SELF-SELECTION UNDER NON-ORDERED VALUATIONS: TYPE-SPLITTING, ENVY-CYCLES, RATIONING AND EFFICIENCY

By **Babu Nahata**,
Department of Economics, University of Louisville
Louisville, Kentucky 40292, USA. nahata@louisville.edu,
Serguei Kokovin and Evgeny Zhelobodko
Department of Economics, Novosibirsk State University
Novosibirsk 630090, Russia. kokovin@math.nsc.ru, ezhel@ieie.nsc.ru

Abstract

We analyze self-selection problem when valuations are non-ordered. The corresponding package-pricing solution has specific graph structure. It is helpful in deriving weak sufficient conditions for both partial efficiency and Pareto-efficiency. Unlike the ordered valuations case, Pareto efficiency is shown to be a non-pathological case. Pareto efficiency and positive consumer surplus are mutually exclusive. Under costs separability optimal package-pricing scheme is shown implementable by small rewards. Counter-examples show that our assumptions are essential. In certain non-ordered situations, package-optimization setting with rations is more appropriate than the standard setting.

Key Words: Principal-agent, self-selection, nonlinear pricing, package pricing, Pareto efficiency, implementation, graph structure, envy cycles

JEL Codes: D42, L11

SELF-SELECTION UNDER NON-ORDERED VALUATIONS: TYPE-SPLITTING, ENVY-CYCLES, RATIONING AND EFFICIENCY *

By Babu Nahata, Serguei Kokovin and Evgeny Zhelobodko †

Abstract

We analyze self-selection problem when valuations are non-ordered. The corresponding package-pricing solution has specific graph structure. It is helpful in deriving weak sufficient conditions for both partial efficiency and Pareto-efficiency. Unlike the ordered valuations case, Pareto efficiency is shown to be a non-pathological case. Pareto efficiency and positive consumer surplus are mutually exclusive. Under costs separability optimal package-pricing scheme is shown implementable by small rewards. Counter-examples show that our assumptions are essential. In certain non-ordered situations, package-optimization setting with rations is more appropriate than the standard setting.

Key Words: Principal-agent, self-selection, nonlinear pricing, package pricing, Pareto efficiency, implementation, graph structure, envy cycles

JEL Codes: D42, L11

1 INTRODUCTION

Consider a general principal-agent problem with several hidden types of agents, also known as 'self-selection problem.' This problem arises in designing Pareto-optimum taxation schemes, in optimal compensation decisions, and in many optimal contracts design. For the sake of simplicity, we formulate and analyze this general problem in a

*Kokovin and Zhelobodko gratefully acknowledge the financial assistance from the Russian Foundation for Basic Research (grants 00-15-98884 and 01-01-00896), the Russian Foundation for Humanitarian Research (grant 99-02-00141) and by the Integracia Program for Russian Universities (grant -274). Nahata gratefully acknowledges financial support from the University of Louisville (IRIG-RIG 277858 and PCG 277858). Part of the paper (Theorem 1) was presented at the 28th meeting of the European Association for Research in Industrial Economics at Trinity College, Dublin and at the 16th meeting of the European Economic Association at the University of Lausanne, Switzerland.

†We are grateful to Denis Anoshin for cooperation and to Victor Polterovich, Sergei Guriev and Constantin Sonin for valuable comments.

specific setting but perhaps in its most common application, namely in the context of a *nonlinear pricing* problem or the so called second-degree price discrimination.

We analyze the case when a monopolistic seller (a principal) of a *single* good uses a set of several *packages* as the self-selection devices to handle finitely-many hidden types of consumers (agents). A *package* is a quantity-outlay couple and *package-plan* is a menu of several packages, suggested to consumers. Although a consumer can select any package of her choice, she can take only one and cannot negotiate ('take-it-or-leave-it'). Given the probabilities or quantities of all consumer types, the *optimal* package-plan is one that maximizes total profit.[1]

Most known results related to this standard packages problem are derived assuming *ordered valuations* of agents (valuation function, or 'valuation' is the integral of inverse-demand function) implying that the demand functions do not cross ('no-crossing condition', or *Spence-Mirrlees condition)*. Under this assumption, together with concavity and differentiability of valuation functions, well-known theorem establishes a 'chain-rule' solution structure, enabling a simple solution and derivation of its properties (see Katz, 1983, Salanie, 1997). Chain-rule means that at the optimal solution each consumer 'almost-envies' *only* her closest lower-demand neighbor and no one else, i.e., among the incentive-compatibility constraints, the only one related to her lower-demand-neighbor's package are active and binding. The standard result is that the lowest-demand agent has zero consumer surplus, while other agents have some positive surplus which is informational rent. Only the highest-demand consumer's package is partially-efficient (i.e., sum of this agent's and principal's objective functions cannot be increased by modifying this package, *ceteris paribus*). Similar results have been obtained by Guesnerie and Laffont (1984) when the distribution of consumer types is continuous (see also Varian (1989), Salanie (1997)).

[1]Essentially, package pricing is just a non-linear outlay function which can either be step-wise or continuous. Our interest in package pricing is because use of packages is quite prevalent for numerous types of goods. From a theoretical standpoint, under usual assumptions, including no arbitrage, package pricing is the best non-linear pricing scheme in the sense that it is *at least as profitable* as any other outlay function (see Katz, 1984). For many real life examples and treatment of more general nonlinear pricing, see Wilson (1993).

In contrast, the problem of non-ordered valuations has drawn much less attention because the analytic framework is more complex and tedious, even though it is as much realistic as the ordered ones.[2] The only paper familiar to us that does not require valuations to be ordered is by Guesnerie and Seade (1982). The paper considers the standard finite-packages problem using general, non-quasi linear, strictly concave, differentiable utility functions, having 'locally different' derivatives. Costs are linear. Without ordering, the solution structure becomes rather complex because the chain-rule is no more applicable. Nonetheless, it was found that the graph of active constraints must have at least one envy-free node, i.e., this package chosen by some consumers is *strictly* not preferred by others. Consequently, this agent's consumption is partially-efficient.

Under similar framework, we assume quasi-linear utility functions, but impose no other restrictions on preferences or costs, at least for the structure of solutions. Valuation functions may be non-ordered, non-increasing (satiable), non-continuous, and non-concave, thus encompassing the case when quantities are *discrete* (e.g., cars, houses etc.). The solution is shown to have at least a specific tree-structure. In a special case of ordered valuations it has the usual chain-structure. Thus the solution structure is shown not to depend upon concavity or continuity (our advanced chain-rule claim generalizes the usual claims). Tree-structure enables us to derive some solution properties. Under additional assumptions like cost separability (e.g., linearity) or valuation concavity, partially-efficient package exists. Counter-examples show that these assumptions are essential. Thus we show that partial efficiency is dependent upon concavity/separability, but not on 'local differences' among consumers. Generally, in ordered case there is consumer surplus and there is no Pareto-efficiency, with unnatural exceptions. In contrast, *Pareto-efficiency* of solutions is shown to be rather *normal* (non-degenerate) outcome under non-ordered

[2] As a practical situation, suppose consumer **A** being very thirsty is willing to pay \$1 for the first half-glass of soft drink, but feels satiated with only 0.5 liter, while another consumer **B** would pay only \$0.5 for the first half-glass of soft drink but is willing to buy a liter. This example shows non-orderedness of demands (valuations). Many pricing practices used to sell soft drinks through packages can be explained only with non-ordered valuations. Although it may appear irrational, but $0.5l$ and $1.0l$ bottles sometimes are sold for the *same* price! Such practice not only can be a seller's optimal strategy, but all such outcomes are Pareto-efficient (see Nahata, Kokovin, Zhelobodko, 2002).

valuations. It typically occurs when solution's graph is star-like (though there are counter-examples), that excludes consumer surplus. In this case interests of consumers conflict with social efficiency.

It is worth mentioning that for all propositions we seek sufficient conditions which are close to necessary ones. Further, our results are general enough to be applicable to all situations of self-selection, including both deterministic and probabilistic settings.

The usually employed standard analysis in finding optimal solutions for ordered case faces conceptual challenges under non-ordered valuations because of *type-splitting* and *envy-cycles* creating an *implementation puzzle* (which so far remains unexplored).[3] A package-plan is called non-implementable in some self-selection game, when the principal cannot enforce the game plan to earn the planned profit. In essence, the 'implementation puzzle' questions the conventional assumption of a 'friendly consumer,' who among her most-preferred packages chooses the package most profitable for the seller. In the ordered case this assumption can be supported by the idea that small rewards can be given to consumers for such a behavior. But the rewards do not help in some non-ordered situations because 'cycles of envy' can occur. Besides, the standard optimization assumes designing as many packages or less as the number of consumer types. As our examples show, this can result in non-optimal solution in non-ordered case where type-splitting occurs. As a result, the standard package-optimization setting can become inadequate in handling the game in question, either because of non-implementability reason or due to type-splitting.

What is then the appropriate setting? How should a principal design a package scheme? In Section 2, we propose 'allocated-packages' setting to handle type-splitting. It implies '*rationing.*' It means that the seller limits the supply of certain packages. Such practice is frequently observed in real life and our arguments justify its use. Rationing also helps in implementing standard package-plans, at least in regular situations. But, in some degenerate cases even rationing also fails (Example 1). We suggest a specific cure, a 'preference- revealing menu' for such cases. Interestingly these non-standard tools (rations and menu) not only can be used to implement standard package-plan, but they may also

[3]Explanation of these terms and problems are given in Section 2, after formal setting.

bring higher profit (especially when combined with allocations) because rationing may *replace* some incentive-compatibility constraints in the optimization problem. In general, however the answer to the question of most appropriate setting remains unclear.

For markets where rations cannot be used, we identify conditions for implementation of standard package-plan through rewards. In particular, 'essential' envy-cycles, 'essential' splitting and non-implementability are excluded when *costs are separable* (e.g., linear), or under 'strict gross convexity' of optimization problem. Therefore, the standard package solution is implementable for rather broad class of situations, though exceptions are non-degenerate (see Example 5).

Section 2 contains formulation of the problem and a brief discussion of non-implementability. Section 3 states conditions for implementability. Section 4 deals with efficiency. Appendix contains most proofs, some examples and figures.

2 SELF-SELECTION MODEL AND IMPLEMENTATION PROBLEM

Consider the standard setting. A monopolist (principal) producing a homogeneous good offers several packages (quantity-outlay bundles $(x_1, t_1), ..., (x_n, t_n) \in R^2$) to n types of consumers (agents), who are indexed as $i \in I := \{1, ..., n\}$, and $m_i > 0$ denotes population of each type. The valuations of possible types are known to the seller, but 'who is who' is hidden, or personal discrimination is impossible. Each agent selects a single package, and both multiple purchases and arbitrage are prevented. To simplify the analysis, all consumers of a given type choose the same package ('no splitting-of-types is allowed). In certain circumstances this assumption may appear non-restrictive, since several similar consumers may be treated as different types. In some other situations, we shall show, it is restrictive. As broadly recognized, under no-splitting, it is sufficient for the monopolist to design exactly n packages for n consumer types, although some packages may be identical in size or contain quantity equal to zero. Thus the actual number of packages may be fewer than n. Following another common convention, we suppose that among all equivalent options an agent chooses the package preferred by the principal (a 'friendly agent'). As shown later both these assumptions become questionable under non-ordered valuations.

The number $m_i > 0$ of consumers of type i, can either be an integer or a fraction of population. In a probabilistic interpretation of the model, each m_i can be treated as probability of type i so that $\Sigma_i m_i = 1$. Utility functions are assumed to be quasi-linear and depend on quantity and tariff (outlay): $u_i(x_i, t_i) = V_i(x_i) - t_i$, so income effects are ruled out. Each valuation function $V_i(.)$ is normalized as $V_i(0) = 0 \, \forall i$. In the deterministic setting cost function $C : R^{2n} - > R$ may depend upon the total output in the form $C(m, x) = c(\sum_i m_i x_i)$, (where $c : R^{2n} - > R$). Alternatively, it may take the form $C(m, x) = \sum_i m_i c(x_i)$ in probabilistic case, denoting *expected* production cost (thus appearing *separable*).[4] Under the above assumptions, our two-stage leader-follower game can be reduced to standard package-optimization problem as follows.

$$\pi(t, x) \;=\; \sum_{i=1}^{n} m_i t_i - C(m, x_1, ..., x_n) \rightarrow \max_{(t,x)\in(R^n,\, X^n)} \;, \quad \textbf{s.t.} \tag{1}$$

$$V_i(x_i) \;-\; t_i \geq V_i(x_k) - t_k \quad \forall i \in I, \forall k \in I \cup \{0\} \setminus \{i\}, \tag{2}$$

$$(x_0, t_0) := \quad (0, 0).$$

Here $X^n := X \times ..... \times X \subset R^n$ is an admissible consumption set, consisting of several sets $X \subset R$ assumed to be similar for all agents, for simplicity. The novelty in our formulation is that consumption set X may either be convex or non-convex, continuous or discrete, the only restriction is the normalization assumption $0 \in X$. For notational convenience, we have introduced a dummy agent #0 whose package by definition is $(0, 0)$. With dummy agent the 'participation constraints' $V_i(x_i) - t_i \geq V_i(0) - 0 = 0$ can be treated as special cases of incentive-compatibility or 'self-selection' constraints $V_i(x_i) - t_i \geq V_k(x_k) - t_k$. All such constraints imply that no consumer wishes to switch to anyone else's package, including the package #0, i.e., no i 'strictly envies' anyone. The situation when a constraint (i, k) is active (i.e., being equality) can be interpreted, following Wilson (1993), as consumer i 'almost-envying' consumer k's package. Note, that no positivity

[4]Here, in each period the seller expects *one* client of type i with probability m_i. The cost functions are more complex for situations when *several* customers can be served in a period.

or increasing-functions assumptions are made, so the model can be *directly* applied to a principal having a revenue function $(-C(m, x_1, ..., x_n)) \geq 0$ and paying rewards $(-t_i) \geq 0$ to hired agents, whose utility from working x_i hours is $V_i(x_i)$, it may decrease and become negative. Besides, decreasing valuations may express demand satiability.

