Oligopolistic Business-to-Business E-Market and Welfare

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Abstract

We examine the effect of an oligopolistic upstream electronic market on upstream and downstream prices. The analysis highlights the two sources of competition that a firm that source from an electronic market (e-market firm) face: competition with less efficient firms that source traditionally (t-market firms) and competition among e-market firms. When size of the upstream e-market is small, the first effect dominates and there is higher profits with lower upstream prices in the e-market. When size of the e-market becomes very large, the second effect makes e-market firms less profitable than t-market firms even though e-market price may start to increase (as market size increases). As consequence, e-market will never completely eliminate the upstream t-market and downstream price can increase when e-market grows beyond a certain size.

Key Words: business-to-business electronic commerce, oligopoly, vertical restraints, e-markets

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1 Introduction

Although not as visible as business to consumer electronic commerce, business to business (B2B) electronic commerce is growing rapidly and is by far the larger (around 80% of total electronic transactions, The Economist(1999)). Its significance among all forms of business to business relationships is increasing also (Lucking-Reiley and Spulber (2000)). For instance, the formation of Convisint initiated by the big three U.S. automobile manufacturers for automobile parts procurement is changing the traditional vertical relationships of the industry. It also had the effect of attracting anti-trust interest to electronic markets (U.S. Federal Trade Commission(2000)).

There is general agreement among economists that electronic markets should present efficiency gains beyond the direct effect of lower transaction costs . Such gains are also possible in business to consumer markets. What is unique to introduction of electronic markets to business to business is that it will change vertical relationships of firms (Lucking-Reiley and Spulber (2000)). Formation of business to business electronic markets for upstream transactions will not only directly effect the upstream market but will also indirectly effect performance of the downstream market.

This paper analyzes these direct and indirect effects of upstream electronic markets (e-markets) with a simple model of one homogeneous input and one homogeneous output. Both goods are produced with constant returns to scale of production. Thus the only factor that influences upstream and downstream prices are market structures of upstream markets. Upstream transaction can be through either traditional bilateral relationship (t-market) or oligopolistic e-market. We characterize how size of e-market upstream influences downstream competition and how prices and profits are effected.

Our analysis shows that existence of e-market of any size is always good for downstream market competition. Upstream prices are reduced in both e-market and t-markets. Traditional market price must be reduced to make the downstream firms that buy from it competitive. When e-market is small, this effect is quite striking: downstream prices are reduced while profit of e-market upstream firms change very little.

As the size of the e-market grows, marginal revenue from increasing output becomes smaller for each upstream e-market firm. This is because in order to increase output downstream, greater price reduction upstream becomes necessary since there are less high cost firms (those firms that buy in high price t-markets) from which outputs can be appropriated. As result, although there are more firms, contraction of individual firm output is so large that total e-market transaction decreases. This means upstream e-market price increases as size of e-market increases. If size of e-market increases further, t-market firms face less competition downstream and upstream t-market prices will also start to increase. Accordingly downstream price will also begin to increase when e-market expands beyond a certain point.

Our approach can be best described as a model of upstream oligopoly. Until now, study of vertical relationships have focused on how various transaction costs determine vertical restraints (Tirole (1992)). Significant search costs and informational asymmetries meant it was more realistic to consider a firm dealing only with one or two firms in vertical relationships.[1] However technology has made it possible to establish markets with multiple sellers and buyers for products previously limited to sourcing by contracts or vertical integration. This has made a straightforward yet unexplored framework of upstream oligopoly markets relevant.[2]

In the next section we introduce the basic framework. We define what we mean by a traditional relationship (t-market) and analyze the situation when e-market is perfectly competitive. Section 3 is the main interest of this paper, the analysis of oligopolistic e-market. We compare the prices and profits in downstream and upstream markets by simulation. In Section 5 we discuss possible extensions to our approach.

2 Traditional and Electronic Relationships

2.1 Traditional Relationships

There are N identical upstream firms and N identical downstream firms. An upstream firm has marginal cost of production c. Downstream firms produce a homogeneous product with total demand $p = a - Q$. Downstream firm has constant returns to scale production. The upstream firms produce a homogeneous product which is the only input that downstream firms use. Let p_w denote the upstream price that an upstream firm charges a downstream firm.

In the final good market (downstream market) firms compete in quantities (Cournot). If all firms have the same input cost, p_w, then each

[1]In fact the textbook example (most recently , Church and Ware (2000)) of opportunistic behaviour, GM and Fisher Body, is that of procurement in automobile industry.

