be oversupplied and the number of firms to be too large.

The introductions of multiproduct firms breaks the link between extra variety and new entry. It is now possible to increase variety without increasing at the same time the intensity of competition (measured by the number of firms). This suggests that, ceteris paribus, allowing for multiproduct firms will result in more variety. What is more, multiproduct firms

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can try to internalize the negative intra-firm externality by forcing some of their rivals to exit and, at the same time, changing their product range; in the extreme case of a single multiproduct firm serving the whole market, the negative externality is completely internalized. On the other hand, though, there are now new forces favoring a reduction in variety. Each firm introducing a new product has to consider two facts: the new product will reduce demand for other products supplied by the same firm; and the resources required to produce the new product could have been used to reduce the unit cost of producing existing products. Furthermore, concentration of production in the hands of fewer firms reduces inefficiencies due to intra-firm externalities but increases inefficiencies due to market power; and, by increasing profits, concentration invites entry.

The purpose of the exercise, therefore, is to find out the relative importance of these effects. The results obtained are clear cut, due to the stylized nature of the model; for the same reason, though, they are unlikely to survive unqualified in less stylized models.

2. The model

2.1. Preferences and technology

The economy consists of n identical agents, each endowed with one unit of labor, the sole primary factor. There is a continuum of final goods, and the commodity space is $(0, \infty)$. If $c(t)$ is the consumption level of good t, then

$$U(c)=\int_0^\infty c(t)^\beta \,\mathrm{d}t, \quad 0<\beta<1, \tag{1}$$

is the utility level of an agent who consumes c.

The main feature of technology in this model is that there is a menu of techniques that can be used in the production of each commodity t. Techniques are parameterized by the amount of fixed (labor) cost they involve. To produce $x(t)$ units of good t with technique $\gamma(t)$ takes $\gamma(t)+\Psi(\gamma(t))x(t)$ units of labor. To produce a vector x of goods with a vector γ of techniques takes

$$L(\gamma, x)=\int_0^\infty [\gamma(t)+\Psi(\gamma(t))x(t)]\,\mathrm{d}t, \tag{2}$$

units of labor.

The unit variables cost function ψ satisfies the following.

Assumption 1 (A.1) There is some $\bar{\gamma}>0$ such that $\psi(\gamma)$ is finite if $\gamma>\bar{\gamma}$ and $\Psi(\gamma)=+\infty$ if $0\leqq\gamma\leqq\bar{\gamma}$.

This assumption implies that the product range will be finite in finite economies, with length bounded by $n/\bar{\gamma}$, because for each commodity produced a firm has to pay a fixed cost greater than $\bar{\gamma}$. ψ is twice continuously differentiable, strictly decreasing and strictly convex on $(\bar{\gamma},\infty)$. Furthermore, we assume the following

Assumption 2 (A.2)

$$\lim_{\gamma\to\bar{\gamma}}\Psi(\gamma)=+\infty,\qquad \lim_{\gamma\to+\infty}\Psi(\gamma)=\Psi_{\infty}>0.$$

The tradeoff between fixed and unit variable cost is captured by the assumed negative slope of ψ. Strict convexity of ψ implies that the cost function

$$c(q)\equiv\min_{\gamma\geqq 0}\{\gamma+\Psi(\gamma)q\}$$

is well-defined for all $q>0$. The fact that unit variable cost is bounded below by a positive number will imply that prices are uniformly bounded away from zero.

Let $e(\gamma)=-\gamma[\Psi'(\gamma)/\Psi(\gamma)]$ be the elasticity of the unit variable cost function.

This elasticity, e, is strictly decreasing on $(\bar{\gamma},\infty)$, and another assumption is made.

Assumption 3 (A.3)

$$\lim_{\gamma\to\bar{\gamma}+}e(\gamma)=+\infty,\qquad \lim_{\gamma\to+\infty}e(\gamma)=\alpha.$$

This is a diminishing-returns type of assumption. In fact, the convexity, of ψ and $\psi_{\infty}>0$ imply that $\alpha=0$.

Assumption 4 (A.4)

$$\frac{\Psi(\gamma)\Psi''(\gamma)}{(-\Psi'(\gamma))^2}\neq\frac{1}{1-\beta}\quad\text{for all } \gamma>\bar{\gamma}.$$

This is a regularity condition that will guarantee uniqueness of solutions of a set of equations that characterize a firm's optimal plan.