The existence of solutions is a standard result in the literature and therefore we do not discuss it here.

What merits some discussion is the standard assumption (in essence incorporated right into formulation of the optimization problem 1) about agents being 'friendly' to the seller. In the traditional ordered-valuation situation this assumption is justifiable, because of the chain-structure of solutions. More generally, when a solution structure is transitive, i.e., when it lacks 'envy-cycles', then the seller can (almost-) *implement* (see ε-*implementability* definition below) optimal package scheme $(\bar{x}, \bar{t})$ by offering arbitrarily small reward (i.e., reduction in tariff ε_i) to each consumer i for choosing exactly the seller-preferred package $(\bar{x}_i, \bar{t}_i - \varepsilon_i)$ out of her set of most-preferred packages. The rewards thus break all envy-links (i.e., constraints-equalities) converting them into strict inequalities and enforcing 'friendly behavior'.

However, under non-ordered valuations this tool may appear insufficient and the standard optimization setting (1) may become inadequate to the game of selling. To grasp this problem, the essence of non-ordered valuations and envy-cycles, consider the example below.

[FIGURE 1]

Example 1: (package scheme non-implementable by rations/rewards).

Suppose that three two-humped camels (Bactrians are good examples, see Figure 1) are willing to buy up to 6 apples from a monkey. Suppose the first consumer's valuations are: $V_1(1) = \$2$ for 1 apple, $V_1(2) = \$3.1$ for 2 apples, while greater quantities do not increase satisfaction: $V_1(x) = \$3.1$ for $x \geq 2$. The second consumer's valuations are: $V_2(1) = \$2$ for 1 apple, $V_2(2) = \$2.5$ for 2, and $V_2(x) = \$4.2$ for $x \geq 3$. The third consumer's valuations are: $V_3(1) = \$1.5$ for 1 apple, $V_3(2) = \$3.1$ for 2 apples, and $V_3(x) = \$4.2$ for $x \geq 3$. For all three $V_i(0) = 0$. Thus the three consumers has satiable demands, pair-wise coinciding at certain points. Cost function is $C(x_1, x_2, x_3) = c(x_1 + x_2 + x_3) = (x_1 + x_2 + x_3) + 100(x_1 + x_2 + x_3 - 6 + |x_1 + x_2 + x_3 - 6|)$. It shows that costs are linear (with factor 1) for less than 6 apples, but drastically increase when quantities are higher than 6. Subtracting the flat linear component $1x$ from the cost function ($\tilde{c}(x) := c(x) - x$) and from the valuation functions, we can express the problem in terms of *reduced valuations*, defined as $v_i(x) := V_i(x) - x$, see Figure 1 (it is similar to *net valuations*, defined as $v_i(x) := V_i(x) - c(x)$).[5] The ordinate in Figure 1 is the reduced valuation or net-tariff (net of cost). It shows profit and/or consumer surplus that can be obtained from any package. The three net-utility indifference curves namely $v_1(x) - t$ (dotted), $v_2(x) - t$ (dashed), and $v_3(x) - t$ (solid) starting from the origin show active participation constraints, arrows showing directions of increase (gradients). We use here some continuous curves interpolating points 1, 2 and 3, assuming that fractions of an apple are available. It is more expressive, while logic of example is the same as for discrete consumer set $X = \mathbf{Z}_+$. What matters, is that each indifference curve has 2 humps and both local maxima can become optimal packages.

In order to find here an optimal package scheme, suppose (unreasonably) that the monkey has full information about each camel's willingness-to-pay, and the seller can prevent each consumer from taking a package designed for someone else. Then, obviously, optimal packages are $\bar{x}_1 = 1$, $\bar{x}_2 = 2$, $\bar{x}_3 = 3$, the related net tariffs (profits) being $t_1 = 1$, $t_2 = 1.1$, $t_3 = 1.2$, and gross tariffs are $T_1 = 2$, $T_2 = 3.1$, $T_3 = 4.2$. Note that it is the unique optimal solution, because increasing any of the packages $\bar{x}_i$ would result in great additional cost, while decreasing would reduce profit. This scheme $(\bar{x}, t)$ is incentive compatible giving a first-best and Pareto-efficient solution to the standard package-optimization problem (1).

[5] Another (simpler) way to introduce this example is to assume just zero costs $c(x) = 0$ for the available 6 apples (the monkey already has them and no more), and assume initial valuations like $V_1(1) = \$2 = V_2(1)$, $V_1(2) = \$3.1 = V_3(2)$, $V_2(3) = \$4.2 = V_3(3)$, equal to our 'reduced' valuations $v_i(x)$ obtained by cost subtraction. The point of starting with positive costs was to explain net valuations/tariffs used in all examples, and to show that this and of all other examples are not due to zero-cost or satiable-demand assumptions. Operating in backward direction, i.e., adding a sufficiently steep linear function to all valuations and costs, one can transform any of our examples into non-satiable-demand positive-cost examples, *keeping all effects* in place!

However, one can check that there is no way to implement (or ε-implement) it, either through small rewards or by simple rationing!

Simple rationing means that the monkey suggests a menu of three packages: 1 apple for \$2, or 2 apples for \$3.1, or 3 apples for \$4.2. It is made known that when someone takes any one of the three packages then that package is gone, so the next selection must be made from the remaining two packages. This strategy is observed in practice, for example, when a waiter in a restaurant tells the guest that a particular item from the menu is already sold out and therefore not available.

Possibility of such rationing can be formally incorporated into some non-standard package-optimization problem. It can also help in implementing standard solutions in some situations, but not in this case. Indeed, the monkey can obtain the desired first-best profit of \$3.3 when the first camel takes $x_1 = 1$, second takes $x_2 = 2$, and the third takes what remains. Or when the first camel takes $x_1 = 2$, the second takes $x_2 = 3$, and the third what remains. But when the first camel takes $x_1 = 1$, and the second takes $x_2 = 3$, then the third takes $x_3 = 0$, in other words, just goes away, leaving the monkey with a profit of only \$2 instead of \$3.[6] Small rewards also cannot help, since the three packages are linked together in a cycle of almost-envy. If everybody behaves 'right,' each camel still 'almost-envies' the other one in a circle $\#1 \to \#2 \to \#3 \to \#1$, and everybody 'almost-envies' to zero: $\#1 \to 0, \#2 \to 0, \#3 \to 0$. If all are behaving wrong from seller's perspective (another profitable situation), the cycle is backward $\#3 \to \#2 \to \#1 \to \#3$. In both situations it is evident that nothing can break the cycle and a considerable (about \$0.5 on average) loss of expected profit is inevitable in implementing a solution to the standard optimal package!

However, non-standard profitable alternative is not to offer the second package at all, thus loosing \$0.1 for sure (instead of loosing \$1 with probability 0.5) but breaking the cycle: $\tilde{x} = (1,\ 1,\ 3)$, $\tilde{T} = (2,\ 2,\ 4.2)$. One apple remains unsold (recall, purchasing

[6]The probability of such 'bad' outcome for the monkey is 1/2, because after somebody takes any 'right,' or 'wrong' package, the next selection would also be either 'right,' or 'wrong' one. Since for the seller the most desired sequence is ('right', 'right') or ('wrong', 'wrong'), average expected loss is not less than 1/6 of profit or about \$1/2.

twice is forbidden, for alternative assumption see Katz, 1984). In this package scheme consumers #1, and #2 are 'bunched together' for one package and the scheme becomes implementable.||

Can valuations concavity guard against non-implementable cycles? If it is not combined with cost convexity and some additional restrictions, the answer is 'no', see Example 2 below.

In essence, Examples 1and 2 show that non-orderedness of valuations and 'essential' cycle of envy can make the standard optimization setting and the resulting solution unreasonable (irrelevant to reality). 'Essential' cycle (defined below as 'non-reducible') means that there is no way to restructure the designed package scheme keeping the same profit. In contrast, the '3-camels' example modified by rounding numbers $3.1 \to 3$, $4.2 \to 4$ would have 'nonessential' cycle, reducible to implementable package scheme $\tilde{x} = (1, 1, 3)$ without sacrificing \$0.1 of profit.

We can compare the 'implementation problem' described here with similar problems in ideal price discrimination, in two-part tariff and in other situations, where monopolist takes all consumer surplus by relying upon the 'friendly' behavior of the consumer, i.e., in her staying in the market. In most such situations this problem is solved trivially by small rewards and escaped much attention. But it is serious here.

Now consider 'splitting'. What if 10 camels of each type came to this monkey seller having 6 apples and no more? Again this monkey monopolist can get his profit $\approx \$3$, but now the assumption of no-splitting becomes invalid. Since rationing *must* occur, its possibility should be included into the optimization problem. Example 5 in the Appendix (see Figure 5 also) shows that *optimal splitting* of essentially-similar consumer types between different packages (and the resulting cycle of envy) can happen even when costs are convex and valuations are concave (but non-ordered).

Since these examples show splitting becoming optimal, the question arises: how many packages are necessary? In continuous-type settings the usual answer is to design a continuous curve of contracts, but sometimes it does not result in one-to-one mapping of agents' types, so the essence of the question remains. In our finite-type setting one solution is to handle each consumer as a separate type (formally), then no more than $\nu := \sum_{i=1}^{n} m_i$ packages are needed. However, this formalization is inconvenient for discussing optimal splitting (information on agents' similarity disappears). Besides, in stochastic setting we have $m_i : \sum_{i=1}^{n} m_i := 1$, so this trick with *predetermined* maximum ν is unavailable.

Instead, to find ν and optimal splitting we can formulate the following 'allocated-packages' optimization problem, implying, in essence, some kind of rationing.[7]

'Allocated-packages' setting (compare with Katz, 1984). The principal chooses a number of packages $\nu \in \mathbf{Z}_+$ and some $n \times \nu$ 'allocation matrix' $A = (a_{11}, ..., a_{1\nu}, ..., a_{n1}, ..., a_{n\nu})$. In a deterministic setting each $a_{ik} \in Z_+$ shows how many consumers of type i buy the package-type k. In stochastic setting $a_{ik} \in [0, 1]$ means the probability of how often such consumer buys such a package. The one-man-one-package assumption is expressed as $\sum_{k \leq \nu} a_{ik} = m_i \; \forall i$ and total production becomes $x_{all} := \sum_{i \leq n} \sum_{k \leq \nu} a_{ik} x_k = 1^n A x$. Incentive-compatibility constraints become more numerous: $v_i(x_k) - t_k \geq v_i(x_j) - t_j \;\; \forall j \; \forall (i, k) : a_{ik} > 0$. Everything else remains the same as in the standard setting.

Note, that standard setting is similar to this new one, except for two additional constraints: 1) standard $\nu := n$, and 2) standard allocation matrix should be a diagonal $n \times n$ matrix with m_i on the diagonal. So, the 'allocated packages' optimization problem has fewer constraints than the standard problem. Therefore, it must bring the same or more profit, than the standard one. Specifically, profit can be shown to be exactly the same for ordered valuations, there is no need for splitting. But for non-ordered case our examples show the allocated setting being better (more profitable). Hence 'allocated packages' are *conceptually more adequate* when they are implementable. However, two conceptual hardships remain.

First, it is easy to see that implementation of solution still remains puzzling under 'allocated' optimization. It may even seem becoming worse. How can the seller enforce a 'friendly behavior' from the splitted consumer types? But actually it is not worse, indeed it is the same. After an allocated solution is found, it can be rearranged as the standard

[7] By the way, in the case of optimal splitting we necessarily see cycles of envy among formally-different, but really-similar consumers. Thus, implementability by rewards always becomes questionable under optimal splitting and hence there is need for rations.

one. Just treat allocated consumers as $n \times \nu$ formally-different types of consumers indexed by (i, k), each with quantities $\tilde{m}_{(i,k)} = a_{ik}$. Therefore, all the properties of solutions of the standard optimization setting also apply to the solutions with allocated packages. This to some extent supports the conceptual framework in our analysis of standard solutions below. Still, it is applicable only to the situations where cycles are absent or reducible, or rationing can be practiced and allocations are implementable as in Example 5. We have no idea about the optimization setting that would, generally, be reasonable when rationing *cannot* be used (what are the examples of such industries?), or what should the principal do normally in cases such as Example 1 ('3-camels') to maximize expected profit.

