# More on Bernanke's "Bad News Principle"

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**Abstract**

The role that Bernanke's Bad News Principle plays in the modern theory of investment under uncertainty is analyzed. The analysis shows that the actual investment dilemma is that by delaying investment firms trade off a higher present value of earnings for a lower present value of the investment cost, in contrast to previous interpretations of this dilemma. The economic interpretation of the Smooth Pasting Condition is clarified too and found to be representing the trade-off mentioned above. I also show that investment triggers stay intact despite changes in the profit process, if the changes are restricted to the range of sufficiently high profits.

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# 1. Introduction

Studying the source of investment cycles, Bernanke (1983) has formulated the "Bad News Principle" which concerns the interaction among the irreversibility of the investment in a certain project, the uncertainty about its future rewards and the option to delay it. According to the principle: *Given the current return, the willingness to invest in the current period depends only on the severity of bad news that may arrive. Just how good is the potential future good news for the investment does not matter at all.* (Bernanke, 1983, page 91).

The rationale for the principle is that for any project the policy "*enter now*" is optimal only if it dominates "*delay entry until Good News would arrive*", where "Good News" is the future event in which the value of the project becomes higher than it is now. Since the stream of profits generated once the Good News finally arrives is collected under both policies, the only thing that does make a difference between them is what happens during the Bad News period until the Good News arrives. During that time *enter now* would generate a stream of profits while the delay policy might lower the present value of the entry cost. Therefore, *enter now* is optimal only if during the considered delay time the value of potential stream of profits exceeds the potential reduction in the present value of the entry cost. This is not merely a necessary condition for *enter now* to be optimal but also a sufficient one because if during the delay time the value of the stream of Bad News profits is more than how much the entry cost is discounted, then the value of an infinite stream of such profits exceeds the entire entry cost. In that case the value of the project exceeds the entry cost since this value is based on an infinite stream of profits that is not restricted to the Bad News range.

Since the late 1980s, an extensive literature has returned to the study of the interaction between uncertainty, irreversibility and the option to delay investment, this time with a focus on the impact that uncertainty exerts on investment.[1] Despite its relevance to this newer literature, the Bad News Principle has been mentioned in it only cursorily: Most articles have not referred to it at all; Some articles have discussed the principle non-technically, but ignored it in their formal analysis;[2] The handful of articles that have explicitly acknowledged the principle in them are not representative of this newer literature since they assume too simple stochastic processes, with only two periods or two states of nature.[3] Filling this void, I analyze here in detail the role that the Bad News Principle plays in the typical cases of the literature using a discrete-time version of Dixit's 1989 influential continuous-time investment model.[4,5] The use of the Bad News Principle yields several results:

First, it deepens our understanding of the problem of investment under uncertainty. So far, the economic rationale for the mathematically derived main results of the relevant literature has not been fully understood. In the most recent effort to attend to this lack, Dixit, Pindyck and Sodal (1999) have presented a new approach that highlights the similarity of the investment decision to the pricing decision of a firm facing a downward slopping demand curve. They find that the economic meaning of the optimal decision rule is that it represents "a trade-off between a larger versus a later net benefit". However, this result is restricted to the case where the investment yields a one-time reward. In the more common case, where

[1]For surveys of this literature see Pindyck (1991), Dixit (1992) and Dixit and Pindyck (1994).

[2]See for example the introduction of Bar-Ilan and Strange (1996) and Dixit (1992, page 3445).

[3]Dixit and Pindyck (1994) and Abel, Dixit, Eberly and Pindyck (1995) show the Bad News Principle via a two-period model. Drazen and Sakellaris (1994) use a two-state process for that purpose.

[4]See Cox, Ross and Rubinstein (1976), Dixit (1991a) and Dixit (1993) for the presentation of the continuous-time Itô processes as limiting cases of discrete-time processes of the type used in this paper.