A pair (ψ, β) that satisfies (A.1)–(A.4) is given by

$$\Psi(\gamma) = \Psi_\infty \frac{\gamma}{\gamma - \bar{\gamma}}, \; \gamma > \bar{\gamma}, \Psi_\infty > 0; \quad 0 < \beta < \tfrac{1}{2}.$$

2.2. Demand

An agent with income M, facing a vector of prices p, will choose a consumption vector c that solves

$$\max \int_0^\infty (c(t))^\beta \, \mathrm{d}t, \qquad \text{subject to} \quad \int_0^\infty p(t)c(t)\,\mathrm{d}t \leqq M. \tag{3}$$

The unique solution to this problem is

$$d(p, M)(t) = \frac{Mp(t)^{-1/(1-\beta)}}{\int_0^\infty p(t)^{-\beta/(1-\beta)}\,\mathrm{d}t}. \tag{4}$$

2.3. Production and entry decisions

Agents play a game in two stages. In the first stage, they choose their occupation. Those that choose to be employers spend their labor endowment in setting production up and cannot, therefore, earn income in the second stage by selling labor. On the contrary, those that choose to be workers can sell their labor endowment to employers in the second stage. Let $\sigma\colon I \to \{1, 2\}$ be a function that summarizes first-stage choices: $\sigma(i) = 1$ if i chooses to be a worker, $\sigma(i) = 2$ if i chooses to be an employee. Let $m(\sigma)$ be the number of workers at σ. Note that if $m(\sigma) = 0$ or $m(\sigma) = n$ no production (or consumption) is possible. We set all payoffs equal to zero in these two cases, and we proceed to the second stage to compute payoffs for the cases $0 \leqq m(\sigma) \leqq n-1$.

In the second stage, employers decide on what to produce, how to produce, and how much to charge, taking demand functions and first-stage decisions as given. We assume that labor has no disutility, and normalized its price to unity. Workers' aggregate demand function is then $m(\sigma)\,\mathrm{d}(p, 1)$. On the other hand, an employer's income equals her profits, and profits depend on demand and hence on employers' income. To get around this circularity, we first fix each employer's income at some level Y_i, $(\sigma(i)=2)$, and then we compute the second-stage equilibrium payoffs $f_i(Y_1,\dots,Y_{n-m})$. Finally, we compute equilibrium income levels by solving the fixed-point equations

$$Y_i=f_i(Y_1,\dots,Y_{n-m}),\quad \sigma(i)=2.$$

Each employer i chooses a vector p_i of prices and a vector γ_i of fixed costs. Let $\boldsymbol{p}=(p_i\colon \sigma(i)=2)$, $p(t)=\min\{p_i(t);\sigma(i)=2\}$, $Y=\sum\{Y_i;\sigma(i)=2\}$, $m=m(\sigma)$. Aggregate demand for t is, by eq. (4),

$$D(\boldsymbol{p})(t)=d(p,m+Y)(t). \tag{5}$$

How is the demand for t at $\boldsymbol{p}$ split among employers? We assume that employer i will attract the whole demand for t if no other employer supplies t at a finite price, but that i will attract zero demand otherwise. Formally

$$D_i(\boldsymbol{p})(t)=\begin{cases}D(\boldsymbol{p})(t) & \text{if } j\neq i \text{ implies } p_j(t)=\infty,\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

Eq. (6) is equivalent to 'no-mill-price undercutting' in the terminology of Hotelling (1929), Eaton (1972) and Eaton and Lipsey (1978), or to 'modified zero conjectural variations' in the terminology of Novshek (1980). The story behind (6) is that each firm will protect its market area $A_i(\boldsymbol{p})=\{t\colon p_i(t)\prec\infty\}$ by underselling any firm $j\neq i$ that tries to sell $t\in A_i(\boldsymbol{p})$ at a price lower than $p_i(t)$.[1]

At the aggregate strategy vector $(\boldsymbol{p},\gamma)$, then, each employer i attains payoff

$$\pi_i(\boldsymbol{p},\gamma)=\int_0^\infty[[p_i(t)-\Psi(\gamma_i(t))]D_i(\boldsymbol{p},t)-\gamma_i(t)]\,\mathrm{d}t. \tag{7}$$

We will compute the symmetric Nash equilibrium of the game[2] defined by (7). A (second-stage) equilibrium $(\boldsymbol{p},\gamma)$ us symmetric if:

[1]Such behavior is sometimes observed: Firms advertise 'we will not be undersold', or promise to pay consumers 'double the difference' between the firm's price and any competitor's (lower) price. It is clearly unsatisfactory, though, to assume such behavior rather than derive it from primitives. I was unable to construct a model that does this and stays close to the Dixit–Stiglitz formulation.

