Risk, utility-enhancing government expenditure, and the world economy

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Abstract

This paper analyzes the influence of risk and the expenditure policy of the public sector in a two-country stochastic AK growth model where public spending is utility-enhancing. Having characterized the macroeconomic equilibrium first we study the impact of risk and the public sector on consumption-wealth ratio, growth and welfare, given the exogenous size of the public sector. A higher weight of public consumption in the utility function raises the rate of growth due to a fall in the consumption-wealth ratio. Then we show that consumption-wealth ratio and welfare are higher in an open economy than in a closed economy and we study whether open economies grow more than closed economies. Next, the welfare-maximizing size of the public sector is derived and compared it to the size that maximizes growth. We analyze the impact of exogenous parameters, risk specially, on the optimal size. Then we establish that a higher weight of public consumption in the utility function reduces private consumption-wealth ratio leaving the rate of growth unchanged when the size of the public sector is optimally chosen. Finally, we show that more open economies should have a higher size of the public sector under more general conditions than those established in Turnovsky (1999).

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Keywords: Risk; utility-enhancing government expenditure; consumption; growth; welfare; optimal size of the public sector

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Contents

1 INTRODUCTION

The role of government expenditure policy in the long run behavior of the economy has received considerable attention in recent years, specially due to the advent of endogenous growth models. That is not surprising since in the Solow-Swan neoclassical growth model "conventional macroeconomic policy had no influence on long-run growth performance" (Turnovsky, 2003, p. 1). Barro (1990) pioneered the analysis based on a closed economy deterministic AK growth model where public spending influences utility.[1] This has led to others, Turnovsky (1996, 1999) for example, to incorporate small open economy features and risk into endogenous growth models where public spending is utility-enhancing. Thus substantial conclusions have been derived regarding the impact of risk and the expenditure policy of the public sector on the economy, and the optimal size of the public sector, provided that the spending of the public sector enhances utility. However, analysis based on two-country stochastic models are badly needed, specially when financial markets are becoming increasingly integrated.

This paper analyzes the influence of risk and the expenditure policy of the public sector by incorporating utility-enhancing public spending [see Barro (1990)] into a two-country stochastic AK growth model developed by Turnovsky (1997, Ch. 11). Then the size of the public sector that maximizes welfare can be endogenously derived, instead of exogenously given as in Turnovsky (1997, Ch. 11). Previous papers introduced risk into endogenous growth models, but public spending was neither utility-enhancing nor productive [see, for example, Eaton (1981)]. Turnovsky (1996) extended Barro's (1990) closed economy model by incorporating utility-enhancing government expenditure into a deterministic endogenous growth small open economy. Turnovsky (1999) added risk to a small open economy. Therefore our model has been built up combining the main characteristics of the core literature[2]:

- It is an AK growth model, as the rest of the models.
- It is a two-country model, following the framework set out by Turnovsky (1997, chap. 11), whereas the rest of the models are one-country models (either a closed economy or small open economy).
- Public consumption is utility enhancing, following the original work by

[1] Barro (1990) also analyzed the role of productivity-enhancing government expenditure.

[2] We denominate "core literature" to the set of papers that have analyzed the impact of risk and/or the expenditure policy of the public sector on the economy based on AK growth models, provided that public spending is utility-enhancing, and to the model developed by Turnovsky (1997, Ch. 11).

Barro (1990). Thus the model is able to determine the size of the public sector that maximizes the welfare of the representative agent, as most of the models of the core literature do. Turnovsky (1997) is the only model that cannot analyze the magnitude of such a size, since public spending is neither utility enhancing nor productive, so that "it can be interpreted as being a real drain on the economy or, alternatively, as some public good that does not affect the marginal utility of private consumption or the productivity of private capital" (Turnovsky, 1997, p. 338). Turnovsky (1996, 1999) extend Barro´s (1990) model from a closed economy to a small open economy setting.

- The model is stochastic. The only models of the core literature that are not stochastic are Barro (1990) and Turnovsky (1996). In this respect Turnovsky (1999) extends the deterministic models in Barro (1990) and Turnovsky (1996) to a stochastic setting.

Table 2.1. encapsulates the relationship between the model of this paper and the core literature.

Table 2.1. An overview of the model

The different models	AK growth	Two countries	Size of the public sector	Stochastic shocks
Barro (1990)	X		X	
Turnovsky (1996)	X		X	
Turnovsky (1997, chap. 11)	X	X		X
Turnovsky (1999)	X		X	X
This model	X	X	X	X

We think that this model can be specially useful in the present moment of the European Economic and Monetary Union (EMU). First, countries of the euro area have adopted the Stability and Growth Pact (SGP) from 1st January 1999 onwards, whose objective is that countries of the euro area must attain budget balance, in the medium or in the long run, so that the assumption of continuous budget balance that we make in this paper seems reasonable. Second, the emphasis of this paper is the long run and, therefore, it does not focus on the influence of business cycles, important as they may be. Third, there exists a recurrent preoccupation regarding whether the shocks that affect European countries are becoming more idiosyncratic (asymmetric) or not and the consequences of such a pattern. In this paper we pay special attention to the influence that the pattern of correlation between domestic and foreign productivity shocks, and public spending shocks generate on the world economy, whereas the core literature has not analyzed such an issue. Fourth, there is a permanent debate about whether the size of the public sector should be bigger or smaller and, more specifically, whether more open economies should have bigger governments or not. Rodrik (1998) showed that economies that are more open to international trade have bigger governments and argues that it is due to the fact that government spending provides social insurance against external risk. However, Alesina and Wacziarg (1998) show that the link between the size of the public sector and openness can be explained alternatively on the grounds that a higher size of the public sector is related to small economies (due to the economies of scale involved in the provision of public goods) and that small economies are usually more open to trade. Then country size is the variable that can account for the positive relation between the size of the public sector and the openness to trade. This model sheds some light on the issue, since it compares the size of the public sector that maximizes the welfare in an open economy with that in a closed economy.

We start analyzing the impact of risk and the public sector on consumption-wealth ratio, the rate of growth of assets and welfare, once the macroeconomic equilibrium has been characterized. Then we compare the results of an open economy in contrast to those of a closed economy. Next, we derive the welfare-maximizing size of the public sector, we discuss whether maximizing growth is equivalent to maximizing welfare and we analyze the impact of exogenous parameters, risk specially, on the optimal size. We discuss whether more open economies should have a higher size of the public sector. Finally, we conclude indicating possible avenues for future research.

2 THE WORLD ECONOMY

2.1 The basic structure

The world economy is composed of two countries, each of them producing only one homogeneous good. In each country there exists a representative agent and the public sector, both with infinite time horizon. This economy is a real one, that is, there are no nominal assets, such as money, different financial assets, etc. Unstarred variables refer to the domestic economy, whereas the starred variables refer to the foreign economy. Developing this model we focus on the domestic economy since the results for the foreign economy are very similar.

The homogeneous good produced by both countries can be either consumed or invested in capital without having to incur in any kind of adjustment costs. We are going to suppose that domestic production can be obtained using only domestic capital, K, through an AK function, and that it can be expressed through a first order stochastic differential equation, so that production flow dY (the variation of the state variable) is not completely determined, but it is subject to a stochastic disturbance

$$dY = \alpha K dt + \alpha K dy,$$

where $\alpha > 0$ is the (constant) marginal physical product of capital and dy represents a proportional domestic productivity shock. More precisely, dy is the increment of a stochastic process y. Those increments are temporally independent and are normally distributed, satisfying that $E(dy) = 0$ and $E(dy^2) = \sigma_y^2 dt$.[3] We omit, for convenience, the formal references to time, although those variables depend on time. We must note that dY indicates the flow of production, instead of Y, as is ordinarily done in stochastic calculus.

