Risk, productive government expenditure, and the world economy

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Abstract

This paper analyzes the expenditure policy of the public sector and risk in a two-country stochastic AK growth model, provided that public spending is productivity- and volatility-enhancing. First we derive the world macroeconomic equilibrium. Then we study the impact of changes in exogenous variables on consumption, growth, and welfare. Next, we show that consumption-wealth ratio and welfare should be higher in an open economy than in a closed economy. We discuss whether open economies grow more than closed economies. Then the optimal size of the public sector is derived in two different scenarios in an open economy. We get that the size of the public sector which maximizes welfare is lower than that which maximizes growth. Finally, we analyze whether more open economies are associated with a higher optimal size of the public sector. The optimal size in an open economy can be higher than that in a closed economy for two reasons: different marginal impact of public spending on productivity and risk diversification.

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Contents

1 INTRODUCTION

The impact of government expenditure policy on long run growth is an important policy issue. The emergence of endogenous growth models in the 80's have provided an useful approach to analyze how government expenditure policy can influence on the long run trajectory of the economy. Barro (1990) pioneered the analysis based on a closed economy deterministic AK growth model where public spending is productive[1]. Others, Turnovsky (1998, 1999) for example, have followed suit incorporating small open economy features, risk and other issues (such as congestion, for example) into endogenous growth models where public spending is productive. Thus substantial conclusions have been derived concerning the impact of risk and the expenditure policy of the public sector on the economy, and the optimal size of the public sector, provided that public spending enhances productivity and volatility. However, there is a recurrent shortage of analysis based on two-country stochastic models, specially when the integration of financial markets is becoming more complete.

This paper analyzes the influence of risk and the expenditure policy of the public sector by incorporating productive public spending [see Barro (1990)] into a two-country stochastic AK growth model developed by Turnovsky (1997, Ch. 11). Thus we derive the size of the public sector that maximizes welfare endogenously, instead of assuming an exogenous size, as in Turnovsky (1997, Ch. 11). Previous analysis introduced risk into endogenous growth models, but public spending was neither utility-enhancing nor productive [see Eaton (1981), for example]. Turnovsky extended Barro's (1990) closed economy model by introducing productivity- and volatility-enhancing public spending into a stochastic endogenous growth small open economy. In addition, Turnovsky (1998) analyzed the impact of productive public spending (subject to congestion) in a risky closed economy. Therefore, our model has been constructed combining the main characteristics of the core literature[2]:

- It is an AK growth model, as the rest of the models.
- It is a two-country model, following the framework set out by Turnovsky

[1] Barro (1990) also analyzed the role of utility-enhancing government expenditure.

[2] We denominate "core literature" to the set of papers that have analyzed the impact of risk and/or the expenditure policy of the public sector on the economy based on AK growth models, provided that public spending is productive and no congestion arises, and to the model developed by Turnovsky (1997, Ch. 11).

(1997, Ch. 11), whereas the rest of the models are one-country models (either a closed economy or small open economy).

- Public spending is productive, pioneered by Aschauer (1989) and incorporated originally into endogenous growth models by Barro (1990). Thus the model is able to determine the size of the public sector that maximizes the welfare of the representative agent, as most of the models of the core literature do. Turnovsky (1997) is the only model that cannot analyze the magnitude of such a size, since the public spending is neither utility enhancing nor productive, so that "it can be interpreted as being a real drain on the economy or, alternatively, as some public good that does not affect the marginal utility of private consumption or the productivity of private capital" (Turnovsky, 1997, p. 338). Turnovsky (1998, 1999) extend Barro (1990) from a closed economy to a model with congestion and to a small open economy setting, respectively.

- The model is stochastic. Barro (1990) is the only model of the core literature that is not stochastic. Turnovsky (1998, 1999) extend the deterministic model in Barro (1990) to a stochastic setting.

Table 3.1. encompasses the relationship between the model of this paper and the core literature.

Table 3.1. An overview of the model

The different models	AK growth	Two countries	Size of the public sector	Stochastic shocks
Barro (1990)	X		X	
Turnovsky (1997)	X	X		X
Turnovsky (1998)	X		X	X
Turnovsky (1999)	X		X	X
This model	X	X	X	X

This model can be specially useful in the present moment of the European Economic and Monetary Union (EMU). First, countries of the euro area have adopted the Stability and Growth Pact from 1st January 1999 onwards, whose objective is that countries of the euro area must attain budget balance, in the medium or in the long run, so that the assumption of continuous budget balance that we make in this chapter seems reasonable. Second, the emphasis of this paper is the long run and, therefore, it does not focus on the influence of business cycles, important as they may be. Finally, there is a permanent debate about whether the size of the public sector should be bigger or smaller and, more specifically, whether more open economies should have bigger governments or not. This model sheds some light on the issue, since it compares the size of the public sector that maximizes welfare in an open economy with that in a closed economy.

This paper is organized as follows. We first obtain the world macroeconomic equilibrium, given that the size of the public sector is exogenously given. Then we study the impact of changes in exogenous variables on key economic variables such as consumption-wealth ratio, the rate of growth of wealth and welfare. We compare the results derived from an open economy with those of a closed economy. Next, we derive the welfare-maximizing size of the public sector. We discuss the differences arising from maximizing growth and welfare. Additionally, we analyze whether more open economies are associated with a higher size optimal of the public sector, even when public spending is productive-only. Finally, we conclude indicating possible avenues for future research.

2 THE WORLD ECONOMY

2.1 The basic structure

The world is a real economy composed of two countries, each of them producing only one homogeneous good. In each country there exists a representative agent with infinite time horizon. The homogeneous good produced by both countries can be either consumed or invested in capital without having to incur in any kind of adjustment costs. There are two assets: domestic capital and foreign capital. Unstarred variables refer to the domestic economy, whereas the starred variables refer to the foreign economy. Both the domestic capital, K, and the foreign capital, K^*, can be owned by the domestic representative agent or the foreign representative agent. The subscript d denotes the holdings of assets of the domestic representative agent and the subscript f denotes the holdings of assets of the foreign representative agent. So it must be satisfied that

$$\begin{aligned} K &= K_d + K_f \\ K^* &= K_d^* + K_f^*. \end{aligned}$$

The wealth of the domestic representative agent, W, and the wealth of the foreign representative agent, W^*, therefore will be

$$W = K_d + K_d^* \quad (1)$$
$$W^* = K_f + K_f^*. \quad (2)$$

The public sector purchases part of the private flow of production and utilizes it to supply a productive pure public good to the private representative agent. Public spending, dG, increases with wealth, so that we can achieve a balanced growth path[3]. We specify public spending as follows

$$dG = gWdt + Wdz, \quad (3)$$

where $g = G/W$ denotes the size of the public sector, and dz is the increment of a stochastic process z. Those increments are temporally independent and are normally distributed. They satisfy that $E(dz) = 0$ and $E(dz^2) = \sigma_z^2 dt$.

Domestic production can be obtained using only domestic capital, K, through a somewhat modified AK function and it is expressed through a first order stochastic differential equation

$$dY = \alpha K dt + \alpha K dy, \quad (4)$$

where

$$\alpha = \overline{\alpha} + \delta g - 0.5\theta g^2. \quad (5)$$

The term $\overline{\alpha} > 0$ is the (constant) physical marginal product of private capital when the size of the domestic public sector is zero and dy represents a

[3] Other rules can also achieve a balanced growth path. See Turnovsky (1996) for more details.

proportional domestic productivity shock. More precisely, dy is the increment of a stochastic process y. Those increments are temporally independent and are normally distributed. They satisfy that $E(dy) = 0$ and $E(dy^2) = \sigma_y^2 dt$.[4] We omit, for convenience, the formal references to time, although those variables depend on time. We must note that dY indicates the flow of production, instead of Y, as is ordinarily done in stochastic calculus.

The production function incorporates the influence of the public sector on the physical marginal product of private capital and on the magnitude of the stochastic domestic productivity shock by means of a quadratic term in g. The modified marginal physical product of private capital, α, is based on Gallaway and Vedder (1995) and Vedder and Gallaway (1998, p. 4). They refer to what US Congress Representative Richard Armey (1995) termed the Armey Curve, which is a figure *à la Laffer* relating the size of the public sector with the rate of growth of the economy (Vedder and Gallaway, 1998, p. 1). However, the curve relating both variables can, in fact, be found before in Barro (1990, pp. S110 and S118), or in Barro and Sala-i-Martin (1995, p. 155) and thus the Armey Curve should be more conveniently renamed as the Barro-Sala-i-Martin-Armey (BSiMA) Curve. Here we have converted that relationship into another one between the size of the public sector and the marginal physical product of private capital.