Second, in situations where rationing *can* be practiced, it raises an additional question. In some variations of the game, as can be seen in Examples 4 and 6 (see Figures 4, 6 and Appendix), rationing can replace some incentive-compatibility constraints from the allocated-packages optimization setting! It can further increase profit, in essence by discouraging an agent not to switch to an unprofitable package. So far we have not found the general formulation and solution to this puzzle.

However, when rationing devices can be used, some other approach, a '*preference revealing menu,*' can be suggested to increase profit further. The idea is simple with practical implications. The principal suggests a menu of packages and asks each agent to identify her *all* most preferred packages. The principal then offers her one out of these packages of *his* choice. Agents should not have any incentive to lie, and the principal now has a new tool revealing consumer types and implementing best solutions. This is not just a mechanism to implement the standard package scheme, it may also bring higher profit, since generally some incentive constraints may be *violated* under rations. For some situations this approach could be more appropriate than the standard one.

In the next section we study implementation *only by rewards* for situations when rations are unnecessary or unavailable. We just identify different envy-cycles and cases when rewards are sufficient for implementing *standard* solutions, which is also the same for implementing allocated packages.

3 SOLUTION STRUCTURE AND IMPLEMENTATION

We first introduce some common graph notions, and some specific terms for describing active constraints structures and partial order in the "envy-space."

Commonly, a directed graph, or "digraph" $G =< N, E >$ is a collection of nodes (vertices) $N(G)$ and collections of arcs (oriented edges) $E(G)$. Somewhat loosely we use notations $i \in G \Leftrightarrow i \in N(G)$, and $(i \to j) \in G \Leftrightarrow (i \to j) \in E(G)$. Pairs like $(i \to j)$ or (i, j) denote an arc *from* node i (arc's *tail*, or adjacent *predecessor* of j) *to* node j (arc's *head*, or adjacent *successor* of i). A node without predecessors is a *source*, while a node without successors is a *sink*. *(Di-)chain* is a sequence of edges with distinct edges, it is a *(di-)path* when vertices are distinct also. *Dicircuit* or *(di-)cycle* is a closed path, i.e., one starting and ending with the same node. Acyclic (lacking di-cycles) digraph is an *in-tree*, if it has a "root" node, such that every other node is connected *to* it by a unique dipath (i.e., "root" is *reachable* from every node). More generally, digraph having a single sink ("root") will be called here an *in-rooted (or single-sink) digraph* if the root is reachable from any other node. Obviously, in-tree is in-rooted and any in-rooted digraph contains a *spanning* in-tree, i.e., such in-tree that contains all digraph's nodes. A *source* of an *in-rooted* graph is a *leaf*. Digraph is *strongly connected* when all its vertices are reachable from each other.

We express partial order within digraphs in terms of nodes "height" relative to each other. Predecessors are considered to be *not lower* than successors. A node i is *strictly higher* than its successor j when i is not a successor of j. In cycles or other strongly connected subgraphs all nodes are supposed *equally high.* A sink is the lowest node in the in-rooted graph.

Our problem (1)–(2) generates different digraph structures (see Figures). "All-arcs" digraph G_{all} contains all $n \times n$ incentive-compatibility constraints (2) treated as oriented arcs, and all $n+1$ agents' numbers (names) treated as nodes. An arc $(i \to j)$ denoting the constraint $V_i(x_i) - t_i \geq V_i(x_j) - t_j$, implies that agent i does not envy agent j's package. Node #0 is the "root" or the "sink" and is also the "lowest" node, while $G_{all} \setminus \{\#0\}$ is strongly connected.

Further, any package plan (x,t) admissible in our problem (1)–(2) generates two interrelated subgraphs of G_{all}: *active agent-graph* and *active package-graph.* Functions V_i being fixed, for a given (x,t), its "*envy-graph*" or *active agent-graph* $\bar{\bar{G}}(x,t) = \bar{\bar{G}}_V(x,t) \subset G_{all}$ is defined as the collection of all nodes $\{0, 1, 2, ..., n\}$ and of all active constraints, i.e., those becoming equality at point $(x,t) \in R^{2n}$. Thus an arc $(i \to j) \Leftrightarrow V_i(x_i) - t_i = V_i(x_j) - t_j$ is interpreted as 'consumer i almost-envying consumer j's choice,' and $\bar{\bar{G}} : R^{2n} \mapsto G_{all}$ is a well-defined function, relating any package-plan (x,t) to its active agent-graph $\bar{\bar{G}}(x,t)$. Orientation of any arc $(i \to j)$ is *downwards* in the "envy-space". This convention becomes convenient in comparing the envy-space (graph order) and the 'profit-space,' often keeping the same direction (see our examples and figures).

Active package-graph $\bar{\bar{G}}_P(x,t)$ is the "*bunching*" of the agent-graph $\bar{\bar{G}}(x,t)$. Bunching means modifying graph such that any couple of agent-nodes i, j having the same package (in the sense $(x_i, t_i) = (x_j, t_j)$) becomes a single package-node with name (ij), while arcs $(i \to j)$ and $(j \to i)$ are eliminated. Similarly *several* agents "*bunched*" together become one package-node.

We are ready to state our main results. 'Tree-rule' Theorem 1 determines the solution structure for both ordered and non-ordered valuations, and is used in proving Theorem 2 and Propositions.

In the statement of the theorem, we denote the adjacent successor of the node i in the in-tree G by $b(i,G)$, and the set of *all* successors of i, up to the root (all nodes "below" i) by $B(i,G)$. So $B(.)$ is a mapping, while $b(.) \in B(.)$ is a function.

THEOREM 1 ("Tree-rule")**:** *Let bundle* $(\bar{x}, \bar{t}) \in R^{2n}$ *be a solution to the optimization problem given by (1)-(2). Then (*A): *All active constraints at* $(\bar{x}, \bar{t})$ *represent an in-rooted (single-sink) digraph* $\bar{\bar{G}} = \bar{\bar{G}}(\bar{x}, \bar{t})$ *with root #0.* (B): *There exists a spanning in-tree* $G_T \subseteq \bar{\bar{G}}$ *of digraph* $\bar{\bar{G}}$, *describing a subset of active incentive constraints, such that* $(\bar{x}, \bar{t})$ *is also a solution to the reduced optimization problem (3)-(4), and all solutions to the reduced problem (3)-(4) are also the solutions to initial problem (1)-(2). Any spanning* *in-tree* $G_T \subseteq \bar{\bar{G}}$ *of digraph* $\bar{\bar{G}}(\bar{x}, \bar{t})$ *fits this condition.*[8]

$$\pi(x, G_T) \quad : \quad = \sum_{i=1}^{n} m_i \, \mathbf{T}_i(x, G_T) - C(m, x) \rightarrow \max_{x \in X^n} \quad \text{s.t.} \tag{3}$$

$$V_i(x_i) - \mathbf{T}_i(x, G_T) \quad \geq \quad V_i(x_j) - \mathbf{T}_j(x, G_T) \quad (\forall i, j \in I), \quad x_0 := 0, \tag{4}$$

where function

$$\mathbf{T}_k(x, G_T) := \sum_{i \in B(k, G_T) \cup \{k\}} [V_i(x_i) - V_i(x_{b(i, G_T)})] \; (\forall k \in I), \; \mathbf{T}_0(.) \equiv 0 \quad .$$

A particular case of 'tree-rule' is stated below.

PROPOSITION 1 (**"**Chain-rule**"**): *Assume valuations obeying the Spence-Mirrlees ordering condition:* $[x, z \in X, \;\; x > z \Rightarrow V_i(x) - V_i(z) > V_{i-1}(x) - V_{i-1}(z) \;\; \forall i = 2, ..., n]$. *Then the active graph (guaranteed by Theorem 1) of a solution* $(\bar{x}, \bar{t})$ *amounts to the chain* $G_T = G^C := \bar{\bar{G}}(\bar{x}, \bar{t}) = (n \rightarrow n-1 \rightarrow n-2... \rightarrow 0)$, *and all not-in-chain incentive constraints (4) can be replaced by constraints* $x_{i-1} \leq x_i$ $(\forall i \in I)$, $x_0 = 0$ *in the optimization problem (3)-(4).*[9]

PROOF: see the Appendix.

To appreciate the usefulness of these results, note that Theorem 1 reduces the number of optimization variables by half and provides a clear algorithm for optimization. It is sufficient to try all trees, then compare the resulting tree-specific solutions. For each tree one can optimize as follows. First find an *unconstrained* optimum of the tree-specific objective function (3*)*, then add the violated constraints (if any) and solve again, and

[8]Caution: different solutions to initial problem may result from different tree-structures, so only full collection of tree-specific problems can be called "equivalent" to the initial problem.

[9]Proposition 1 is essentially the 'chain-rule theorem', generalizing similar claims (Katz, 1973) in some respects. Cost function may have both probabilistic and deterministic interpretations, we neither require continuity nor convexity (or concavity). Besides, increasing condition $[\, x \geq z \Rightarrow V_1(x) - V_1(z) \geq 0]$ turns out to be unnecessary. In contrast, it is easy to prove that some valuations ordering is essential for chain-rule: when any incentive-compatible package plan obeys chained order of quantities $[\, \bar{x}_{i-1} \leq \bar{x}_i \;\; (i \in I)]$, then valuations satisfy weak Spence-Mirrlees ordering condition on these quantity couples: $[\, i > k \Rightarrow V_i(\bar{x}_i) - V_i(\bar{x}_k) \geq V_k(\bar{x}_i) - V_k(\bar{x}_k) \;\; \forall i, k \in I]$. Instead of direct proof, our generalized chain-rule can also be obtained from Milgrom and Shannon's (1993) theorems on Spence-Mirrlees condition and comparative statics, by interpreting package size and tree-position as monotone functions of agent's number.

so on. Our experience shows that this optimization sequence algorithmically is the most efficient. For optimization, as well as for deriving properties of solutions in Proposition 2, the following reformulation of objective function is helpful.

$$\pi(x, G_T) \quad : \quad = [\sum_{i=1}^{n} Q_i(x_i, G_T)] - C(m, x) \quad \text{where} \tag{5}$$

$$Q_i(x_i, G_T) := \qquad S_i V_i(x_i) - \sum_{j \in A_1(i, G_T)} S_j V_j(x_i), \tag{6}$$

$$S_i := \qquad \sum_{j \in A(i, G_T) \cup \{i\}} m_j \quad (\forall i \in I),$$

where $A(i, G)$ denotes the set of all nodes lying above the node i (predecessors), while $A_1(i, G) \subseteq A(i, G)$ denotes the set of only adjacent predecessors of the node i (derivation of Q_i can be checked directly using recursion).

This formulation of tree-specific problem is advantageous when the cost function is separable, then one can often just solve n independent problems $Q_i(x_i, G) \to \max_{x_i}$, for each x_i *separately,* at least when none of the not-in-tree constraints are binding.

For another special (ordered) case, Proposition 1 reduces the number of constraints significantly and, the only possible graph becomes known, *a priori.* It is the chain $(n \to n-1 \to ... \to 0)$. Unlike the non-ordered case, it also implies that higher valuation function (agent's number) relates to higher consumption and occupies a higher position in the envy-graph.

Now consider implementation issue. For this we introduce notions related to bunching and different cycles in active digraphs (see examples and figures to comprehend the ideas).

Definition 1: A feasible bundle $(\bar{x}, \bar{t}) \in R^{2n}$ in (1)-(2) is called *transitive-reduced* when its package-graph $\bar{\bar{G}}_P(\bar{x}, \bar{t})$ is acyclic. A bundle $(\tilde{x}, \tilde{t}) \in R^{2n}$ is called a *transitive-reduced version* of a feasible bundle $(\bar{x}, \bar{t}) \in R^{2n}$, when its package-graph $\bar{\bar{G}}_P(\tilde{x}, \tilde{t})$ is acyclic and for every component $(\tilde{x}_i, \tilde{t}_i)$ there is a prototype $(\bar{x}_j, \bar{t}_j) = (\tilde{x}_i, \tilde{t}_i)$ (with the same or other number $j \neq i$).[10] A feasible bundle $(\bar{x}, \bar{t})$ is *transitive-reducible* when it has a transitive-reduced version.

[10] In essence, reduction of a package-plan to its transitive version consists in just *deleting* some packages and reallocating related agents to some remaining packages.

The idea of small rewards for implementing principal's package plan can now be formalized as follows.

Definition 2: A package bundle $(\bar{x}, \bar{t})$ is *ε-implementable* when any neighborhood of $\bar{t} \in R^n$ contains some tariffs $t^\varepsilon \in R^n$ resulting in an incentive-compatible bundle $(\bar{x}, t^\varepsilon)$ which is "*envy-free*" (strictly incentive compatible). It means that all constraints (2) are strict inequalities at $(\bar{x}, t^\varepsilon)$, except bunched couples $(i, j) : (x_i, t_i) = (x_j, t_j)$. I.e., package-graph $\bar{\bar{G}}_P(\bar{x}, t^\varepsilon)$ has all nodes disconnected.