The foreign economy is structured symmetrically to the domestic economy. Thus, foreign production is carried out using capital domiciled abroad, K^*, with a production function very similar to the one in the domestic economy

$$dY^* = \alpha^* K^* dt + \alpha^* K^* dy^*,$$

where $\alpha^* > 0$ is the marginal physical product of capital and dy^* represents a proportional foreign productivity shock. More precisely, dy^* is the increment of a stochastic process y^*. Those increments are temporally independent and are distributed normally, satisfying that $E(dy^*) = 0$ and that $E(dy^{*^2}) = \sigma_{y^*}^2 dt$.

Both the domestic capital, K, and the foreign capital, K^*, can be owned by the domestic representative agent or the foreign representative agent. The subscript d denotes the holdings of assets of the domestic representative agent and the subscript f denotes the holdings of assets of the foreign representative agent. So it must be satisfied that

$$\begin{aligned} K &= K_d + K_f \\ K^* &= K_d^* + K_f^*. \end{aligned}$$

The wealth of the domestic representative agent, W, and the wealth of the foreign representative agent, W^*, therefore will be

[3] That is, the production flow follows a Brownian motion with drift αK and with variance $\alpha^2 K^2 \sigma_y^2$.

$$
\begin{aligned}
W &= K_d + K_d^* \qquad (1)\\
W^* &= K_f + K_f^*. \qquad (2)
\end{aligned}
$$

2.2 The domestic economy

2.2.1 The problem

The preferences of the domestic representative agent are represented by a constant elasticity of substitution (or isoelastic) intertemporal utility function where she obtains utility from private consumption, C, and from public consumption, G

$$
E\int_0^\infty U(C,G)e^{-\beta t}dt = E\int_0^\infty \frac{1}{\gamma}(CG^\eta)^\gamma e^{-\beta t}dt \qquad (3)
$$
$$
-\infty < \gamma < 1; \eta > 0; \gamma\eta < 1; \gamma(1+\eta) < 1.
$$

The welfare of the domestic representative agent in period 0 is the expected value of the discounted sum of instantaneous utilities, conditioned on the set of disposable information in period 0. The parameter β is a positive subjective discount rate (or rate of time preference). For the isoelastic utility function the Arrow-Pratt coefficient of relative risk aversion is given by the expression $1-\gamma$. When $\gamma = 0$ this function corresponds to the logarithmic utility function. The empirical evidence suggests a high degree of relative risk aversion, so that $\gamma < 0$ (Campbell, 1996). The parameter η measures the influence of public consumption on the welfare of the domestic representative agent. We suppose that both private consumption and public consumption generate a positive marginal utility, so that $\eta > 0$. The other restrictions on the utility function are necessary to ensure concavity with respect to private consumption and public consumption.

The domestic representative agent consumes at a deterministic rate $C(t)dt$ in the instant dt and must pay the corresponding taxes and thus the dynamic budget restriction can be expressed in the following way

$$
dW = [\alpha K_d + \alpha^* K_d^*]\,dt + [\alpha K_d dy + \alpha^* K_d^* dy^*] - Cdt - dT, \qquad (4)
$$

where dT denotes the taxes the domestic representative agent must pay to the public sector. The structure of taxes will be detailed below.

Besides the domestic representative agent there is a public sector. Public sector spending, dG, increases with wealth, so that we can achieve a balanced growth path[4]. Public spending evolves according to

$$dG = gWdt + Wdz, \tag{5}$$

where $g = G/W$ is the size of the public sector and dz is the increment of a stochastic process z. Those increments are temporally independent and are normally distributed, satisfying that $E(dz) = 0$ and $E(dz^2) = \sigma_z^2 dt$. Public sector spending is financed solely via tax collection: the public sector equilibrates its budget continuously, which seems reasonable in the long run, as is the focus of this paper. Therefore, public deficits are not allowed, that is,

$$dT = dG. \tag{6}$$

Combining equations (6), (5), and substituting them into (4), we get the following restriction for the resources of the domestic economy

$$dW = [\alpha K_d + \alpha^* K_d^* - C - gW]\, dt + [\alpha K_d dy + \alpha^* K_d^* dy^* - Wdz]\,. \tag{7}$$

Let us remember that the holding of assets by the domestic representative agent is subject to the domestic wealth equation (1). If we define the following variables for the domestic representative agent

$$\begin{aligned} n_d &\equiv \frac{K_d}{W} = \text{ share of the domestic portfolio materialized} \\ &\quad \text{in domestic capital} \\ n_d^* &\equiv \frac{K_d^*}{W} = \text{ share of the domestic portfolio materialized} \\ &\quad \text{in foreign capital,} \end{aligned}$$

equation (1) can be expressed in a more convenient way

$$1 = n_d + n_d^* \tag{8}$$

[4] Other rules can also achieve a balanced growth path. See Turnovsky (1996) for more details.

and substituting those variables into the budget constraint (7) we obtain the following dynamic restriction for the resources of the domestic economy

$$\frac{dW}{W} = \left[\alpha n_d + \alpha^* n_d^* - \frac{C}{W} - g\right] dt + \left[\alpha n_d dy + \alpha^* n_d^* dy^* - dz\right]. \tag{9}$$

This equation can be more conveniently expressed as

$$\frac{dW}{W} = \psi dt + dw, \tag{10}$$

where the deterministic and stochastic parts of the rate of accumulation of assets, dW/W, can be expressed in the following way

$$\psi \equiv n_d\left[\alpha - \alpha^*\right] + \alpha^* - g - \frac{C}{W} \equiv \rho - g - \frac{C}{W} \tag{11}$$

$$dw \equiv n_d\left[\alpha dy - \alpha^* dy^*\right] + \alpha^* dy^* - dz, \tag{12}$$

where $\rho \equiv \alpha n_d + \alpha^* n_d^* \equiv n_d\left[\alpha - \alpha^*\right] + \alpha^*$ denotes the gross rate of return of the asset portfolio.

2.2.2 The equilibrium

The objective of the domestic representative agent consists in choosing the path of private consumption and portfolio shares that maximize the expected value of the intertemporal utility function (3), subject to $W(0) = W_0$, (10), (11), and (12). This optimization is a stochastic optimum control problem.[5] Initially we are going to suppose that the government establishes an arbitrarily exogenous size of the public sector, g. We analyze the case in which such a size is chosen optimally in section 4.

It is important to bear in mind that the domestic agent takes as given the rates of return of different assets, as well as the corresponding variances and covariances. However, these parameters will endogenously be determined in the macroeconomic equilibrium we are going to obtain.

The first step in order to solve this optimization problem is to introduce a value function, $V(W)$, which is defined as

[5] To solve problems of stochastic optimum control see, for example, Kamien and Schwartz (1991, section 22), Malliaris and Brock (1982, ch. 2), Obstfeld (1992), or Turnovsky (1997, ch. 9; 2000, ch. 15).

$$V(W)=\underset{\{C,n_d\}}{Max}\ E\int_0^\infty \frac{1}{\gamma}(CG^\eta)^\gamma e^{-\beta t}dt, \tag{13}$$

subject to the restrictions (10), (11), and (12) and given initial wealth. The value function in period 0 is the expected value of the discounted sum of instantaneous utilities, evaluated along the optimal path, starting in period 0 in the state $W(0)=W_0$.