Both parameters δ and θ in equation (5) are positive, so that the function is concave in the size of the public sector, g, and we restrict ourselves to the case $g < \delta/\theta < 1$. Then we assume that the marginal impact of the public sector on the marginal physical product of private capital is positive, at a diminishing rate[5]. In addition, an increase in the size of the public sector amplifies the magnitude of the impact of domestic productivity shocks, at a diminishing rate. We could easily introduce into the model the alternative assumption that an increase in the size of the public sector reduces the impact of domestic productivity shocks, just changing the signs for the parameters δ and θ for the stochastic component in equation (5) above. In case $\delta = \theta = 0$ then we return to a standard AK production function.

This production function captures essentially, albeit in an different al-

[4] That is, the production flow follows a Brownian motion with drift αK and with variance $\alpha^2 K^2 \sigma_y^2$.

[5] It can be easily obtained that the marginal physical product of private capital, α, becomes a maximum when $g = \delta/\theta$. Here we are assuming that the marginal impact of public spending on α becomes negative for some value of the size of the public sector $g < 1$ and, therefore, that it is negative for $g = 1$ as well.

ternative way, the basic features of models with a productive public sector, such as Barro (1990), or Turnovsky (1998, 1999) in stochastic settings, and it makes possible to extend the analysis and to compare our results to theirs. We have chosen this alternative way of modeling because it is easier to adapt to a two-country world economy and additionally it can be easily extended to encompass volatility-reducing features. We should note that here we introduce the flow of production goods provided by the public sector and not the stock of accumulated public capital stock.[6] Thus, even though we should postulate it as a stock (being the spending in public physical structures), that would lead to a transitional dynamics equilibrium (Turnovsky, 1998. p. 6), and then the literature has usually opted to postulate it as a flow to be analytically tractable.[7] Additionally, we should note that our formulation implies that only the deterministic component of the public spending in production goods is productive.

The same structure applies to the foreign economy. Foreign public spending is given by

$$dG^* = g^* W^* dt + W^* dz^*,$$

where $g^* = G^*/W^*$ denotes the size of the foreign public sector, and dz^* is the increment of a stochastic process z^*. Those increments are temporally independent and are normally distributed. They satisfy that $E(dz^*) = 0$ and $E(dz^{*^2}) = \sigma_{z^*}^2 dt$.

The foreign economy is structured symmetrically to the domestic economy. Thus foreign production is obtained using only foreign capital, K^*, through a modified AK function

$$dY^* = \alpha^* K^* dt + \alpha^* K^* dy^*,$$

such that

$$\alpha^* = \overline{\alpha}^* + \delta^* g^* - 0.5\theta^* g^{*^2},$$

[6] As Turnovsky puts it (1998, p. 5), "In introducing productive government expenditure one must choose between formulating it as a flow or as a stock, a choice that involves a tradeoff between tractatibility and realism".

[7] See, for example, Barro (1990) and Turnovsky (1998, 1999).

where $\overline{\alpha}^* > 0$ is the marginal physical product of capital when the size of the foreign public sector is zero and dy^* represents a proportional foreign productivity shock. The term dy^* is the increment of a stochastic process y^*. Those shocks are temporally independent and are distributed normally, satisfying that $E(dy^*) = 0$ and that $E(dy^{*^2}) = \sigma_{y^*}^2 dt$.

2.2 The domestic economy

2.2.1 The problem

The preferences of the domestic representative agent are represented by an isoelastic intertemporal utility function where she obtains utility from consumption, C

$$E \int_0^\infty \frac{1}{\gamma} C^\gamma e^{-\beta t} dt; -\infty < \gamma < 1. \tag{6}$$

The welfare of the domestic representative agent in period 0 is the expected value of the discounted sum of instantaneous utilities, conditioned on the set of disposable information in period 0. The parameter β is a positive subjective discount rate (or rate of time preference). The utility function becomes logarithmic when $\gamma = 0$. The empirical evidence suggest a high degree of risk aversion (Campbell, 1996).[8] The restrictions on the utility function are necessary to ensure concavity with respect to consumption.

The domestic representative agent consumes at a deterministic rate $C(t)dt$ in the instant dt and must pay the corresponding taxes and thus the dynamic budget restriction can be expressed as

$$dW = [\alpha K_d + \alpha^* K_d^*]\, dt + [\alpha K_d dy + \alpha^* K_d^* dy^*] - Cdt - dT, \tag{7}$$

where dT denotes the taxes the domestic representative agent must pay to the public sector. We assume that the collection of taxes exactly offset public spending

$$dT = dG, \tag{8}$$

that is, the public sector balances budget continuously.

[8] The Arrow-Pratt coefficient of risk aversion is given by $1 - \gamma$.

Combining equations (3) and (8), and substituting them into (7), the restriction for the resources of the domestic economy is given by

$$dW = [\alpha K_d + \alpha^* K_d^* - C - gW]\, dt + [\alpha K_d dy + \alpha^* K_d^* dy^* - W dz] \,. \qquad (9)$$

Going back to equation (1), if we define the following variables for the domestic representative agent

$$n_d \equiv \frac{K_d}{W} = \text{ share of the domestic portfolio materialized in domestic capital}$$
$$n_d^* \equiv \frac{K_d^*}{W} = \text{ share of the domestic portfolio materialized in foreign capital,}$$

equation (1) can be expressed in a more convenient way as

$$1 = n_d + n_d^*$$

Substituting those variables into the budget constraint (9) we obtain the following dynamic restriction for the resources of the domestic economy

$$\frac{dW}{W} = \left[\alpha n_d + \alpha^* n_d^* - \frac{C}{W} - g\right] dt + [\alpha n_d dy + \alpha^* n_d^* dy^* - dz] \,.$$

This equation can be more conveniently expressed as

$$\frac{dW}{W} = \psi dt + dw, \qquad (10)$$

where the deterministic and stochastic parts of the rate of accumulation of assets, dW/W, can be expressed in the following way

$$\psi \equiv n_d [\alpha - \alpha^*] + \alpha^* - g - \frac{C}{W} \equiv \rho - g - \frac{C}{W} \qquad (11)$$
$$dw \equiv n_d [\alpha dy - \alpha^* dy^*] + \alpha^* dy^* - dz, \qquad (12)$$

where $\rho \equiv \alpha n_d + \alpha^* n_d^* \equiv n_d [\alpha - \alpha^*] + \alpha^*$ denotes the gross rate of return of the asset portfolio.

2.2.2 The equilibrium

The objective of the domestic representative agent consists in choosing the path of consumption and portfolio shares that maximizes the expected value of the intertemporal utility function (6), subject to $W(0) = W_0$, (10), (11) and (12). This optimization is a stochastic optimum control problem.[9] Initially we assume that the government sets an arbitrarily exogenous size of the public sector, g. We analyze the case in which such a size will be chosen optimally in section 4.

It is important to bear in mind that the domestic agent takes as given the rates of return of different assets, as well as the corresponding variances and covariances. However, these parameters will endogenously be determined in the macroeconomic equilibrium we are going to obtain. We look for values of the endogenous variables that are not stochastic in equilibrium and then we show that the results validate the initial assumption that equilibrium values are not stochastic.

We introduce a value function, $V(W)$, which is defined as

$$V(W) = \underset{\{C,n_d\}}{Max}\ E\int_0^\infty \frac{1}{\gamma}C^\gamma e^{-\beta t}dt, \tag{13}$$

subject to the restrictions (10), (11) and (12), and given initial wealth. The value function in period 0 is the expected value of the discounted sum of instantaneous utilities, evaluated along the optimal path, starting in period 0 in the state $W(0) = W_0$.

Starting from equation (13) the value function must satisfy the following equation, known as the Hamilton-Jacobi-Bellman equation of stochastic control theory or, for short, the Bellman equation

$$\beta V(W) = \underset{\{C,n_d\}}{Max}\left[\frac{1}{\gamma}C^\gamma + V'(W)W\psi + 0.5V''(W)W^2\sigma_w^2\right], \tag{14}$$

where ψ can be found in (11) and σ_w^2 denotes the variance of the stochastic element of the rate of accumulation of assets, given by equation (12).

[9] To solve problems of stochastic optimum control see, for example, Kamien and Schwartz (1991, section 22), Malliaris and Brock (1982, ch. 2), Obstfeld (1992), or Turnovsky (1997, ch. 9; 2000, ch. 15).