[5] See Leahy (1993), Bar-Ilan and Strange (1996), Kngsted (1996) and Sabarwal (2004) as examples for the numerous articles that contain models based on the main features of Dixit's 1989 model.

the investment yields a flow of profits, the application of the Bad News Principle reveals that this higher but later net benefit is Good News that is irrelevant to the firm's dilemma and that the firm's actual trade-off is between a larger benefit and later cost. By delaying investment, the firm gives up the flow of operative profit that could be gained during that time, but lowers the present value of the irreversible investment cost. I also show that this dilemma is in fact what the Smooth Pasting condition represents in the typical continuous-time models of this literature. Although rigorously derived, so far there have been no attempts to assign an economic meaning to this condition.[6]

Second, using the Bad News Principle I show that investment thresholds may remain intact even when the original profit process is changed, as long as the changes are restricted to the good news range. This result may be particularly relevant to the study of the effect of changes in the government regulations on firms that make sufficiently large returns or, in the individual level, the effect of changes in the upper tax brackets. Dixit (1991b) and Dixit and Pindyck (1994, pages 296-303) have identified a particular case of this result, namely the case of an exogenously enforced price ceiling. Another possibility for changes that are restricted to the Good News range is the endogenous truncation of the price process caused by free entry under competition. Studying such a case, Leahy (1993) has shown that the optimal entry triggers under competition are identical to those of the case where the price process is not truncated by competitive entry. As the analysis preformed here shows, these results by Dixit (1991b), Dixit and Pindyck (1994) and Leahy (1993) are closer than

[6]For rigorous derivations of the smooth pasting condition see, among others, McKean's 1965 appendix to Samuelson (1965), Merton (1973) and Dixit and Pindyck (1994, pp. 130-132). More recently, Sodal (1998) has simplified the derivation of this condition. Yet, he too has abstracted from its economic meaning: the non-technical explanations he provides are restricted to the necessary condition for optimum of differentiating the value function with respect to the investment thresholds.

previously noticed, to the Bad News Principle.

In section 2 I develop the model. In section 3 I use this model to show the applications described above. Some technical proofs are left to the appendix.

## 2. The Model

Consider a risk-neutral firm with an infinite planning horizon that has an option to enter a certain production project. The entry cost to the project is *I*. Once the firm enters the project it cannot exit it. Production yields an operating profit, denoted by $P_i$, in each period. The index *i* stands for the size of the profit and it is not a time index.

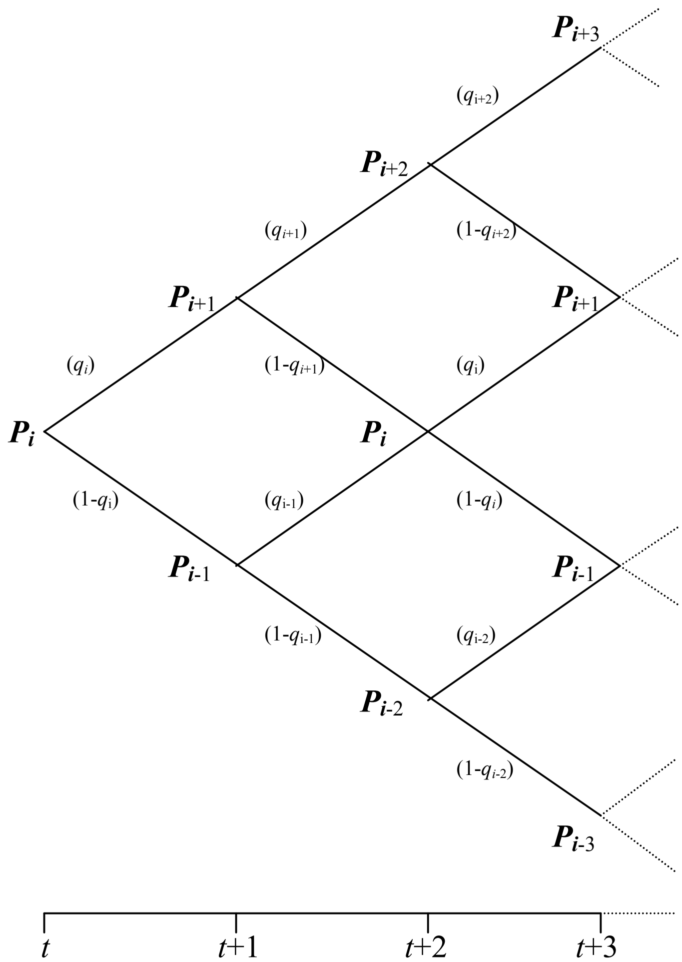


**Figure 1:** The profit process, starting at period *t* with the profit $P_i$.

The time index is omitted since I assume that the profit process is time-stationary. If at a certain period the profit is $P_i$, then at the next period the profit is $P_{i+1}$ with probability $q_i$, or $P_{i-1}$ with probability $1-q_i$. *Figure 1* is a scheme of this generalized random walk.