Second, starting from equation (13) the value function must satisfy the following equation, known as the Hamilton-Jacobi-Bellman equation of stochastic control theory or, for short, the Bellman equation

$$\beta V(W)=\underset{\{C,n_d\}}{Max}\left[\frac{1}{\gamma}(CG^\eta)^\gamma+V'(W)W\psi+0.5V''(W)W^2\sigma_w^2\right]. \tag{14}$$

Third, we differentiate partially (14) with respect to C and n_d in order to get the first order optimality conditions of this problem

$$\begin{aligned} C^{\gamma-1}G^{\eta\gamma}-V'(W) &= 0 \qquad (15)\\ V'(W)W(\alpha-\alpha^*)+V''(W)W^2cov\left[dw,\alpha dy-\alpha^* dy^*\right] &= 0. \qquad (16)\end{aligned}$$

The solution to this problem is obtained through trial and error. We seek to find a value function $V(W)$ that satisfies, on the one hand, the first order optimality conditions and, on the other, the Bellman equation. In the case of isoelastic utility functions the value function has the same form of the utility function [Merton (1969), generalized in Merton (1971)]. Thus we suggest the guess solution

$$V(W)=AW^{\gamma(1+\eta)}, \tag{17}$$

where the coefficient A will be determined below. This guess solution implies that

$$\begin{aligned} V'(W) &= A\gamma(1+\eta)W^{\gamma(1+\eta)-1}\\ V''(W) &= A\gamma(1+\eta)\left[\gamma(1+\eta)-1\right]W^{\gamma(1+\eta)-2}.\end{aligned}$$

Substituting these expressions in the first order optimality conditions (15) and (16) we get that

$$C^{\gamma-1}G^{\eta\gamma} = A\gamma(1+\eta)W^{\gamma(1+\eta)-1} \quad (18)$$

$$(\alpha-\alpha^*)\,dt = [1-\gamma(1+\eta)]\,cov\,[dw, \alpha dy - \alpha^* dy^*]\,. \quad (19)$$

Both are typical equations in stochastic models in continuous time. Equation (18) indicates that at the optimum, the marginal utility derived from private consumption must be equal to the marginal change in the value function or the marginal utility of wealth. Equation (19) shows us that the optimal choice of portfolio shares of the domestic representative agent must be such that the risk-adjusted rates of return of both domestic and foreign capital are equalized.

Combining (18) and (19), and substituting them in the equation (14), we can calculate, after some algebra, the equilibrium portfolio shares and the consumption-wealth ratio in the domestic open economy

$$n_d = \frac{\alpha-\alpha^*}{[1-\gamma(1+\eta)]\,\Delta} + \frac{\alpha^{*^2}\sigma^2_{y^*} - \alpha\alpha^*\sigma_{yy^*} + \alpha\sigma_{yz} - \alpha^*\sigma_{y^*z}}{\Delta} \quad (20)$$

$$n_d^* = 1 - n_d \quad (21)$$

$$\left(\frac{C}{W}\right)_o = \frac{1}{(1-\gamma)(1+\eta)}\,[\beta - \gamma(1+\eta)\,(\rho-g) + 0.5\gamma(1+\eta)[1-\gamma(1+\eta)]\sigma^2_{w,o}]\,, \quad (22)$$

where

$$\Delta = \alpha^2\sigma^2_y - 2\alpha\alpha^*\sigma_{yy^*} + \alpha^{*^2}\sigma^2_{y^*} \quad (23)$$

$$\sigma^2_{w,o} = n_d^2\alpha^2\sigma^2_y + 2n_d n_d^*\alpha\alpha^*\sigma_{yy^*} + n_d^{*^2}\alpha^{*^2}\sigma^2_{y^*} + \sigma^2_z - 2n_d\alpha\sigma_{yz} - 2n_d^*\alpha^*\sigma_{y^*z}. \quad (24)$$

Please note that neither the expression Δ nor the variance of the rate of accumulation of domestic assets, $\sigma^2_{w,o}$, can be negative and the variables with the subscript o refer to values in an open economy.

Then the equilibrium rate of wealth accumulation of the open domestic economy follows the stochastic process

$$\frac{dW}{W} = \psi_o dt + dw_o, \quad (25)$$

where the deterministic and stochastic components are, respectively

$$
\begin{aligned}
\psi_o &= \frac{1}{(1-\gamma)(1+\eta)} \{(1+\eta)(\rho - g) - \beta \\
& \quad -0.5\gamma(1+\eta)[1-\gamma(1+\eta)]\sigma_{w,o}^2\} \qquad (26) \\
dw_o &= n_d \alpha dy + n_d^* \alpha^* dy^* - dz. \qquad (27)
\end{aligned}
$$

Even though with more general utility functions portfolio shares and consumption-wealth ratio will be functions of time, in this model all those variables are constants because the utility function exhibits constant relative risk aversion, the production function is linear, and the mean and variances of the underlying stochastic processes are stationary: the equilibrium is characterized by balanced real growth, where all the (real) assets grow at the same rate, and by constant consumption-wealth ratio and portfolio shares. Additionally we should observe that portfolio shares do not depend on the size of the public sector, but they do depend on the degree of relative risk aversion. The result is very similar to Turnovsky (1997, ch. 11). However, we should note that portfolio shares depend, in addition, on the parameter that reflects the influence of public consumption in the utility function of the domestic representative agent, η. The same is also true for the foreign economy, as we shall see below.

Now we are going to describe the behavior of the domestic economy if it were closed in order to compare the results of an open economy with those of a closed economy later on. In a model of perfect capital mobility such as this, where domestic and foreign assets are traded without restrictions, we use the shares of the domestic portfolio materialized in domestic and foreign capital, n_d and n_d^* respectively, to approximate the degree of openness of the domestic economy. Since our emphasis is on the trade of assets, then we are calling closed economy to the situation where there is no trade of assets. However, we should bear in mind that what we call closed economy is compatible with positive amounts of exports and imports, but subject to the restriction that the trade of goods must be balanced. For the case of a closed economy the equilibrium solution will be given by the expressions

$$\left(\frac{C}{W}\right)_c = \frac{1}{(1-\gamma)(1+\eta)}\{\beta - \gamma(1+\eta)(\alpha - g) + 0.5\gamma(1+\eta)[1-\gamma(1+\eta)]\sigma^2_{w,c}\} \quad (28)$$

$$\sigma^2_{w,c} = \alpha^2\sigma^2_y + \sigma^2_z - 2\alpha\sigma_{yz} \quad (29)$$

$$\psi_c = \frac{1}{(1-\gamma)(1+\eta)}\{(1+\eta)(\alpha - g) - \beta - 0.5\gamma(1+\eta)[1-\gamma(1+\eta)]\sigma^2_{w,c}\} \quad (30)$$

$$dw_c = \alpha dy - dz,$$

where the variables with the subscript c refer to values in a closed economy.