[2]It is more convenient to let firm i choose $q_i(t)=l/p_i(t)$ rather than $p_i(t)$. The strategy space of firm i is the set of pairs (q_i,γ_i) where $q_i=(0,\infty)\to[0,\Psi]$ *and* $\gamma_i\colon(0,\infty)\to[0,\infty)$ are Lebesque integrable functions. Firm i chooses (q_i,γ_i) to maximize profits.

(1) Labor supply is equally shared by firms

$$\int_0^\infty p_i(t)D_i(\boldsymbol{p}, t)\,\mathrm{d}t = \frac{m}{n-m} \quad \text{if} \quad \sigma(i)=2; \tag{8}$$

(2) each firm offers the same degree of variety

$$\lambda(A_i(\boldsymbol{p})) = \lambda(A_j(\boldsymbol{p})) \quad \text{if} \quad \sigma(i)=\sigma(j)=2, \tag{9}$$

where λ is the Lebesque measure.

3. Symmetric Nash equilibria

The symmetry of preferences and technology implies that only the size of market areas matters (not their location on the line) and that all goods in a firm's market area will be produced with the same technique and sold at the same price. The no-mill-price undercutting assumption implies that market areas will be disjoint. We can then show the following.

Fact 1. For each $1 \leqq m \leqq n-1$, *and for each* $Y \geqq 0$, *there is a symmetric second-state equilibrium* $({}^*\boldsymbol{p}, {}^*\gamma)$. *It is given by*

$$ {}^*\gamma_i(t) = \begin{cases} \hat{\gamma} & \text{if} \quad t \in A_i({}^*\boldsymbol{p}), \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

$$\lambda(A_i({}^*\boldsymbol{p})) = \hat{T} = \frac{(m+Y)(1-\beta)(n-m-1)}{\hat{\gamma}(n-m)(n-m-\beta)}, \tag{11}$$

$$-\hat{\gamma}\frac{\Psi'(\hat{\gamma})}{\Psi(\hat{\gamma})} = \frac{1-\beta}{\beta}, \tag{12}$$

$$ {}^*p_i(t) = \begin{cases} \psi(\hat{\gamma})\dfrac{n-m-\beta}{\beta(n-m-1)} & \text{if} \quad t \in A_i({}^*\boldsymbol{p}), \\ \infty & \text{otherwise}. \end{cases} \tag{13}$$

A proof of this fact is available from the author on request.

From these equations and (7) we can compute the equilibrium payoff of each employer

$$\pi_i(Y) = \frac{(1-\beta)(m+Y)}{(n-m)(n-m-\beta)}. \tag{14}$$

Finally, recalling the definition of Y, we solve $(n-m)\pi_i(Y) = Y$ and we obtain $Y = [(1-\beta)m]/[n-m-1]$.
The product range of each firm is then, by (11),

$$\hat{T} = \frac{m(1-\beta)}{(n-m)\hat{\gamma}}, \tag{15}$$

and the profit of each firm is

$$^*\pi(m) = \frac{(1-\beta)m}{(n-m)(n-m-1)}. \tag{16}$$

The equilibrium utility levels for workers and firms are, respectively, $u_{\mathrm{w}}(m) = U(d(^*\boldsymbol{p}, 1))$, $u_{\mathrm{f}}(m) = U(d(^*\boldsymbol{p}, {}^*\pi(m))$, or equivalently,

$$u_{\mathrm{w}}(m) = \left(\frac{m(1-\beta)}{\hat{\gamma}}\right)^{1-\beta} \left[\frac{\beta(n-m-1)}{\Psi(\hat{\gamma})(n-m-\beta)}\right]^{\beta}, \tag{17}$$

$$u_{\mathrm{f}}(m) = \frac{m(1-\beta)}{\hat{\gamma}} \left[\frac{1-\beta}{-\Psi'(\hat{\gamma})(n-m)(n-m-\beta)}\right]^{\beta}. \tag{18}$$

We can now go to the first stage of the game and analyze occupational choice. Suppose there are m workers in (first-stage) equilibrium. A worker will not switch occupations if his utility as a worker ($u_{\mathrm{w}}(m)$) is greater than his utility as an employer ($u_{\mathrm{f}}(m-1)$). An employer will not switch occupations if her utility as an employer ($u_{\mathrm{f}}(m)$) is greater than her utility as a worker ($u_{\mathrm{w}}(m+1)$). To find out whether such an m exists, let $B(m) = u_{\mathrm{f}}(m)/u_{\mathrm{w}}(m+1)$. The equilibrium conditions are given by

$$B(m) \geqq 1, \quad B(m-1) \leqq 1. \tag{19}$$

By inspection of (17) and (18), and by (12),

$$B(m) = \left(\frac{m(1-\beta)}{(n-m)(n-m-\beta)} \, \frac{n-m-1-\beta}{n-m-2}\right)^{\beta}. \tag{20}$$

B is strictly increasing in m in the interval $[2, n-2)$. (Note that each of the two fractions inside the bracket increases in m). What is more, $B(n-2) = +\infty$, and $B(2) \prec 1$ for n sufficiently large. Hence the following is true.