[2]Kamien and Tauman (1986) uses a similar approach. In their framework, upstream demand is also determined by the downstream oligopoly market. However there was only one seller (patent owner selling patented technology) upstream.

output (q), total output (Q) and final good price (p) are

$$q = \frac{a - p_w}{N+1}, \quad Q = \frac{N(a - p^w)}{N+1}, \quad p = \frac{a + p_w}{N+1}.$$

Because of constant returns to scale production, q is also an upstream firm's factor demand function.

Under the traditional relationship, each upstream firm sells to one downstream firm. We assume that upstream firm has local monopoly power: the upstream price with the traditional vertical relationship is, that of monopoly with constant marginal cost c and inverse demand $p_w = a - (N+1)q$. The upstream price and firm output are,

$$p_w^T = \frac{a + c}{2}, \quad q^T = \frac{(a - c)}{2(N+1)}.$$

The final good (downstream) equilibrium price and outputs are,

$$P^T = \frac{(N+2)a + Nc}{2(N+1)}, \quad Q^T = \frac{(a - c)N}{2(N+1)}.$$

Upstream and downstream firms' profits are,

$$\pi_U^T = \frac{(a - c)^2}{4(N+1)}, \quad \pi_D^T = \frac{(a - c)^2}{4(N+1)^2}.$$

2.2 Electronic Market

Now we consider a situation where k pair of firms form an e-market for the intermediate good. We denote by p_w^e and p_w^t the upstream prices in traditional and electronic markets. Downstream market is now a Cournot oligopoly with k firms with marginal cost p_w^e and $N-k$ firms with marginal cost p_w^t. The corresponding outputs q_w^t and q_w^e take into account the fact that if one of the costs are very low, then the high cost firm must stay out of the market,

$$q_w^t = \begin{cases} \frac{a - p_w^t}{N-k+1} & p_w^t \leq \frac{(N-k+1)p_w^e - a}{N-k} \\ \frac{a - p_w^t(k+1) + kp_w^e}{N+1}, & \text{otherwise} \\ 0 & p_w^t \geq \frac{a + kp_w^e}{k+1} \end{cases} \tag{1}$$

$$q_w^e = \begin{cases} \frac{a - p_w^e}{(k+1)} & p_w^e \leq \frac{(k+1)p_w^t - a}{k} \\ \frac{a - (N-k+1)p_w^e + (N-k)p_w^t}{N+1} & \text{otherwise} \\ 0 & p_w^e \geq \frac{a + (N-k)p_w^t}{N-k+1} \end{cases} \tag{2}$$

Again, they are also firms' factor demand functions.

2.3 E-Market is Perfectly Competitive

Extreme case is when the e-market has the perfectly competitive outcome, i.e., $p_w^{e0} = c$. In the downstream market, t-market firms face competitors that buy the input at marginal cost. The demand in the t-market is

$$q_w^t(p_w^t) = \begin{cases} \frac{a-p_w^t}{N-k+1} & p_w^t \leq \frac{(N-k+1)c-a}{N-k} \\ \frac{a-p_w^t(k+1)+kc}{N+1}, & \text{otherwise} \\ 0 & p_w^t \geq \frac{a+kc}{k+1} \end{cases}.$$

Upstream t-market firms will never price so low that e-market firms are driven out of the market.

Proposition 1. *When there are k upstream firms in e-market, and the e-market is perfectly competitive, upstream e-market and t-market equilibrium prices are,*

$$p_w^{e0} = c, \quad p_w^{t0} = \frac{a + (2k+1)c}{2(k+1)}.$$

The corresponding equilibrium outputs are,

$$q^{e0} = \frac{(N+k+2)(a-c)}{2(k+1)(N+1)} \quad q^{t0} = \frac{a-c}{2(N+1)}.$$

The total output is $kq^{e0} + (N-k)q^{t0}$. Because of constant returns to scale, the output is for both upstream and downstream markets. Final good (downstream market) equilibrium prices and outputs are,

$$p^0 = \frac{(N+k+2)a + (N+k+2Nk)c}{2(k+1)(N+1)}, \quad Q^0 = \frac{(N+2kN+k)(a-c)}{2(k+1)(N+1)}.$$

Corollary 1. *$p^0 < p^T$ and p^0 decreasing in k.*

Consumers benefit from e-market of any size and the benefit increases with number of firms in the e-market.