To guarantee that consumption is positive in the domestic open economy we impose the feasibility condition that the marginal propensity to consume out of wealth must be positive since wealth does not become negative

$$\frac{1}{(1-\gamma)(1+\eta)}\{\beta - \gamma(1+\eta)(\rho - g) + 0.5\gamma(1+\eta)[1-\gamma(1+\eta)]\sigma^2_{w,o}\} > 0.$$

For the first order optimality conditions to characterize a maximum, the corresponding second order condition must be satisfied, that is, the Hessian matrix associated to the maximization problem and evaluated at the optimal values of the choice variables

$$\begin{bmatrix} (\gamma-1)(V'(W))^{\frac{\gamma-2}{\gamma-1}} & 0 \\ 0 & V''(W)W^2\Delta \end{bmatrix}$$

must be negative definite,[6] which implies that

$$\begin{aligned} (\gamma-1)(V'(W))^{\frac{\gamma-2}{\gamma-1}} &< 0 \\ V''(W)W^2\Delta &< 0, \end{aligned}$$

where $\Delta > 0$ (in a risky economy) was already defined in equation (23). To evaluate those conditions first we obtain the value of the coefficient A in equation (18)

[6] See Chiang (1984, pp. 320-323), for example.

$$A = \frac{g^{\eta\gamma}}{\gamma(1+\eta)} \left(\frac{C}{W}\right)^{\gamma-1}, \tag{31}$$

where C/W is the optimal value pointed out by equation (22). Then we substitute (31) into the value function (17). Noting that $g = G/W$, the value function is given, after some algebra, by

$$V(W) = \frac{g^{\eta\gamma}}{\gamma(1+\eta)} \left(\frac{C}{W}\right)^{\gamma-1} W^{\gamma(1+\eta)}, \tag{32}$$

where we can observe that, given the restrictions on the utility function, $V'(W) > 0$ and $V''(W) < 0$ provided that $C/W > 0$.

In addition, we impose that the macroeconomic equilibrium must satisfy the transversality condition so as to guarantee the convergence of the value function

$$\lim_{t\to\infty} E\left[V(W)\, e^{-\beta t}\right] = 0. \tag{33}$$

Now let us show that should the feasibility condition be satisfied then that would be equivalent to satisfy the transversality condition.[7] To evaluate (33), we start expressing the dynamics of the accumulation of wealth

$$dW = \psi W dt + W dw. \tag{34}$$

The solution to equation (34), starting from the initial wealth $W(0)$, is[8]

$$W(t) = W(0) e^{(\psi - 0.5\sigma_w^2)t + w(t) - w(0)}.$$

Since the increments of w are temporally independent and are normally distributed then[9]

$$\begin{aligned} E[AW^{\gamma(1+\eta)}e^{-\beta t}] &= E[AW(0)^{\gamma(1+\eta)} e^{\gamma(1+\eta)(\psi-0.5\sigma_w^2)t + \gamma(1+\eta)[w(t)-w(0)] - \beta t}] \\ &= AW(0)^{\gamma(1+\eta)} e^{\left[\gamma(1+\eta)(\psi-0.5\sigma_w^2) + 0.5\gamma^2(1+\eta)^2\sigma_w^2 - \beta\right]t}. \end{aligned}$$

[7] See Merton (1969). Turnovsky (2000) provides, for example, the proof of the transversality condition as well.

[8] See Malliaris and Brock (1982, pp. 135-136), for example.

[9] See Malliaris and Brock (1982, pp. 137-138), for example.

The transversality condition (33) will be satisfied if and only if

$$\gamma(1+\eta)\left\{\psi - 0.5\gamma(1+\eta)\left[1-\gamma(1+\eta)\right]\sigma_w^2\right\} - \beta < 0.$$

Now substituting equations (11) and (22), it can be shown that this condition is equivalent to

$$\frac{C}{W} > 0, \tag{35}$$

and thus feasibility guarantees convergence as well.

Finally, we should note that since the public sector equilibrates its budget continuously then the intertemporal budget constraint of the public sector is satisfied trivially.

2.3 The foreign economy

2.3.1 The problem

The problem facing the foreign representative agent can be formulated in an analogous way. Her preferences are represented by the following intertemporal utility function

$$\begin{aligned} & E\int_0^\infty \frac{1}{\gamma^*}(C^* G^{*\eta^*})^{\gamma^*} e^{-\beta^* t} dt \\ -\infty &< \gamma^* < 1; \eta^* > 0; \gamma^*\eta^* < 1; \gamma^*(1+\eta^*) < 1. \end{aligned}$$

The equation of the rate of accumulation of wealth of the foreign representative agent can be expressed as

$$\frac{dW^*}{W^*} = \psi^* dt + dw^*,$$

where

$$\begin{aligned} \psi^* &\equiv n_f\alpha + n_f^*\alpha^* - g^* - \frac{C^*}{W^*} \equiv \rho^* - g^* - \frac{C^*}{W^*} \\ dw^* &\equiv n_f\alpha dy + n_f^*\alpha^* dy^* - dz^*. \end{aligned}$$

2.3.2 The equilibrium

The equilibrium portfolio shares and consumption-wealth ratio in the foreign economy are

$$
\begin{aligned}
n_f &= \frac{\alpha - \alpha^*}{[1-\gamma^*(1+\eta^*)]\,\Delta} + \frac{{\alpha^*}^2\sigma_{y^*}^2 - \alpha\alpha^*\sigma_{yy^*} + \alpha\sigma_{yz^*} - \alpha^*\sigma_{y^*z^*}}{\Delta} \\
n_f^* &= 1 - n_f \\
\left(\frac{C^*}{W^*}\right)_o &= \frac{1}{(1-\gamma^*)(1+\eta^*)}\left\{\beta^* - \gamma^*(1+\eta^*)(\rho^* - g^*)\right. \\
&\quad \left. -0.5\gamma^*(1+\eta^*)\left[\gamma^*(1+\eta^*)-1\right]\sigma_{w^*,o}^2\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{w^*,o}^2 &= n_f^2\alpha^2\sigma_y^2 + 2n_f n_f^*\alpha\alpha^*\sigma_{yy^*} + {n_f^*}^2{\alpha^*}^2\sigma_{y^*}^2 \\
&\quad + \sigma_{z^*}^2 - 2n_f\alpha\sigma_{yz^*} - 2n_f^*\alpha^*\sigma_{y^*z^*}.
\end{aligned} \tag{36}
$$

The equilibrium rate of accumulation of wealth in the foreign economy follows the stochastic process

$$
\frac{dW^*}{W^*} = \psi_o^* dt + dw_o^*
$$

where its deterministic and stochastic components are, respectively

$$
\begin{aligned}
\psi_o^* &= \frac{1}{(1-\gamma^*)(1+\eta^*)}\left\{(1+\eta^*)(\rho^* - g^*) - \beta^*\right. \\
&\quad \left. -0.5\gamma^*(1+\eta^*)\left[\gamma^*(1+\eta^*)-1\right]\sigma_{w^*,o}^2\right\} \\
dw_o^* &= n_f\alpha dy + n_f^*\alpha^* dy^* - dz^*.
\end{aligned}
$$

3 EQUILIBRIUM ANALYSIS

In this section first we review the impact of changes in exogenous variables on the consumption-wealth ratio, the rate of growth of wealth of the domestic economy, and welfare, since most of the results are standard[10]. Then we compare the results of an open economy with those of a closed economy.

[10] We refer to Turnovsky (1997, Ch. 11) for the analysis of the impact of production risk and public spending on portfolio shares and on the variance of the rate of growth of the domestic economy.