To better comprehend the definitions, note that a graph structure related to *binding* constraints (those relevant for solution) may differ from the structure of *active* constraints defined above. Consequently (see figures), an envy-free package bundle may be incentive-incompatible!

REMARK 1: *Any transitive-reduced optimal solution* $(\tilde{x}, \tilde{t})$ *is ε-implementable.*

It follows trivially from the definitions (see also Proof of Theorem 2).

We are ready to establish implementability and lack of essential splitting/cycles. It can be guaranteed without restricting valuation functions, we just assume cost function to be *separable.* To evaluate the usefulness of this assumption, recall counter-examples to Theorem 2 *(*Examples 1, 5, Figures 1 and 5*).* In spite of cost convexity, these examples show non-reducible envy-cycles, lack of implementability, optimal splitting of agent types – all these because of non-separable costs. The key idea in establishing implementability for separable costs is that higher/lower node's position in envy-space (in digraph) correlates with similar position in 'profit-space' (defined in the proof), so all cycles have the *same level of profit.* This enables eliminating cycles from package-graph without losses, by a *bunching procedure* described in the proof (it is the second key idea).

THEOREM 2 ("Implementability"):[11] *Assume the cost function is separable:* $C(m, x) = \sum_{i=1}^n m_i c(x_i)$ (some $c : R \mapsto R$)*. Then* *(*A): *For any solution* $(\bar{x}, \bar{t})$ *, any node i succeeding*

[11]We are grateful to Denis Anoshin for his contribution in proving this theorem and some propositions below. In particular, for checking the hypothesis that generalizes Theorem 2 from linear to separable functions. We are grateful to Victor Polterovich for suggesting this generalization that enables us to encompass all typical stochastic settings.

a node j in the solution graph $\bar{\bar{G}}(\bar{x}, \bar{t}) : (j \to ...i)$, brings profit no more than j in the sense $(\bar{t}_i - c(\bar{x}_i)) \leq (\bar{t}_j - c(\bar{x}_j))$. (B): Any solution has a transitive-reduced version $(\tilde{x}, \tilde{t})$ being ε-implementable optimal solution and lacking type-splitting .[12]

Proof: Separability allows us to relate the solution's in-rooted graph $\bar{\bar{G}}(\bar{x}, \bar{t})$ of active constraints to *profit-space* placing the nodes with higher 'per-package profit' $(\bar{t}_i - c(\bar{x}_i))$ relatively higher in the graph. Suppose the graph $\bar{\bar{G}}(\bar{x}, \bar{t})$ had a profit-ascending arc $(j \to i) : (\bar{t}_i - c(\bar{x}_i)) > (\bar{t}_j - c(\bar{x}_j))$. Then transforming (only) package $(\bar{x}_j, \bar{t}_j)$ into package $(\hat{x}_j, \hat{t}_j) = (\bar{x}_i, \bar{t}_i)$ we obtain a new package plan $(\hat{x}, \hat{t}) \in R^{2n}$, which improves objective function by bringing more profit. At the same time, it remains incentive-compatible (feasible), since no new packages appears and, except j, all other agents remain unaffected, and j has exactly the same payoff as before, i.e., $V_j(\hat{x}_j) - \hat{t}_j = V_j(\bar{x}_j) - \bar{t}_j \geq V_j(\bar{x}_k) - \bar{t}_k \, \forall k$. So, the solution $(\bar{x}, \bar{t})$ was not optimal, resulting in a contradiction. Hence profit-ascending arcs or chains do not exist in $\bar{\bar{G}}(\bar{x}, \bar{t})$ and any envy-cycle does have the same level of profit at all of its nodes.

This helps in proving reducibility and implementation. Let us construct the transitive-reduced version of the solution. Take any cycle $S \subseteq \bar{\bar{G}}(\bar{x}, \bar{t})$ and delete in it any quantity-descending arc $(j \to i) \in S : \bar{x}_i < \bar{x}_j$, replacing its tail-package $(\bar{x}_j, \bar{t}_j)$ by the smaller package $(\tilde{x}_j, \tilde{t}_j) = (\bar{x}_i, \bar{t}_i)$ (head-package of this arc) in the package plan. By repeating this procedure we can delete (not uniquely?) *all* quantity-descending arcs having the same profit levels. As a result, we reach a new package plan $(\tilde{x}, \tilde{t})$ whose package-graph $\bar{\bar{G}}_P(\tilde{x}, \tilde{t})$ is acyclic, since a cycle cannot have only quantity-ascending arcs, while quantity-descending arcs are deleted (when quantities of two packages coincide, tariffs coincide also, so only bunched or quantity-descending arcs remain in $\bar{\bar{G}}(\tilde{x}, \tilde{t})$). No new package-nodes appear, and none of the agents become worse off because all transformations are along indifference curves. Therefore, the new transitive-reduced plan $(\tilde{x}, \tilde{t})$ is incentive-compatible. At the same time, the new plan $(\tilde{x}, \tilde{t})$ gives the same profit as the optimal $(\bar{x}, \bar{t})$ one. Therefore, reducibility is proved. We also see that splitting (if any) was redundant, since it is excluded in the cycle-free plan $(\tilde{x}, \tilde{t})$.

[12] In this context 'splitting' means that formally different consumers with coinciding tastes ($V_i = V_j$) choose different packages. For 'no-splitting' they must choose the same package.

Let us construct exact ε-implementation of any transitive-reduced solution $(\tilde{x}, \tilde{t})$ by changing tariffs t_i so that $(\tilde{x}, \tilde{t})$ is transformed into a similar envy-free bundle $(\tilde{x}, \hat{t}) \approx (\tilde{x}, \bar{t})$. Define constraint's slack as $\delta_{ij} := (V_j(\tilde{x}_j) - \tilde{t}_j) - (V_j(\tilde{x}_i) - \tilde{t}_i)$ and choose $\hat{\delta} := \min_{\{\forall i,j:\delta_{ij}>0\}}\{\delta_{ij}\} > 0$ from all non-active constraints. Then starting from any envy-free package-node k (the summit of any branch, including possibly several agent-nodes), take arbitrarily-small positive quantity $\varepsilon_1 < \hat{\delta}/2$, and reduce the related tariff in the sense $\hat{t}_k := \tilde{t}_k - \varepsilon_1$. This eliminates envy-arc(s) going from k, so some tree below the package node(s) becomes envy-free. Sufficiently small ε_1 ensures that no arcs going *to* k emerge from reduction in $\tilde{t}_k$. Repeat this operation on another envy-free node with $\varepsilon_2 < \varepsilon_1/2$ and so on. Thus the envy-free approximation of $(\tilde{x}, \tilde{t})$ can be constructed for all $\varepsilon_1 < \hat{\delta}/2$. *Q.E.D.*

To see the broad application of this theorem, note that a typical stochastic setting has separable cost function.[13] So splitting and implementation problems are serious mainly in deterministic settings, or in some non-trivial stochastic situations with complex costs. For these situations we try to establish no-cycle conditions by restricting costs and valuations *together* as formulated in the following definition of convexity and gross convexity.

DEFINITION 3: *Gross convexity* of the problem *(3)* for a given solution $(\bar{x}, \bar{t})$ and its tree $G_T(\bar{x}, \bar{t})$ takes place when an admissible set X is convex, cost function $C(.)$ is convex w.r.t. x on its domain and all $(\bar{x}, \bar{t})-$related partial objective functions $Q_i(\cdot)$ defined in *(5)* are concave on X. Gross convexity is called 'strict' when all $Q_i(\cdot)$ are strictly concave. (Strict) *convexity* is defined similarly by considering valuations $V_i(\cdot)$ instead of $Q_i(\cdot)$.

Unfortunately, Examples 2 and 5 show that neither strict convexity nor non-strict gross convexity are sufficient for eliminating non-reducible cycles. It is *strictness* of gross convexity that can eliminate cycles completely, as stated in the proposition below.

PROPOSITION 2: *Assume cost function as aggregate-type in the sense* $C(m, x) = c(\sum_{i=1}^{n} m_i x_i)$, *where* $c(.)$ *is a differentiable function* $(c : R \mapsto R)$. *Functions* V_i *are continuous. Strict gross convexity of the problem at solution* $(\bar{x}, \bar{t})$ *takes place and all agent types show the same quantities:* $m_i = m_j \ \forall i, j \in I$. *Then solution's package-graph* $\bar{\bar{G}}_P(\bar{x}, \bar{t})$ *is*

[13] Recall, situations where *several* customers can be served in a period, can have more complex cost functions.

acyclic and $(\bar{x}, \bar{t})$ *is an* ε*-implementable solution without type-splitting.*

PROOF: in the Appendix (it is based on cycle-reduction ideas mentioned above and in Proposition 4).

Proposition 2 is illustrated by Example 2, where cycles cannot be eliminated under non-strict gross convexity. How realistic is the assumption of 'gross convexity' at the solution? To guarantee this property in an *ad hoc* fashion without knowing the solution tree, we should restrict valuations too much. Indeed, most often a function that should be concave for this property, looks like $Q_i(x_i, G) := (S_{i+1} + m_i)V_i(x_i) - S_{i+1}V_{i+1}(x_i)$. So all neighboring in the tree valuation functions V_i must have *not-too-different second derivatives*,[14] or more concave functions must precede by number (succeed in the in-tree). This sequence and gross convexity can be violated in solutions: see Example 5. Can we weaken other assumptions of Proposition 2? Differentiability is essential (see Example 2). "Coinciding quantities" assumption is not really restrictive, since in a simple deterministic case one can treat each consumer as a separate type.

Regarding other possible statements about implementation, we conjecture some condition *only* on valuations V_i that excludes non-reducible cycles for *any* cost function. Another possible extension is to prove that any example with essentially-different agents and with cycles 'non-implementable by rations' is *degenerate* in some sense. It seems that any cycle can be eliminated with probability 1 by a slight perturbation of valuation functions, if this perturbation is *different* for different agent-nodes. If this is true then such bad cycles as in Example 1 'almost never' exist. In contrast, it is not the case with splitted cycles (often occurring among similar agents) like in Examples 2 and 5. Here it is more natural to study perturbations of valuations among agent types, not among agent-nodes. In particular, by slightly varying common valuation function $v_3 = v_4 = v_5$ of the entire agent type (agents #3, #4, #5) in Example 2, one can keep the splitting-effect

[14]For instance, one can prove by contradiction that gross convexity must hold for any solution to 3-agents problem when second derivatives satisfy condition $2|\ddot{v}_i(x)| \leq 3|\ddot{v}_j(x)|\ \forall x, i, j$. This condition on derivatives is sufficient also to ensure concavity of Q_i on any chain of a solution graph, but not at branching points.

and the related cycle, so the example is non-degenerate!

Now from no-cycle conditions we turn to related efficiency issue.

4 ENVY-FREE SUMMITS AND EFFICIENCY

We now derive some properties of solutions. The interesting one is 'partial efficiency' of at least some packages among the optimal package scheme. Roughly, partial efficiency of some agent's package $(\bar{x}_i, \bar{t}_i)$ means that consumer i would not wish to increase her package size $\bar{x}_i$, for the price equal to marginal costs, and similar decrease is also unwanted. As revealed in Guesnerie and Seade (1984, Propositions 2-4), partial efficiency is tightly, but non-trivially connected with these packages being 'envy-free' (being 'leaves' of related graphs) and with lack of cycles at 'summits'. We now extend these results by dropping 'locally- different valuations' assumption (Examples 1and 5 violate it) that is too restrictive. Instead we assume valuations being quasi-linear. [15]

DEFINITION 4: Given $X^n, V_i(.), C(.)$, a package quantity $\bar{x}_i$ is called "*partially-efficient*" in a situation $\bar{x} = (\bar{x}_1, ..., \bar{x}_n) \in X^n$ when partial welfare function $W_{i,\bar{x}}(x_i) := m_i V_i(x_i) - C(m, \bar{x}_1, ..., x_i, ..., \bar{x}_n)$ attains its maximum on X at point $\bar{x}_i$.[16]

DEFINITION 5: A subgraph $S \subseteq G$ of digraph G is an *(in-)branch* if it is connected and no arc leads to S from $G \setminus S$. A subgraph $S \subseteq G$ is called a *branch-summit* of G, when S is a branch not containing other branches, so that all nodes of S are equally high. A single-node branch is a *leaf* or source.

[15] Compare our below propositions with its following alternative formulation, almost-coinciding with Propositions 2-4 in Guesnerie and Seade (1982). Suppose cost function is linear, valuation functions $V_i(.)$ are strictly concave and differentiable on R_+. Suppose additionally, that all agents have locally-different derivatives at the solution (x_i, t_i), i.e. $\dot{v}_i(x_i, t_i) \neq \dot{v}_j(x_i, t_i)$. Then there is at least one 'envy-free' agent i in tree G. She has partially-efficient package (x_i, t_i), and no other agents choose it (no bunching).

[16] For a better intuition and to allow generalizations (non-quasi-linearity), this definition can be reformulated as follows. For efficiency, weakly-more-preferred set of packages $L_i := L_i(\bar{x}, \bar{t}) := \{(a, t) \in X \times R | V_i(a) - t \geq V_i(\bar{x}_i) - \bar{t}_i\}$ must not intersect with the higher-profit set $M_i(\bar{x}, \bar{t}) := \{(a, t) \in X \times R | t - C(m, \bar{x}_1, ..., \bar{x}_{i-1}, a, \bar{x}_{i+1}, ...\bar{x}_i) > \bar{t}_i - C(m, \bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, ...\bar{x}_i)\}$.