We differentiate partially equation (14) with respect to C and n_d in order to obtain the first order conditions of this optimization

$$C^{\gamma-1} - V'(W) = 0 \tag{15}$$
$$V'(W)W(\alpha - \alpha^*) + V''(W)W^2 cov\left[dw, \alpha dy - \alpha^* dy^*\right] = 0. \tag{16}$$

The solution to this problem is obtained through trial and error. We seek to find a value function $V(W)$ that satisfies, on the one hand, the first order optimality conditions and, on the other, the Bellman equation. In the case of isoelastic utility functions the value function has the same form of the utility function [Merton (1969), result generalized in Merton (1971)]. Thus we suggest the guess solution

$$V(W) = AW^{\gamma}, \tag{17}$$

where the coefficient A will be determined below. This guess solution implies that

$$\begin{aligned} V'(W) &= A\gamma W^{\gamma-1} \\ V''(W) &= A\gamma(\gamma-1)W^{\gamma-2}. \end{aligned}$$

Substituting these expressions into the first order optimality conditions (15) and (16) we get that

$$C^{\gamma-1} = A\gamma W^{\gamma-1} \tag{18}$$
$$(\alpha - \alpha^*)\, dt = (1-\gamma)\, cov\left[dw, \alpha dy - \alpha^* dy^*\right] \quad . \tag{19}$$

These are typical equations in stochastic models in continuous time. Equation (18) indicates that at the optimum, the marginal utility derived from consumption must be equal to the marginal change in the value function or the marginal utility of wealth. Equation (19) shows us that the optimal choice of portfolio shares of the domestic representative agent must be such that the risk-adjusted rates of return of both assets are equalized.

Combining (18) and (19), and substituting them into the equation (14), we can calculate, after some algebra, the equilibrium portfolio shares and the consumption-wealth ratio in the domestic open economy

$$n_d = \frac{\alpha - \alpha^*}{(1-\gamma)\Delta} + \frac{{\alpha^*}^2 \sigma_{y^*}^2 - \alpha\alpha^* \sigma_{yy^*} + \alpha\sigma_{yz} - \alpha^*\sigma_{y^*z}}{\Delta} \tag{20}$$

$$n_d^* = 1 - n_d$$

$$\left(\frac{C}{W}\right)_o = \frac{1}{1-\gamma}\left\{\beta - \gamma(\rho - g) + 0.5\gamma(1-\gamma)\sigma_{w,o}^2\right\}, \tag{21}$$

where

$$\Delta = \alpha^2\sigma_y^2 - 2\alpha\alpha^*\sigma_{yy^*} + {\alpha^*}^2\sigma_{y^*}^2 \tag{22}$$

$$\sigma_{w,o}^2 = n_d^2\alpha^2\sigma_y^2 + 2n_d n_d^*\alpha\alpha^*\sigma_{yy^*} + {n_d^*}^2{\alpha^*}^2\sigma_{y^*}^2 + \sigma_z^2 - 2n_d\alpha\sigma_{yz} - 2n_d^*\alpha^*\sigma_{y^*z}. \tag{23}$$

Please note that neither Δ nor the variance of the rate of growth of assets, $\sigma_{w,o}^2$, can be negative and the subscript o refer to values in an open economy.

The equilibrium is characterized by a balanced real growth. The equilibrium rate of wealth accumulation of the domestic economy follows the stochastic process

$$\frac{dW}{W} = \psi_o dt + dw_o,$$

where the deterministic and stochastic components are, respectively

$$\psi_o = \frac{1}{1-\gamma}\left\{\rho - g - \beta - 0.5\gamma(1-\gamma)\sigma_{w,o}^2\right\} \tag{24}$$

$$dw_o = n_d\alpha dy + n_d^*\alpha^* dy^* - dz. \tag{25}$$

Now we can get the equilibrium solution in a closed economy by setting $n_d = 1$ and $n_d^* = 0$ in equations (21), (23), (24), and (25). We will use the shares of the domestic portfolio materialized in domestic and foreign capital, n_d and n_d^* respectively, to approximate the degree of openness of the domestic economy. The equilibrium of the domestic economy if it were closed is given by

$$\left(\frac{C}{W}\right)_c = \frac{1}{1-\gamma}\left\{\beta-\gamma\left(\alpha-g\right)+0.5\gamma\left(1-\gamma\right)\sigma_{w,c}^2\right\} \tag{26}$$

$$\sigma_{w,c}^2 = \alpha^2\sigma_y^2+\sigma_z^2-2\alpha\sigma_{yz} \tag{27}$$

$$\psi_c = \frac{1}{1-\gamma}\left\{\alpha-g-\beta-0.5\gamma\left(1-\gamma\right)\sigma_{w,c}^2\right\} \tag{28}$$

$$dw_c = \alpha dy - dz,$$

where the variables with the subscript c refer to values in a closed economy. We should note that in case there is no risk, that is, $\sigma_{w,c}^2=0$, differentiating equation (28) with respect to the size of the public sector, g, we can see that the rate of growth of assets first increases with the size of the public sector but then diminishes with the size of the public sector, thus implying the BSiMA Curve referred above.

To guarantee that consumption is positive in the domestic open economy we impose the feasibility condition that the marginal propensity to consume out of wealth must be positive since wealth does not become negative

$$\frac{1}{1-\gamma}\left\{\beta-\gamma\left(\rho-g\right)+0.5\gamma\left(1-\gamma\right)\sigma_{w,o}^2\right\}>0.$$

For the first order optimality conditions to characterize a maximum, the corresponding second order condition must be satisfied, that is, the Hessian matrix associated to the maximization problem and evaluated at the optimal values of the choice variables

$$\begin{bmatrix} \left(\gamma-1\right)\left(V'(W)\right)^{\frac{\gamma-2}{\gamma-1}} & 0 \\ 0 & V''(W)W^2\Delta \end{bmatrix}$$

must be negative definite,[10] which implies that

$$\begin{aligned} \left(\gamma-1\right)\left(V'(W)\right)^{\frac{\gamma-2}{\gamma-1}} &< 0 \\ V''(W)W^2\Delta &< 0, \end{aligned}$$

[10] See Chiang (1984, pp. 320-323), for example.

where $\Delta > 0$ (in a risky economy) was already defined in equation (22). To evaluate those conditions first we obtain the value of the coefficient A in equation (18)

$$A = \frac{1}{\gamma}\left(\frac{C}{W}\right)^{\gamma-1}, \tag{29}$$

where C/W is the optimal value pointed out by equation (21). Then we substitute (29) into the value function (17). Then the value function is given, after some algebra, by

$$V(W) = \frac{1}{\gamma}\left(\frac{C}{W}\right)^{\gamma-1} W^{\gamma}, \tag{30}$$

where we can observe that, given the restrictions on the utility function, $V'(W) > 0$ and $V''(W) < 0$ provided that $C/W > 0$.

In addition, we impose that the macroeconomic equilibrium must satisfy the transversality condition so as to guarantee the convergence of the value function

$$\lim_{t\to\infty} E\left[V\left(W\right)e^{-\beta t}\right] = 0. \tag{31}$$

Now let us show that should the feasibility condition be satisfied then that would be equivalent to satisfy the transversality condition.[11] To evaluate (31), we start expressing the dynamics of the accumulation of wealth

$$dW = \psi W dt + W dw. \tag{32}$$

The solution to equation (32), starting from the initial wealth $W(0)$, is[12]

$$W(t) = W(0)e^{(\psi-0.5\sigma_w^2)t+w(t)-w(0)}.$$

Since the increments of w are temporally independent and are normally distributed then[13]

[11] See Merton (1969). Turnovsky (2000) provides, for example, the proof of the transversality condition as well.

[12] See Malliaris and Brock (1982, pp. 135-136), for example.

[13] See Malliaris and Brock (1982, pp. 137-138), for example.

$$
\begin{aligned}
E[AW^{\gamma}e^{-\beta t}] &= E[AW(0)^{\gamma}e^{\gamma(\psi-0.5\sigma_w^2)t+\gamma[w(t)-w(0)]-\beta t}] \\
&= AW(0)^{\gamma}e^{\left[\gamma(\psi-0.5\sigma_w^2)+0.5\gamma^2\sigma_w^2-\beta\right]t}.
\end{aligned}
$$

The transversality condition (31) will be satisfied if and only if

$$\gamma\left\{\psi-0.5\gamma\left(1-\gamma\right)\sigma_w^2\right\}-\beta<0.$$

Now substituting equations (24) and (21), it can be shown that this condition is equivalent to

$$\frac{C}{W}>0,$$

and thus feasibility guarantees convergence as well.

Finally, we should note that since the public sector equilibrates its budget continuously then the intertemporal budget constraint of the public sector is satisfied trivially.