The value of the project is the expected sum of the discounted profits. The project's value when the profit is $P_i$ will be denoted as $V(P_i)$. The following Proposition 1 establishes that $V(P_i)$ is an increasing function of $i$.

*Proposition 1:* $V(P_{i+1}) > V(P_i) \ \forall i$.

*Proof*: In the appendix. □

**2.1 The optimal policy**

I now turn to characterizing the optimal policy of the firm. Let the current profit be $P_m$. Assume that the firm has not entered the project yet and it considers delaying entry until the profit is $P_n$. At the entry time the expected return from this policy is $V(P_n)-I$, but in the current period the firm does not know the date in which it will enter, and therefore - what is the value of the discount factor that multiplies this expected return. The firm also must consider what is the return that can be collected until the profit reaches $P_n$. These variables are defined and denoted below:

$E_{m,n}$ - The present value of the stream of profits that the project yields while the profit evolves from its current level, $P_m$, until it reaches $P_n$ for the first time (excluding the profit $P_n$ received when the process finally hits $P_n$).[7]

$B_{m,n}$ - The value of the discount factor, $(1+r)^{-T}$, where $T$ is the number of periods until the profit is $P_n$ for the first time, starting at $P_m$ (excluding the period in which the process finally hits $P_n$).

These definitions enable the following dynamic programming presentation of the project's value function for each $m$ and $n$:

$$(1) \qquad V(P_m) \equiv E_{m,n} + B_{m,n}V(P_n).$$

Suppose that the current profit is $P_i$. The standard N.P.V. rule tells us that the policy "enter now" is better than the policy "never enter" if $V(P_i)$-$I$>0. However, other policies are possible too. One example is the policy "delay entry until the profit is $P_{i+1}$". The expected return from this policy is denoted by $F(P_i)$ and satisfies:

$$(2) \qquad F(P_i) = B_{i,i+1}[V(P_{i+1}) - I].$$

"Enter now" is better than this last policy only if:

$$(3) \qquad V(P_i) - I > F(P_i).$$

$F(P_i)$, therefore is an alternative cost that can be added to the direct cost of entry, $I$. It follows from (2) and Proposition 1 that if indeed $V(P_i)$-$I$>0, then this

[7]There is a positive probability that the price process will never reach $P_n$.

alternative cost is positive too. Applying (2) in (3), immediate entry is preferred if:

(4) $$V(P_i) - I > B_{i,i+1}[V(P_{i+1}) - I].$$

Looking at an equation similar to (4) has led Dixit, Pindyck and Sodal (1999) have concluded that economic meaning of the optimal decision rule is that it represents "a trade-off between a larger versus a later net benefit". However, their result is restricted to the case where the investment yields a one-time reward. Here, modifying this investment rule through the Bad News Principle, using (1), we can rewrite it as:

(4') $$E_{i,i+1} + B_{i,i+1}V(P_{i+1}) - I > B_{i,i+1}[V(P_{i+1}) - I].$$

$V(P_{i+1})$ appears on both sides of (4'), multiplied by the same term and therefore cancels out. Rearranging this expression we get the following condition for "enter now" to be better than "delay entry until the profit is $P_{i+1}$":

(5) $$E_{i,i+1} > (1-B_{i,i+1})I.$$

Thus, when the profit is $P_i$, the magnitudes of the profits that are higher than $P_i$, and of the transition probabilities related to them are irrelevant to the question whether "enter now" is better than "delay entry until the profit is $P_{i+1}$". The only relevant factors are the cost of the delay – the forgone profits during the delay time, on the left side of (5), and the benefit form the delay – the expected decrease in the present value of the entry cost. Both are related only to the parameters of the profit process below $P_{i+1}$. Note that the case where $P_{i+1}$ is not reached at all and the delay saves the entire entry cost is a just one component of the reduction, due to the delay, in the present value of the entry cost.

So far I have only discussed the choice between "enter now" and "delay entry until the profit is $P_{i+1}$", given that the value of the current profit is $P_i$. Next, I show that it is unnecessary to discuss policies that suggest a longer delay of entry. To do so I rewrite condition (5) as:

(5')   $M_i > I,$

where the function $M_i$ is defined by:

(6)   $$M_i \equiv \frac{E_{i,i+1}}{1 - B_{i,i+1}}.$$

To gather an economic meaning for $M_i$ note from (6) that:

(7)   $$M_i = E_{i,i+1} + B_{i,i+1}E_{i,i+1} + B_{i,i+1}{}^2 E_{i,i+1} + B_{i,i+1}{}^3 E_{i,i+1} + ... = E_{i,i+1} \sum_{j=0}^{\infty} B_{i,i+1}{}^j .$$

(7) leads to interpreting $M_i$ as the value of the project in the case where there are no better news than the current profit, $P_i$. More specifically, assume that the original profit process is replaced by a new one in which: (i) for all $j \le i$ all the values of the pairs $\{P_j, q_j\}$ are the same as in the original process; (ii) when the profit is $P_i$ in a certain period then in the next period with probability $q_i$ the value of the profit remains $P_i$, instead of rising to $P_{i+1}$.