Propositions 3 and 4 show different conditions for partial efficiency of branch-summits and 'profit-summits' (defined below), together with additional properties.

PROPOSITION 3: *Let cost function $C(.)$ be separable, then*

(a). *There exist partially-efficient packages among components of any solution $(\bar{x}, \bar{t})$. Moreover, the solution's profit-summit (the collection S of the highest-profit nodes k : $(\bar{t}_k - c(\bar{x}_k)) \geq (\bar{t}_i - c(\bar{x}_i))\ \forall i \forall k \in S$) has only partially-efficient nodes and consists of one or more branches.*[17]

(b). *When each personalized welfare function $W_{i,\bar{x}}(x_i) = m_i(V_i(x_i) - c(x_i) + const)$ is unimodal on admissible set X, then in the profit-summit of solution's package-graph G_P there are only leaves, i.e., no cycles exist in the profit-summit of G_P.*[18]

PROPOSITION 4: *Assume convex $X \subseteq R$, cost function is of aggregate-type: $C(m, x) = c(\sum_{i=1}^{n} m_i x_i)$* where $c : R \mapsto R$ is *continuous. Let all partial welfare functions $V_i(x_i) - c(\sum_{j=1}^{n} m_j x_j)$ be continuous, concave w.r.t. x_i on their domain. Let a branch-summit $S \subset \bar{\bar{G}}_P$ of a solution's package-graph $\bar{\bar{G}}_P(\bar{x}, \bar{t})$ include only agents with the same quantities: $m_i = m_j\ \forall i, j \in S$. Then:*

(a). *S has only partially-efficient packages.*

(b). *Suppose additionally that $c(.)$ is convex and differentiable and all V_i $(\forall i \in S)$ are concave. Then S has uniform relative-profit level in the sense $[t_i - \bar{c}x_i = t_j - \bar{c}x_j$ $(\forall i, j \in S)$, where $\bar{c} = \frac{1}{m_i}\partial c(\sum_i m_i \bar{x}_i)/\partial x_j$ $(\forall j \in S)]$. Besides, upper envelope of active indifference curves of S locally coincide with the related linear interval $[(\bar{x}_{\hat{k}}, \bar{t}_{\hat{k}}), (\bar{x}_{\check{k}}, \bar{t}_{\check{k}})]$, where $\bar{x}_{\hat{k}} := \min_{i \in S}\{\bar{x}_i\}$, $\bar{x}_{\check{k}} := \max_{i \in S}\{\bar{x}_i\}$.*

COROLLARY 1: *Under assumptions of Proposition 4 each solution has a partially-efficient node.*

COROLLARY 2: *In the case* (b) *under additional assumption of strict concavity of all V_i, each branch-summit S is a leaf (so has no cycles).*

PROOFS: see Appendix.

[17] There are examples when profit-summit is a chain, not a branch-summit. These two notions may differ even for linear costs.

[18] Unimodality is a weaker condition than strict concavity of V and convexity of $c(.)$. It prevents cycles within branches like the one in Example 3.

Counter-examples to Propositions 3 and 4 (Examples 2, 3 and 6) illustrate the related ideas and show why the assumptions are essential.

[FIGURE 2]

Example 2 (Figure 2): illustrates Propositions 4 and 2. It shows that in spite of solution's gross convexity and Pareto-efficiency, *type-splitting, envy-cycle and non-implementability* may occur, resulting in some loss of profit, without strict convexity.

Consider a package problem in terms of *net* valuations and costs, or just a problem with satiable demand.[19] The net cost function is $C(x_{sum}) = 0.0001/(6.442878 - \min\{x_{sum}, 6.442878\})$, where $x_{sum} := \sum x_i$. Thus production below 6.3 is almost costless, while production $x_{sum} \geq 6.442878$ is infinitely costly. Net valuations of buyers are:

$v_1(x_1) = 4\,x_1 - 3.2\,x_1^2$, $v_2(x_2) = 1.32\,x_2 - 0.36\,x_2^2$,

$v_i(x_i) = \min\{\ 5\,x_i\,;\ 1 + 0.01x_i\,;\ 12 - 4\,x_i\}\ (i = 3, 4, 5)$.

The solution is: $\bar{x}_1 = 0.623438$, $\bar{x}_2 = 1.819440$, $\bar{x}_3 = 0.25$; $\bar{x}_4 = 0.95$; $\bar{x}_5 = 2.7$, $t \approx (1.25, 1.21, 1.0025, 1.0095, 1.027)$. Output is $x_{sum} = 6.342878$. This plan is optimal because all the consumer surplus goes to the principal and the solution is Pareto-efficient. Indeed, all agents get their most-preferred quantities, all derivatives being $v_i'(x_i) = C'(6.342878) = 0.01$ (see Figure 2). Hence it is a 'first-best' solution giving revenue (almost equal to profit) like $\sum_i t_i \approx 5.499$. It is a unique solution (if permutations of #3, #4 and #5 are not considered), because increasing any of the packages $\bar{x}_i$ (without changing other packages) would result in a great additional cost, while decreasing packages would reduce profit.[20]

Therefore, to keep the same profit, 'splitting' is inevitable, i.e., different packages occur for essentially similar agents #3, #4 and #5. Hence, envy-cycle is inevitable and so is non-implementability by rewards.

[19] Like in Example 1, we can add here linear component $Nx, (\forall N > 4)$ to cost function and to all valuations. It keeps the essence of the example, but eliminates agents' satiability (used only for convenient exposition).

[20] In contrast, taking completely flat valuations $v_i(x_i) = \min\{\ 5\,x_i\,;\ 1\,;\ 12 - 4\,x_i\}\ \ (i = 3, 4, 5)$ we would arrive at approximately the same solution, which is not unique, because three packages could be made smaller: $\bar{x}_i = 0.623438$ $(i = 3, 4, 5)$, so envy-cycle is eleminated without losing profit.

This optimal package plan seems implementable by rations. But it is sufficient to modify function $v_5(.) := \min\{\ 0.95x_5\,;\ 1 + 0.01x_5\,;\ 12 - 4\,x_5\}$ as shown by the dotted line in Figure 2 to eliminate "complete" digraph among agents #3, #4 and #5, since agent #5 stops almost-envying #3. Then the same solution becomes *non-implementable by rations*. Indeed, suppose that agents #3, and #4 choose packages #5, and #4, which are in a limited supply. Afterwards when the fifth agent appears he is forced to take $(0, 0)$,and profit is reduced. Here non-active constraints $[(\#i \rightarrow \#1), (\#i \rightarrow \#2)\ (i = 3, 4, 5)]$ prevent elimination of undesirable cycle!

Now, let us see whether *differentiability is essential* for Propositions 2 and 4 to guarantee uniform profit-level (important for the proof). Slightly modifying the example, we take a piece-wise linear cost function $C(x_{sum}) = \max\{0,\ \ 1000(x_{sum} - 6.\,342878)\}$, being completely flat below $x_{sum} = 6.\,342878$. Almost nothing changes in Figure 2 and in the optimal solution, which remains the same and unique. But, unlike the previous case, now the relative profit level $t_i - x_i C'(6.342878) = t_i$ differs among the cycled nodes #3, #4, #5. Thus we have Pareto-efficient branch-summit cycle with non-uniform profit-level. Further modification of indifference curves (moving (x_4, t_4) slightly up) may also disturb the "flat-roof" property of branch summit (#3, #4 and #5), i.e., linearity of the indifference curve envelope connecting $(x_3, t_3), (x_4, t_4), (x_5, t_5)$. ||

[FIGURE 3]

Example 3 (Figure 3): It shows the *absence of partially-efficient agents*, inefficiency in spite of star-graph (due to costs *non-convexity* and non-separability).

Suppose only 4 discrete quantities of total production are really feasible: $x_{sum} = 0,\ 7,\ 7.1,\ 7.2$, or 7.3. For these quantities the cost function is $C(0) = 0$, $C(7) = C(7.1) = C(7.2) = C(7.3) = 3$. For other quantities costs are prohibitively high: $x_{sum} \neq 0, 7, 7.1, 7.2, 7.3 \Rightarrow C(x_{sum}) = 1000$. There are two agents of type 2 ($m_2 = 2$) and three ($m_1 = 3$) agents of type 1, with valuations as follows.

$V_1(0) = 0$, $V_1(1) = 5$, $V_1(1.1) = 5.1$, $V_1(2) = 6$, $V_1(2.1) = 7.2$, $m_1 = 3$.

$V_2(0) = 0$, $V_2(1) = 4$, $V_2(1.1) = 5.2$, $V_2(2) = 7$, $V_2(2.1) = 7.1$, $m_2 = 2$.

Package bundle $\{(x_1,t_1),(x_2,t_1)\} = \{(1,5),(2,7)\}$ is strictly incentive compatible (no envy occurs). It yields $x_{sum} = 3x_1 + 2x_2 = 7$ and profit $\pi(1,2) = 3*5+2*7-3 = 26$. It is optimal. Indeed, among other combinations (x_1,x_2), technologically feasible (without type-splitting) ones are only $(0,0) \to x_{sum} = 0$, $(1.1,\ 2) \to x_{sum} = 3*1.1+2*2 = 7.3$ and $(1,\ 2.1) \to x_{sum} = 7.2$. Note that technology enables only to increase either the first or the second package by 0.1. This would increase related payoff, tariff and profit, so current package scheme is inefficient. However, it creates incentive for another agent to switch to this new package. Therefore both 'improvements' $(1.1,\ 2)$ and $(1,\ 2.1)$ are incentive incompatible, yielding actually non-feasible demand $(x_1,x_2) = (1.1,\ 1.1) \to x_{sum} = 5$ and $(x_1,x_2) = (2.1,\ 2.1) \to x_{sum} = 10$, respectively. Thus, even though the optimal package scheme is envy-free, partial efficiency is absent because *non-active constraints are binding.* Can *rationing* solve this problem and increase profit? We do not find any such improvement. For instance, by using package scheme $\{(x_1,t_1),(x_2,t_1)\} = \{(1,\ 5),(2.1,\ 7.2)\}$ and restricting purchase to no more than 1 package of size 2.1, the seller could be better off, but agent of type 1 can switch to the new package. Note also that different numbers $m_1 = 3 \neq m_2 = 2$ of consumers are important for our conclusions.||

Now turn from partial efficiency to general *Pareto-efficiency*, which is important for public policy. Should price discrimination be restricted? In contrast to the ordered-valuations case, Pareto-efficiency is *not* a degenerate case under non-ordered valuations as stated in Remark 2 below. In fact, the situations leading to Pareto efficiency are quite probable.

REMARK 2: *Assume costs are separable. Then, for some package scheme $(\bar{x},\bar{t})$ to be optimal and Pareto-efficient at the same time (i.e., first-best), it is sufficient that the related net-valuation functions satisfy the following conditions.*

$V_i(x_j) - V_j(x_j) \leq 0, \forall i,j$

$x_i = \arg\max_{x \in X}(V_i(x) - c(x))$.

PROOF: Maximization condition implies partial efficiency for all i, which entails Pareto-efficiency here. No-envy condition on $V_i(x_j)$ implies tariffs $t_i = V_i(x_i)$ and zero consumer surplus. This results in maximal profit.||

In essence, these conditions only require that each net valuation curve should have maximum lying above all other curves (implying star-like solution graph). Hence the class of functions/packages satisfying these conditions is sufficiently broad.[21] These conditions are also necessary for Pareto-efficiency in two-consumers case. In general, however, they are not, as shown by Example 4 below (see Figure 4). But such examples represent rather degenerate cases. Similar (not exactly the same) condition is that solution should have a star-like graph. This is also close to necessary and sufficient condition for Pareto-efficiency, but it is really not. Example 4 contradicts necessity, while Example 3 contradicts sufficiency (being non-degenerate, but very peculiar, non-convex). Under problem convexity star-like graph is sufficient for Pareto-efficiency, which follows from Proposition 4.

By the way, the same star-graph condition is closely related with another interesting question: Is there any *consumer surplus* (the so-called '*informational rent*') arising from monopolist's inability to distinguish consumers' types? Remark below answers the question.

REMARK 3: *Agent(s) connected to the root of the solution graph* $\bar{\bar{G}}$ *have* zero consumer surplus, *and there is at least one such agent i with zero surplus* $(V_i(\bar{x}_i)-\bar{t}_i=0)$. *Conversely, agent(s) (if any) not connected to the root of the tree G do have positive consumer surplus (positive informational rent for those whose participation constraints are not active).*

So, Pareto-efficiency and informational rent are almost conversely related. Roughly, star-graph condition ensures efficiency and excludes rent. Normally Pareto-efficiency does not allow consumer surplus (informational rent), except in some peculiar or degenerate counter-examples. Such examples can be constructed either from non-differentiable valuation functions, or from counter-veiling effect, as illustrated below.