The profits of firms in the t-markets are,

$$\pi_U^{t0} = \frac{(a-c)^2}{4(k+1)(N+1)}, \quad \pi_D^{t0} = \frac{(a-c)^2}{4(N+1)^2}.$$

The profits of the firms in e-market are,

$$\pi_U^{e0} = 0, \quad \pi_D^{e0} = \frac{(N+k+2)^2(a-c)^2}{4(k+1)^2(N+1)^2}.$$

Corollary 2. *If the e-market is perfectly competitive:*

1. *Downstream firm profit is greater in e-market. Upstream firm profit is larger in t-market.*
2. *Existence of e-market does not change profit of a downstream firm in t-market ($\pi_D^{t0} = \pi_D^T$) but reduces the profit of upstream firm in t-market ($\pi_U^{t0} < \pi_U^T$).*

Although the final product price is reduced by existence of a perfectly competitive e-market, profit remains the same for the downstream firm since upstream price is also reduced.

Downstream and upstream firms have opposite ranking of profits. The interesting question is how aggregate profit (sum of an upstream and a downstream firm in the same market) changes with existence of e-markets. We have,

$$\begin{aligned} ps^T &= \pi_D^T + \pi_U^T = \frac{(a-c)^2(N+2)}{4(N+1)^2}, \\ ps^{t0} &= \pi_D^{t0} + \pi_U^{t0} = \frac{(a-c)^2(N+k+2)}{(k+1)(N+1)^2}, \\ ps^{e0} &= \pi_D^{e0} + \pi_U^{e0} = \frac{(a-c)^2(N+k+2)^2}{(k+1)^2(N+1)^2}. \end{aligned}$$

Corollary 3. *Aggregate profit is larger for the e-market firm. The difference in profits ($ps^{e0} - ps^{t0}$) is decreasing in k.*

The upstream firm's profit is zero in e-market. But the upstream-downstream pair gains by joining the e-market. There are two sources of this gain. Firstly, downstream firm becomes more competitive because its cost (upstream price) is lower. Secondly, distortion from double marginalization is eliminated. Unlike the usual solution in vertical restraints, upstream marginalization is eliminated. If there is possibility of transfer payments between upstream and downstream firms, firms would want to join the e-market. However the gain from cost efficiency decreases as number of low cost e-market firms increases. Thus if there is some fixed cost of switching to e-market, there is a maximum size of e-market beyond which firms have no incentive to join.

The total aggregate surplus is reduced by existence of e-market,

$$N\pi^T - \left(k\pi^{e0} + (N-k)\pi^{t0}\right) = \frac{(Nk-k-2)k(a-c)^2}{4(N+1)(k+1)^2}.$$

This difference is increasing in k.

Finally, the social surplus with only t-market (SS^T) and with both markets (SS^{et0}) are,

$$SS^T = \frac{N(a-c)^2(3N+4)}{8(N+1)^2}, \quad SS^{et0} = \frac{(3k+2Nk+3N+4)(k+2Nk+N)(a-c)^2}{8(k+1)^2(N+1)^2}.$$

The difference is,

$$SS^{et0} - SS^T = \frac{(Nk+2N+4+3k)k(a-c)^2}{8(N+1)(k+1)^2},$$

which is positive and increasing in k.

Corollary 4. *Social surplus is greater when there is an e-market and the difference increases with size of e-market.*

3 E-Market is Cournot Oligopoly

Now we consider a case where e-market is oligopolistic and firms choose outputs. According to Kreps and Scheinkman (1983), this would also be situation if e-market sellers first committed to capacity (which is becomes common knowledge) and then competed in prices by choosing them simultaneously.

We consider the e-market first. Given a t-market upstream price p_w^t, a downstream firm's demand for the intermediate product in e-market is given by equation (2). Using the fact that all firms are identical, we can obtain the inverse e-market demand function,

$$p_w^e = P^t(Q^t) \begin{cases} \frac{k(a+(N-k)p_w^t - Q^e(N-k)}{k(N-k+1)} & Q^t \leq a - p_w^t \\ a - \frac{(k+1)Q^t}{k} & Q^t \geq a - p_w^t, \end{cases}$$

where Q^t is the total output by e-market downstream firms. If the output is very large, the market clearing price must be very low. This makes the firms in the t-market produce nothing.