Under this process truncated at $P_i$, it is impossible to continue focusing the analysis on the event in which the profit process hits $P_{i+1}$ for the first time, starting at $P_i$. For this truncated process, the equivalent event is that the value of the profit is $P_i$ for a second consecutive period for the first time, starting at $P_i$. Note that with probability $q_i$ this event occurs in the period next to the current one. The present value of the stream of profits collected until this event occurs is exactly $E_{i,i+1}$ as was defined in the previous section. Denoting the number of periods until this event occurs by $s$, the expected value of $(1+r)^{-s}$ is exactly $B_{i,i+1}$ as was defined too in the previous section. Thus, as (7) shows, $M_i$ is the value of the project if the current value of the profit is $P_i$ and the profit process is truncated at this value in the manner described above. This interpretation of $M_i$ implies that it must be smaller than $V(P_i)$. Proposition 2 establishes this result:

<u>Proposition 2:</u> $V(P_i) > M_i$ for all $i$.

<u>Proof:</u> Evaluating (1) at $P_i$ and $P_{i+1}$ and then replacing $V(P_{i+1})$ with $V(P_i)$ yields:

$$(8) \qquad V(P_i) = E_{i,i+1} + B_{i,i+1}V(P_{i+1}) > E_{i,i+1} + B_{i,i+1}V(P_i),$$

where the inequality follows from Proposition 1. Simplifying (8) yields:

$$(9) \qquad V(P_i) > \frac{E_{i.i+1}}{\left(1-B_{i,i+1}\right)} \equiv M_i. \qquad \square$$

Interpreting $M_i$ as the value of the project if the current value of the profit is $P_i$ and the profit process is truncated at this value also implies that $M_i$ is increasing in $i$. The reason for that is that $M_{i+1}$ is the present value of a stream of profits that starts at a higher profit level and also is based on a process truncated at a higher profit level, compared to $M_i$. Propositions 3 establishes this result.

*Proposition 3:* The function $M_i$ is increasing in $i$.

*Proof:* See in the appendix. □

Using Proposition 3 I now show in Proposition 4 that if "enter now" dominates "delay entry until the profit is $P_{i+1}$", given that the value of the current profit is $P_i$, then "enter now" is also the optimal policy.

*Proposition 4:* If the profit is $P_i$ and the policy "enter now" is better than the policy "delay entry until the profit is $P_{i+1}$", then "enter now" is better than any other policy of the form "delay entry until the profit is $P_n$".

*Proof:* Assume that when the profit is $P_i$ "enter now" is better than "delay entry until the profit is $P_{i+1}$", implying that $M_i>I$. I start in the case where $n>i+1$. It follows from Proposition 3 that $M_k>I$ for every $i+1<k$. Note that, starting at $P_i$, the profit process can reach $P_n$ only after it reaches all the values $P_k$ where $i<k<n$. Thus, delaying entry until the profit is $P_n$, cannot be an optimal policy because $M_{n-1}>I$, implying that it is better to delay the entry only until the profit hits $P_{n-1}$. For the same reasons, delaying entry until the value of the profit is $P_{n-2}$ dominates delaying it until the profit would reach $P_{n-1}$ and therefore also dominates delaying entry until the profit would reach $P_n$. Repeating this argument for all $i<k<n$ shows that for each $k$ satisfying $i\leq k<n$ entry when the value of the profit is $P_k$ dominates delaying the entry until the value of the profit is $P_n$.