[FIGURE 4]

[21]This Remark seems extendable to the case of non-separable cost functions. Complete study of efficiency and *all* kinds of inefficiencies for two linear-demand consumers, see our another paper (2002, www.math.nsc.ru/~mathecon/kokovin).

Example 4 (Figure 4): It shows a Pareto-efficient solution, still having positive consumer surplus (informational rent). There are 3 consumers. Costs are assumed to be zero and demands are satiable (or it is a picture of 'net valuations'). All three packages are located at satiation points, and hence they are efficient. This tree is (2→1, 3→1, 1→#0) and other trees cannot bring an optimal package scheme when type 1 consumers are much more numerous than the other two. The second and the third consumers envy the first one, and not #0, so both of them enjoy *informational rent*. In spite of informational rent, efficiency remains, because they symmetrically outweigh each other (a degenerate case) in the sense that $\frac{\partial}{\partial x_1}[V_2(\bar{x}_1) + V_3(\bar{x}_1)] = 0$. As a result, at point $\bar{x}_i$, argmaximum of the function $Q_i(x_i) - C(m, x)$ (required for profitability of $\bar{x}_i$) coincide with the argmaximum of $m_i V_i(x_i) - C(m, x)$, which is required for efficiency of $\bar{x}_i$.

Obviously, such coincidence of argmaxima is a rare and degenerate cases. Normally informational rent precludes efficiency.

5 CONCLUSIONS

For the standard package problem we have analyzed solution structure, envy-cycles, types splitting and implementation of solutions. All these considerations become important and relevant when valuations are non-ordered. Non-standard "rationing" approach for implementation is suggested.

Under weak restrictions, the standard solution has some graph structure that can be used for optimization and for deriving properties. Some conditions for both partial and Pareto-efficiency are established. Under reasonable assumptions, existence of Pareto-efficiency and consumer surplus (informational rent) are mutually exclusive. Conditions for implementability of standard-optimal pricing scheme are derived, counter-examples showing the importance of such conditions.

APPENDIX

Proof OF THEOREM 1:

A. We must show that consumers' set I can be partitioned into a sequence of some $N \leq n$ non-intersecting sets (layers) $\hat{I} = I_0 \cup I_1 \cup ... \cup I_N$ in a special manner. Specifically, each layer I_k must encompass all nodes (consumers) connected to the root with some k-long chain of arcs, which is the shortest way to the root. Let us construct such sequence of layers starting from the root: $I_0 := \{0\}$ and making sure that the next layer is also non-empty. Indeed, if I_1 were empty, one could improve the solution $(\bar{x}, \bar{t})$ by increasing all tariffs $t_i : i \geq 1$ simultaneously for some positive quantity. This is obvious from (1)-(2). So, the first layer is non-empty, i.e., $\emptyset \neq I_1 := \{i : V_i(x_i) - t_i = 0\}$. By similar arguments, for some agent k not in $I_0 \cup I_1$, her tariff t_k must be bounded from above by some *active* incentive constraint (see inequalities (1)-(2)). Moreover, some of such active constraints must connect agent k to some agent j from the *previous* layer I_1, otherwise one could increase all tariffs simultaneously for agents not in $I_0 \cup I_1$ (connected to I_1 means that the tariff t_j in the right-hand side of some incentive constraint relates to some agent $j \in I_1$). Thus this node k must belong to $I_2 := \{i : V_i(x_i) - t_i = V_i(x_j) - t_j \;\; \exists j \in I_1\}$ and this set is non-empty. Arguing similarly for higher layers, one can further construct the needed non-empty, non-intersecting sequences of layers until I is exhausted. This results in the needed root-connected subgraph.

B. Since a tree can always be chosen from the already found root-connected subgraph, some tree $G = G_T = G(\bar{x}, \bar{t})$ of active-constraints exists. Take any such tree. To prove its properties, one must compare the solutions of the new $G-$specific problem with the optimum of the initial problem. In essence, the only difference between the new (3) and the initial problem (1)-(2) lies in replacing active inequalities $(i, j) \in G$ by similar equalities, thus making the constraints set more restrictive in deriving tariffs $\mathbf{T}_k(x_i) := \sum_{i \in B(k,G) \cup \{k\}} [V_i(x_i) - V_i(x_{b(i,G)})]$. So at point $(\bar{x}, \bar{t})$, the new objective function takes the optimal value of the old function and the new constraints are satisfied. The new constraints-set is more restrictive than the old one, so no admissible $(\tilde{x}, \tilde{t})$ can be better than $(\bar{x}, \bar{t})$ in the new problem. Therefore, $(\bar{x}, \bar{t})$ is optimal. At the same time, any solution $(\tilde{x}, \tilde{t})$ to the new problem must be as good as $(\bar{x}, \bar{t})$ for both (coinciding) objective functions. Therefore, it is also a solution to the old problem (1)-(2).||

Proof of Proposition 1 ("Chain-Rule," compare with Katz, 1983):

The Proposition states that, at the solution, every agent $i \in I$ "almost-envies" her closest lower-demand neighbor, and no one else. For constraints that "almost-envy" the lower-demand neighbor, we should prove that all these constraints are binding (being equalities) at any solution, while all other remaining constraints are not binding in the sense that they can be eliminated or replaced with constraints $x_i \geq x_{_{i-1}}$ without altering the solution.

1) Consider any incentive-compatible scheme $(\bar{x}, \bar{t})$. By rearranging any pair i, k of incentive-compatibility constraints we get relation $V_k(\bar{x}_i) - V_k(\bar{x}_k) \leq \bar{t}_i - \bar{t}_k \leq V_i(\bar{x}_i) - V_i(\bar{x}_k)$. Note that $\bar{x}_i = \bar{x}_k \Rightarrow \bar{t}_i = \bar{t}_k$. For $i > k$, such inequality together with the Spence-Mirrlees assumption of ordered valuations (OV) entails $\bar{x}_i \geq \bar{x}_k$. Therefore, constraints $[x_i \geq x_{_{i-1}} \forall i]$ are satisfied by any admissible (x, t), and adding them to initial constraints do not change the admissible set. When using ordering $[\bar{x}_i \geq \bar{x}_{_{i-1}} \forall i]$, note that when some consumer i does not (strictly) envy someone whose consumption level (and to related tariff) is lower $\bar{x}_k < \bar{x}_i$; then any higher-demand agent $j > i$ also does not envy $(\bar{x}_k, \bar{t}_k)$. Indeed, from $[\bar{t}_i - \bar{t}_k \leq V_i(\bar{x}_i) - V_i(\bar{x}_k),\ \bar{x}_k < \bar{x}_i] \Rightarrow$ (using OV under $j > i, \bar{x}_k < \bar{x}_i \leq \bar{x}_j$) $[\bar{t}_i - \bar{t}_k \leq V_j(\bar{x}_i) - V_j(\bar{x}_k)] \Rightarrow$ (using incentive constraint $\bar{t}_i \geq \bar{t}_j - V_j(\bar{x}_j) + V_j(\bar{x}_i)$ and replacing $\bar{t}_i$ with smaller expression) $[(\bar{t}_j - V_j(\bar{x}_j) + V_j(\bar{x}_i)) - \bar{t}_k \leq V_j(\bar{x}_i) - V_j(\bar{x}_k)] \Rightarrow$

$[\bar{t}_j - \bar{t}_k \leq V_j(\bar{x}_j) - V_j(\bar{x}_k)]$. Thus, without changing the admissible set, we can eliminate all constraints that represent envying a lower number agent's $(k < j)$ constraints (they are nonessential after constraints $[x_i \geq x_{_{i-1}} \forall i]$ are added), except the neighboring constraints $(k = j - 1 < j)$.

2) Now suppose that $(\bar{x}, \bar{t})$ not only is feasible but it is also a solution. Consider the chain of "neighboring" constraints $[0 \leq V_1(\bar{x}_1) - \bar{t}_1; V_2(\bar{x}_1) - \bar{t}_1 \leq V_2(\bar{x}_2) - \bar{t}_2; \ldots$ $\ldots V_n(\bar{x}_{n-1}) - \bar{t}_{n-1} \leq V_n(\bar{x}_n) - \bar{t}_n]$. If the first inequality were strict, we could increase tariff t_1, thereby improving the objective function, other inequalities are satisfied. But, this contradicts the supposition that $(\bar{x}, \bar{t})$ was a solution. Similarly, all relations of this chain are equalities at any solution of the initial (and of the modified) problem. Combining these equalities with ordering $[\bar{x}_i \geq \bar{x}_{i-1}]$ and the assumption of OV, entails $[V_i(\bar{x}_i) - V_i(\bar{x}_{i-1}) = \bar{t}_i - \bar{t}_{i-1} \geq V_{i-1}(\bar{x}_i) - V_{i-1}(\bar{x}_{i-1}) \geq V_{i-2}(\bar{x}_i) - V_{i-2}(\bar{x}_{i-1})]$, implying

that any agent $i-1$ does not strictly-envy her higher-number neighbor i. Now take the similar relation $[\bar{t}_{i-1} - \bar{t}_{i-2} \geq V_{i-2}(\bar{x}_{i-1}) - V_{i-2}(\bar{x}_{i-2})]$ for agent $i-2$. By substituting here the left side of former-obtained expression $\bar{t}_i - V_{i-2}(\bar{x}_i) + V_{i-2}(\bar{x}_{i-1}) \geq \bar{t}_{i-1}$ for $\bar{t}_{i-1}$, we get $[\bar{t}_i - V_{i-2}(\bar{x}_i) \geq \bar{t}_{i-2} - V_{i-2}(\bar{x}_{i-2})]$. Thus all constraints that represent not envying the higher-number agents are redundant in the modified problem. They follow from ordering $\bar{x}_i \geq \bar{x}_{i-1}$, OV and equalities $[V_i(\bar{x}_i) - V_i(\bar{x}_{i-1}) = \bar{t}_i - \bar{t}_{i-1},\ \bar{x}_i \geq \bar{x}_{i-1}\ \forall i]$.

Thus, the set of solutions to initial problem is equivalent to the set of solutions to the modified problem. *Q.E.D.*

We can add that when functions increase, then the implications is $[\bar{x}_i \geq \bar{x}_k \Leftrightarrow \bar{t}_i \geq \bar{t}_k]$.

Proof of Proposition 3:

(a) Because the cost function is separable, the notion of profit-summit S for a solution $(\bar{x}, \bar{t})$ is definite. Let S include the first consumer, i.e., $(\bar{t}_1 - c(\bar{x}_1)) \geq (\bar{t}_i - c(\bar{x}_i))\, \forall i \in \bar{\bar{G}}$. Assume this node $(\bar{x}_1, \bar{t}_1)$ is partially-inefficient, implying that it could be replaced by an alternative node $(\tilde{x}_1, \tilde{t}_1) \in L_1(\bar{x}, \bar{t}) := \{(a,t) \in X \times R | V_1(a) - t \geq V_1(\bar{x}_1) - \bar{t}_1\}$ which is weakly-preferred to $(\bar{x}_1, \bar{t}_1)$ and gives more profit than $(\bar{x}_1, \bar{t}_1)$ and other packages: $(\tilde{t}_1 - c(\tilde{x}_1)) > (\bar{t}_i - c(\bar{x}_i))\, \forall i \in \bar{\bar{G}}$ (by transitivity). When we replace first agent's package $(\bar{x}_1, \bar{t}_1)$ with the new one $(\tilde{x}_1, \tilde{t}_1)$, she has no incentive to switch from $(\tilde{x}_1, \tilde{t}_1)$ to any other package. Since $(\tilde{x}_1, \tilde{t}_1) \in L_1$, her incentive-compatibility constraints are not violated. However, other agents $i \neq 1$ may now switch to this new package $(\tilde{x}_1, \tilde{t}_1)$ (i.e., their old packages may become incentive incompatible). But this switching is profitable. More formally, the initial bundle $(\bar{x}, \bar{t})$ can be transformed into a new bundle $(\tilde{x}, \tilde{t})$, such that $(\bar{x}_1, \bar{t}_1)$ is replaced by $(\tilde{x}_1, \tilde{t}_1)$ and $(\tilde{x}_1, \tilde{t}_1)$ also replaces all packages $(\bar{x}_i, \bar{t}_i)$ envying $(\tilde{x}_1, \tilde{t}_1)$, i.e., for all $i : V_i(\tilde{x}_1) - \tilde{t}_1 > V_i(\bar{x}_i) - \bar{t}_i$. This new bundle $(\tilde{x}, \tilde{t})$ satisfies all constraints and brings more profit, so the initial package could not have been optimal. A contradiction.

Similarly we prove that each profit-summit S is envy-free, otherwise an agent who almost-envies a package from S could be given this package resulting in a higher profit.

(b) Unimodality assumption means that the *unconstrained* maximum of personalized welfare function (that can be reduced to $V_i(x) - c(x)$ for maximization) on X is unique.