Fact 2. For all sufficiently large n, there is a unique equilibrium number of workers m(n), and $2 \leqq m(n) \leqq n-2$.

Clearly, $m(n)$ is the smallest integer larger than the unique root of the equation $B(x) = 1$.

We can now go back to eqs. (13), (15), (17) and (18) and replace m by its equilibrium value $m(n)$. The resulting expressions, together with (10) and (12), describe the symmetric equilibrium. To see why equilibrium obtains, fix m for a moment and go back to the second stage of the game. Each firm can spend the labor it buys in three different ways: To expand its product range, to reduce its unit variable cost, or to increase the quantities supplied. Because of global increasing returns, increasing the quantities supplied reduces total average costs, but eventually demand prices decline faster than average costs. Increasing the product range increases revenue because the special form of the utility function implies that consumers are willing to pay a lot for the first few units of a new good [$(c^{\beta})' = \infty$ if $c = 0$], but it also raises unit costs, because the firm produces less of each good, and/or invests less in unit variable cost reduction. In equilibrium, the costs of change exceed the benefits.

When labor supply m increases, firms increase the price–marginal cost ratio [eq. (13)] because they now have greater market power vis-à-vis consumers; and expand their product range [eq. (15)] keeping their unit variable costs fixed [eq. (12)], because of the strong preference of consumers for variety. This is clearly a special result.

We now look at the entry decision. If there is only one firm ($m = n-1$), economies of scale are fully exploited and intrafirm externalities are absent, but the firm has a lot of market power [p/ψ in eq. (13) is maximized when $m = n-1$] and enjoys a high level of utility (u_f is maximized at $m = n-1$). Workers, on the other hand, enjoy zero utility [eq. (17)]. There is clearly an incentive to enter. On the other hand, when there is only one worker ($m = 1$), firms have very little market power (p/ψ is minimized when $m = 1$), economies of scale are dissipated, and firms attain a low utility level (u_f is minimized when $m = 1$). There is clearly an incentive to exit. In equilibrium, the costs of switching occupations exceed the benefits.

4. Asymptotic properties of equilibria

The main result in this section is that the ratio of workers in the population converges to unity as population increases to infinity. To see this, let $y_n = m(n)/n$. If $\langle y_n \rangle$ does not converge to 1, it contains a subsequence converging to some $y \in [0, 1)$.

By (19), $\langle y_n \rangle$ must satisfy

$$\frac{y_n(1-\beta)}{n(1-y_n)(1-y_n-\beta/n)} \cdot \frac{1-y_n-(1+\beta)/n}{1-y_n-2/n} \geqq 1.$$

The limit of the left-hand side along the subsequence is zero, a contradiction. Hence

$$\lim_{n\to\infty} \frac{m(n)}{n} = 1. \tag{21}$$

The number of firms $x_n = n - m(n)$ diverges to infinity as population increases to infinity. For if not, $\langle x_n \rangle$ contains a bounded subsequence; if $\bar{x}$ is an upper bound, then $2 \leqq x_n \leqq \bar{x}$ along the subsequence. Then the equilibrium condition $B(m(n)-1) \leqq 0$ yields

$$m(n) - 1 \leqq \frac{(x_n-1)(x_n+1)(x_n+1-\beta)}{(1-\beta)(x_n-\beta)}.$$

This is clearly a contradiction, since the left-hand side diverges to infinity while the right-hand side is bounded along the subsequence. Hence

$$\lim_{n\to\infty} (n - m(n)) = \infty. \tag{22}$$

Combining the two equilibrium conditions (19) we obtain information about the speed of convergence of $y_n = m(n)/n$.