I now turn to the case where $n<i$. Since $M_i$ is assumed to exceed $I$ then, by proposition 2, $V(P_i)–I>0$. This implies that:

$$(10) \qquad V(P_i) - I > B_{i,n}[V(P_n) - I],$$

which follows from $0<B_{i,n}<1$ and from $V(P_n)<V(P_i)$ which follows from Proposition 1. (10) shows that when the value of the profit is $P_i$ the value of immediate entry, in the left-hand side of (10), exceeds the value of delaying entry until the profit reaches $P_n$ for the first time, in the right-hand side of (10). □

From Proposition 2 it also follows that if (5) is satisfied for a certain $P_i$, then, for this $P_i$, "immediate entry" is better not only from a policy of a delayed entry but also from the policy of "never enter". This happens since giving up on the investment altogether is actually a specific case of delaying entry. Thus, (5) is not only a necessary condition for investment but also a sufficient one. The following definition concludes the analysis of the optimal policy:

*Definition*: The Entry Threshold, denoted by $P_H$, is the smallest value of $P_i$ that satisfies (5).

By this definition and the analysis so far the firm will not enter if the profit is below this value, and will enter the project immediately if the profit is higher than it. It follows from Proposition 4 that if an entry threshold exists, then it is unique.

It also follows from this definition that all the values of $P_i$ and $q_i$ in the range $i>H$ do not affect $P_H$. This is the form the Bad News Principle takes in this model.

### 2.2 The Bad News Principle and the Smooth Pasting Condition

In the standard continuous-time version of this model $P_H$ is found in the following way: First, functions that correspond to $V(P_i)$ and $F(P_i)$ of this model are defined. Then, the value of $P_H$ is found using the two following conditions:

$$(11) \qquad V(P_H) - I = F(P_H),$$

$$(12) \qquad V'(P_H) = F'(P_H).$$

(11) is known as the "Value Matching Condition" and (12) as the "Smooth Pasting Condition." Dumas (1991) has shown that the value matching condition is not an optimality condition. This condition follows directly from the definition of $F(P)$ and holds for every value of $P_H$, not necessarily the optimal one. In contrast, the economic meaning of the smooth pasting condition has remained not well understood so far, although the condition has been rigorously derived. Enhancing the fog around the role of this condition, Brekke and Oksendal (1991) have shown that although it is based on a differentiation of the value function, in most of the relevant cases it is not merely a necessary condition for optimum but also a sufficient one. Here I show that condition (5) for the optimality of immediate entry is actually the discrete generalization of the smooth pasting condition. To see that, note that adding the term $(1\text{-}B_{i,i+1})V(P_{i+1})$ to both sides of (5) and rearranging the resulting inequality yields:

$$(13) \qquad (1\text{-}B_{i,i+1})[V(P_{i+1}) - I] > (1\text{-}B_{i,i+1})V(P_{i+1}) - E_{i,i+1}.$$

Rearranging (13), it becomes:

(13') $$[V(P_{i+1}) - I] - B_{i,i+1}[V(P_{i+1}) - I] > V(P_{i+1}) - [E_{i,i+1} + B_{i,i+1}V(P_{i+1})].$$

Applying (1) and (2) in (13'), and noting, from the definition of $F(\cdot)$, that $F(P_{i+1}) = V(P_{i+1}) - I$ yields that it is optimal to enter at $P_i$ if and only if:

(14) $$F(P_{i+1}) - F(P_i) > V(P_{i+1}) - V(P_i).$$

Thus, the derivation of $P_H$ using (5) is a discrete generalization of the smooth pasting condition. Note that, as was shown by *Proposition 4* and the paragraph preceding it, satisfying (5) is both necessary and sufficient for optimality of immediate entry, just as Brekke and Oksendal (1991) have shown for the smooth pasting condition.

## 3. Applications

By definition of $P_H$, all the values of $P_i$ and $q_i$ for which $i>H$ do not affect $P_H$. In this section I show how this implies that the value of $P_H$ may not be sensitive to the changes in the profit process induced by incorporating competition or government regulation in the model.

### 3.1 Government regulation

A variety of government regulations are imposed on firms that are making sufficiently high returns. To incorporate the infliction of such regulation in the model assume that it changes the profit process in the following way: There exists a certain $k$ such that for each $j<k$ the value of $P_j$ is the same as in the original process, and for each $j\geq k$ the

value of $P_j$ is lower than in the original process. Also assume that even after the change $P_j>P_{j-1}$ for each $j\geq k$ and that for each $j$ the probability $q_j$ is the same as in the original process. Let $P_H$ denote the entry threshold corresponding to the original process. If $k$ is larger than $H$ then $P_H$ is also the entry threshold that corresponds to the case in which the regulation alters the profit process because it is still the smallest value of $P_i$ that satisfies (5). Note that unless keeping the Bad News Principle in mind, the result that the regulation does not alter the investment threshold may seem somewhat surprising because the value of the investment is indeed lowered by the regulation. Dixit (1991b) and Dixit and Pindyck (1994) have identified one particular case of this general result, namely the case in which a ceiling is enforced on the price of the output generated by the project, i.e., the case in which $P_j=P_k$ for each $j\geq k$.

