

Optimal Consumption Choice under Uncertainty with Intertemporal Substitution[†]

Peter Bank*
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin

Frank Riedel**
Institut für Wirtschaftstheorie I
Humboldt-Universität zu Berlin
Spandauer Straße 1
D-10178 Berlin

Abstract

We extend the analysis of the intertemporal utility maximization problem for Hindy-Huang-Kreps utilities reported in Bank and Riedel (1998) to the stochastic case. Existence and uniqueness of optimal consumption plans are established under arbitrary convex portfolio constraints, including the cases of both complete and incomplete markets. For the complete market setting, Kuhn-Tucker-like necessary and sufficient conditions for optimality are given. Using this characterization, we show that optimal consumption plans are obtained by reflecting the associated level of satisfaction on a stochastic lower bound. When uncertainty is generated by a Lévy process and agents exhibit constant relative risk aversion, closed-form solutions are derived. Depending on the structure of the underlying stochastics, optimal consumption occurs at rates, in gulps, or singular to Lebesgue measure.

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Introduction

Standard intertemporal choice theory has long assumed that agents derive their satisfaction directly from the *rate of consumption*. Hindy, Huang, and Kreps (1992) (and, for the stochastic case, Hindy and Huang (1992)) strikingly show that such a specification of intertemporal preferences does not capture the substitutability of consumption over time, the reason being that the rate of consumption reacts too sensitively to small changes of the lifetime consumption plan. As a remedy Hindy, Huang, and Kreps (1992) proposed to replace the rate of consumption with some *level of satisfaction* which keeps track of past consumption. Indeed, preferences based on the latter quantity treat consumption of one and the same good at different but nearby times as substitutes.

Once this new approach to intertemporal choice theory has been accepted, it is important to understand the consumption behavior such preferences induce. This is the aim of the present paper, which extends our previous work Bank and Riedel (1998) on Hindy-Huang-Kreps-preferences under certainty to the choice problem under uncertainty.

Approaching the utility maximization problem at first from a general perspective, we establish existence and uniqueness of optimal consumption plans. Existence of a solution to the utility maximization problem is an issue in the stochastic framework, since budget sets are no longer compact as in the deterministic setting. Using a new method which is based on a theorem of Komlós (1967) and its extension by Kabanov (1999), we are able to give a short proof of existence of an optimal policy under convex portfolio constraints. This includes the cases of complete as well as incomplete markets, and extends the result of Jin and Deng (1997) who prove existence for the special case of short-sale constraints in a diffusion model.

Moving on, we study the characterization and construction of optimal consumption plans when markets are complete. In this context, Hindy and Huang (1993) use the *Bellman methodology* to derive sufficient conditions for optimality based on the Hamilton-Jacobi-Bellman equation; in a special case, this allows them to give an explicit solution. Instead of using the Bellman approach, we extend the *Kuhn-Tucker-like theorem* of Bank and Riedel (1998) from the deterministic to the stochastic framework, thereby obtaining necessary and sufficient conditions for optimality. This way of attacking the problem is similar to the Cox and Huang (1989) method in the time-additive case.

The explicit construction of optimal plans is more difficult than in the deterministic case, where the optimal level of satisfaction is a smooth time-dependent function of the current price for consumption. Since the present context allows for price processes of unbounded variation, the optimal level of satisfaction can no longer be of this form, because — as an average — it typically has bounded variation.

Proceeding from our characterization of optimal plans, we derive an equation (cf. (17)) characterizing what we call the *minimal level of satisfaction*, and show that the investor optimally consumes just enough to keep his level of satisfaction always above this minimum. This allows us to reduce the utility optimization problem to finding a solution to the *minimal level equation*, which, therefore, plays here the same role as the Hamilton-Jacobi-Bellman equation in the dynamic programming approach.

In a homogeneous setting where in particular prices are driven by a Lévy process, we provide an explicit solution to the minimal level equation. This gives the explicit description of the optimal consumption plan and allows us to calculate the indirect utility in

closed-form.

We carry out several case studies illustrating the considerable flexibility of the Hindy-Huang-Kreps framework. The whole variety of consumption patterns can arise, depending only on the structure of the respectively chosen stochastics. If state prices are driven by Brownian motion, optimal consumption is singular with respect to Lebesgue measure, as already pointed out by Hindy and Huang (1993). If prices are driven by a Poisson process, the occurring price shocks or — in the terminology of Hindy and Huang (1992) — the corresponding information surprises induce the investor to consume in gulps, if there is a 'nice' downward price shock. If prices jump upward, he refrains from consumption for a while, until he has got 'unsatisfied' or rich enough to make him willing to consume again.

An outline of the paper is as follows. In Section 1 we introduce the general technical framework and formulate the utility maximization problem. Section 2 proves existence and uniqueness of a solution. In Section 3 we give necessary and sufficient conditions for optimality when the considered financial market is complete. Furthermore, we investigate the general structure of optimal consumption plans and motivate the concept of the 'minimal level of satisfaction'. Finally, Section 4 provides some explicit solutions.

1 Formulation of the Utility Maximization Problem

Consider an investor who wishes to consume his initial wealth $w \geq 0$ over a fixed finite time period $[0, T]$. Assume he can invest in a money market account with bounded interest rate $(r(t), 0 \leq t \leq T)$ and at least one risky security. Uncertainty is described by a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t, 0 \leq t \leq T), \mathbb{P})$ satisfying the usual conditions of right continuity and completeness; \mathcal{F}_0 is \mathbb{P} -a.s. trivial. A priori, the consumption plans at the investor's deposit are given by

$$\mathcal{X}_+ \triangleq \{C \mid C \text{ is the distribution function of an optional random measure}^1\},$$

while his budget-feasible set is

$$\mathcal{A}(w) \triangleq \{C \in \mathcal{X}_+ \mid \Psi(C) \leq w\}.$$

Here, $\Psi(C) \in [0, \infty]$ denotes the minimal initial capital needed to finance a given consumption plan $C \in \mathcal{X}_+$ by investing in the assets of the financial market. In fairly general situations (see Remark 1.1 below), this quantity can be expressed in the form ²

$$(1) \quad \Psi(C) \triangleq \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \int_0^T y(t) dC(t) \quad (C \in \mathcal{X}_+)$$

where $y(t) \triangleq \exp\left(-\int_0^t r(s) ds\right)$ and \mathcal{P} is a fixed nonempty set of probability measures on (Ω, \mathcal{F}_T) . The specific choice of this set is determined by the risk-structure of the considered financial market. The elements of \mathcal{P} are called risk-neutral measures.

¹Equivalently, $C : \Omega \times [0, T] \rightarrow \mathbb{R}$ is a nonnegative, adapted process with increasing, rightcontinuous paths.

²By convention, integration over time intervals is carried out including the involved finite boundaries. We let any consumption stream start in $C(0-) \triangleq 0$; a positive value at time 0 indicates an initial consumption gulp and corresponds to a point mass $C(0) > 0$ of the random measure dC at time $t = 0$. Similarly, we henceforth assume that any other integrator B starts from some initial value $B(0-)$ assumed to be zero unless otherwise stated.