3.1 Consumption

The optimal consumption-wealth ratio shown in equation (22) is standard in the literature[11]: the consumption function is a linear function of wealth. First, we review how consumption responds to changes in exogenous variables that are not directly related to risk or to the influence of the public sector. Thus a higher subjective discount rate, β, increases consumption-wealth ratio, because the domestic representative agent finds more attractive to dedicate a higher proportion of wealth to consumption, thus reducing investment. In addition, a higher gross rate of return of the asset portfolio, ρ, raises (reduces) consumption-wealth ratio if $\gamma < (>)0$ and does not change if $\gamma = 0$. That is the overall result of two opposite effects, substitution and income effects. A higher gross rate of return of the asset portfolio has always a negative substitution effect since consumption becomes less attractive whereas investment is more attractive. The income effect on the consumption-wealth ratio originated by a higher gross rate of return of the asset portfolio is equal to unity: it makes possible to raise both actual and future consumption. If $\gamma < (>)0$ then income (substitution) effect dominates substitution (income) effect and if $\gamma = 0$ the two effects compensate each other. From here onwards whenever we get that the result depends on the sign of the parameter γ only we focus on the case where $\gamma < 0$, for being the most relevant situation empirically.

Second, we study the influence of variables related to risk, but not affected by the behavior of the public sector. Thus the effect of a higher coefficient of risk aversion, γ, on consumption is ambiguous. Additionally, a higher variance of the rate of growth, $\sigma_{w,o}^2$, reduces consumption-wealth ratio if $\gamma < 0$. Substitution and income effects arise again: totally differentiating equation (22) we can easily show that an increase of the variance of the rate of growth is equivalent to a fall in the gross rate of return of the asset portfolio, ρ, of $0.5\left[1-\gamma(1+\eta)\right]$. Analogous conclusion applies to the impact of a higher variance of domestic productivity shocks, σ_y^2, a higher variance of foreign productivity shocks, $\sigma_{y^*}^2$,or a higher covariance between domestic and foreign productivity shocks, σ_{yy^*}, on consumption-wealth ratio.

Third, we analyze the role of the public sector. Consumption-wealth ratio decreases as the size of the public sector, g, increases, for $\gamma < 0$. An increase in the size of the public sector is equivalent to a fall in the gross rate of return of the asset portfolio of 1. In addition, an increase in the variance of public spending shocks, σ_z^2, diminishes consumption-wealth ratio when

[11] See Merton (1969) for the pioneer work in continuous time with uncertainty. We refer to Turnovsky (1996; 1997, Ch. 11; 1999) for more details on the impact of changes in exogenous variables on consumption-wealth ratio.

$\gamma < 0$. An increase in the variance of public spending shocks is equivalent to a fall in the gross rate of return of the asset portfolio of $0.5\,[1-\gamma(1+\eta)]$, since the variance of the rate of growth increases. In contrast, if either the covariance between domestic productivity shocks and domestic public spending shocks, σ_{yz}, or the covariance between foreign productivity shocks and domestic public spending shocks, σ_{y^*z}, increase then consumption-wealth ratio increases for $\gamma < 0$. That is due to a reduction in the variance of the rate of growth of the domestic economy.

For the case that the utility function is logarithmic the consumption function becomes much simpler

$$\frac{C}{W} = \frac{\beta}{1+\eta}, \tag{37}$$

already found in Turnovsky (1996, 1999). This implies that a higher weight of public consumption in the utility function, η, reduces unambiguously the consumption-wealth ratio. A higher value of η increases the attractiveness of public consumption *in relation to* private consumption, given the exogenous size of the public sector. In addition, any other variable (risk, for example) does not change consumption-wealth ratio and the consumption function in an open economy is equal to that in a closed economy.

3.2 Growth

The mean rate of growth of assets achieved in equilibrium, given by (26), is standard in the literature[12]. First, we focus on the impact of variables that do not refer either to risk or to the public sector on the rate of growth of assets. Thus a higher subjective discount rate, β, reduces unambiguously the rate of growth since dedicating resources to consumption becomes more attractive whereas investment is discouraged. In addition, a higher gross rate of return of the asset portfolio, ρ, increases the rate of growth, even though consumption-wealth ratio may rise.

Second, we study the influence of variables related to risk, but not affected by the behavior of the public sector. Thus a change in the parameter γ generates an ambiguous effect on the growth rate. Departing from $\psi_o = \rho - g - (C/W)_o$, this model shows that an increase in the variance of domestic productivity shocks, σ_y^2, shifting investment towards foreign capital, tends to increase the rate of growth, on the one hand, if $\alpha^* > \alpha$. On the other hand,

[12] We refer to Turnovsky (1996; 1997, Ch. 11; 1999) again for more details on the impact of changes in exogenous variables on the rate of growth of wealth.

the growth-enhancing effect is reinforced when $\gamma < 0$ since consumption-wealth ratio falls due to an increase in σ_y^2 (Turnovsky, 1997, p. 442). Similarly, an increase in the variance of the foreign productivity shocks, $\sigma_{y^*}^2$, making domestic capital more attractive, tends to increase the rate of growth if $\alpha > \alpha^*$. Again the positive effect on the rate of growth is strengthened if $\gamma < 0$: consumption-wealth ratio falls due to an increase in $\sigma_{y^*}^2$.

Third, we analyze the impact of the public sector on the rate of growth. It is easy to show that a higher size of the public sector, g, reduces unambiguously the rate of growth of the economy, even though consumption-wealth ratio may fall. A higher variance of domestic public spending, σ_z^2, increases the rate of growth of the economy for $\gamma < 0$, because consumption-wealth ratio falls (Turnovsky, 1997, p. 444). In contrast, we get the opposite conclusions when either the covariance of domestic productivity and public spending shocks, σ_{yz}, or the covariance of foreign productivity shocks and public spending shocks, σ_{y^*z}, increases.

Fourth, in the case of a logarithmic utility function the growth rate is given by the expression

$$\psi_o = \rho - g - \frac{\beta}{1+\eta}.$$

Thus a higher value of the parameter η increases unambiguously the rate of growth of assets of the domestic economy. Even though it seems counter-intuitive at first glance, the reason behind is that a higher weight of public consumption reduces consumption-wealth ratio, as we saw in the previous section, thus increasing the rate of accumulation of assets of the economy, given the exogenous size of the public sector. In addition, the rate of growth of domestic wealth does not have to be equal in an open economy compared to a closed economy, as we will see below in more detail.

Finally, we conclude that most of the results are standard in the literature, even though they must be adjusted to include utility-enhancing public consumption. As Turnovsky (1997, p. 432) puts it, "With identical preferences and portfolios, differences in the international growth rates of wealth and therefore of consumption are due entirely to differences in the respective size of government, $g - g^*$, in the two economies. If the size of government is uniform, then the equilibrium growth rates, ψ and ψ^*, will be identical". However, the parameter η plays here an important role in the model to account for differences in the rates of growth as well. This implies that, having the representative agents of both economies identical preferences, portfolios and sizes of the public sector, differences in the growth rates of both economies can be explained in terms of differences in the weight of public consumption in the utility functions of both economies. Additionally we have shown that economies which assign a higher weight to public consumption in their utility function will have higher growth rates due to lower consumption-wealth ratios.

3.3 Welfare

Economic welfare is measured by the value function we have used to solve the problem of intertemporal optimization, given by equation (32). Taking the total differential of equation (32) we obtain, after some algebra, that[13]

$$\frac{dV}{V} = (\gamma - 1)\frac{d(C/W)}{C/W} + \gamma \left[\eta - \frac{g}{C/W}\right] \frac{dg}{g}, \tag{38}$$

where we can observe that changes in the optimal consumption-wealth ratio and the (exogenous) size of the public sector have an impact on welfare.