### 3.2 Competitive Equilibrium

Another possibility for a change in the firm's profit process that occurs only in the range of sufficiently large profits is the endogenous truncation of this process by other firms' entry under competition. In this sub-section I incorporate competition to the model in a manner similar to Leahy (1993). In Leahy's model (1993) competition is characterized by the existence of an infinite number of identical firms with the option to enter the market. For each firm entry involves paying a fixed entry cost. Following Dixit (1989) each firm's instantaneous output is on a fixed scale normalized to one unit. The market price of this output is a stochastic process.

When the price is high enough to trigger one firm's entry, all other firms find entry optimal too since all the firms are identical. Leahy assumes that repeated entry by the firms prevents the price process from going beyond $P_H$. Other assumptions in his model are that the firms have perfect knowledge of the price level, and that they are sufficiently small so that their repeated entry cannot make the price process decline to less than $P_H$. Thus, due to these assumptions, the competition among identical firms truncates the price process at the entry trigger. *Figure 2* describes the effect of the firms' entry on the price process. Owing to this truncation, returns are lower in competition compared to the case of a single firm facing a process with no truncation. The entry threshold, however, is the same in both cases, as Leahy proves.

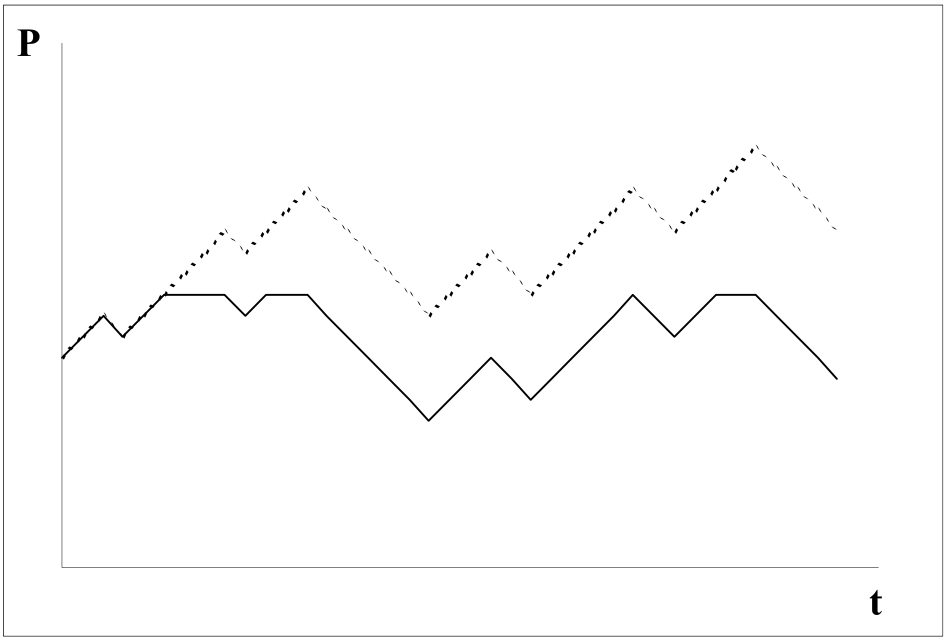


**Figure 2** (adopted from Leahy, 1993)**:** The price process in the case of competition. The dashed line shows this process had there not been a truncation due to the competition.

The model in the current paper can be interpreted as a discrete version of Leahy's model. A version that is more general in every aspect except for the fact that here, for simplicity sake, there is no exit possibility. Beside the additional generality, the contribution of the analysis here is the simpler proof of the result that the entry trigger is not affected by the truncation of the profit process. To stay close to Leahy's model I shall regard $P_i$ as the output price, instead of the profit, in the following section. The properties of the optimal investment policy, as derived in the previous sections and particularly by propositions 1, 2, 3 and 4 and the definition of the entry threshold, are robust to this transformation. The robustness occurs since in these models the output price is the only stochastic variable relevant to the firm and the profit in each period is an increasing function of this price.