Remark 1.1 Note that the above formulation allows for incomplete markets and, more generally, even for markets under convex constraints; see, e.g., Föllmer and Kabanov (1998), Föllmer and Kramkov (1997), Cvitanic and Karatzas (1993).

To illustrate this further, let us consider a model of a security market consisting of a riskless bond and a stock. Assume short selling of the stock is prohibited. Föllmer and Kramkov (1997) show that this economic setting may be captured by choosing

$$\mathcal{P} \triangleq \{\mathbb{P}^* \sim \mathbb{P} \mid \mathbb{P}^* \text{ is a supermartingale measure for each } S \in \mathcal{S}\},$$

where \mathcal{S} denotes the set of all gain processes which are attainable by some admissible strategy without short selling. More precisely, they prove that

$$\sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[\gamma(T)H]$$

is the minimal amount needed to hedge a given contingent claim $H \geq 0$ with maturity T . For a consumption plan $C \in \mathcal{X}_+$, this induces formula (1) for the minimal budget the investor needs to finance it.

With a given consumption plan $C \in \mathcal{X}_+$ the investor associates the utility

$$U(C) \triangleq \int_0^T u(t, Y(C)(t)) dt$$

where $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ denotes a continuous felicity function which is increasing and concave in its second argument, and where

$$Y(C)(t) \triangleq \gamma(t) + \int_0^t \theta(t, s) dC(s)$$

is the investor's level of satisfaction obtained from his consumption up to time $t \in [0, T]$. We assume that the functions $\gamma : [0, T] \rightarrow \mathbb{R}$ and $\theta : [0, T] \times [0, T] \rightarrow \mathbb{R}$ are continuous and nonnegative. $\theta(t, s)$ describes the weight assigned at time t to consumption made at time $s \leq t$; $\gamma(t)$ may be interpreted as an exogenously given level of satisfaction for time t .

Remark 1.2 A standard choice for θ and γ is $\theta(t, s) \triangleq \beta e^{-\beta(t-s)}$ and $\gamma(t) \triangleq \eta e^{-\beta t}$ with constants $\beta, \eta > 0$; compare, e.g., Sundaresan (1989), Constantinides (1990).

The investor's problem is to maximize his expected utility over all budget-feasible consumption plans, i.e.,

$$(2) \quad \max_{C \in \mathcal{A}(w)} \mathbb{E}U(C).$$

2 Existence and Uniqueness

This section proves existence and uniqueness of a solution for the utility maximization problem (2) under the following

Assumption 2.1 *The felicity function u is bounded, or it satisfies the growth-condition*

$$(3) \quad u(t, y) \leq A(1 + y^\alpha) \quad \forall 0 \leq t \leq T, y \geq 0$$

for some constants $A > 0, \alpha \in (0, 1)$ and there is a risk-neutral measure $\hat{\mathbb{P}} \in \mathcal{P}$ with density $\hat{Z} \triangleq \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$ satisfying

$$(4) \quad \hat{Z}^{-1} \in \mathcal{L}^{\hat{\mathbb{P}}}(\mathbb{P})$$

for some $\hat{p} > \frac{\alpha}{1-\alpha}$.

Remark 2.2 *Similar assumptions on felicity functions have been made for the case of time additive functionals in Cox and Huang (1991) and Aumann and Perles (1965). The example in Kramkov and Schachermayer (1998) suggests that it is in general impossible to weaken our assumption. An integrability condition similar to (4) can be found in Cuoco (1997).*

The following is the main result of this section.

Theorem 2.3 *Under Assumption 2.1, the utility maximization problem (2) has a solution. This solution is unique if, in addition, $u(t, \cdot)$ is strictly concave for every $t \in [0, T]$ and $C \mapsto Y(C)$ is injective up to \mathbb{P} -indistinguishability.*

Remark 2.4 *Injectivity of $C \mapsto Y(C)$ follows, e.g., if $\theta(t, s) = \xi(t)\zeta(s)$ for some strictly positive, continuous functions ξ and ζ .*

Let us prepare the proof of Theorem 2.3 by the following technical

Lemma 2.5 (i) *There is a constant $B > 0$ such that*

$$Y(C)(t) \leq B(1 + C(t)) \quad (0 \leq t \leq T)$$

for all $C \in \mathcal{X}_+$.

(ii) *If $C^n \in \mathcal{X}_+$ ($n = 1, 2, \dots$) converge weakly to $C \in \mathcal{X}_+$ then*

$$Y(C^n)(t) \rightarrow Y(C)(t)$$

for every point t of continuity of C .

(iii) *$\mathcal{A}(w)$ is norm-bounded uniformly in $L^1(\mathbb{P}^*)$ ($\mathbb{P}^* \in \mathcal{P}$), i.e.,*

$$\sup_{\mathbb{P}^* \in \mathcal{P}} \sup_{C \in \mathcal{A}(w)} \mathbb{E}^* C(T) < +\infty.$$

(iv) *$\{U(C), C \in \mathcal{A}(w)\}$ is uniformly \mathbb{P} -integrable.*

PROOF : (i) and (ii) follow immediately from our assumptions on $y(\cdot)$ and $\theta(\cdot, \cdot)$. The boundedness of the interest rate process r implies (iii). It remains to prove (iv). This assertion is trivial if u is bounded. For unbounded u we show that $\{U(C), C \in \mathcal{A}(w)\}$ is bounded in $L^p(\mathbb{P})$ with $p \triangleq \frac{\hat{p}}{\alpha(1+\hat{p})} > 1$. Indeed, in this case Assumption 2.1 and (i) yield

$$\mathbb{E}[U(C)^p] \leq \mathbb{E} \left(\int_0^T A(1 + Y(C)(t)^\alpha) dt \right)^p \leq c(1 + \mathbb{E}[C(T)^{\alpha p}])$$

for some constant $c > 0$. Note that $\alpha p < 1$, and apply Hölder's inequality to get

$$\mathbb{E}[C(T)^{\alpha p}] \leq \mathbb{E}[C(T)\hat{Z}]^{\alpha p} \mathbb{E}\left[\hat{Z}^{-\frac{\alpha p}{1-\alpha p}}\right]^{1-\alpha p} \leq cw^{\alpha p} \mathbb{E}[\hat{Z}^{-\hat{p}}]^{1-\alpha p},$$

for some constant $c > 0$ by (iii). In connection with the above estimation and (4), this yields the desired $L^p(\mathbb{P})$ -boundedness. \square

Remark 2.6 *Since uniform integrability implies \mathcal{L}^1 -boundedness, part (iv) of the above Lemma yields in particular that the value of our maximization problem (2) is finite.*