First, a higher optimal consumption-wealth ratio can improve or deteriorate the welfare of the domestic representative agent. That is due to the fact that the value function can take either positive or negative values, depending on the sign of the coefficient γ. Since C/W and g are positive in equation (32) then $\gamma V(W) > 0$. For the case $\gamma < 0$ then anything that increases the optimal consumption-wealth ratio elevates the welfare of the representative agent. Thus, for example, a higher subjective discount rate, increasing the optimal consumption-wealth ratio, generates higher welfare if $\gamma < 0$.

Second, the size of the public sector is an important factor influencing the welfare of the representative agent. Thus a higher size of the public sector can increase or reduce the welfare of the domestic representative agent, even though it reduces unambiguously the rate of growth. The crucial point lies on whether $g \lesseqgtr \eta C/W$. If $g < \eta C/W$, then an increase of the size of the public sector augments the welfare of the representative agent. That is due to the fact that the marginal utility derived from public consumption is higher than the marginal utility derived from private consumption. If $g = \eta C/W$, then an increase of the size of the public sector does not alter the welfare of the representative agent because the marginal utility derived from public consumption is equal to the marginal utility derived from private consumption: it is the size of the public sector that maximizes welfare, as we will see in the next section. Finally, if $g > \eta C/W$, then an increase of the size of the public sector reduces the welfare of the representative agent

[13] Please note that the optimal consumption-wealth ratio, given by equation (22), depends on the size of the public sector, g, as well.

because the marginal utility derived from public consumption is lower than the marginal utility derived from private consumption. These results can be related with the conclusions established in Turnovsky (2000, p. 438): "Thus we infer that increasing the growth rate by reducing government expenditure is not necessarily welfare improving. This will be the case only if initially g is above its optimum". We will see below that this is completely consistent with the analysis of the size of the public sector that maximizes the welfare of the representative agent.

3.4 Open economy versus closed economy

In order to compare the results of an open economy with those of a closed economy it is convenient to calculate the difference between the variance of the growth rate in an open economy and that in a closed economy. Thus if we substract equation (29) from equation (24) we get, after some algebra, that

$$\sigma_{w,o}^2 - \sigma_{w,c}^2 = \Delta n_d^* \left(n_d^* - 2\widetilde{n}_d^* \right), \tag{39}$$

where

$$\widetilde{n}_d^* = \frac{\alpha^2 \sigma_y^2 - \alpha\alpha^* \sigma_{yy^*} - \alpha\sigma_{yz} + \alpha^* \sigma_{y^*z}}{\Delta},$$

is the share of the domestic portfolio materialized in foreign capital that minimizes the variance of the growth rate given by equation (24).

First, we can compare the consumption-wealth ratio in an open economy with that in a closed economy. If we substract equation (28) from equation (22) we get, using equation (39), that, after some algebra,

$$\left(\frac{C}{W}\right)_o - \left(\frac{C}{W}\right)_c = -\frac{1}{1-\gamma} \left\{ 0.5\gamma \left[1 - \gamma \left(1 + \eta\right)\right] \Delta n_d^{*^2} \right\}. \tag{40}$$

We see the difference between both consumption-wealth ratios depends on the sign of the parameter γ only. Thus if $\gamma < 0$ then the consumption-wealth ratio will be higher in an open economy than that in a closed economy, assuming an interior solution for the value of portfolio shares. An easy way to explain that result can be found focusing on the case $n_d = \widetilde{n}_d$, where

$$\widetilde{n}_d = 1 - \widetilde{n}_d^* = \frac{\alpha^{*^2} \sigma_{y^*}^2 - \alpha\alpha^* \sigma_{yy^*} + \alpha\sigma_{yz} - \alpha^* \sigma_{y^*z}}{\Delta}, \tag{41}$$

denotes the share of the domestic portfolio materialized in domestic capital that minimizes the variance of the rate of growth of wealth. In such a situation we get from equation (39) that the variance of the growth rate in an open economy is lower than that in a closed economy, $\sigma^2_{w,o} < \sigma^2_{w,c}$. As we saw above, a reduction of the variance of the growth rate is equivalent to an increase in the gross rate of return of the asset portfolio. That, in turn, originates a negative substitution effect and a positive income effect on the consumption-wealth ratio. If $\gamma < 0$ the income effect is stronger than the substitution effect and the consumption-wealth ratio in an open economy is higher than that in a closed economy. Additionally, the higher the value of the optimal share of the domestic portfolio materialized in foreign capital, n^*_d, the higher the difference between the results of an open economy with those of a closed economy.

Second, we can compare the rate of growth in an open economy with that in a closed economy departing from the equation (11) corresponding to an open economy and substracting from it that corresponding to a closed economy

$$\psi_o - \psi_c = n^*_d(\alpha^* - \alpha) - \left[\left(\frac{C}{W} \right)_o - \left(\frac{C}{W} \right)_c \right]. \tag{42}$$

We see that the rate of growth in an open economy can be higher than, equal to or lower than that in a closed economy, depending on the signs of the two terms in (42). For example, we can establish focusing on the case where $\gamma < 0$, that:

- If $\alpha \geq \alpha^*$ then the rate of growth in an open economy will be lower than that in a closed economy. The reason behind it is that the consumption-wealth ratio in an open economy is higher than that in a closed economy and, additionally, if $\alpha \geq \alpha^*$ then the gross rate of return of the asset portfolio in an open economy, ρ, is lower than or equal to the marginal physical product of the domestic capital.

- If $\alpha < \alpha^*$ then the rate of growth in an open economy can be higher than, equal to or lower than that in a closed economy.

Table 2.2. sums up the comparison between the rate of growth in an open economy with that in a closed economy given by equation (42).

Table 2.2. Comparing rates of growth			
	$\gamma > 0$	$\gamma = 0$	$\gamma < 0$
$\alpha > \alpha^*$	$\psi_o \lesseqqgtr \psi_c$	$\psi_o < \psi_c$	$\psi_o < \psi_c$
$\alpha = \alpha^*$	$\psi_o > \psi_c$	$\psi_o = \psi_c$	$\psi_o < \psi_c$
$\alpha < \alpha^*$	$\psi_o > \psi_c$	$\psi_o > \psi_c$	$\psi_o \lesseqqgtr \psi_c$

Finally, we can compare the welfare of the domestic representative agent in an open economy with that in a closed economy. Since we have shown above in equation (40) that the consumption-wealth ratio in an open economy is higher than that in a closed economy for $\gamma < 0$, then going back to the value function given by equation (32) we can establish that the welfare of the domestic representative agent is higher in a risky open economy than in a risky closed economy. This result adds insights to those shown in Obstfeld (1994) and Turnovsky (1997, Ch. 11), where they analyze the impact on welfare when changing from a domestic closed economy with low-yield and no risk (or relatively low risk) assets to an open economy with high-yield and high-risk assets, among other things. Obstfeld (1994, p. 1326-27) showed that "international risk-sharing can yield substantial welfare gains through its positive effect on expected consumption growth. The mechanism linking global diversification to growth is the attendant world portfolio shift from safe, but low-yield, capital into riskier, high-yield capital". Additionally, Turnovsky (1997, p. 439) showed that for a logarithmic utility function "the higher growth rate more than offsets the additional risk, and the opportunity to invest in a higher return, higher risk foreign asset improves welfare". However, we should note that our conclusion is not based on low risk-high risk considerations, but on closed economy-open economy considerations. In addition, our result hinges on the sign of the parameter γ again: we get the opposite result about welfare if $\gamma > 0$, for example.