$$\lim_{n\to\infty} \frac{y_n}{n(1-y_n)^2} = \frac{1}{1-\beta}. \tag{23}$$

It is then routine to check that as market size n increases to infinity, the product range $\hat{T}_n$ of each firm increases to infinity, at a rate given by

$$\lim_{n\to\infty} \frac{\hat{T}_n}{n} = \frac{1}{\hat{\gamma}}, \tag{24}$$

and that the equilibrium utilities of workers and employers increase to infinity, while their ratio converges to unity. Finally if $W_n = m(n)1$ is total wage income and $\Pi_n=(n-m(n))\pi(m(n))$ is total profit income, then (16) implies

$$\lim_{n\to\infty}\frac{W_n}{W_n+\Pi_n}=1. \tag{25}$$

As market size n increases, the incentive to concentrate production (fuller exploitation of scale economies and/or internalization of intrafirm externalities) is strengthened relative to the incentive to decentralize production (avoidance of monopolistic exploitation due to firms' market power). As a result, the ratio of employers in population declines. The strong perference of agents for variety dictates to firms to spend the additional labor supply to expand their product range rather than reduce unit costs. What is more, the absolute number of firms has to increase, because if there was an upper bound on the number of firms their profits would eventually become large enough to invite entry. The special form of the utility function implies that there is 'room for entry' if uncumbent profits are sufficiently high, because new entrants can always offer new varieties and compete demand away from the incumbents' brands (recall that direct price competition is disallowed by the no-mill-price undercutting assumption). Finally, equilibrium utilities of workers and employers are asymptotically equal because in large economies the negative effect of a new entrant on profits is close to zero, and therefore workers will not switch occupations if and only if their utility is almost equal to the utility of employers. The share of labor in total income converges to unity because the proportion of employers in population approaches zero and employers' income is asymptotically equal to workers' income.

5. Efficiency of equilibria

As market-size increases to infinity, market power (price over marginal cost) decreases to $1/\beta$ [see eq. (13)]. Firms retain some market power in the limit because of global increasing returns to scale; if $p=\mathrm{MC}$ in the limit firms would make losses, and exit would raise prices above MC.

To check whether the product range is Pareto optimal in large finite economies, we compute the minimum amount of labor that is necessary to allow each agent to enjoy his/her equilibrium utility

$$\min\int_0^\infty\left[\gamma(t)+\Psi(\gamma(t))\sum_{i=1}^{n}c_i(t)\right]\mathrm{d}t+1,$$

$$\text{subject to}\quad \int_0^\infty (c_i(t))^\beta\,\mathrm{d}t\geqq u_{\mathrm{w}}(m(n))\quad\text{if}\quad \sigma(i)=1,$$

$$\int_0^{\infty} (c_i(t))^{\beta}\,\mathrm{d}t \geqq u_{\mathrm{f}}(m(n)) \quad \text{if} \quad \sigma(i)=2.$$

By the symmetry of the problem, this is equivalent to computing a product range δ, a fixed cost γ, and two consumption levels $c_{\mathrm{w}}, c_{\mathrm{f}}$ to

$$\min L=\delta(\gamma+\Psi(\gamma)(mc_{\mathrm{w}}+(n-m)c_{\mathrm{f}}))+1,$$

$$\text{subject to} \quad \delta c_{\mathrm{w}}^{\beta}=u_{\mathrm{w}}(m(n)),$$

$$\delta c_{\mathrm{f}}^{\beta}=u_{\mathrm{f}}(m(n)).$$

The solution of this minimization problem is

$$\gamma=\hat{\gamma}, \tag{26}$$

$$\delta=\frac{m(n)(1-\beta)}{\hat{\gamma}}, \tag{27}$$

and the minimized value of the objective function is $L_n=m(n)+1$, while the welfare loss due to imperfect competition is

$$n-L_n=n-m(n)-1. \tag{28}$$

Hence, finite economies are inefficient because there are too many firms in equilibrium (each paying an entry fee equal to its owner's labor endowment), while efficiency dictates production by a sigle firm, to fully exploit scale economies. While the total welfare loss increases to infinity with market size, the per capita loss converges to zero because the ratio of employers to population converges to zero. On the other hand, the product range is Pareto efficient in large finite economies [compare (26) and (27) with (10) and (15)]. This is a special result, but it does indicate that allowing for multiproduct firms can make a difference in the efficiency conclusions one draws from the Dixit–Stiglitz model. Similar asymptotic results were obtained by Guesnerie and Hart (1985) and Ushio (1985).

6. Concluding remarks

This note has extended the Dixit–Stiglitz model of product differentiation in two directions: Career choice is endogenous and firms are allowed to produce more than one product. We have obtained an equilibrium existence result and we have explicitly described the symmetric equilibrium. As market size increases to infinity, the ratio of employers to population, and per capital welfare loss, converge to zero.

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