Now we can give the

PROOF OF THEOREM 2.3 : Choose a maximizing sequence $\tilde{C}^n \in \mathcal{A}(w)$ ($n = 1, 2, \dots$) for (2). By Lemma 2.5 (iii) and Kabanov's version of Komlós' Theorem (Kabanov (1999), Lemma 3.5; Komlós (1967)), there exists a subsequence, again denoted by (\tilde{C}^n) , which is almost surely weakly Cesaro convergent to some $C^* \in \mathcal{X}_+$, i.e., almost surely we have

$$C^n(t) \triangleq \frac{1}{n} \sum_{k=1}^n \tilde{C}^k(t) \rightarrow C^*(t) \quad (n \uparrow +\infty)$$

for every point t of continuity of C^* . By concavity, (C^n) is again a maximizing sequence. We claim that C^* is optimal for (2). Indeed, since γ is continuous, we have

$$\int_0^T \gamma(t) dC^*(t) = \lim_n \int_0^T \gamma(t) dC^n(t) \mathbb{P}\text{-a.s.}$$

Hence, by Fatou's Lemma,

$$\mathbb{E}^* \int_0^T \gamma(t) dC^*(t) \leq \liminf_n \mathbb{E}^* \int_0^T \gamma(t) dC^n(t) \leq w,$$

for every $\mathbb{P}^* \in \mathcal{P}$, i.e., $C^* \in \mathcal{A}(w)$. Furthermore, Lemma 2.5 (i) and (ii) yield $U(C^n) \rightarrow U(C^*)$ for $n \uparrow +\infty$ \mathbb{P} -a.s. by dominated convergence. In connection with part (iv) of the same lemma, this implies optimality of C^* by Lebesgue's Theorem.

If two solutions \tilde{C} and C^* are not indistinguishable, then, by assumption, neither are their respective levels of satisfaction $\tilde{Y} \triangleq Y(\tilde{C})$ and $Y^* \triangleq Y(C^*)$. Optimality excludes that they only defer at time $t = T$ because this would imply a (suboptimal) final jump by one of the policies. Thus, on a set with positive probability \tilde{Y} and Y^* defer on an open time interval. Hence, by strict concavity of $u(t, \cdot)$ for any $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}U\left(\frac{1}{2}\{\tilde{C} + C^*\}\right) &= \mathbb{E} \int_0^T u\left(t, \frac{1}{2}\{\tilde{Y}(t) + Y^*(t)\}\right) dt \\ &> \mathbb{E} \int_0^T \frac{1}{2} \{u(t, \tilde{Y}(t)) + u(t, Y^*(t))\} dt \\ &= \frac{1}{2} \{\mathbb{E}U(\tilde{C}) + \mathbb{E}U(C^*)\} \\ &= \max_{C \in \mathcal{A}(w)} \mathbb{E}U(C) \end{aligned}$$

in contradiction to $\frac{1}{2}\{\tilde{C} + C^*\} \in \mathcal{A}(w)$ and to the optimality of \tilde{C} and C^* in this set. \square

3 Solutions in the Complete Case

Assume now that the financial market is complete in the sense that $\mathcal{P} = \{\hat{\mathbb{P}}\}$ is a singleton. Thus there is precisely one risk neutral measure $\mathbb{P}^* = \hat{\mathbb{P}}$. Set $\hat{Z}(t) \triangleq \mathbb{E}[\hat{Z} | \mathcal{F}_t]$ ($0 \leq t \leq T$) and suppose that \hat{Z} satisfies the integrability condition of Assumption 2.1. Let $\psi(t) \triangleq y(t)\hat{Z}(t)$ ($0 \leq t \leq T$) denote the associated state-price density. From now on, we require, in addition to Assumption 2.1, the following

Assumption 3.1 *The felicity function $u = u(t, y)$ is strictly concave and differentiable in y .*

Since a strictly concave and increasing function is strictly increasing, by the above assumption the investor's utility function is non-satiated and he will therefore always exhaust his budget.

3.1 Necessary and Sufficient Conditions for Optimality

In the described context we can formulate and prove the following analogue of the Kuhn-Tucker Theorem for the utility maximization problem (2). It provides a characterization of the solution to (2) in terms of the functional

$$(5) \quad \phi(C)(t) \triangleq \mathbb{E} \left[\int_t^T \partial_y u(s, Y(C)(s)) \theta(s, t) ds \middle| \mathcal{F}_t \right] \quad (0 \leq t \leq T, C \in \mathcal{X}_+).$$

Remark 3.2 $\phi(C)(t)$ may be interpreted as marginal utility resulting from an additional infinitesimal consumption at time t , otherwise following the consumption plan $C \in \mathcal{X}_+$. Note furthermore that $\phi(C)(t)$ is always well-defined as a random variable taking values in $[0, +\infty]$ because $\partial_y u$ and θ are nonnegative. Mathematically, $\phi(C)$ may be interpreted as the Riesz representation of the utility gradient at C , as pointed out by Duffie and Skiadas (1994) in their Example 5.

Theorem 3.3 *A consumption plan $C^* \in \mathcal{X}_+$ solves (2) if and only if the following conditions hold true for some Lagrange multiplier $M \geq 0$:*

- (i) $\Psi(C^*) = w$,
- (ii) $\phi(C^*) \leq M\psi$ \mathbb{P} -a.s.,
- (iii) C^* is flat off $\{\phi(C^*) = M\psi\}$ \mathbb{P} -a.s., i.e.,

$$\mathbb{E} \int_0^T 1_{\{\phi(C^*)(t) \neq M\psi(t)\}} dC^*(t) = 0.$$

PROOF : Let us first prove sufficiency. Take a budget-feasible $C \in \mathcal{A}(w)$ and let $Y \triangleq Y(C)$, $Y^* \triangleq Y(C^*)$. By concavity of u and definition of Y and Y^* , one has

$$\mathbb{E}U(C^*) - \mathbb{E}U(C) = \mathbb{E} \int_0^T \{u(s, Y^*(s)) - u(s, Y(s))\} ds$$

$$\begin{aligned}
&\geq \mathbb{E} \int_0^T \{\partial_y u(s, Y^*(s))(Y^*(s) - Y(s))\} ds \\
&= \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \theta(s, t) [dC^*(t) - dC(t)] \right\} ds.
\end{aligned}$$

We split up the last expectation in two terms:

$$I^* \triangleq \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \theta(s, t) dC^*(t) \right\} ds$$

and

$$I \triangleq \mathbb{E} \int_0^T \left\{ \partial_y u(s, Y^*(s)) \int_0^s \theta(s, t) dC(t) \right\} ds.$$

For the first term Fubini's Theorem yields

$$I = \mathbb{E} \int_0^T \left\{ \int_t^T \partial_y u(s, Y^*(s)) \theta(s, t) ds \right\} dC(t).$$