4 THE OPTIMAL SIZE OF THE PUBLIC SECTOR

We have analyzed the equilibrium of the world economy assuming an exogenous size of the public sector so far. Now we obtain the size of the public sector that maximizes the welfare of the domestic representative agent or, for short, the optimal size of the public sector. We discuss whether maximizing welfare implies maximizing growth or not. Then we analyze the effect of changes in exogenous parameters on the optimal size of the public sector, on consumption-wealth ratio, on growth, and on welfare, provided that the size of the public sector is optimal. Finally, we compare the results of an open economy with those of a closed economy.

Formally, we have to differentiate partially the expression in the right hand side of the Bellman equation (14) with respect to g, where $G = gW$, to calculate the optimal size of the public sector

$$\frac{\eta}{g}C^{\gamma}\left(gW\right)^{\eta\gamma} - V'(W)W = 0,$$

which combining with the first order condition equation (15) implies that the optimal size of the public sector, $\widehat{g}$, must satisfy the following condition

$$\hat{g} = \eta\frac{C}{W}, \tag{43}$$

which is identical to Turnovsky (1996, p. 60; 1999, p. 888).[14] Equation (43) implies that the marginal utility of public consumption must be equal to the marginal utility of private consumption when both public and private consumption are optimally chosen.

Combining equation (43) with (22) we calculate the optimal size of the public sector, the consumption-wealth ratio, and the growth rate when public consumption is optimally chosen in an open economy

$$\begin{aligned}
\hat{g}_o &= \frac{\eta}{\left[1-\gamma(1+\eta)\right](1+\eta)}\{\beta - \gamma(1+\eta)\rho \\
&\quad +0.5\gamma(1+\eta)\left[1-\gamma(1+\eta)\right]\sigma_{w,o}^2\} \\
\left(\frac{C}{W}\right)_o &= \frac{1}{\left[1-\gamma(1+\eta)\right](1+\eta)}\{\beta - \gamma(1+\eta)\rho \\
&\quad +0.5\gamma(1+\eta)\left[1-\gamma(1+\eta)\right]\sigma_{w,o}^2\} \\
\psi_o &= \frac{1}{1-\gamma(1+\eta)}\left\{\rho - \beta - 0.5\gamma(1+\eta)\left[1-\gamma(1+\eta)\right]\sigma_{w,o}^2\right\}.
\end{aligned} \tag{44}$$

Please note that whenever we refer to the optimal size of the public sector in general we will use the term $\widehat{g}$ and whenever we refer only to the optimal size in an open economy we will use $\widehat{g}_o$.

Additionally, we obtain the optimal size of the public sector, the consumption-wealth ratio, and the rate of growth rate when public consumption is optimally chosen in a closed economy

[14] We should note that the optimal size of the public sector, $\hat{g}$, is not exactly identical to that shown in Turnovsky (1999). However, it is identical in the sense that in both cases the optimal ratio of public consumption to private consumption is given by $G/C = \eta$.

$$
\begin{aligned}
\hat{g}_c &= \frac{\eta}{[1-\gamma(1+\eta)]\,(1+\eta)}\{\beta-\gamma(1+\eta)\alpha \\
&\quad +0.5\gamma(1+\eta)\,[1-\gamma(1+\eta)]\,\sigma^2_{w,c}\} \qquad (45)\\
\left(\frac{C}{W}\right)_c &= \frac{1}{[1-\gamma(1+\eta)](1+\eta)}\{\beta-\gamma(1+\eta)\alpha \\
&\quad +0.5\gamma(1+\eta)\,[1-\gamma(1+\eta)]\,\sigma^2_{w,c}\} \\
\psi_c &= \frac{1}{1-\gamma(1+\eta)}\left\{\alpha-\beta-0.5\gamma(1+\eta)\,[1-\gamma(1+\eta)]\,\sigma^2_{w,c}\right\}
\end{aligned}
$$

Finally, in the case of a logarithmic utility function we find that the optimal size of the public sector is given by

$$\hat{g} = \frac{\eta\beta}{(1+\eta)}, \qquad (46)$$

which is equal to the (deterministic) optimal size of the public sector obtained by Turnovsky (1996, p. 60) and very similar to Turnovsky (1999, p. 888). Then it is easy to show that private consumption-wealth ratio plus the optimal size of the public sector is given by

$$\frac{C}{W} + \hat{g} = \beta, \qquad (47)$$

where optimal consumption-wealth ratio is given by equation (37) above. Therefore, we get a standard result in the literature again: (private plus public) consumption-wealth ratio is equal to the subjective discount rate.

4.1 Growth vs. welfare maximizing

Now we can compare the optimal size of the public sector with the size that maximizes the rate of growth. Going back to equation (26) it is straightforward to calculate that the size of the public sector that maximizes the rate of growth is zero. The intuition behind the result is immediate. Public spending is utility-enhancing but it does not affect the productivity of the economy. Therefore, since public spending does not enhance growth directly but it imposes a sacrifice, then the size of the public sector that maximizes growth should be zero. The optimal size of the public sector is clearly higher than that the size that maximizes the rate of growth. Both objectives are not equivalent.

4.2 Analysis of the optimal size

First, we focus on the influence of changes in exogenous variables that do not refer either to risk or the public sector. Differentiating equation (44) with respect to β

$$\frac{\partial \hat{g}_o}{\partial \beta} = \frac{\eta}{[1-\gamma(1+\eta)]\,(1+\eta)} > 0,$$

we can observe that a higher subjective discount rate increases the optimal size of the public sector, because public consumption becomes more attractive. In addition, the effect of a higher gross rate of return, ρ, on the optimal size of the public sector is given by the expression

$$\frac{\partial \hat{g}_o}{\partial \rho} = -\frac{\eta\gamma}{[1-\gamma(1+\eta)]},$$

where a higher gross rate of return of the asset portfolio will raise the optimal size of the public sector for $\gamma < 0$. An increase in the gross rate of return originates a positive income effect on public consumption (allowing to dedicate more resources to public consumption) stronger than the negative substitution effect (public consumption becoming less attractive while investing more attractive).

Second, we analyze the impact of changes in exogenous variables that are related to risk, but not related to the behavior of the public sector. An increase in the parameter γ has an ambiguous effect on the optimal size of the public sector. In addition, differentiating (44) with respect to $\sigma^2_{w,o}$

$$\frac{\partial \widehat{g}_o}{\partial \sigma^2_{w,o}} = 0.5\gamma\eta, \tag{48}$$

we show that a higher variance of the growth rate, $\sigma^2_{w,o}$, reduces the optimal size of the public sector if $\gamma < 0$. A higher variance of the rate of growth is equivalent to a fall in the gross rate of return of the asset portfolio, ρ, as we showed above. That conclusion can be easily extended for the impact of a higher variance of domestic productivity shocks, σ^2_y, a higher covariance between domestic and foreign productivity shocks, σ_{yy^*}, or a higher variance of foreign productivity shocks, $\sigma^2_{y^*}$. For example, if shocks become less idiosyncratic in the EMU (that is, σ_{yy^*} increases) then the optimal size of the public sector should be lower for $\gamma < 0$. These results are in clear contrast to those found in Turnovsky (1999, pp. 888-889). He finds that, for a logarithmic utility function, a higher domestic risk increases unambiguously the optimal size of the public sector, whereas the impact of a higher foreign risk depends on whether the domestic economy holds positive stocks of foreign capital or not.