2.3 The foreign economy

2.3.1 The problem

The structure of the foreign economy and the problem facing the foreign representative agent can be formulated in an analogous way to the domestic economy. Her preferences are represented by the following intertemporal utility function

$$E\int_0^{\infty}\frac{1}{\gamma^*}C^{\gamma^*}e^{-\beta^*t}dt;\,-\infty<\gamma^*<1.$$

The dynamics of foreign wealth are given by

$$\frac{dW^*}{W^*}=\psi^*dt+dw^*, \tag{33}$$

where

$$
\begin{aligned}
\psi^* &\equiv n_f\alpha+n_f^*\alpha^*-g^*-\frac{C^*}{W^*}\equiv\rho^*-g^*-\frac{C^*}{W^*} &(34)\\
dw^* &\equiv n_f\alpha dy+n_f^*\alpha^*dy^*-dz^*. &(35)
\end{aligned}
$$

2.3.2 The equilibrium

The equilibrium portfolio shares and the consumption-wealth ratio in the foreign economy are given by

$$\begin{aligned} n_f &= \frac{\alpha - \alpha^*}{(1-\gamma^*)\,\Delta} + \frac{{\alpha^*}^2 \sigma_{y^*}^2 - \alpha\alpha^*\sigma_{yy^*} + \alpha\sigma_{yz} - \alpha^*\sigma_{y^*z}}{\Delta} \\ n_f^* &= 1 - n_f \\ \left(\frac{C^*}{W^*}\right)_o &= \frac{1}{1-\gamma^*}\left\{\beta^* - \gamma^*\left(\rho^* - g^*\right) + 0.5\gamma^*\left(1-\gamma^*\right)\sigma_{w^*,o}^2\right\}, \end{aligned}$$

where

$$\begin{aligned} \sigma_{w^*,o}^2 &= n_f^2\alpha^2\sigma_y^2 + 2n_f n_f^*\alpha\alpha^*\sigma_{yy^*} + {n_f^*}^2{\alpha^*}^2\sigma_{y^*}^2 + \sigma_z^2 \\ &\quad -2n_f\alpha\sigma_{yz} - 2n_f^*\alpha^*\sigma_{y^*z}, \end{aligned}$$

and Δ was already defined in equation (22) above.

The equilibrium rate of accumulation of wealth in the foreign economy follows the stochastic process

$$\frac{dW^*}{W^*} = \psi_o^* dt + dw^*,$$

where its deterministic and stochastic components are, respectively

$$\begin{aligned} \psi_o^* &= \frac{1}{(1-\gamma^*)}\left\{\rho^* - g^* - \beta^* - 0.5\gamma^*(1-\gamma^*)\sigma_{w^*,o}^2\right\} \\ dw_o^* &= n_f\alpha dy + n_f^*\alpha^* dy^* - dz^*. \end{aligned}$$

3 EQUILIBRIUM ANALYSIS

Now we review briefly the impact of changes in exogenous variables on the consumption-wealth ratio, the rate of growth of wealth of the domestic economy, and welfare, since most of the results are standard.[14] Next, we compare the results of an open economy with those of a closed economy, provided that the size of the public sector is exogenously given.

[14] See Turnovsky (1997, Ch. 11), for example.

3.1 Consumption

The optimal consumption-wealth ratio obtained in equation (21) is standard in the literature: domestic consumption is a linear function of domestic wealth.[15] To start with, we review the impact of changes in exogenous variables that are not directly related to risk or public spending on consumption. Thus a higher subjective discount rate, β, increases consumption-wealth ratio, because the domestic representative agent finds more attractive to dedicate a higher proportion of wealth to consumption, thus reducing investment. In addition, the impact of a higher gross rate of return of the asset portfolio, ρ, on consumption-wealth ratio depends on the sign of the parameter γ. That is the overall result of two opposite effects, substitution and income effects. A higher gross rate of return of the asset portfolio has always a negative substitution effect since consumption becomes less attractive whereas investment is more attractive. The income effect on the consumption-wealth ratio originated by a higher gross rate of return of the asset portfolio is equal to unity: it makes possible to raise both actual and future consumption. For example, if $\gamma < 0$ then income effect dominates substitution effect. Thus increasing the gross rate of return of the asset portfolio, ρ, raises consumption-wealth ratio. From here onwards whenever we get that the result depends on the sign of the parameter γ only, we focus on the case where $\gamma < 0$, for being the most relevant situation empirically [Campbell (1996)].

Second, the impact of variables related to risk, but not affected by the behavior of the public sector, is reviewed. Thus the effect of a higher coefficient of risk aversion, γ, on consumption is ambiguous. Additionally, a higher variance of the rate of growth, $\sigma^2_{w,o}$, reduces consumption-wealth ratio if $\gamma < 0$. Substitution and income effects arise again: totally differentiating equation (21) it can be easily shown that an increase of the variance of the rate of growth is equivalent to a fall in the gross rate of return of the asset portfolio, ρ, of $0.5\,[1-\gamma(1+\eta)]$. Similar conclusion applies to the impact of a higher variance of domestic productivity shocks, σ^2_y, a higher variance of foreign productivity shocks, $\sigma^2_{y^*}$, or a higher covariance between domestic and foreign productivity shocks, σ_{yy^*}, on consumption-wealth ratio.

Third, we review the role of the public sector. A higher size of the public sector, g, originates a positive productive effect plus a negative volatility

[15] Pioneered by Merton (1969[1992]) within a context of uncertainty using in continuous time. We refer to Turnovsky (1997, 2000a) for details more related to our model.

effect: the net effect depends on which of the effects dominate. For example, if the net effect is positive, which is equivalent to a rise in the gross rate of return of the asset portfolio, ρ, then consumption-wealth ratio increases for $\gamma < 0$. Next, an increase in the variance of public spending shocks, σ_z^2, diminishes consumption-wealth ratio when $\gamma < 0$. An increase in the variance of public spending shocks is equivalent to a fall in the gross rate of return of the asset portfolio of $0.5\,[1-\gamma(1+\eta)]$, since the variance of the rate of growth increases. In contrast, if either the covariance between domestic productivity shocks and domestic public spending shocks, σ_{yz}, or the covariance between foreign productivity shocks and domestic public spending shocks, σ_{y^*z}, increase then consumption-wealth ratio increases for $\gamma < 0$. That is due to a reduction in the variance of the rate of growth of the domestic economy.

For the case that the utility function is logarithmic, that is, $\gamma = 0$, then we find the familiar result that $C/W = \beta$. Only changes in the subjective discount rate alter consumption-wealth ratio.

3.2 Growth

The equilibrium mean rate of growth of assets, shown in (24), is standard in the literature[16]. First, we review the impact of variables that do not refer either to risk or public spending on the rate of growth of assets. Thus a higher subjective discount rate, β, reduces unambiguously the rate of growth since dedicating resources to consumption becomes more attractive whereas investment is discouraged. In addition, a higher gross rate of return of the asset portfolio, ρ, increases the rate of growth, even though consumption-wealth ratio may rise.

Second, we study the impact of variables related to risk, but not affected by the behavior of the public sector. Thus a change in the parameter γ generates an ambiguous effect on the growth rate. Next, an increase in the variance of domestic productivity shocks, σ_y^2, shifting investment towards foreign capital, tends to increase the rate of growth, on the one hand, if $\alpha^* > \alpha$. On the other hand, the growth-enhancing effect is reinforced when $\gamma < 0$ since consumption-wealth ratio falls due to an increase in σ_y^2 (Turnovsky, 1997, p. 442). Similarly, an increase in the variance of the

[16]See Turnovsky (1997, Ch. 11) for more details on the impact of changes in exogenous variables on the rate of growth of wealth.

foreign productivity shocks, $\sigma^2_{y^*}$, making domestic capital more attractive, tends to increase the rate of growth if $\alpha > \alpha^*$. The positive effect on the rate of growth is strengthened if $\gamma < 0$: consumption-wealth ratio falls due to an increase in $\sigma^2_{y^*}$.

Third, the impact of the public sector on the rate of growth is analyzed. A higher size of the public sector, g, increases, on the one hand, unambiguously the rate of growth since it raises the gross rate of return of the asset portfolio, ρ. On the other hand, consumption-wealth ratio can fall or raise, as we showed in section 3.1 above. Then the overall effect of increasing the size of the public sector on the rate of growth depends on which of the two effects (on ρ or C/W) dominate. For example, if consumption-wealth ratio falls or does not change, then a higher size of the public sector enhances growth. Next, a higher variance of domestic public spending, σ^2_z, increases the rate of growth of the economy for $\gamma < 0$, because consumption-wealth ratio falls (Turnovsky, 1997, p. 444). In contrast, we get the opposite conclusions when either the covariance of domestic productivity and public spending shocks, σ_{yz}, or the covariance of foreign productivity shocks and public spending shocks, σ_{y^*z}, increases.

Fourth, in the case of a logarithmic utility function the growth rate is given by the expression

$$\psi_o = \rho - g - \beta.$$

Then it is easy to show that, for example, as long as the impact of a higher size of the public sector, g, on the gross rate of return of the asset portfolio, ρ, is higher than unity then increasing the size of the public sector is growth-enhancing.

Finally, we should note that most of the literature shows that the impact of a higher size of the public sector on growth depends basically on whether $\partial\rho/\partial g$ is higher than unity or not. However, our model shows that, once risk is introduced, then we get a more complex relationship between the size of the public sector and the rate of growth of wealth.