Since C is adapted, we may replace the $\{\dots\}$ -term in the above expression by its conditional expectation which, by definition, is $\phi(C^*)$; confer Lemma I.3.12 of Jacod and Shiryaev (1987). Hence,

$$I = \mathbb{E} \int_0^T \phi(C^*)(t) dC(t) \leq M \mathbb{E} \int_0^T \psi(t) dC(t) \leq Mw$$

where the first inequality follows from condition (ii) and the last inequality is due to the budget constraint. By conditions (i) and (iii), the above calculation carried out for C^* instead of C shows

$$I^* = \mathbb{E} \int_0^T \phi(C^*)(t) dC^*(t) = M \mathbb{E} \int_0^T \psi(t) dC^*(t) = Mw.$$

Summing up, we find

$$\mathbb{E}U(C^*) - \mathbb{E}U(C) \geq I^* - I \geq Mw - Mw = 0$$

establishing sufficiency. Necessity follows from Lemma 3.4 and Lemma 3.5 below. \square

Before we attack the necessity part of the proof of Theorem 3.3, let us shortly sketch the argument. The idea is to proceed along the same lines as in the proof of the finite-dimensional Kuhn-Tucker theorem. In a first step, we show that the candidate policy C^* solves the problem linearized around C^* . This is done in Lemma 3.4 below. Solutions of the linear problem are easily characterized (Lemma 3.5), and it follows that C^* has to satisfy the conditions given in the theorem.

Lemma 3.4 *Let $C^* \in \mathcal{A}(w)$ be optimal for (2) and let $\phi^* \triangleq \phi(C^*)$. Then C^* solves the linear problem*

$$(6) \quad \max_{C \in \mathcal{A}(w)} \mathbb{E} \int_0^T \phi^*(t) dC(t),$$

and the value of this problem is finite.

PROOF : Consider $C \in \mathcal{A}(w)$ and let $C^\epsilon \triangleq \epsilon C + (1 - \epsilon)C^*$ ($0 \leq \epsilon \leq 1$). By optimality of C^* and concavity of $u(t, \cdot)$ ($0 \leq t \leq T$), we have for $Y^\epsilon \triangleq Y(C^\epsilon)$, $Y \triangleq Y(C)$, $Y^* \triangleq Y(C^*)$

$$\begin{aligned}
0 &\geq \frac{1}{\epsilon} \{\mathbb{E}U(C^\epsilon) - \mathbb{E}U(C^*)\} \\
&= \mathbb{E} \int_0^T \frac{1}{\epsilon} \{u(s, Y^*(s) + \epsilon(Y(s) - Y^*(s))) - u(s, Y^*(s))\} ds \\
&\geq \mathbb{E} \int_0^T \{\partial_y u(s, Y^\epsilon(s))(Y(s) - Y^*(s))\} ds \\
&= \mathbb{E} \int_0^T \{\partial_y u(s, Y^\epsilon(s)) \int_0^s \theta(s, t) [dC(t) - dC^*(t)]\} ds \\
&= \mathbb{E} \int_0^T \left\{ \int_t^T \partial_y u(s, Y^\epsilon(s)) \theta(s, t) ds \right\} [dC(t) - dC^*(t)] \\
&= \mathbb{E} \int_0^T \phi^\epsilon(t) [dC(t) - dC^*(t)],
\end{aligned}$$

where $\phi^\epsilon(t) \triangleq \int_t^T \partial_y u(s, Y^\epsilon(s)) \theta(s, t) ds$ ($0 \leq t \leq T$). Let furthermore $\phi^* \triangleq \phi^0$. By Fatou's Lemma we have

$$(7) \quad \liminf_{\epsilon \downarrow 0} \mathbb{E} \int_0^T \phi^\epsilon(t) dC(t) \geq \mathbb{E} \int_0^T \phi^*(t) dC(t).$$

We claim that

$$(8) \quad \lim_{\epsilon \downarrow 0} \mathbb{E} \int_0^T \phi^\epsilon(t) dC^*(t) = \mathbb{E} \int_0^T \phi^*(t) dC^*(t)$$

whence we deduce our assertion as follows. Letting $\epsilon \downarrow 0$ in the above series of estimations, (7) and (8) give us

$$\mathbb{E} \int_0^T \phi^*(t) dC(t) \leq \mathbb{E} \int_0^T \phi^*(t) dC^*(t).$$

Since C and C^* are adapted, Lemma I.3.12 of Jacod and Shiryaev (1987) allows us to replace the term $\phi^*(t)$ in the above inequality by its conditional expectation $\phi^*(t)$, and we obtain indeed optimality of C^* for the linear problem (6).

It remains to prove (8). For this it suffices to show that

$$I^\epsilon \triangleq \int_0^T \phi^\epsilon(t) dC^*(t) \quad (0 \leq \epsilon \leq \frac{1}{2})$$

has a \mathbb{P} -integrable upper bound. In the following chain of inequalities, we use the estimate $\partial_y u(t, y)y \leq u(t, y) - u(t, 0)$ for the concave felicity function and the estimates $Y^\epsilon(s) \geq \frac{1}{2}Y^*(s)$, $\int_0^s \theta(s, t) dC^*(t) \leq Y^*(s)$ for the level of satisfaction:

$$\begin{aligned}
I^\epsilon &= \int_0^T \left\{ \partial_y u(s, Y^\epsilon(s)) \int_0^s \theta(s, t) dC^*(t) \right\} ds \\
&\leq \int_0^T \partial_y u(s, Y^\epsilon(s)) Y^*(s) ds \\
&\leq 2 \int_0^T \partial_y u(s, \frac{1}{2}Y^*(s)) \frac{1}{2}Y^*(s) ds
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_0^T \left(u(s, \frac{1}{2} Y^*(s)) - u(s, 0) \right) ds \\
&\leq 2 \int_0^T \left(u(s, Y^*(s)) - u(s, 0) \right) ds \\
&= 2 \left\{ U(C^*) - \int_0^T u(s, 0) ds \right\}.
\end{aligned}$$

Since $U(C^*)$ is integrable by Lemma 2.5 (iv), we have found the required upper bound for I^ϵ . \square

Let us now discuss the linear problem (6).