Each e-market upstream firm takes output of all other $k-1$ firms as given and decides on output q_i to maximize profit,

$$\left(P^t(\sum_{j \neq i} q_j + q_i) - c \right) q_i.$$

Noting that each firm's output (q_i) should be optimal given output of other firms ($\sum_{j \neq i} q_j$) and symmetry ($\sum_{j \neq i} q_j = (k-1)q_i$), we can characterize a firm's equilibrium output for different levels of p_w^t. When p_w^t is close to c, then e-market firms produce nothing. When p_w^t is just high enough so that e-market firms actually produce,

$$q_i = \frac{k(a + p_w^t(N-k) - c(N-k+1)}{(N+1)(k+1)}.$$

Both e-market and t-market firms are selling. As p_w^t becomes larger, e-market firms produce just enough to force t-market firms to produce nothing,

$$q_i = \frac{k(a-c)}{(k+1)^2}.$$

When p_w^t is even larger, the e-market firms behave as if they were the only firms,

$$q_i = \frac{a - p_w^t}{k}.$$

The t-market firms are so inefficient, they are not a threat at all. Using $Q^t = kq_i$, we obtain the relationship between the t-market upstream price and the corresponding e-market upstream price when e-market firms are behaving optimally (Nash equilibrium of Cournot game). Summarizing we have,

$$p_w^e = \begin{cases} \frac{a+k(N-k+1)c+p_w^t(n-k)}{(N-K+1)(k+1)}, & p_w^t \leq p^{t1} \equiv \frac{a(N+1+k+kN-k^2)+ck^2(N+1-k)}{k^2N-k^3+kN+k+N+1} \\ \frac{-a+(k+1)p_w^t}{a}, & p^{t1} < p_w^t \leq p^{t2}, \\ \frac{kc+a}{k+1} & p_w^t > p^{t2} \equiv \frac{a(1+2k)+ck^2}{(k+1)^2}. \end{cases} \tag{3}$$

Now we find the relationship between an e-market upstream price, p_w^e, and the corresponding t-market upstream price when t-market firms are behaving optimally. Equation (1) is the demand that a t-market upstream firm faces. Each upstream firm chooses p_w^t to maximize its profit. The basic relationship between rival upstream market price and demand is the same as in the e-market. When p_w^e is very small, firms from both markets will be producing downstream. For slightly higher p_w^e, t-market firms produce so that p_w^t is just low enough to drive e-market firms out of downstream market. When p_w^t is even higher, it becomes Cournot competition with just t-market firms.

$$p_w^t = \begin{cases} \frac{a+kp_w^e+(k+1)c}{2(k+1)}, & p_w^e \leq p^{e1} \equiv \frac{a(N+k+2)+c(k+1)(N-k)}{kN-k^2+2N+2} \\ \frac{-a+(N-k+1)p_w^e}{N-k}, & p^{e1} < p_w^e \leq p^{e2}, \\ \frac{a+c}{2} & p_w^e > p^{e2} \equiv \frac{a(N-k+2)-c(N-k)}{2(N-k+1)}. \end{cases} \tag{4}$$

In equilibrium both equations (3) and (4) must be satisfied. It is not an equilibrium for firms from one market to be shut out of downstream market. For instance, given p_w^t, it is it is possible for upstream e-market firms price to force t-market downstream firms to produce nothing. However then the final product price will be high enough for t-market upstream firms to lower price below p_w^t. Therefore such price pair is not an equilibrium. All firms will be producing downstream in equilibrium. The equilibrium upstream prices satisfy the first segment (everyone producing) of both price relationships,

Proposition 2. *The equilibrium upstream prices when e-market is a Cournot oligopoly are,*

$$p_w^{e*} = \frac{(N+k+1)a + (k+1)(N+k+2kN-2k^2)c}{3Nk+2+2k+2k^2N-2k^3+2N-k^2}$$
$$p_w^{t*} = \frac{(k+kN+N-k^2+1)a + (-2k^3+2k^2N+k+2kN+N+1)c}{3Nk+2+2k+2k^2N-2k^3+2N-k^2}.$$

We have the following characterization of the equilibrium prices,

Proposition 3. *1. $p_w^{e*} < p_w^{t*} < p_w^T$ for $1 \leq \forall k \leq N$.*

2. There is a $k^e(N)$ such that

$$\frac{\partial p_w^{e*}}{\partial k} \lesseqqgtr 0 \Leftrightarrow k \lesseqqgtr k^e(N).$$

$k^e(N)$ is decreasing in N and $k^e(N) < N$ for sufficiently large N.