3.3 Welfare

Economic welfare is given by the value function used to solve the problem of intertemporal optimization, shown in equation (30). Thus if we totally differentiate equation (30), we get, after some algebra, that

$$\frac{dV}{V} = (\gamma - 1)\frac{d(C/W)}{C/W},$$

where we observe that only changes in the optimal consumption-wealth ratio (influenced by the public sector, risk, and so on) have an impact on economic welfare. A higher optimal consumption-wealth ratio can improve or deteriorate the welfare of the domestic representative agent. Since C/W is positive in equation (30), the value function can take either positive or negative values, depending on the sign of the coefficient γ, subject to $\gamma V(W) > 0$. For the case $\gamma < 0$ then anything that increases the optimal consumption-wealth ratio raises welfare. Thus, for example, a higher size of the public sector, if it increases the optimal consumption-wealth ratio, generates higher welfare if $\gamma < 0$. However, we should note that increasing growth does not necessarily raise welfare.

3.4 Open versus closed economy

It turns up convenient to obtain the difference between the variance of the growth rate in an open economy, shown in equation (23), and the same variance in a closed economy, shown in equation (27). The difference between both variances can be given, after some algebra, by

$$\sigma^2_{w,o} - \sigma^2_{w,c} = \Delta n^*_d \left(n^*_d - 2\widetilde{n}^*_d\right). \tag{36}$$

where

$$\widetilde{n}^*_d = \frac{\alpha^2\sigma^2_y - \alpha\alpha^*\sigma_{yy^*} - \alpha\sigma_{yz} + \alpha^*\sigma_{y^*z}}{\Delta},$$

is the share of the domestic portfolio materialized in foreign capital that minimizes the variance of the growth rate given by equation (23).

First, if we substract equation (26) from equation (21) we get after some algebra, via equation (36), that

$$\left(\frac{C}{W}\right)_o - \left(\frac{C}{W}\right)_c = -\frac{1}{1-\gamma}\left\{0.5\gamma\left(1-\gamma\right)\Delta n^{*^2}_d\right\}. \tag{37}$$

The difference between both consumption-wealth ratios depends critically upon the value of the parameter γ. Thus if $\gamma < 0$ then the consumption-wealth ratio is higher in an open economy than that in a closed economy, assuming an interior solution for the value of portfolio shares. An easy way to explain that result can be found focusing on the case $n_d = \widetilde{n}_d$, where

$$\widetilde{n}_d = 1 - \widetilde{n}_d^* = \frac{{\alpha^*}^2 \sigma_{y^*}^2 - \alpha\alpha^* \sigma_{yy^*} + \alpha\sigma_{yz} - \alpha^* \sigma_{y^*z}}{\Delta},$$

is the share of the domestic portfolio materialized in domestic capital that minimizes the variance of the growth rate shown in equation (23). Then we get, from equation (36), that the variance of the growth rate in an open economy is lower than that in a closed economy, $\sigma_{w,o}^2 < \sigma_{w,c}^2$. A reduction of the variance of the growth rate is equivalent to an increase in the gross rate of return of the asset portfolio. That, in turn, originates a negative substitution effect and a positive income effect on the consumption-wealth ratio. For example, if $\gamma < 0$ then the income effect is stronger than the substitution effect and the consumption-wealth ratio in an open economy is higher than that in a closed economy. Additionally, the higher the value of the optimal share of the domestic portfolio materialized in foreign capital, n_d^*, the higher the difference between the results of an open economy with those of a closed economy.

Second, we can compare the rate of growth in an open economy with that in a closed economy departing from the equation (24) corresponding to an open economy and substracting from it that corresponding to a closed economy

$$\psi_o - \psi_c = n_d^*(\alpha^* - \alpha) - \left[\left(\frac{C}{W} \right)_o - \left(\frac{C}{W} \right)_c \right].$$

Thus the rate of growth in an open economy can be higher than, equal to or lower than that in a closed economy, depending on the signs of two terms. For example, for $\gamma < 0$:

- If $\alpha \geq \alpha^*$ then the rate of growth of wealth in an open economy is lower than that in a closed economy. Consumption-wealth ratio in an open economy is higher than that in a closed economy and, additionally, if $\alpha \geq \alpha^*$ then the gross rate of return of the asset portfolio in an open economy is lower than or equal to the marginal physical product of the domestic capital.

- If $\alpha > \alpha^*$ then the rate of growth of assets in an open economy can be higher than, equal to or lower than that in a closed economy.

Table 3.2. sums up the comparison between the rate of growth in an open economy with that in a closed economy

Table 3.2. Comparing rates of growth			
	$\gamma > 0$	$\gamma = 0$	$\gamma < 0$
$\alpha > \alpha^*$	$\psi_o \lesseqgtr \psi_c$	$\psi_o < \psi_c$	$\psi_o < \psi_c$
$\alpha = \alpha^*$	$\psi_o > \psi_c$	$\psi_o = \psi_c$	$\psi_o < \psi_c$
$\alpha < \alpha^*$	$\psi_o > \psi_c$	$\psi_o > \psi_c$	$\psi_o \lesseqgtr \psi_c$

Finally, focusing on welfare, since consumption-wealth ratio in an open economy should be higher than that in a closed economy for $\gamma < 0$, as shown above in equation (37), then welfare should be higher in a risky open economy than in a risky closed economy. Please note that welfare, which is given by the value function in (30), depends mainly on consumption-wealth ratio. This result adds insights to those shown in Obstfeld (1994) and Turnovsky (1997, Ch. 11), where they analyze the impact on welfare when changing from a domestic closed economy with low-yield and no risk (or relatively low risk) assets to an open economy with high-yield and high-risk assets, among other things. Obstfeld (1994, p. 1326-27) showed that "international risk-sharing can yield substantial welfare gains through its positive effect on expected consumption growth. The mechanism linking global diversification to growth is the attendant world portfolio shift from safe, but low-yield, capital into riskier, high-yield capital". Additionally, Turnovsky (1997, p. 439) showed that for a logarithmic utility function "the higher growth rate more than offsets the additional risk, and the opportunity to invest in a higher return, higher risk foreign asset improves welfare". However, we should note that our conclusion is not based on low risk-high risk considerations, but on closed economy-open economy considerations, provided that the size of the public sector is exogenously given. In addition, we should point out that this result depends exclusively on the sign of the parameter γ.

4 THE OPTIMAL SIZE OF THE PUBLIC SECTOR

Now we turn to the size of the public sector g that maximizes welfare or, for short, the optimal size of the public sector. A crucial characteristic of the model is that domestic productive government expenditure generates an externality on the foreign economy, and viceversa. That leads us to consider two different scenarios in an open economy. In the first scenario we assume that the domestic public sector only takes into account the impact of public spending on the domestic economy and not that impinged on the foreign economy: the domestic productive public sector does not internalize the externality. Thus we get a unilateral (or one-sided) optimal size of the public sector, which is perceived individually as optimal, but it is not for the world as a whole. In the second scenario we assume that the domestic public sector takes into account the impact of public spending on both domestic and foreign economies, that is, the domestic public sector internalizes the externality. Thus we obtain a harmonized size of the public sector which is optimal for the world as a whole.

In addition, we obtain the optimal size for a domestic closed economy. Next, we discuss whether the size of the public sector that maximizes welfare coincides with that which maximizes growth. Finally, we analyze whether more open economies are associated with a higher optimal size of the public sector, first on the more simple case where public spending only influences productivity, but not volatility, and later on the more general case. For simplicity, we assume in this section that $\sigma_{yy^*} = \sigma_{yz} = \sigma_{y^*z} = \sigma_{yz^*} = \sigma_{y^*z^*} = 0$.

4.1 Open economy: the unilateral optimal size

In the first scenario in an open economy we assume that the domestic public sector takes into account the impact of public spending on the domestic economy only. Thus the domestic public sector does not internalize the externality caused in the foreign economy. To obtain the unilateral optimal size of the public sector we differentiate partially the expression in the right hand side of the Bellman equation (14) with respect to g, and then substituting the value function (30) into the result obtained, we get that

$$\left(\delta - \theta \hat{g}_{o,u}\right) n_d - (1 - \gamma) \left(\delta - \theta \hat{g}_{o,u}\right) n_d^2 \sigma_y^2 - 1 = 0, \tag{38}$$

where $\hat{g}_{o,u}$ denotes the unilateral optimal size of the public sector. Equation (38) means that, at the optimal size, the marginal return of an additional unit of public spending, $(\delta - \theta\hat{g}_{o,u})\, n_d - (1-\gamma)(\delta - \theta\hat{g}_{o,u})\, n_d^2\sigma_y^2$, must be equal to the marginal cost, 1. The marginal return of public spending includes, in turn, the productive effect, $(\delta - \theta\hat{g}_{o,u})\, n_d$, plus the volatility effect, $-(1-\gamma)(\delta - \theta\hat{g}_{o,u})\, n_d^2\sigma_y^2$ in the domestic economy only.[17]

From equation (38) the unilateral optimal size of the public sector in an open economy is implicitly derived as

$$\hat{g}_{o,u} = \frac{n_d\delta\left[1-(1-\gamma)\, n_d\sigma_y^2\right] - 1}{n_d\theta\left[1-(1-\gamma)n_d\sigma_y^2\right]}, \tag{39}$$

where both productive and volatility effects determine the optimal size of the public sector. The terms in the numerator capture the first order effects on growth and volatility and the terms in the denominator the second order effects.