Third, we focus on the impact of changes in variables related to the behavior of the public sector. Then we easily show that whatever increases the variance of the rate of growth, be a higher variance of domestic public spending, σ_z^2, be a lower covariance between domestic (foreign) productivity shocks and domestic public spending, σ_{yz} (σ_{y^*z}), should reduce the optimal size of the public sector if $\gamma < 0$, as we showed in (48).

Finally, focusing on the logarithmic case, we find that differentiating equation (46) with respect to the parameter η we get that

$$\frac{\partial \hat{g}}{\partial \eta} = \frac{\beta}{(1+\eta)^2} > 0, \tag{49}$$

which intuitively seems straightforward.

4.3 Consumption and growth

If we analyze the influence of changes in different exogenous parameters on consumption-wealth ratio and growth when the size of the public sector is optimal, then most of the qualitative results obtained when the size of the public sector was exogenously given do not change at all, even though the quantitative results do change. However, some results deserve attention.

Restricting ourselves to the case of a logarithmic utility function, then it can be easily shown that an increase in the parameter η, in addition to raising the optimal size of the public sector unambiguously [see equation (49)], reduces in the same amount the private consumption-wealth ratio, given by equation (37). Going back to equation (47) above, then a change in the parameter η modifies the distribution of total consumption spending between private and public spending. Increasing the optimal size of the public sector "crowds out" private consumption-wealth ratio one-to-one. Only variations in the subjective discount rate change private plus public consumption-wealth ratio. Changes in any other variable do not modify either total consumption spending or the distribution between both types of consumption. Therefore, a change in the parameter η does not change the rate of growth of wealth, in contrast to the conclusions we got in Section 3.2 above.

4.4 Open economy versus closed economy

Now we can compare the optimal size of the public sector in an open economy with that in a closed economy, as well as the consumption-wealth ratio, the rate of growth, and welfare in the same way that we did when the size of the public was exogenously given in Section 3.4.

First, if we substract equation (45) from equation (44) we get using equation (39), after some algebra, that

$$\hat{g}_o - \hat{g}_c = -0.5\eta\gamma\Delta n_d^{*^2}.$$

We focus on the case $n_d = \widetilde{n}_d$, where $\widetilde{n}_d$ is the variance-minimizing share of the domestic portfolio, given by equation (41). However, the results do not depend on that assumption. Then we get from equation (39) that the variance of the growth rate in an open economy is lower than that in a closed economy, $\sigma^2_{w,o} < \sigma^2_{w,c}$. As we saw above, a reduction of the variance of the growth rate is equivalent to an increase in the gross rate of return of the asset portfolio. That, in turn, originates a stronger positive income effect than the negative substitution effect: the optimal size of the public sector in an open economy is higher than that in a closed economy, which is what the empirical evidence suggests (Rodrik, 1998; Alesina and Wacziarg, 1998). Additionally, the higher the value of the optimal share of the domestic portfolio materialized in foreign capital, n_d^*, the higher the difference between the optimal size of the public sector in an open economy with that in a closed economy. The result we have obtained is equal to that shown in Turnovsky (1999), but differs significantly in the conditions that are necessary to reach that conclusion: we get that result for $\gamma < 0$, which is what the empirical evidence suggests, and he shows that the optimal size in an open economy is higher than that in a closed economy if the utility function is logarithmic, provided that the domestic economy holds positive stocks of foreign capital in a small open economy.

Second, similarly we can get the difference between the consumption-wealth in an open economy compared to that in a closed economy

$$\left(\frac{C}{W}\right)_o - \left(\frac{C}{W}\right)_c = -0.5\gamma\Delta n_d^{*^2}.$$

We can show again if $\gamma < 0$ then the consumption-wealth ratio in an open economy is higher than that in a closed economy. Therefore, provided that the size of the public sector is optimal, we get the same qualitative results as those obtained when the size of the public sector was exogenously given. However, the quantitative results do change slightly.

Third, comparing the rate of growth in an open economy with that in a closed economy, the results will be qualitatively identical to those obtained in the case where the size of the public sector was exogenously given. Thus we will not pursue the analysis further.

Finally, we can easily show that welfare is higher in a risky open economy than in a risky closed economy if $\gamma < 0$, as we showed above in Section 3.4.

5 CONCLUSIONS

The impact of risk and utility-enhancing public spending on the economy is a topic that have been analyzed extensively. However, the models have been focused almost exclusively on closed or small open economies. In this paper we have analyzed a two-country stochastic AK growth model, based on Turnovsky (1997, Ch. 11), where the consumption good provided by the public sector is utility-enhancing [Barro (1990)]. The results obtained can be divided into four groups.

First, having characterized the world equilibrium, we have analyzed the impact of changes in different exogenous variables on the consumption-wealth ratio, the rate of growth of wealth, and welfare. Most of the results are standard in the literature. However, we have shown that a higher weight of public consumption in the utility function raises the rate of growth, due to a reduction in the consumption-wealth ratio, given the exogenous size of the public sector. Therefore, different preferences towards utility-enhancing government expenditure produce different rates of growth, other things being equal. Additionally, even though increasing the size of the public sector is always growth-reducing, it is welfare-augmenting when the size of the public sector is below its optimal size.

Second, we have compared the results in an open economy with those of a closed economy. Thus we have shown that consumption-wealth ratio in an open economy should be higher than that in a closed economy. In the simplest case where the portfolio share is equal to that which minimizes the variance of the rate of growth, an open economy achieves a lower variance of the rate of growth thus encouraging consumption. Then we have discussed whether the rate of growth of assets in an open economy should be higher than in a closed economy. Even though the model does not offer clear-cut results, we have shown that an open economy will unambiguously grow slower than a closed economy if the marginal physical product of domestic capital is higher than that of foreign capital. In addition, since welfare depends basically on the consumption-wealth ratio, welfare is higher in a risky open economy than in a risky closed economy, thus extending the results in Obstfeld (1994) and Turnovsky (1997, Ch. 11).

Third, we have derived the welfare-maximizing size of the public sector and compared it to the size that maximizes growth. Then we have analyzed the impact of changes in different exogenous variables on the optimal size of the public sector. Thus we have shown that whatever increases the variance of the rate of growth (a higher covariance between domestic and foreign productivity shocks, for example) reduces the optimal size of the public sector, in contrast to the results found in Turnovsky (1999). In addition, a higher value of the parameter η increases the optimal size of the public sector just in the same amount private consumption-wealth ratio falls, so that public plus private consumption-wealth ratio and the rate of growth of wealth do not change. That changes substantially our conclusions with respect to those when the size of the public sector was exogenously given. Next, we have established that the optimal size of the public sector in an open economy is higher than that in a closed economy under more general conditions than those established in Turnovsky (1999). The lower variance of the growth rate obtained in an open economy tends to raise public consumption.

Finally, we should point out possible avenues for future research. We could relax the assumption of continuous budget equilibrium and introduce public bonds in the model. However, that would increase enormously the complexity of the model. Introducing money is also an interesting element that could be integrated in a two-country world economy. Additionally, public spending could be productive also, and not only utility enhancing.

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