Defining the firms' entry trigger in the case of competition by $P_c$, I now show, following Leahy, that $P_c$=$P_H$. To account for the truncation that the firms' entry cause at $P_c$ I assume that if the current price is $P_c$ then, the price in the next period, instead of rising to $P_{c+1}$, stays $P_c$ with probability $q_c$, and becomes $P_{c-1}$ with probability 1-$q_c$.

To see that $P_c$ cannot be higher than $P_H$ note that this cannot be a Nash equilibrium. Since in that case (5) still holds at $P_H$, it will always be optimal for a single firm to diverge from this equilibrium and enter when the price is $P_H$.

To prove that $P_c$ cannot be lower than $P_H$ note that the value of entry when the value of the profit is $P_c$ and the profit process is truncated at this level in the manner described above is given by $M_c$ which is smaller than $I$ as follows from $c<H$, the definition of $P_H$ and Proposition 3.

$P_H$, therefore, is the optimal entry threshold for a competitive firm too. In competition profits should not exceed the normal return, and this is the case here too. It follows from proposition 3 and its proof that, if (5) is an equality, $V(P_H) = I$ and profits for a competitive firm in this model are zero. If (5) is a strict inequality, then $P_H$ is the entry threshold that yields the smallest non-negative profits.

Extending the analysis to a case where the parameters of the price process depend on the market size, as was also done by Leahy, is possible in the current framework as well. In this case, in addition to truncating the price process, entry by the firms also changes the market size and therefore changes the price process.

Modifying the model to account for this property requires adding an index that shows the market size and increases whenever the entry trigger is hit. It is also necessary to follow Leahy in assuming that each firm is sufficiently small such that its own entry has only an infinitely small effect on the price process' parameters. Because of this assumption the market size index has the same value in all the relevant equations of the current model, rendering its results robust to this modification of the model.

## 4. Concluding Remarks

In this paper I have demonstrated Bernanke's Bad News Principle in a framework in which the profit flow is a process characterized by infinite-time, infinite state space and gradual changes, typical assumptions in the literature on investment under uncertainty. The model was then used to show that investment triggers might stay unaffected even when the original profit process undergoes a truncation. Leahy (1993) and Dixit (1991b) have already identified two cases related to this rather general result and the analysis preformed here shows their results are closer than previously noticed to the Bad News Principle.

In the current paper it was assumed that once the firm has entered the project, it cannot exit it. Incorporating exit to the analysis will lead to a "Good News Principle" with regard to the exit decision: at any level of the profit process, the decision to exit at that level does not depend on any of the parameters of the profit process below that level. It only depends on the parameters related to higher profit levels. In such a case there are going to be two thresholds level, the higher one for entry and the lower one for exit, and their values will depend only on the parameters of the profit process related to the profit levels between these two thresholds. Abel, Dixit, Eberly and Pindyck (1995) analyze such a case in a two-period example.

## Appendix

The following definitions and relations are necessary to the proofs of the lemmas and propositions in this appendix:

$f_{m,n,j}$ - the probability that it takes $j$ periods until the profit process reaches the level $P_n$ for the first time, starting at the level $P_m$ (excluding the period in which the process finally hits $P_n$).

$E_{m,n,j}$ - The present value of the sum of the profits that the project yields while the profit evolves from its current level, $P_m$, until it reaches $P_n$ for the first time (excluding the profit $P_n$ received when the process finally hits $P_n$) given that it takes $j$ periods (excluding the period in which the process finally hits $P_n$)

From these definitions it follows that:

(a3) $$E_{i,i+1,1} = E_{i,i-1,1} = P_i.$$

(a4) $$E_{m,n} = \sum_{j=1}^{\infty} f_{m,n,j} E_{m,n,j} .$$

(a5) $$B_{m,n} = \sum_{j=1}^{\infty} f_{m,n,j} \frac{1}{(1+r)^j} .$$

*Lemma 1* below is required for the proof of Proposition 1 and Proposition 3:

*Lemma 1:* (i) $E_{i,i+1} \le P_i \dfrac{1+r}{r}(1 - B_{i,i+1})$, (ii) $E_{i+1,i} \ge P_{i+1} \dfrac{1+r}{r}(1 - B_{i,i+1})$.