Lemma 3.5 *Let ϕ, ψ be two strictly positive, rightcontinuous and adapted processes. Then every solution C^* to the linear optimization problem*

$$(9) \quad \max_{C \in \mathcal{X}_+} \mathbb{E} \int_0^T \phi(t) dC(t) \quad \text{s.t.} \quad \mathbb{E} \int_0^T \psi(t) dC(t) \leq w$$

satisfies

$$(10) \quad \mathbb{E} \int_0^T 1_{\{\phi(t) \neq M\psi(t)\}} dC^*(t) = 0,$$

where

$$M \triangleq \text{ess sup}_{\Omega} \sup_{t \in [0, T]} \frac{\phi(t)}{\psi(t)}.$$

PROOF :

1. We first show that the value v of the linear problem (9) is Mw . Indeed, it is easy to see that $v \leq Mw$. On the other hand, for every $K < M$ the set

$$\left\{ \omega \in \Omega \mid \left(\sup_{t \in [0, T]} \frac{\phi(t)}{\psi(t)} \right) (\omega) > K \right\}$$

has positive probability. Therefore, letting

$$\tau^K \triangleq \inf \left\{ t \in [0, T] \mid \frac{\phi(t)}{\psi(t)} > K \right\}$$

we can find $c \geq 0$ such that $C^K \triangleq c 1_{[\tau^K, T]} \in \mathcal{X}_+$ satisfies $\mathbb{E} \int_0^T \psi(t) dC^K(t) = w$. We have

$$\begin{aligned}
Mw \geq v &\geq \mathbb{E} \int_0^T \phi(t) dC^K(t) = \mathbb{E} \left[c \phi(\tau^K) 1_{\{\tau^K < +\infty\}} \right] \\
&\geq \mathbb{E} \left[c K \psi(\tau^K) 1_{\{\tau^K < +\infty\}} \right] = K \mathbb{E} \int_0^T \psi(t) dC^K(t) \\
&= Kw.
\end{aligned}$$

Letting $K \uparrow M$ in the above inequality yields $v = Mw$.

2. Suppose that C^* is a solution to (9). Then by 1. and the definition of M

$$Mw = \mathbb{E} \int_0^T \phi(t) dC^*(t) \leq M \mathbb{E} \int_0^T \psi(t) dC^*(t) \leq Mw$$

implying (10). \square

3.2 The Structure of Optimal Consumption Plans

As the finite-dimensional Kuhn-Tucker Theorem, our infinite-dimensional version (Theorem 3.3) does not yield an explicit description of the optimum. However, we can use the obtained characterization to analyze the general structure of the solution, as we will show in this section. The main result of this analysis will be Theorem 3.12. This theorem provides an equation characterizing what we call the 'minimal level of satisfaction'. This is an adapted process $I = (I(t), 0 \leq t \leq T)$ which gives us a canonical lower boundary at which the investor should optimally 'reflect' his level of satisfaction. In our non-Markovian setup, this equation plays the same role as the Hamilton-Jacobi-Bellmann equation in Dynamic Programming.

As a first application of Theorem 3.3, we prove a version of the Dynamic Programming Principle:

Proposition 3.6 *A consumption plan which is optimal at time zero is its best continuation at any other time $t > 0$. More precisely, if $C^* \in \mathcal{X}_+$ is a solution to (2) then, \mathbb{P} -a.s., it also solves the problem*

$$\text{Maximize } \mathbb{E}[U(C) | \mathcal{F}_t] \text{ subject to } C \equiv C^* \text{ on } [0, t) \text{ and } \Psi_t(C) \leq \Psi_t(C^*)$$

where

$$\Psi_t(C) \triangleq \frac{1}{y(t)} \mathbb{E} \left[\int_t^T y(s) dC(s) \middle| \mathcal{F}_t \right] \quad (C \in \mathcal{X}_+)$$

is the price-functional at time t .

PROOF : Using the first-order conditions satisfied by C^* , this can be shown by the same calculation as for the sufficiency part of Theorem 3.3, now carried out for conditional expectations instead of ordinary expectations. \square

Let us now study the dependency of the optimal consumption plan on the exogenous level of satisfaction $y(\cdot)$. To make this precise, let us specify the following dynamics for the level of satisfaction:

Assumption 3.7 *For a consumption plan $C \in \mathcal{X}_+$, the corresponding level of satisfaction $Y(C)$ evolves according to the ODE*

$$(11) \quad Y(C)(0-) = \eta, \quad dY(C)(t) = \beta(t) (dC(t) - Y(C)(t-) dt) \quad (0 \leq t \leq T)$$

where $\beta(\cdot)$ is a strictly positive, continuous function $[0, T] \rightarrow \mathbb{R}$ and $\eta \geq 0$ is a constant.

Remark 3.8 *The (unique) solution to (11) is given by*

$$Y(C)(t) = \eta e^{-\int_0^t \beta(s) ds} + \int_0^t \beta(s) e^{-\int_s^t \beta(v) dv} dC(s) \quad (0 \leq t \leq T).$$

Thus, the choice

$$y(t) \triangleq \eta e^{-\int_0^t \beta(s) ds}, \quad \theta(t, s) \triangleq \beta(s) e^{-\int_s^t \beta(v) dv} \quad (0 \leq s \leq t \leq T)$$

imbeds the above specification of $Y(\cdot)$ into the context of the preceding sections.

Under Assumption 3.7, $C \mapsto Y(C)$ is injective and, therefore, we may apply Theorem 2.3 to obtain existence and uniqueness of an optimal consumption plan for every choice of initial level of satisfaction $\eta \geq 0$. In order to stress the dependency of this plan on this parameter, let us for the moment denote it by $C^{M,\eta}$; $M > 0$ is the Lagrange multiplier induced by our Kuhn-Tucker Theorem 3.3.

Now, we may ask: How does the optimal plan $C^{M,\eta}$ depend on the initial level of satisfaction η ?

Lemma 3.9 *Let $Y(\cdot)$ and $\tilde{Y}(\cdot)$ denote the functionals for the level of satisfaction with initial value η and $\tilde{\eta}$, respectively. Suppose $0 \leq \eta \leq \tilde{\eta}$.*

Then the respective optimal levels of satisfaction $Y^ \triangleq Y(C^{M,\eta})$, $\tilde{Y}^* \triangleq \tilde{Y}(C^{M,\tilde{\eta}})$ with the same Lagrange parameter $M > 0$ are related by*

$$(12) \quad \tilde{Y}^*(t) = \tilde{\eta} e^{-\int_0^t \beta(s) ds} \vee Y^*(t) \quad (0 \leq t \leq T).$$

In particular, we have

$$(13) \quad dC^{M,\tilde{\eta}}(t) = 1_{\{\tau < t \leq T\}} dC^{M,\eta}(t) + \tilde{\Delta} \delta_{\{\tau\}}(dt)$$

where the second summand is the Dirac measure with point mass

$$\tilde{\Delta} \triangleq Y^*(\tau) - \tilde{\eta} e^{-\int_0^\tau \beta(s) ds}$$

at time

$$\tau \triangleq \inf \left\{ t \geq 0 \mid \tilde{\eta} e^{-\int_0^t \beta(s) ds} \geq Y^*(t) \right\}.$$

PROOF : Let $\tilde{C} \in \mathcal{X}_+$ be the consumption plan defined by the right side of (13). From the dynamics for the level of satisfaction specified in Assumption 3.7, it may easily deduced that $\tilde{Y}(\tilde{C})$ coincides with the right side of (12). Moreover, we see that $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, T]$. We will show that \tilde{C} is optimal for the problem with initial level of satisfaction $\tilde{\eta}$ and that it has Lagrange parameter $M > 0$. By uniqueness of this plan, we then obtain Equation (12).