It coincides with the constrained maximum $\bar{x}_i$ for $i \in S$, because all the constraints in (4) can be shown to be non-binding. Indeed, as explained in the proof of (a), if this welfare function had a better value at some point $\tilde{x}_i \in X$, then $\bar{x}_i$ could be replaced by $\tilde{x}_i$ improving profit (due to maximum-profit assumption for $\bar{x}_i : i \in S$). Suppose there existed two different package nodes $k, k' \in S : (\bar{x}_{k'}, \bar{t}_{k'}) \neq (\bar{x}_k, \bar{t}_k)$ connected with an arc of envy: $\exists i \in S : (\bar{x}_i, \bar{t}_i) = (\bar{x}_k, \bar{t}_k)$, $V_i(\bar{x}_{k'}, \bar{t}_{k'}) - \bar{t}_{k'} = V_i(\bar{x}_k, \bar{t}_k) - \bar{t}_k$. Taking into account that profit level of nodes $k, k' \in S$ is the same, this then means that the objective function $V_i(x) - c(x)$ has two different maxima $(\bar{x}_{k'}, \bar{t}_{k'}) \neq (\bar{x}_k, \bar{t}_k)$ on X. This contradicts the unimodality assumption.||

PROOF OF PROPOSITION 4: Take any package-node $k \in S$ from branch-summit S (the only non-trivial case is when S contains several cycled nodes) and any agent-node i belonging to k : $(\bar{x}_k, \bar{t}_k) = (\bar{x}_i, \bar{t}_i)$. Define agent i's strictly-higher-profit set $M_i \subset R^2$ as $M_i(\bar{x}, \bar{t}) := \{(\xi, \tau) \in X \times R | \tau - C(m, \bar{x}_1, ..., \bar{x}_{i-1}, \xi, \bar{x}_{i+1}, ...\bar{x}_i) > \bar{t}_i - C(m, \bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, ...\bar{x}_i)\}$. Note that, for all agents $(i : (\bar{x}_k, \bar{t}_k) = (\bar{x}_i, \bar{t}_i))$ from k, all such higher-profit sets do coincide: $M_i = M_j$ $(j : (\bar{x}_k, \bar{t}_k) = (\bar{x}_j, \bar{t}_j))$. This is because of the assumption that $m_i = m_j$ and construction of costs. Define also the set $L_i \subset R^2$ consisting of all packages weakly-preferred to $(\bar{x}_i, \bar{t}_i)$ as $L_i := L_i(\bar{x}, \bar{t}) := \{(\xi, \tau) \in X \times R | V_i(\xi) - \tau \geq V_i(\bar{x}_i) - \bar{t}_i\}$. Define the set $K \subset S$ of all agents (including all bunched within k) almost-envying this package $(\bar{x}_k, \bar{t}_k)$. Define their joint set $L_K := \cup_{i \in K} L_i$ containing all weakly-preferred sets of packages. Suppose there exists an intersection point $(\tilde{x}_i, \tilde{t}_i) \in L_K \cap M_i$, so there exists a partially non-efficient agent j: $(\tilde{x}_i, \tilde{t}_i) \in L_j \cap M_j = L_j \cap M_i$. When both L_j, M_i are convex then the whole semi-open linear interval $I_j = (\,(\bar{x}_i, \bar{t}_i), (\tilde{x}_i, \tilde{t}_i)] \subset L_j \cap M_i$ belongs to the named intersection, i.e., brings the same or more payoff to agent j from K (maybe $j = i$) and strictly more profit to principal than $(\bar{x}_j, \bar{t}_j)$. In more general case, instead of straight interval I_j, we have at least some connected set I_j (whose closure $\bar{I}_j$ connects points $(\bar{x}_i, \bar{t}_i), (\tilde{x}_i, \tilde{t}_i) \in \bar{I}_j$) with the same property: $I_j \subset L_j \cap M_i$. Indeed, by concavity assumption on $V_i(.) - C(.)$, the difference-function $\varphi(\xi) := V_i(\xi) - C(m, \bar{x}_1, ..., \bar{x}_{i-1}, \xi, \bar{x}_{i+1}, ...\bar{x}_i) - V_i(\bar{x}_i) - C(m, \bar{x})$ is zero at point $\xi = \bar{x}_i$ and it is positive at point $\tilde{x}_i$, therefore it is positive on the whole interval

$(\bar{x}_i, \tilde{x}_i)$. This enables building the needed connected set I_j. It explains why $L_j \cap M_i$ has points arbitrarily close to $(\bar{x}_i, \bar{t}_i)$. Then $L_K \cap M_i$ has them as well.

Branch-summit, choice of K, and continuity of V_i guarantee that there exists a sufficiently small $\varepsilon > 0$ and closed $\varepsilon-$ neighborhood $B_{\varepsilon K} = B_{\varepsilon K}(\bar{x}_i, \bar{t}_i) := \{(\xi, \tau) : ||(\xi, \tau) - (\bar{x}_i, \bar{t}_i)|| \leq \varepsilon\}$ around the initial point $(\bar{x}_i, \bar{t}_i)$ such that, except for agents from K (whose active indifference curves are crossing at node k) nobody else can envy the points in the neighborhood, i.e., $V_j(\xi) - \tau < V_j(\bar{x}_j) - \bar{t}_j$ $(\forall j \not\in K, \forall (\xi, \tau) \in B_{\varepsilon K})$. By using the established non-emptiness of intersection $L_K \cap M_i \cap B_{\varepsilon K} \neq \emptyset$, we can now construct a new intersection point $(\hat{x}_{\hat{\jmath}}, \hat{t}_{\hat{\jmath}}) \in L_K \cap M_i$ $(\exists \hat{\jmath} \in K)$ belonging also to 'no-envy' vicinity $B_{\varepsilon K}$. In particular, among such points, we can take the highest one in the sense that $(\hat{x}_{\hat{\jmath}}, \hat{t}_{\hat{\jmath}}) : \hat{t}_{\hat{\jmath}} \geq \forall \tau : (\xi, \tau) \in L_K \cap M_i \cap B_{\varepsilon K}$. The broader intersection $L_K \cap \bar{M}_i \cap B_{\varepsilon K}$ (where $\bar{M}_i$ denotes the closure of M_i) is compact by continuity assumption, hence the highest point exists within it. In particular, it should be one of the two intersections between the boundary of L_K and the boundary of $B_{\varepsilon K}$. By definition of M_i , the boundary $\bar{M}_i \setminus M_i$ is the *lower* boundary of this set, so the distinction between $\bar{M}_i$ and M_i does not matter for choosing $(\hat{x}_{\hat{\jmath}}, \hat{t}_{\hat{\jmath}})$. Here $\hat{\jmath}$ denotes an agent whose indifference curve is active at point $(\hat{x}_{\hat{\jmath}}, \hat{t}_{\hat{\jmath}})$, lying on the boundary of L_K (which is always the upper boundary). By construction, there is such an agent $\hat{\jmath} \in K : V_{\hat{\jmath}}(\hat{x}_{\hat{\jmath}}) - \hat{t}_{\hat{\jmath}} = V_{\hat{\jmath}}(\bar{x}_{\hat{\jmath}}) - \bar{t}_{\hat{\jmath}}$, and no one strictly envies this point: $V_i(\hat{x}_{\hat{\jmath}}) - \hat{t}_{\hat{\jmath}} \leq V_i(\bar{x}_i) - \bar{t}_i$ $\forall i = 1, ..., n$. Further take $\hat{\jmath} \equiv j$ for notation simplicity.

Suppose, this j is in the same package-node with initial $i : (\bar{x}_j, \bar{t}_j) = (\bar{x}_i, \bar{t}_i)$. Then, by replacing her package $(\bar{x}_j, \bar{t}_j)$ by the new package $(\hat{x}_j, \hat{t}_j)$, none of incentive-compatibility constraints can be violated (using the 'highest' position of $(\hat{x}_j, \hat{t}_j)$ in $L_K \cap B_\varepsilon$), while the profit strictly increases (we use here the fact that higher-profit set $M_i = M_j$ is the same for both i, j, due to restrictions on $C(.)$ and $m_i = m_j$). This contradicts optimality of $(\bar{x}, \bar{t})$.

In the opposite case, when $(\bar{x}_j, \bar{t}_j) \neq (\bar{x}_i, \bar{t}_i)$, we first move all cycled agents around the cycle connecting i and j, in particular, move one step in the direction of almost-envy. Since this transformation of package scheme means that replacing the packages of almost-envying-agents by the envied packages, it neither disturbs the incentive-compatibility constraints nor the total output and costs (we use assumption $m_i = m_j$). The set of active indifference curves remains the same. After this transformation we come to a contradiction similar to the one in case $i : (\bar{x}_j, \bar{t}_j) = (\bar{x}_i, \bar{t}_i)$

In essence, we have proved not only partial efficiency ($L_i \cap M_i = \emptyset$), but even more than needed: $L_{K(k)} \cap M_i = \emptyset \quad (\forall k \in G_P(S), \forall i \in K(k))$. It is useful further.

Now we prove flatness (within the cycle) of the unified set $L_S := \cup_{i\in S} L_i$ (see Example 2). Under differentiability of $c(.)$, we can use the marginal cost $\bar{c} := \frac{1}{m_i}\partial c(\sum_i m_i \bar{x}_i)/\partial x_j$ ($\exists j \in S$), which is the same $\forall j \in S$, because $m_i = m_j$for separating L_S from M_i. Because of partial-efficiency of $i \in S$, convexity of more-profitable sets M_i and more-utility sets L_i, we can separate each M_i from L_i by a line $L_{0i} := \{(x,t) | \, t = \bar{c}x + c_{0i}\}$ with some constants c_{0i}, and such lines must have the same slope $\bar{c}$ for all i. Note that without differentiability, a *unique* constant $\bar{c}$ separating each M_i from L_i is not guaranteed (see Example 2). So, we may speak of profit-ascending or descending arcs relative to this unique $(1, -\bar{c})$ direction. Arguing as we did for separable cost function, suppose there existed a profit-ascending arc $(i \to j)$ of envy, then the related indifference curve $V_i(x) = t_i$ would ascend from point (x_i, t_i) towards (x_j, t_j) and going somewhere higher than the separating line L_{0i}. This contradicts the separation of M_i from L_i by L_{0i}. We conclude that no ascending arcs exist and branch-summit S becomes a $\bar{c}-$profit-summit of this subgraph S, having nodes not only with the same slope, but with the same profit-level constants $c_{0i} = c_0$. Consequently, the unified more-utility set L_S appears flat in the interval $[(\bar{x}_{\hat{k}}, \bar{t}_{\hat{k}}), (\bar{x}_{\check{k}}, \bar{t}_{\check{k}})]$ stretching from the left-most node $(\bar{x}_{\hat{k}}, \bar{t}_{\hat{k}}) : \bar{x}_{\hat{k}} \le \bar{x}_i \, (\forall i \in S)$ to the right-most node $(\bar{x}_{\check{k}}, \bar{t}_{\check{k}}) : \bar{x}_i \le \bar{x}_{\check{k}} \, (\forall i \in S)$ in S. Here L_S's border is the line L_{0i}. Moreover, all the arcs of envy within the cycle S are the intervals of this line.

Consider COROLLARY 2 under V_i strict concavity. Now flat border of any more-preferred set L_i is impossible, so left-most and right-most nodes $(\bar{x}_{\hat{k}}, \bar{t}_{\hat{k}}), (\bar{x}_{\check{k}}, \bar{t}_{\check{k}})$ of summit S must coincide. This entails the absence of non-reduced cycles. *Q.E.D.*

PROOF OF PROPOSITION 2*:* We shall use similar arguments for each Q_i as we did for V_i in proving PROPOSITION 4(b).

An optimal solution enables one to optimize w.r.t. some variables when other variables are fixed. We use induction starting from the highest level of the graph. By Proposition 4 all branch-summits of the active graph have no cycles. For this highest level of the tree (tree is guaranteed by Theorem 1), the absence of cycles was proved by optimizing w.r.t. its variables. All incentive-compatibility constraints (arcs of possible envy) leading *to* these nodes were shown to be non-active. Hence they are non-binding or redundant because of problem gross convexity. Arcs in the reverse direction, *from* these nodes, by construction are taken into account in functions Q_i. So, we can consider another similar package-optimizing problem, without agents related to the highest-level nodes. We eleminate highest-level nodes and arcs, but the valuation functions of second-highest level nodes are replaced by new functions $\tilde{V}_i(.) = Q_i(.)$. Highest-level quantities x_i are fixed in this optimization (being partially-efficient), but, as reflected in Q_i, the related tariffs are optimized. Applying Proposition 4 again to the new problem we exclude cycles on second-highest level. We approach another level in a similar fashion and so on, recursively. *Q.E.D.*

EXAMPLES

Example 5 (Figure 5): It shows optimal splitting of consumer types and cycle of envy due to absence of gross convexity (and interesting quantity-descending digraph). There are 2 identical consumers with concave valuation functions $v_i(x_i) := 0.8x_i - 0.4x_i^2\ (i = 1, 3)$ and a different agent with function $v_2(x_2) := 3.0x_2 - 2.0x_2^2$. Convex cost function C depends on output $\bar{X} = x_1 + x_2 + x_3$, according to $C(\bar{X}) := 0.02/(2 - \bar{X})$ when $\bar{X} \leq 2$; and $C(\bar{X}) := \infty$ when $\bar{X} > 2$. So production of more than 2 units is impossible (this is the essence of the example). Optimal package scheme (found by numerical iterative optimization) is: $(x_1, T_1) = (0, 0)$, $(x_2, T_2) \approx (0.487, 0.986)$, $(x_3, T_3) \approx (1.375, 0.344)$ (similar splitting effect can occur in other examples without having $x_i = 0\, \exists i$). Here 'cycle of envy' connects agents 1 and 3, while agents 2 has envy-free and partially-efficient package. Thus *identical* agents numbered 1 and 3 do choose *different packages.* That is, *splitting* of agents type (1 and 3) is optimal. It can be checked that this is strictly better than any other plan.