*Proof:* It follows from the definition of $E_{i,i+1,j}$ that:

(a6) $$E_{i,i+1,j} \le P_i + \frac{P_i}{1+r} + \frac{P_i}{(1+r)^2} + \dots + \frac{P_i}{(1+r)^{j-1}} = P_i \frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^j\right].$$

Using this in (a4) yields:

(a7) $$E_{i,i+1} = \sum_{j=1}^{\infty} f_{i,i+1,j} E_{i,i+1,j} \le P_i \frac{1+r}{r} \sum_{j=1}^{\infty} f_{i,i+1,j}\left[1 - \frac{1}{(1+r)^j}\right] = P_i \frac{1+r}{r}(1\text{-}B_{i,i+1}),$$

where the last equality springs from (a5). This proves (i). The proof of (ii) is similar.□

*Proposition 1:* $V(P_{i+1}) > V(P_i) \;\; \forall i$.

*Proof:* From the two parts of *Lemma 1*, and since $P_{i+1} > P_i$, it follows that:

(a8) $$\frac{E_{i,i+1}}{1 - B_{i,i+1}} \le \frac{1+r}{r} P_i \le \frac{1+r}{r} P_{i+1} \le \frac{E_{i+1,i}}{1 - B_{i+1,i}}.$$

Applying $E_{i,i+1} = V(P_i) - B_{i,i+1} V(P_{i+1})$ and $E_{i+1,i} = V(P_{i+1}) - B_{i+1,i} V(P_i)$ in (a8) yields:

(a9) $$\frac{V(P_i) - B_{i,i+1} V(P_{i+1})}{1 - B_{i,i+1}} \le \frac{V(P_{i+1}) - B_{i+1,i} V(P_i)}{1 - B_{i+1,i}},$$

which simplifies to: $V(P_{i+1}) \ge V(P_\mathrm{i})$. □

*Lemma 2* below is required to prove Proposition 3:

<u>*Lemma 2:*</u> (i) $$E_{i+1,i+2} = \frac{(1+r)P_{i+1} + (1-q_{i+1})E_{i,i+1}}{1+r-(1-q_{i+1})B_{i,i+1}},$$

(ii) $$B_{i+1,i+2} = \frac{q_{i+1}}{1+r-(1-q_{i+1})B_{i,i+1}}.$$

<u>*Proof:*</u> In order to prove (i) note that there are two possibilities by which the profit can move from $P_{i+1}$ to $P_{i+2}$. First, with probability $q_{i+1}$, the value of $E_{i+1,i+2}$ is $P_{i+1}$ because the profit moves directly from $P_{i+1}$ to $P_{i+2}$ at the next period. In the second case, with probability $(1-q_{i+1})$ the profit moves from $P_{i+1}$ to $P_i$. Then, from state $P_i$ the profit must return to $P_{i+1}$ before it can finally reach $P_{i+2}$. The expected value of the stream of the profits received until the price process moves back to $P_{i+1}$ is $E_{i,i+1}$. After the price is again $P_{i+1}$ the present value of profits that will be collected until the profit is finally $P_{i+2}$. This value is, once again, $E_{i+1,i+2}$. This analysis takes the following form:

(a10) $$E_{i+1,i+2} = P_{i+1} + \frac{1-q_{i+1}}{1+r}\left[E_{i,i+1} + B_{i,i+1}E_{i+1,i+2}\right].$$

Simplifying this equation yields (i). Similar considerations lead to:

(a11) $$B_{i+1,i+2} = q_{i+1}\frac{1}{1+r} + \frac{1-q_{i+1}}{1+r}B_{i,i+1}B_{i+1,i+2},$$

which, when simplified, yields (ii). □

*Proposition 3:* The function $M_i$ is increasing in $i$.

*Proof:* It follows from both parts of *Lemma 2* that for each $i$:

(a12)
$$M_{i+1} \equiv \frac{E_{i+1,i+2}}{1-B_{i+1,i+2}} = \frac{\dfrac{(1+r)P_{i+1}+(1-q_{i+1})E_{i,i+1}}{(1+r)-(1-q_{i+1})B_{i,i+1}}}{1-\dfrac{q_{i+1}}{(1+r)-(1-q_{i+1})B_{i,i+1}}}$$

$$\geq \frac{(1+r)P_i+(1-q_{i+1})E_{i,i+1}}{r+(1-q_{i+1})(1-B_{i,i+1})} \geq \frac{\dfrac{rE_{i,i+1}}{1-B_{i,i+1}}+(1-q_{i+1})E_{i,i+1}}{r+(1-q_{i+1})(1-B_{i,i+1})} = \frac{E_{i,i+1}}{1-B_{i,i+1}} \equiv M_i.$$

The second equality follows from *lemma* 2, the first inequality follows from $P_{i+1} \geq P_i$ and the second inequality follows from part 1 of *Lemma 1*. □

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