So let us verify the first-order conditions for \tilde{C} . We have for $0 \leq t \leq T$

$$(14) \quad \begin{aligned} \tilde{\phi}(\tilde{C})(t) &= \mathbb{E} \left[\int_t^T \partial_y u(s, \tilde{Y}(\tilde{C})(s)) \theta(s, t) ds \mid \mathcal{F}_t \right] \\ &\leq \mathbb{E} \left[\int_t^T \partial_y u(s, Y^*(s)) \theta(s, t) ds \mid \mathcal{F}_t \right] = \phi(C^{M,\eta})(t) \end{aligned}$$

$$(15) \quad \leq M\psi(t),$$

where Inequality (14) follows from $\tilde{Y}(\tilde{C}) \geq Y^*$; Inequality (15) is due to the first-order conditions satisfied by $C^{M,\eta}$. The above estimate shows that \tilde{C} satisfies the first-order inequality constraint with Lagrange parameter $M > 0$.

Hence, it remains to check the flat-off condition. Note first that $\text{supp } d\tilde{C} \subset [\tau, T]$. Moreover, we have $\tilde{Y}(\tilde{C}) = Y^*$ and, therefore, also $\tilde{\phi}(\tilde{C}) = \phi(C^{M,\eta})$ on this interval. Hence,

$$\mathbb{E} \int_0^T 1_{\{\tilde{\phi}(\tilde{C}) \neq M\psi\}} d\tilde{C} = \mathbb{E} \int_\tau^T 1_{\{\phi(C^{M,\eta}) \neq M\psi\}} d\tilde{C} = 0$$

where the last inequality is due to the absolute continuity of $d\tilde{C}$ with respect to $dC^{M,\eta}$ and to the flat-off condition satisfied by the latter consumption plan. \square

Remark 3.10 *The preceding lemma shows in particular that it suffices to find the optimal consumption plan for $\eta = 0$. All other cases may be recovered from this one by equations (12) and (13).*

Let us now motivate the announced concept of a 'minimal level of satisfaction' by some heuristics. For every time $t \in [0, T)$, consider an agent K_t who starts consuming and investing from that time on. Assume that, from his specific initial time t on up to the common time horizon T , every investor K_t has the same preference characteristics $u(\cdot, \cdot)$, $\beta(\cdot)$ as investor K_0 (who starts at time zero). Suppose furthermore that all investors start, at their respective initial time, with level of satisfaction zero. This means that $Y_t(t-) = 0$ for all $t \in [0, T)$, where $Y_t(\cdot)$ denotes the evolution of the level of satisfaction for agent K_t if he follows his optimal consumption plan. Assume that every agent K_t is endowed (at his respective initial time t) with the amount needed to buy the optimal plan $C_t = C_t^M$ with constant Lagrange parameter $M > 0$.

The Dynamic Programming Principle allows us to interpret, for every time t , the level of satisfaction $Y_s(t)$, achieved by agent K_s ($s \leq t$) up to that time, as an exogenously given level of satisfaction for his optimization problem at time t . Since, by assumption, all agents have the same Lagrange multiplier $M > 0$, we can apply an analog of Lemma 3.9 (formulated for initial time t instead of initial time zero). From this we may infer that, from time t on, the optimal consumption plan C_s with $s \leq t$ can be recovered from the plan C_t : agent K_s should optimally wait until the level of satisfaction of agent K_t gets larger than his own, and from that time on he should follow the same consumption plan as this agent. For the respective levels of satisfaction, this means that $Y_s(\cdot)$ is larger than $Y_t(\cdot)$ until the first time agent K_s consumes after time t . Moreover, from that time on, $Y_s(\cdot)$ and $Y_t(\cdot)$ coincide. Ignoring some technical difficulties (measurable selection and other regularity problems), we may deduce from this that

$$I(t) \triangleq Y_t(t) \quad (0 \leq t < T)$$

gives us a *universal* lower boundary from which we may recover *all* optimal consumption plans C_t with the same Lagrange multiplier $M > 0$ via 'reflection' of the level of satisfaction. Lemma 3.11 below makes precise what we mean by 'reflecting the level of satisfaction at a given lower boundary'; Figure 1 in Section 4.1 below illustrates the way a consumption plan may be defined by this property.

Lemma 3.11 *Let $I = (I(t), 0 \leq t \leq T)$ be a real valued, adapted RCLL-process. Set*

$$Y^I(t) \triangleq e^{-\int_0^t \beta(s) ds} \left(\eta \vee \sup_{0 \leq s \leq t} \left\{ I(s) e^{\int_0^s \beta(s) ds} \right\} \right) \quad (0 \leq t \leq T)$$

and consider the rightcontinuous process of bounded variation C^I defined by

$$C^I(0-) \triangleq 0, \quad dC^I(t) \triangleq Y^I(t) dt + \beta(t)^{-1} dY^I(t) \quad (0 \leq t \leq T)$$

This process is nondecreasing and adapted, i.e., $C^I \in \mathcal{X}_+$. Moreover, its associated level of satisfaction $Y(C^I)$ coincides with Y^I and is minimal above I :

$$Y(C^I)(t) = Y^I(t) = \min_{C \in \mathcal{X}_+, Y(C) \geq I} Y(C)(t) \quad \text{for all } 0 \leq t \leq T.$$

In particular, if $t \in [0, T]$ is a point of increase of C^I then $Y(C^I)(t) = I(t)$.

We say, C^I reflects its associated level of satisfaction at the process I .

PROOF : Consider a consumption plan $C \in \mathcal{X}_+$. By Assumption 3.7, the process $A(C)$ defined by

$$A(C)(0-) \triangleq \eta \quad \text{and} \quad A(C)(t) \triangleq e^{\int_0^t \beta(s) ds} Y(C)(t) \quad (0 \leq t \leq T)$$

is increasing and adapted. In terms of $A(C)$, the restriction $Y(C) \geq I$ may be rewritten as

$$A(C)(t) \geq e^{\int_0^t \beta(s) ds} I(t) \quad \text{for all } 0 \leq t \leq T.$$

Obviously, the minimal increasing process A^I which starts in $A^I(0-) \triangleq \eta$ and dominates the right side of this inequality is the running supremum

$$A^I(t) \triangleq \sup_{0 \leq s \leq t} \{\eta \vee e^{\int_0^s \beta(v) dv} I(s)\} \quad (0 \leq t \leq T).$$

Note that, by definition, $Y^I(t) = e^{-\int_0^t \beta(s) ds} A^I(t)$ and

$$dC^I(t) = \frac{1}{\beta(t)} e^{-\int_0^t \beta(s) ds} dA^I(t) \quad (0 \leq t \leq T);$$

this yields $C \in \mathcal{X}_+$ and $Y(C^I) = Y^I$ as claimed. Furthermore, minimality of Y^I is inherited from the minimality of A^I . \square

The above arguments suggests that, for a given Lagrange multiplier $M > 0$, there exists a canonical lower boundary process I at which the investor should optimally reflect his level of satisfaction. However, the heuristic way to construct this minimal level described above is far from being constructive, and, therefore, we would like to derive additional properties of this process that allow to characterize it more explicitly.