4.2 Open economy: the harmonized optimal size

In the second scenario in an open economy we begin by assuming that the external effect of domestic public spending is internalized. The preferences of the central planner are represented by the sum of two isoelastic intertemporal utility functions, depending on domestic and foreign consumption, and both having equal weight

$$E\int_0^\infty \left(\frac{1}{\gamma}C^\gamma e^{-\beta t} + \frac{1}{\gamma^*}C^{\gamma^*}e^{-\beta^* t}\right) dt; -\infty < \gamma, \gamma^* < 1. \tag{40}$$

The dynamics of domestic wealth are given by equations (10), (11) and (12). Foreign wealth, in turn, evolves according to equations (33), (34), and (35).

[17] Please note that we assumed above that $g < \delta/\theta < 1$. In addition, for the unilateral optimal size of the public sector $\hat{g}_{o,u}$ to be positive, the marginal return derived from public spending is required to be higher than its marginal cost for $\hat{g}_{o,u} = 0$, that is, $n_d\delta\left[1-(1-\gamma)\,\overline{\alpha} n_d\sigma_y^2\right] > 1$. Furthermore, the second order condition for g, that is, concavity with respect to g, requires that $1-(1-\gamma)\,\alpha n_d\sigma_y^2 > 0$.

The objective of the central planner would consist in choosing the sizes of the public sectors, g and g^*, that maximize (40), subject to $W(0) = W_0$, $W^*(0) = W_0^*$, (10), (11), (12), (33), (34), and (35).

We introduce a value function, $G(W, W^*)$, which is defined as

$$G(W, W^*) = V(W) + V^*(W^*) = \underset{\{g,g^*\}}{Max}\ E \int_0^\infty \left(\frac{1}{\gamma} C^\gamma e^{-\beta t} + \frac{1}{\gamma^*} C^{\gamma^*} e^{-\beta^* t} \right) dt, \qquad (41)$$

subject to the restrictions (10), (11), (12), (33), (34), (35), and given initial wealth.

The value function, given by equation (41), must satisfy the Bellman equation

$$\beta V(W) + \beta^* V^*(W^*) = \underset{\{g\}}{Max} \left[\frac{1}{\gamma} C^\gamma + \frac{1}{\gamma^*} C^{\gamma^*} + V'(W) W \psi \right.$$
$$\left. + 0.5 V''(W) W^2 \sigma_w^2 + V^{*'}(W^*) W^* \psi^* + 0.5 V^{*''}(W^*) W^{*^2} \sigma_{w^*}^2 \right]. \qquad (42)$$

Now we focus on the case where both economies grow at the same rate, or in a broader sense, where the rates of growth of both economies do not differ very much. That has been, in fact, the path followed traditionally in the literature on two-country endogenous growth models. In addition to tractability reasons, the literature has emphasized that if it were not the case that both economies grow at the same rate then one would become infinitely big compared to the other (Razin and Yuen, 1993; Lejour and Verbon, 1996). Thus if we restrict to the case where consumption-wealth ratio, portfolio shares and the size of the public sector are the same in both countries, then domestic wealth and foreign wealth should grow at the same rate and we obtain a clear-cut solution.[18] Before going ahead, we should note that assuming that both economies grow at the same rate implies that the external effect of the domestic economy on the foreign economy is equal to the external effect of the foreign economy on the domestic economy or, alternatively since the public sector budget is balanced, that tax revenues on

[18] Formally, this means that $n_d = n_f$, $n_d^* = n_f^*$, $\frac{C}{W} = \frac{C^*}{W^*}$, $g = g^*$, $dz = dz^*$, $\gamma = \gamma^*$, and $\beta = \beta^*$.

the domestic economy paid by foreigners are equal to tax revenues on the foreign economy paid by domestic residents.

We differentiate partially the right hand side of equation (42) with respect to the harmonized optimal size of the public sector, g, in order to obtain the first order condition of this optimization. Substituting the value function (30) into the result obtained, we have that

$$\begin{aligned}&(\delta - \theta\widehat{g}_{o,h})\, n_d \left[1 - (1-\gamma)\, n_d \alpha \sigma_y^2\right] \\ &+ (\delta^* - \theta^*\widehat{g}_{o,h})\, n_d^* \left[1 - (1-\gamma)\, n_d^* \alpha^* \sigma_{y^*}^2\right] - 1 = 0, \end{aligned} \tag{43}$$

where $\widehat{g}_{o,h}$ denotes the harmonized optimal size of the public sector. Equation (43) shows that, at the optimal size, the marginal return of public spending,

$$(\delta - \theta\widehat{g}_{o,h})\, n_d \left[1 - (1-\gamma)\, n_d \alpha \sigma_y^2\right] + (\delta^* - \theta^*\widehat{g}_{o,h})\, n_d^* \left[1 - (1-\gamma)\, n_d^* \alpha^* \sigma_{y^*}^2\right],$$

must equalize the marginal cost, 1. The marginal return of public spending, in turn, reflects the impact of spending on the domestic economy, $(\delta - \theta\widehat{g}_{o,h})\, n_d \left[1 - (1-\gamma)\, n_d \alpha \sigma_y^2\right]$, plus the impact on the foreign economy, $(\delta^* - \theta^*\widehat{g}_{o,h})\, n_d^* \left[1 - (1-\gamma)\, n_d^* \alpha^* \sigma_{y^*}^2\right]$, both in terms of productivity and volatility.[19] Thus we get from equation (43) that the harmonized optimal size of the public sector in an open economy is implicitly given by

$$\hat{g}_{o,h} = \frac{n_d\delta\left[1-(1-\gamma)\,\alpha n_d \sigma_y^2\right] + n_d^*\delta^*\left[1-(1-\gamma)\, n_d^*\sigma_{y^*}^2\right] - 1}{n_d\theta\left[1-(1-\gamma)\,\theta n_d \sigma_y^2\right] + n_d^*\theta^*\left[1-(1-\gamma)\, n_d^*\sigma_{y^*}^2\right]}. \tag{44}$$

The terms in the numerator capture the first order effects on growth and volatility, weighted by the appropriate portfolio shares, whereas the terms in the denominator reflect second order effects.

4.3 Closed economy

Turning to the domestic closed economy, we could obtain the optimal size of the public sector, $\hat{g}_c$, solving the corresponding problem for the domestic

[19] That the harmonized optimal size of the public sector $\hat{g}_{o,h}$ is positive requires that the marginal return derived from public spending higher than its marginal cost for $\hat{g}_{o,h} = 0$. This implies that $n_d\delta\left[1-(1-\gamma)\,\overline{\alpha} n_d\sigma_y^2\right] + n_d^*\delta^*\left[1-(1-\gamma)\,\overline{\alpha}^* n_d^*\sigma_{y^*}^2\right] > 1$. The second order condition for g requires that $1-(1-\gamma)\,\alpha n_d\sigma_y^2 > 0$ and $1-(1-\gamma)\,\alpha^* n_d^*\sigma_{y^*}^2 > 0$.

representative agent by setting $n_d = 1$ in any of both above equations (38) or (43)

$$(\delta - \theta\widehat{g}_c) - (1-\gamma)(\delta - \theta\widehat{g}_c)\,\alpha\sigma_y^2 - 1 = 0, \tag{45}$$

so that the optimal size of the public sector in a closed economy is implicitly given by

$$\hat{g}_c = \frac{\delta\left[1-(1-\gamma)\,\alpha\sigma_y^2\right]-1}{\theta\left[1-(1-\gamma)\,\alpha\sigma_y^2\right]}. \tag{46}$$

Please note that the optimal size of the public sector in a foreign closed economy can be found, in turn, by setting $n_d = 0$ in equation (44).

Thus in a risk-free closed economy, the optimal size of the public sector is given by

$$\hat{g}_c = \frac{\delta - 1}{\theta}. \tag{47}$$

The mathematical result in (47) cannot be compared to that found in Barro (1990): the optimal size of the public sector in Barro (1990) is equal to the exponent on public spending in a Cobb-Douglas production function that exhibits constant returns to scale.[20] Instead, our approach to incorporate public spending has been different as we showed in Section 2.1, following Gallaway and Vedder (1995) and Vedder and Gallaway (1998). However, the important point is that we can compare the results obtained in equations (39) and (44) to benchmark results in equations (46) or (47) in our model. Therefore, we can *compare* the *conclusions* derived from our model with those of the core literature, even though we *cannot compare* the *mathematical results* in equations (39), (44), (46), and (47) of our model vis-à-vis those of the core literature.