3. There is a $k^t(N)$ such that

$$\frac{\partial p_w^{t*}}{\partial k} \lesseqqgtr 0 \Leftrightarrow k \lesseqqgtr k^t(N).$$

$k^t(N)$ is decreasing in N and $k^t(N) < N$ for sufficiently large N.

4. $k^t(N) > k^e(N)$ for $\forall N$.

Proof is in the Appendix. When N is small, t-market price decreases as k increases. E-market price decreases up to some k and then increases. E-market price is always lower than t-market price (Figure 1(a)).

When N is large, as k increases, both market prices increase in the beginning. Eventually, e-market price starts to increase and then for slightly larger k, so does t-market price. Again, e-market price is always lower than t-market price (Figure 1(b)).

There are two sources of competition that each e-market upstream firm faces. First, there is competition in the downstream market from t-market firms. Secondly, there is competition from other e-market firms both upstream and downstream. Consider the Cournot competition in the upstream market: marginal revenue from increasing output increases as k increases because effect of more output from a single firm on market clearing price is smaller. This effect decreases the equilibrium e-market price as k increases. However as k increases, the aggregate effect of e-market firms downstream becomes larger and when k is very large, marginal revenue in upstream market from increasing output will begin to decline in k. Output of each upstream market firm begins to decrease in k for very large k's.

Of course the equilibrium prices are determined by the interaction of the two markets. As k increases, traditional firm will always produce more in response to competition from e-market. As we just observed, e-market firms may also produce more and lower price when k is small. The equilibrium prices will be decreasing in k. However when k is large, price in e-market will start to increase (for each t-market price). The equilibrium e-market price will eventually begin to increase for sufficiently large k. This increase may eventually lead to increase in the traditional market price also. This reversal effect is larger when N is very large because output of each firm is relatively smaller.

For the rest of the analysis, rather than presenting the complicated equilibrium formulae, we plot how prices, outputs and profits change with size of e-market (k) when there are many downstream firms ($N = 200$) and when there are few ($N = 10$). Other parameter values are $A = 100$ and $c = 10$. In the Figures, plots for t-market are in crosses and e-market plots are in a solid line.

Figures 2 and 3 show how equilibrium firm outputs (q^{t*}, q^{e*}) and total market outputs ($Q^{t*} = (N-k)q^{t*}$, $Q^{e*} = kq^{e*}$) change with size of e-market. Output of each e-market firm is declining and that of t-market firm is increasing in k but e-market firm output is always larger. The firm level trend is magnified by number of firms when we look at the total market outputs. The downstream market (final good) price will be,

$$P^* = a - Q^{t*} - Q^{e*}.$$

Reflecting the upstream price levels, the final good price starts to increase when size of e-market becomes very large when N is large (Figure 4).

Now we examine the profitability of firms. Since the upstream price is lower in the e-market, e-market downstream firms have greater profits than t-market downstream firms ($\pi_D^e > \pi_D^t$, Figure 5). When the size of e-market is small, e-market downstream firms are one of the few firms with cost advantage. This implies large outputs and e-market upstream firms are more profitable than traditional market firms despite the lower upstream price. As size of e-market increases, competition among e-market firms downstream increases and eventually e-market upstream firms become less profitable than t-market upstream firms (π_U^t and π_U^e). Both profits are decreasing in size of e-market but e-market firms' profits decrease more quickly (Figure 6).

The aggregate profits ($\pi_D^t + \pi_U^t$ and $\pi_D^e + \pi_U^e$) are in Figure 7. The e-market makes the downstream firms very competitive. Thus loss to e-market upstream firms from competition in the upstream market is off set by benefit from being competitive in the downstream market when size of e-market is small. As size increases, aggregate profit decreases. Traditional market firms' aggregate profit also decreases with k but at a slower rate. Eventually when there are sufficiently many e-market firms, competition among e-market firms become so fierce that their aggregate profit becomes less than t-market firms. Thus not all of the market will be e-market in equilibrium (if firms had choice of markets).[3]

Finally, we compare the profits when there is an e-market and where there are only t-markets ((π_U^T, π_D^T, $\pi_D^T + \pi_U^T$, Figure 8).[4] Recall from Proposition 3 that upstream prices will be lower in both markets when there is an e-market. This implies that final good price will always be lower with e-market in existence. In the figure, straight horizontal lines are plots when there are only t-markets). Downstream firms in both markets do benefit from e-market of any size. There is a positive externality to t-market firms from competition in the e-market. Benefit to the upstream firms depend on the size of the market. Upstream e-market firm does benefit if the size of e-market is small through competitiveness of the downstream firms in the final good market. However this is no longer the case when e-market is larger because now there is competition among efficient e-market firms. Accordingly, aggregate profit of an e-market pair is greater than t-market only if e-market is small. Upstream t-market firms

[3]Possibility of making type of market to join choice is discussed in Section 4.