To this end, let us suppose that the felicity function u satisfies

$$\partial_y u(t, 0+) = +\infty \quad \text{for all } t \in [0, T].$$

Then our Kuhn-Tucker conditions suggest that every agent K_t immediately starts consuming at time t : otherwise his level of satisfaction would remain zero over an open time-interval, contradicting the inequality restriction $\phi(C^*) \leq M\psi$ for optimal plans. Hence, at time t , the first-order condition is binding for agent K_t and we obtain the following equality for his level of satisfaction:

$$(16) \quad \mathbb{E} \left[\int_t^T \partial_y u(s, Y_t(s)) \theta(s, t) ds \middle| \mathcal{F}_t \right] = M\psi(t).$$

As pointed out above, we conjecture that agent K_t reflects his level of satisfaction at some lower boundary process I . Thus, Lemma 3.11 allows us to rewrite equation (16) in terms of this process I :

$$(17) \quad \mathbb{E} \left[\int_t^T \partial_y u \left(s, \sup_{t \leq v \leq s} \left\{ I(v) e^{-\int_v^s \beta(x) dx} \right\} \right) \theta(s, t) ds \middle| \mathcal{F}_t \right] = M\psi(t).$$

Since I is a universal lower boundary for the level of satisfaction of *all* agents K_t , this equality should hold true for *all* $t \in [0, T]$. It provides us the desired characterization of the 'canonical' minimal level of satisfaction as can be seen by the following

Theorem 3.12 *Suppose I is an adapted RCLL-process satisfying Equation (17) for all $t \in [0, T)$ and $I(T) = 0$. Then the consumption plan C^I which reflects its associated level of satisfaction at I (cf. Lemma 3.11) is optimal given initial capital $w = \Psi(C^I)$. Moreover, its associated Lagrange parameter is given by $M > 0$.*

PROOF : We verify the first-order conditions for C^I :

$$\begin{aligned}
\phi(C^I)(t) &= \mathbb{E} \left[\int_t^T \partial_y u \left(s, e^{-\int_0^s \beta(v) dv} \left(\eta \vee \sup_{0 \leq v \leq s} \{I(v) e^{\int_0^v \beta(x) dx}\} \right) \right) \theta(s, t) ds \middle| \mathcal{F}_t \right] \\
(18) \quad &\leq \mathbb{E} \left[\int_t^T \partial_y u \left(s, \sup_{t \leq v \leq s} \{I(v) e^{-\int_v^s \beta(x) dx}\} \right) \theta(s, t) ds \middle| \mathcal{F}_t \right] \\
&= M\psi(t).
\end{aligned}$$

The first equality follows from the definition of C^I and the explicit formula for $Y(C^I)$ provided by Lemma 3.11. The last equality is precisely Equation (17). Hence, C^I satisfies the inequality constraint $\phi(C^I) \leq M\psi$. Moreover, if, for fixed $\omega \in \Omega$, t is a point of increase for $C^I(\omega, \cdot)$, then we have

$$Y(C^I)(\omega, s) = \sup_{t \leq v \leq s} \{I(\omega, v) e^{-\int_t^v \beta(x) dx}\}$$

for all $s \geq t$. Hence, we have equality in (18) whenever C^I is increasing. This proves the flat-off condition for C^I . \square

Remark 3.13 *In discrete time, it is easy to construct a solution to (the discrete-time analog of) Equation (17) via backwards induction, provided*

$$\partial_y u(t, 0+) = +\infty \quad \text{and} \quad \partial_y u(t, +\infty) = 0$$

for all $t \in [0, T]$. In the present continuous-time framework, this procedure is no longer available. Indeed, the construction and even an existence proof of a solution to (17) seem to be more involved and will, therefore, be discussed elsewhere.

From the above results, we may derive the following method to construct explicit solutions to the utility maximization problem (2):

1. For every $M > 0$, find an adapted RCLL-process process $I = I^M$ which solves Equation (17) for every $t \in [0, T)$.
2. For each $M > 0$, compute the price of the consumption plan $C^M \triangleq C^{I^M}$ which reflects its associated level of satisfaction at I^M .
3. The consumption plan C^M with $\Psi(C^M) = w$ is then the unique solution to the investor's utility maximization problem (2).

4 Explicit Solution in a Homogeneous Setting

In this section, we are going to derive some explicit solutions to the utility maximization problem (2) by applying the method described at the end of the preceding section.

4.1 Heuristics for Computing the Minimal Level of Satisfaction

Let us try to find a solution $I = (I(t), t \geq 0)$ to the minimal level equation (17) by looking for a 'good' candidate for this minimal level of satisfaction.

To this end, we first recall the structure of optimal consumption plans as they are derived in the 'classical' theory based on time-additive von Neumann-Morgenstern utility functionals. In such a setting, utility is obtained from the current *rate* of consumption, rather than from the instantaneous level of satisfaction. Applying methods of convex duality (confer, e.g., Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)), one shows that the marginal felicity of an optimal consumption rate for this problem should equal some fixed multiple of the state-price density. This leads to the absolutely continuous optimal consumption plan $dC^{ac}(t) \equiv i(t, L\psi(t)) dt$, where $i(t, \cdot) \triangleq (\partial_y u(t, \cdot))^{-1}$ is the inverse of marginal felicity and L is a strictly positive constant.

Since, at least formally, the level of satisfaction $Y(C)$ plays the same role for our utility functional $U(C)$ as the rate of consumption for the classic von Neumann-Morgenstern utilities, the above solution suggests to choose $C \in \mathcal{X}_+$ such that $Y(C)(t) \equiv i(t, L\psi(t))$. But typically the right side of this equality will be of unbounded variation, while the left side must have bounded variation for any choice of $C \in \mathcal{X}_+$ under Assumption 3.7. Hence, there might be no $C \in \mathcal{X}_+$ inducing a level of satisfaction of the form suggested above. In any case, however, we can consider the consumption plan C^L which reflects its associated level of satisfaction $Y^L \triangleq Y(C^L)$ at

$$I^L(t) \triangleq i(t, L\psi(t)) \quad (t \geq 0).$$

Figure 1 illustrates the structure of the consumption plans defined this way.

Figure 1: Typical paths for the state-price ψ (light grey line), the level of satisfaction $Y(C^L)$ (black line) and its minimal level I^L (grey line).