We find that the optimal size of the public sector in a closed economy with risk, given by (46), is lower than that with no risk, given by (47). This result was already shown in Turnovsky (1999, p. 899). However, his additional result that if there were no production risk in the domestic economy "the

[20] There are additional terms in the optimal size of the public sector shown in Turnovsky (1998, 1999).

optimal size of the productive government in the stochastic open economy coincides with that in the deterministic closed economy" no longer applies. Thus the harmonized optimal size of the public sector in a stochastic open economy, given by equation (44), is not necessarily equal to the optimal size of the public sector in a deterministic closed economy with no domestic production risk [equation (47)].

4.4 Growth vs. welfare maximizing

Now we can compare the optimal size of the public sector with the size that maximizes growth, where the rate of growth of assets is shown in equation (24). Partially differentiating equation (24) with respect to g, we derive the unilateral size of the public sector that maximizes the rate of growth as

$$\overline{g}_{o,u} = \frac{n_d \delta \left[1 - \gamma \left(1 - \gamma\right) n_d \sigma_y^2\right] - 1}{n_d \theta \left[1 - \gamma (1 - \gamma) n_d \sigma_y^2\right]}. \tag{48}$$

If we substract equation (39) from equation (48), it can be easily shown, after some algebra, that the unilateral size of the public sector that maximizes the rate of growth, $\overline{g}_{o,u}$, is unambiguously higher than the unilateral optimal size of the public sector, $\widehat{g}_{o,u}$. That has been shown already in Turnovsky (1998, p. 16): "The intuition is that the maximization of the growth rate entails more risk than the risk averse agent, concerned with his time profile of consumption, finds to be optimal". The result that welfare maximizing and growth maximizing objectives do not amount to the same thing has already been derived in other contexts, such as models where productive public spending influences costs of adjustment of new investment, where the productive good provided by the public sector is introduced as a stock, instead of as a flow [see Turnovsky (2003, p. 20) for more details and references] or where the representative agent is not risk averse (Turnovsky, 1998, p. 16). In addition, we should note that in case public spending is regarded to be volatility-reducing (besides productivity-enhancing) in opposition to our model, then we would conclude that the unilateral optimal size of the public sector, $\widehat{g}_{o,u}$, is higher than that which maximizes growth, $\overline{g}_{o,u}$. Instead, if there is no risk then both maximizing welfare and growth are equivalent in the case of Cobb-Douglas production function (Barro, 1990, pp. S111-S112)[21]. In

[21] However, if the production function were not Cobb-Douglas "the relative size of government that maximizes utility turns out to exceed the value that maximizes the

addition, the results obtained for the unilateral optimal size can be easily extended to the harmonized optimal size and to the optimal size in a risky closed economy.

4.5 Open versus closed economy

Now we discuss whether higher values of holdings of foreign capital out of domestic wealth, n_d^*, that is, more open economies, are associated with a higher optimal size of the public sector (Turnovsky, 1999).

4.5.1 A digression: productive-only spending

The case where public spending influences productivity only is analyzed. The results of the model become much simpler. If the impact of public spending on volatility is null then we have that the domestic production function in equation (4) becomes

$$dY = \alpha K dt + \overline{\alpha} K dy,$$

where α was given by equation (5) and it is affected by the size of the public sector. Instead, the parameter $\overline{\alpha}$ is a constant and it is not influenced by the size of the public sector. It is easy to show that the optimal size of the public sector in a domestic closed economy is equal to that where there is no risk, given by equation (47). Symmetrically, the optimal size of the public sector in a foreign closed economy, $\hat{g}_c^*$, is given by

$$\hat{g}_c^* = \frac{\delta^* - 1}{\theta^*}. \tag{49}$$

First, we can establish going back to equation (39) that the unilateral optimal size of the public sector is given by

$$\hat{g}_{o,u} = \frac{\delta n_d - 1}{\theta n_d}.$$

growth rate [...] if and only if the magnitude of the elasticity of substitution between g [the quantity of public services provided to each household-producer] and k [capital per worker] is greater than unity" (Barro, 1990, p. S112).

It can be easily checked that the optimal size of the public sector in a closed economy, $\hat{g}_c$, is unambiguously higher than the unilateral optimal size of the public sector, $g_{o,u}$, for interior values of portfolio shares. The reason behind it is that the higher the value of the portfolio share n_d^* is, the lower the level of internalization of the externality is. The domestic public sector finds optimal to reduce the unilateral optimal size since the benefit of public spending for the domestic public sector becomes lower. Additionally, the more open the domestic economy is, that is, the higher the value of the portfolio share n_d^*, then the lower the unilateral optimal size of the public sector is.

Second, we can establish following equation (46) that the harmonized optimal size of the public sector is given by

$$\widehat{g}_{o,h} = \frac{\delta n_d + \delta^* n_d^* - 1}{\theta n_d + \theta^* n_d^*},$$

which implies that the harmonized optimal size of the public sector in an open economy is always between the values of the optimal size of the public sector in both closed economies. That is not surprising since the world economy is a closed economy after all. Then it can be easily shown for the harmonized optimal size that

$$sgn\left(\widehat{g}_{o,h} - \widehat{g}_c\right) = sgn\left(\widehat{g}_c^* - \widehat{g}_c\right).$$

Thus, for example, we find that the harmonized optimal size of the public sector, $\widehat{g}_{o,h}$, is higher than the optimal size in domestic closed economy, $\widehat{g}_c$, if and only if $\widehat{g}_c^* > \widehat{g}_c$, that is, the optimal size of the public sector in a foreign closed economy is higher than that in a domestic closed economy. The reason behind it is that in that case the marginal impact of public spending is higher in a foreign economy.

4.5.2 The general case

Now we turn to the more general case where public spending influences volatility as well as productivity. To begin with, the first order condition for the unilateral optimal size of the public sector in an open economy $\widehat{g}_{o,u}$ is given by equation (38), whereas for the case of a closed economy, $\widehat{g}_c$, is given by equation (45). Comparing equations (38) and (45) we observe that, on the one hand, the impact of public spending on productivity, $(\delta - \theta g)\, n_d$, is lower in an open economy than in a closed economy, $\delta - \theta g$, for the same size of the public sector, g, as we showed above in section 4.5.1. On the other hand, the influence of public spending on volatility is higher (that is, less negative) in an open economy, $-(1-\gamma)(\delta - \theta g)\alpha n_d^2 \sigma_y^2$, than in a closed economy, $-(1-\gamma)(\delta - \theta g)\alpha \sigma_y^2$, for the same size g, since the impact in an open economy depends on the share of the portfolio materialized in domestic capital, n_d. Thus the net impact depends upon which of the effects dominate. For example, obtaining that the unilateral optimal size of the public sector in an open economy $\widehat{g}_{o,u}$ is higher than that in a closed economy $\widehat{g}_c$ implies that if we introduce $\widehat{g}_{o,u}$ in equation (45) for the first order condition of the optimal size in a closed economy then we should have that

$$(\delta - \theta \widehat{g}_{o,u}) - (1-\gamma)(\delta - \theta \widehat{g}_{o,u})\alpha \sigma_y^2 < 1, \tag{50}$$

due to the second order condition required for the size of the public sector g to be a maximum. Combining both equations (38) and (50), and after some algebra, we get that the unilateral optimal size will be higher than the optimal size in a closed economy if and only if

$$(1-\gamma)(\delta - \theta \widehat{g}_{o,u})(1+n_d)\alpha \sigma_y^2 > (\delta - \theta \widehat{g}_{o,u}).$$