[4]Only simulations for $N = 10$ is presented. The plots for $N = 200$ is qualitatively the same. There is so much difference between p_w^t and other two prices so that either p_w^t is out of the graph, or the other two prices are indistinguishable.

always do worse by formation of an e-market. The gain to downstream t-market firms is not enough to offset this loss to upstream firm. Thus the aggregate profit is smaller for t-market firms if there is an e-market of any size.

4 Concluding Remarks

Electronic commerce has made it possible to form markets for products that previously did not have markets and it has changed the form of the market. In this paper we have focused on how change in a vertical relationship can change downstream market performance. Our focus has been the sizes of e-markets and traditional markets.

There are several ways to extend our approach for a more complete analysis. We assumed that when a traditional bilateral relationship is replaced by e-market, both downstream and upstream firms join the e-market. However when one firm discontinues a bilateral relationship the other firm may decide to form a new traditional relationship with another firm. Which market to join is not a choice in our framework.[5]

There is also no reason why a single firm could not have access to both e-market and t-market. A firm might choose to procure both from a long term contractual relationship and a spot e-market (Newberry (1998)). One expects that introduction of electronic commerce has made spot markets for products for which there were none traditionally or existing ones have been enhanced.

5 References

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[5] We note however that, one reason Convisint attracted antitrust interest was because the downstream firms refuse to deal with upstream firms not in the e-market. That is, only downstream firms choose which market to join. The other half of the bilateral relationship was forced to join that same market. In our framework, it suffices to check the incentive of the downstream firm to join the e-market is determined by π_e^D for k and π_t^D for $k-1$.

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Appendix

Proof of Proposition 3

1.

$$p_w^{t*} - p_w^{e*} = \frac{(a-c)(1-kN+k^2)}{-3kN-2-2k-2k^2N+2k^3-2N+k^2}.$$

This is positive when $N \geq 3$ and $1 \leq k \leq N-1$. Similarly,

$$p_w^T - p_w^{t*} = \frac{k(-N-2kN-k+2k^2)(a-c)}{2(-3kN-2-2k-2k^2N+2k^3-2N+k^2)} > 0,$$

for relevant range.

2.

$$\frac{\partial p_w^{e*}}{\partial k} = \frac{F(a-c)}{(3kN+2+2k+2k^2N-2k^3+2N-k^2)^2},$$

where $F = -2-6kN+4k^2N+4k+13k^2-6N+4k^3-3N^2-4N^2k$.

This is negative when $k=1$. Since

$$\frac{\partial F}{\partial k} = 2(k+N+2)(6k-2N+1) \lesseqgtr 0 \Leftrightarrow k \lesseqgtr \frac{2N-1}{6},$$

if F is zero, it will be at a unique $k^e(N)$. If

$$F|_{k=N-1} = 3-12N^2-6N+4N^3 > 0,$$

then $k^e(N) < N-1$. In fact it will be negative for sufficiently large N. To see how $k^e(N)$ depends on N,

$$\frac{\partial F}{\partial N} = -6k+4k^2-6-6N-8kN < 0.$$

3.

$$\frac{\partial p_w^{t*}}{\partial k} = \frac{G(a-c)}{(-3*k*n-2-2*k-2*k^2*n+2*k^3-2*n+k^2)^2},$$

where $G = -6kN + 2k^2N - 2k + 5k^2 - N + 4k^3 - 2k^4 - N^2 + 4k^3N - 4N^2k - 2N^2k^2$.

We do a similar analysis on G.

4.

$$\Delta = G - F = -2k^2N - 6k + 5N - 8k^2 + 4k^3N - 2N^2k^2 - 2k^4 + 2N^2 + 2.$$

One can show that $\frac{\partial\Delta}{\partial k} < 0$ for $1 \le k \le N-1$ and $\Delta < 0$ when $k = 1$. So $G - F < 0$ for $1 \le k \le N-1$. Given monotonicity of G and F in k we observed previously, we have $k^e(N) < k^t(N)$.