Further, one can check, that by removing the constraint of round number for the allocated packages $a_{ik} \in Z_+$ (but keeping constraint $a_{11} + a_{33} = 2$) we should get almost the same solution with optimal splitting. Therefore this effect does not depend upon agents' discreetness or continuity.||

Example 6:[22] Figure 6 shows three packages and the related active indifference curves. Agents #2 and #3 have similar valuations and envy each other, while agent #1 has envy-free but inefficient node. In the deterministic interpretation of Example 6 (as well as in Example 5) rationing may increase expected profit beyond the standard-optimal! Indeed, suppose that agent #1 comes last to buy after agents 2 and 3 have chosen their packages. Suppose, the principal knows the existence of exactly 3 agents of certain types in the market. In this situation he can easily identify that it is the type #1 agent coming now, since the other two largest packages are already sold. So, if the principal is able to use rationing, he should offer now only package #1 to agent #1 stating that other packages are unavailable. This strategy increases profit at the Pareto-efficient level giving the first best outcome!

Similarly, in Example 4, after agent #1 takes her package, thus revealing that only agents #2 and #3 remain in the market, the principal can withdraw package #1 and raise tariffs for packages #2 and #3.||

Department of Economics, University of Louisville, Louisville, Kentucky 40292, USA. nahata@louisville.edu,

Department of Economics, Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090, Russia. kokovin@math.nsc.ru,

Department of Economics, Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090, Russia. ezhel@ieie.nsc.ru

[22]Mainly, it explains profitable rationing. Besides, Example 6 (Fig. 6) shows that: 1) claim (*a)* of Proposition 3 cannot be easily extended to non-summit subgraph; 2) even under separable costs, in the absence of concavity of valuations there can be essential cycles at profit-summit and at envy-summit (branch-summit); 3) when some non-active constraint binds, there can be a separate partially-inefficient leaf-node (envy-free node).

REFERENCES

Guesnerie, R., and J.J. Laffont (1984): "A Complete Solution to a Class of a Principal-Agent Problems with an Application to the Control of a Self-managed Firm," *Journal of Public Economics,* 25, 329-369.

Guesnerie, R., and J. Seade (1982): "Nonlinear Pricing in a Finite Economy," *Journal of Public Economics,* 17, 157-179.

Katz, M.L. (1983): "Nonuniform Pricing, Output and Welfare Under Monopoly," *Review of Economic Studies*, 50, 37-56.

Katz, M.L. (1984): "Nonuniform Pricing, with Unobservable Purchases," *Review of Economic Studies*, 51, 461-70.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica, 62, 157-180.*

Nahata, B., S. Kokovin, and E. Zhelobodko (2002): "Package Sizes, Tariffs, Quantity Discounts and Premiums," *Working paper*, www.math.nsc.ru/ ~mathecon/kokovin.

Salanie, B. (1997): *The Economics of Contracts,* Cambridge, MA: MIT Press.

Varian, H.R. (1989): "Price Discrimination," in *Handbook of Industrial Organization*, Vol. I, ed. by R.Schmalensee and R.D. Willig, Amsterdam: Elsevier Science, 597-654.

Wilson, R.B. (1993): *Nonlinear Pricing,* Oxford: Oxford University Press.

Figures

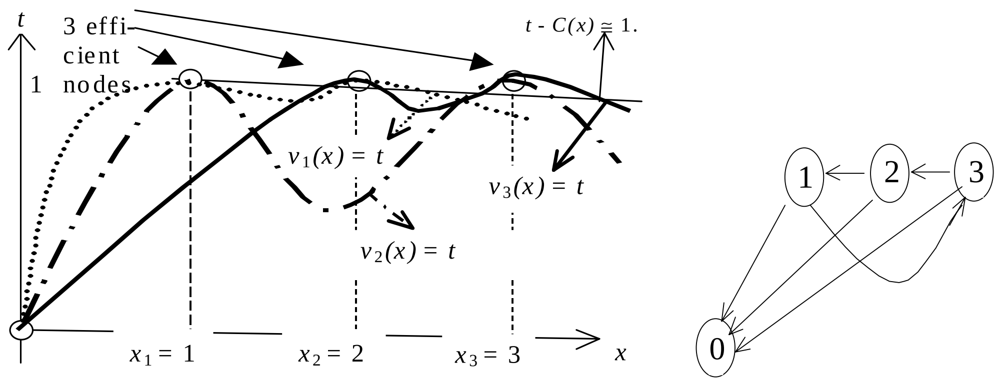

FIGURE 1. Example 1: a cycle non-implementable by rations and rewards.
Left: active net-valuation curves with gradients, active cost curve.
Right: solution's active graph (envy-graph).

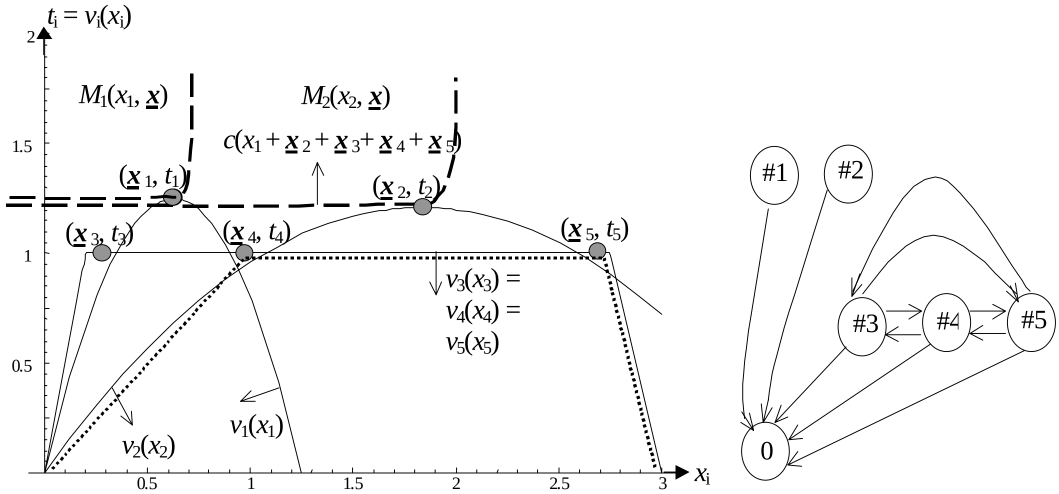

FIGURE **2**. Example 2: splitting and non-implementable cycle, in spite of convexity.
Left: active net valuation curves, gradients, packages.
Right: solution’s active graph.

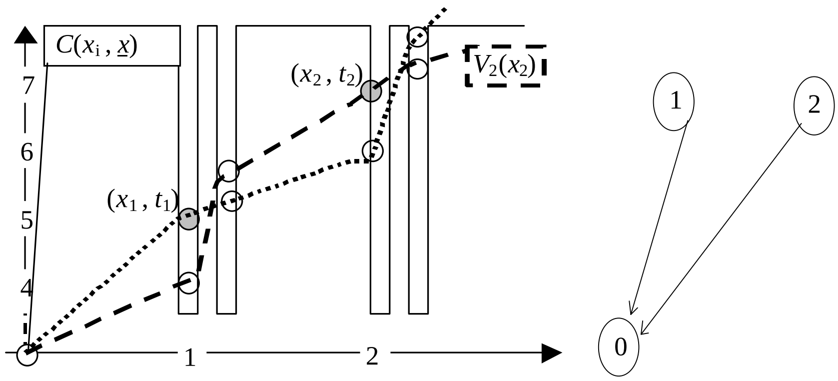

FIGURE 3. Example 3: All nodes envy-free but inefficient (non-active binding constraints).

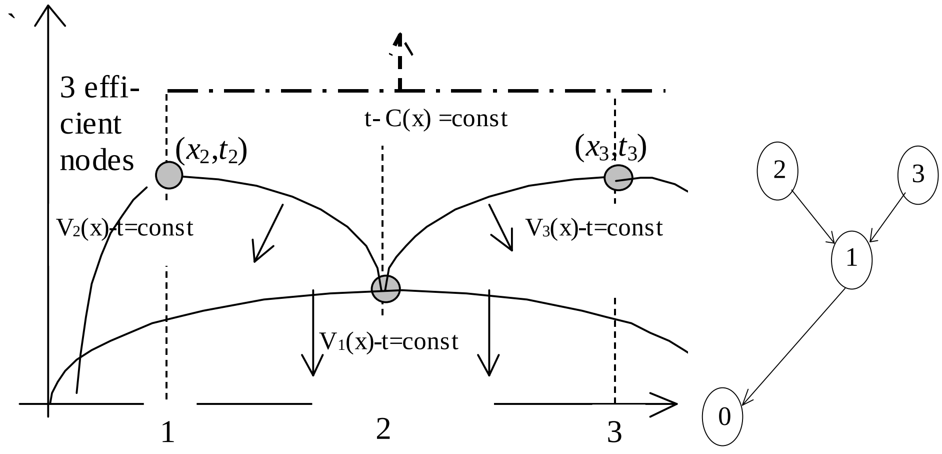

FIGURE 4 Example 4: Informational rent in spite of efficiency.

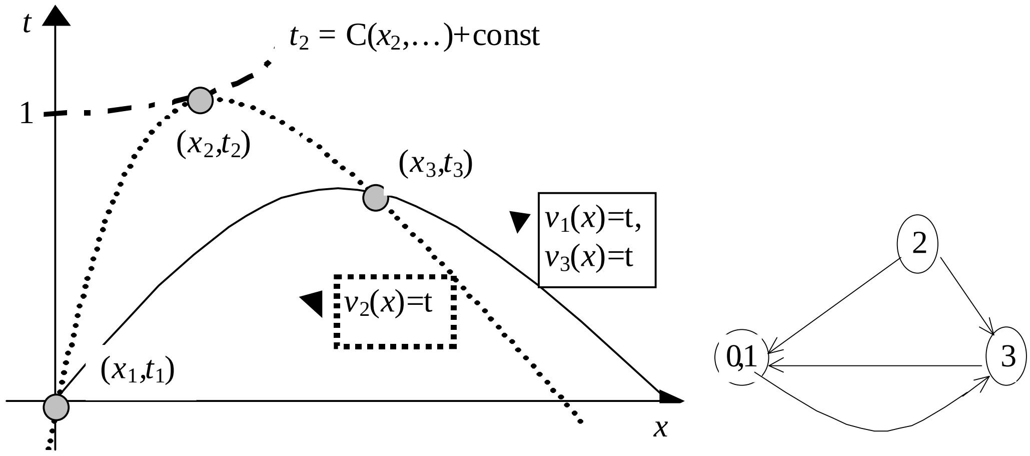

FIGURE 5. Example 5: a cycle non-implementable by rewards.
Left: active valuation curves, active cost curve.
Right: envy-graph: splitting and bunching.

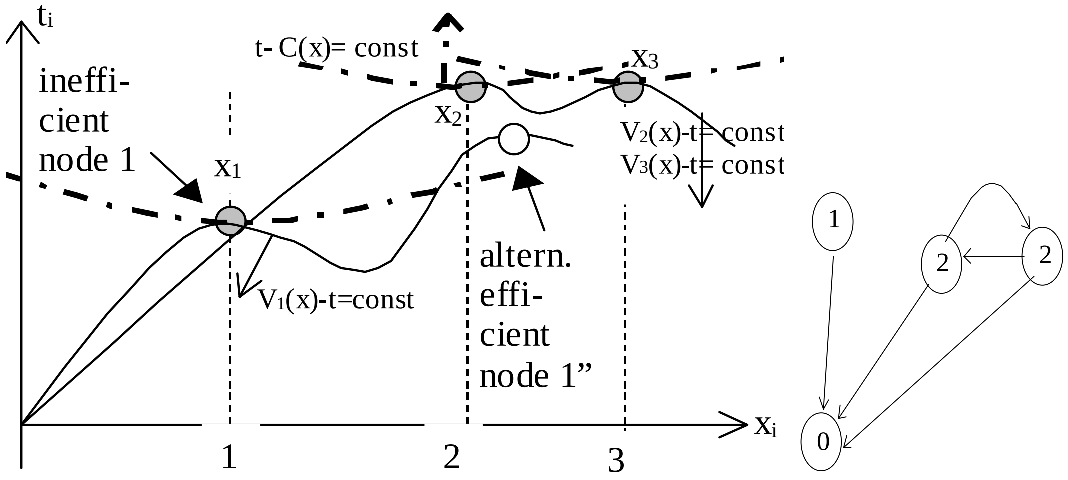

FIGURE 6. Example 6: envy-free but inefficient node (non-active binding constraint). Rationing increases profit.