Second, the harmonized optimal size of the public sector is given by equation (43). If we compare equation (43) with equation (45) we can easily show that, on the one hand, the impact of public spending on productivity in an open economy, $(\delta - \theta g) n_d + (\delta^* - \theta^* g) n_d^*$, compared to that in a closed economy, $\delta - \theta g$, for the same size g, depends on the difference between the optimal size of the public sector in a foreign closed economy $\widehat{g}_c^*$ shown in (49) and that in a domestic closed economy $\widehat{g}_c$ given by (47), as we showed above in section 4.5.1. For example, if $\widehat{g}_c^* > \widehat{g}_c$ then the impact of public spending on productivity is higher in an open economy than in a closed economy. On the other hand, the impact of public spending on volatility in an open economy,

$$-(\delta - \theta g)(1-\gamma)\alpha n_d^2 \sigma_y^2 - (\delta^* - \theta^* g)(1-\gamma)\alpha^* {n_d^*}^2 \sigma_{y^*}^2,$$

in relation to that in a closed economy, $-(\delta - \theta g)(1-\gamma)\alpha \sigma_y^2$, depends mainly on the variances of productivity shocks in both economies and the values of the portfolio shares n_d and n_d^*. For example, to get that the harmonized optimal size of the public sector $\widehat{g}_{o,h}$ is higher than that in a closed economy $\widehat{g}_c$ involves that substituting $\widehat{g}_{o,h}$ in equation (45) for the first order condition for the optimal size in a domestic closed economy implies that

$$(\delta - \theta\widehat{g}_{o,h}) - (1-\gamma)(\delta - \theta\widehat{g}_{o,h})\,\alpha\sigma_y^2 < 1. \tag{51}$$

so that the second order condition for the optimal size g is satisfied. If we combine equation (43) with equation (51), then we obtain, after some algebra, the result that $\widehat{g}_{o,h} > \widehat{g}_c$ provided that

$$\begin{aligned}(\delta^* - \theta^*\widehat{g}_{o,h}) + (1-\gamma)(\delta - \theta\widehat{g}_{o,h})(1+n_d)\,\alpha\sigma_y^2 > \\ (\delta - \theta\widehat{g}_{o,h}) + (1-\gamma)(\delta^* - \theta^*\widehat{g}_{o,h})\,n_d^*\alpha^*\sigma_{y^*}^2.\end{aligned} \tag{52}$$

In equation (52) we observe first the result obtained above in section 4.5.1 for productive-only spending: the harmonized optimal size of the public sector, $\widehat{g}_{o,h}$, will be higher than that in a domestic closed economy, $\widehat{g}_c$, if and only if the optimal size of the public sector in a foreign closed economy $\widehat{g}_c^*$ is higher than that in a domestic closed economy $\widehat{g}_c$. In addition, we have that the impact of public spending on volatility in an open economy can be higher or lower than that in a closed economy. However, under most conditions the impact will be higher (that is, less negative) in an open economy. For example, if we focus for simplicity on the case where the marginal product in both countries is equal, that is, $\alpha = \alpha^*$, the marginal impact of public spending on productivity is equal in both economies, that is, $\delta - \theta\widehat{g}_{o,h} = \delta^* - \theta^*\widehat{g}_{o,h}$, and the variances of productivity shocks are similar, then we know from section 3.4 that the variance of the growth rate in an open economy is lower than that in a closed economy, $\sigma_{w,o}^2 < \sigma_{w,c}^2$, as given by equation (36). In addition, since the impact on volatility is the only factor that can originate a difference between the optimal size in an open economy and that in a closed economy, then we conclude that the harmonized optimal size of the public sector in an open economy is higher than in a closed economy because the impact on volatility is higher (that is, less negative) in an open economy than in a closed economy.

These results offer additional insights to those of Turnovsky (1999), where a more open economy is unambiguously associated with a higher optimal size in an open economy, provided that the domestic economy holds positive stocks of foreign capital in a small open economy. Here we have found two reasons why the harmonized optimal size of the public sector in an open economy should be higher than that in a closed economy. The first one is based on the fact that the optimal size of the public sector in a foreign closed economy $\widehat{g}_c^*$ can be higher than that in a domestic closed economy $\widehat{g}_c$. That depends upon the difference in the marginal impact of public spending on productivity in both countries. The second reason has to do with risk diversification. Since the impact of public spending on volatility is surely higher (that is, less negative) in an open economy than in a closed economy, that makes that the harmonized optimal size of the public sector in an open economy should be higher than that in a closed economy. Turnovsky (1999, p. 889) bases his result on "the country's ability to export its domestic risk, rather than due to insulating the country from foreign risk, as argued by Rodrik [1998]". We should note that our argument based on the higher impact of public spending on volatility due to risk diversification argument more Rodrik's (1998, p. 1011) "insulation function" than Turnovsky's "risk exporting" argument. However, Rodrik emphasizes the central role that the public sector plays in insulating against external risk, whereas here the result is the consequence of the risk diversification achieved through perfect capital mobility.

5 CONCLUSIONS

The impact of productive public spending and risk on long run growth is an important issue for economic policy. However, the analysis has been mostly relegated either to a closed economy or an small open economy. This paper extends a two-country stochastic AK growth model based on Turnovsky (1997, Ch. 11) incorporating a production good that enhances both the productivity of physical marginal product of private capital and volatility [Barro (1990) for the original deterministic model and Turnovsky (1998, 1999) for an extension to a risky closed economy and a small open economy, respectively]. The main conclusions can be

First, having obtained the world equilibrium, we review how consumption-wealth ratio, the rate of growth of assets and welfare respond to changes in exogenous variables, provided that the size of the public sector is exogenously given. Most of the results are familiar. However, we have shown that a higher size of the public sector, enhancing productivity and volatility, implies a richer analysis about the impact on the consumption-wealth ratio and the rate of growth. For example, if the net impact of a higher size of the public sector is positive then consumption-wealth should raise. Since welfare depends basically on consumption-wealth ratio, then public spending influences welfare altering consumption-wealth ratio.

Second, we have compared the behavior of key economic variables in an open economy in contrast to a closed economy. Since an open economy can achieve a lower variance of the rate of growth through risk diversification, then consumption-wealth ratio should be higher in an open economy than in a closed economy, assuming that the size of the public sector is exogenously given. Next, we have shown that the rate of growth in an open economy is lower than that in a closed economy if the marginal physical product of domestic capital is higher than that of foreign capital. Additionally we should note that welfare should be higher in open economy than in a closed economy, since welfare depends upon consumption-wealth ratio.

Third, we have obtained the optimal size of the public sector in an open economy. Since domestic productive government expenditure generates an externality on the foreign economy in the model we have considered two different scenarios in an open economy. In the first scenario we assume that the domestic productive public sector only takes into account the impact of productive government spending on the domestic economy and not that impinged on the foreign economy. In the second scenario we assume that the domestic productive public sector takes into account the impact of productive government spending on both domestic and foreign economies. We should note that obtaining a closed form harmonized optimal size requires assuming that both economies grow exactly at the same rate. It has been derived that the optimal size of the public sector in a closed economy with risk is lower than that with no risk, as is Turnovsky (1999). However, we have shown that in case there is no domestic production risk then the harmonized optimal size of the public sector in an open economy does not have to be equal to the optimal size in a domestic closed economy necessarily, in contrast to Turnovsky (1999). Next, we obtained that the size that maximizes welfare is lower than that which maximizes growth due to risk aversion, as in Turnovsky (1998).

Fourth, we have compared the optimal size of the public sector in an open economy with that of a closed economy. In the case public spending is productive only, we find that the unilateral optimal size of the public sector in an open economy should be unambiguously lower than that in a closed economy since public spending is not fully internalized. In contrast, we conclude that the harmonized optimal size of the public sector should be higher than that in a closed economy provided that the optimal size of the public in a foreign closed economy is higher than in a domestic closed economy. The marginal impact of public spending on productivity is higher abroad than at home. In the more general case where public spending influences volatility as well, we conclude that the unilateral optimal size of the public sector will be higher than that in a domestic closed economy if and only if the marginal impact on volatility is higher (less negative) than that on productivity. As regards the harmonized case, we argue that there are two channels through which the harmonized optimal size of the public sector should be higher than that in a domestic closed economy. The first channel is that the optimal size of the public sector in a foreign closed economy should be higher than that in a domestic closed economy, as argued in the productive-only case. The second channel has to do with the higher positive impact (that is, less negative) of public spending on volatility in an open economy than in a closed economy due to risk diversification. We find that the second channel goes along the same lines of the argument in Rodrik (1998) about insulating an open economy from external risk through the intervention of the public sector rather than that in Turnovsky (1999) about the ability to export domestic risk. However, Rodrik attributes a central role to the public sector, while here our argument is based on the relatively lower impact of public spending on volatility achieved by the risk diversification in a world with perfect capital mobility.

Fifth, we should note that this model can easily be extended so that public spending, instead of being volatility-enhancing, is volatility-reducing. That would reverse most of the conclusions of this paper. For example, it can be easily shown that if public spending is volatility-reducing as well as productivity-enhancing then the size of the public sector that maximizes welfare should be higher than that which maximizes growth. Further, it implies significative changes as regards the conclusions about the optimal size of the public sector in an open economy as compared to those in a closed economy.

Finally, we must note that the model has important limitations. We have analyzed the equilibrium in a world economy where, even though the domestic and the foreign economies are different, the representative agents are identical, and the equilibrium is characterized by identical balanced growth rates. Therefore, it is not suitable to study structural differences between countries, but it could be useful for countries where their rates of growth are quite similar. Additionally, we should point out possible paths for future research. We could relax the assumption of continuous budget equilibrium and introduce public bonds in the model. However, that would increase enormously the complexity of the model. Introducing money is also an interesting element that could be integrated in a two-country world economy. The incorporation of congestion in the provision of the consumption good would be another possibility to extend the model. The inclusion of more complex strategic interaction would be an additional interesting feature.

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