Remark 4.1 *If we choose C^L as described, the last part of Lemma 3.11 shows that for any two times s and t in which consumption occurs the marginal rate of substitution, expressed in terms of the level of satisfaction, is equal to the economic rate of substitution:*

$$\frac{\partial_y u(s, Y^L(s))}{\partial_y u(t, Y^L(t))} = \frac{\psi(s)}{\psi(t)}.$$

Thus, I^L gives us a 'natural' candidate for the minimal level of satisfaction I which we are looking for. However, there is an intuitive caveat against the conjecture $I = I^L$. Indeed, consider equation (17), which characterizes the minimal level of satisfaction, for times t close to the time horizon T . It is easy to see that a solution I to this equation must converge to zero as $t \uparrow T$. This corresponds to the observation that, in our setting, the contribution to total expected utility is considerably smaller for consumption at times close to the time horizon T than for consumption at time zero, say. But, we typically do *not* have $i(t, M\psi(t)) \rightarrow 0$ as $t \uparrow T$ and, therefore, the received candidate plans C^L do not

take into account properly the time horizon T . In particular, the described minimal level will not induce optimal consumption plans, if — as we assumed up to now — the time horizon is finite. But for an investor with *infinite time horizon* this caveat does not apply, and, in fact, we can prove optimality of our candidate policies in a homogeneous setting under this condition; confer Corollary 4.8.

4.2 The Homogeneous Setting

In contrast to the preceding sections let us henceforth assume that the investor's time horizon is infinite: $T = +\infty$. We furthermore suppose his felicity function to have the separable, homogeneous form

$$(19) \quad u(t, y) = e^{-\delta t} \frac{1}{\alpha} y^\alpha \quad (t \geq 0, y \geq 0)$$

for some constant $\alpha \in (-\infty, 1) - \{0\}$ and denote by

$$i(t, z) \triangleq \left(e^{\delta t} z \right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0, z > 0)$$

the inverse of its associated marginal felicity function $\partial_y u(t, \cdot)$.

Remark 4.2 *The case $\alpha = 0$, corresponding to log-felicity, can be treated with the same method as the power-felicities above. However, for ease of exposition, we leave this case to the interested reader.*

We assume the function $\beta(\cdot)$ of Assumption 3.7 to be a strictly positive constant, which, by slight abuse of notation, will also be denoted by β . Hence, the level of satisfaction is a time-homogenous, exponentially weighted average of past consumption:

$$Y(C)(t) = \eta e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)} dC(s)$$

with constants $\beta > 0$ and $\eta \geq 0$.

Furthermore, we assume that the unique state-price density ψ is of the form

$$\psi(t) = \exp(-\theta X(t) - (r + \pi(-\theta))t) \quad (t \geq 0),$$

for some Lévy process X with finite Laplace-exponent $\pi(\xi)$ ($\xi \in \mathbb{R}$) (under \mathbb{P}). Hence, interest rates are constant, $r(t) \equiv r \geq 0$ and uncertainty is introduced by a stochastic process X with stationary, independent increments which possesses all exponential moments

$$\mathbb{E} \exp(\xi X(t)) < +\infty \quad (\xi \in \mathbb{R}, t \geq 0).$$

The Laplace-exponent $\pi(\cdot)$ of X may then be defined via

$$\mathbb{E} \exp(\xi X(t)) = \exp(\pi(\xi)t) \quad \text{for all } \xi \in \mathbb{R}, t \geq 0;$$

see, e.g., Bertoin (1996). The constant $\theta > 0$ may be viewed as the 'market price of risk'. The deterministic case $X(t) \equiv 0$ is treated in Hindy, Huang, and Kreps (1992) and Bank and Riedel (1998), and will, therefore, be excluded implicitly in the following.

Example 4.3 1. For $X = (W(t), t \geq 0)$, a standard Brownian motion, we have $\pi(\xi) = \frac{1}{2}\xi^2$, and the state-price density

$$\psi(t) = \exp\left(-\theta W(t) - \left(r + \frac{1}{2}\theta^2\right)t\right) \quad (t \geq 0)$$

takes the well known form of a geometric Brownian motion. This specification of ψ corresponds to the setup studied in Hindy and Huang (1993).

2. If $X = (\pm N(t), t \geq 0)$ is a Poisson process with upward (downward) jumps and intensity λ , then $\pi(\xi) = \lambda(e^{\pm\xi} - 1)$ and, therefore,

$$\psi(t) = \exp\left(\mp\theta N(t) - \left(r + \lambda(e^{\mp\theta} - 1)\right)t\right) \quad (t \geq 0)$$

is a geometric Poisson process.

Remark 4.4 Note that the above examples describe indeed complete financial markets if \mathbb{F} is the augmented filtration generated by X .

To ensure that the utility maximization problem is well-posed also for an infinite time horizon $T = +\infty$, it is necessary (not sufficient, cf. Theorem 4.9) to introduce

Assumption 4.5 $\delta + \alpha\beta > 0$.

Indeed, in case $\delta + \alpha\beta \leq 0$, it is easy to see that the investor obtains infinite utility $\mathbb{E}U(C) = +\infty$ by consuming all his wealth in one single gulp at time $t = 0$.

In the above setting, we may represent the consumption plans C^L ($L > 0$) defined in the preceding section in the following form:

$$dC^L(t) = \frac{1}{\beta}e^{-\beta t} dA^L(t) \quad (t \geq 0)$$

where, for $t \geq 0$,

$$(20) \quad A^L(0-) \triangleq \eta, \quad A^L(t) \triangleq \eta \vee \left\{L^{-\frac{1}{1-\alpha}}S(t)\right\}$$

with

$$(21) \quad S(t) \triangleq \sup_{0 \leq s \leq t} \left\{\psi(s)^{-\frac{1}{1-\alpha}} e^{(\beta - \frac{\delta}{1-\alpha})s}\right\}.$$

As usual we have

$$Y(C^L)(t) = e^{-\beta t} A^L(t) \quad (t \geq 0).$$

4.3 A Solution to the Level Equation

Let us now prove that the consumption plans C^L described above are indeed optimal in their respective class. Following the method proposed at the end of Section 3.2, we first show that, in the above homogeneous setting, the candidates I^L ($L > 0$) for the minimal level of satisfaction do indeed solve Equation (17) for some $M = M(L) > 0$:

Lemma 4.6 *In the homogeneous setting of Section 4.2, the process*

$$I^L(t) \triangleq i(t, L\psi(t)) = \left(Le^{\delta t}\psi(t)\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0)$$

solves Equation (17) for

$$(22) \quad M \triangleq \mathbb{E} \left[\int_0^{+\infty} e^{-(\delta+\alpha\beta)s} \beta \inf_{0 \leq v \leq s} \{e^{-(\beta(1-\alpha)-\delta)v} \psi(v)\} ds \right] L < +\infty.$$

PROOF : We have

$$\begin{aligned} & \mathbb{E} \left[\int_t^{+\infty} \partial_y u \left(s, e^{-\beta s} \sup_{t \leq v \leq s} \{I^L(v)e^{\beta v}\} \right) \beta e^{-\beta(s-t)} ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\int_t^{+\infty} e^{-(\delta+\alpha\beta)s} \beta e^{\beta t} \inf_{t \leq v \leq s} \{Le^{-(\beta(1-\alpha)-\delta)v} \psi(v)\} ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\int_0^{+\infty} e^{-(\delta+\alpha\beta)s} \beta \inf_{0 \leq v \leq s} \{Le^{-(\beta(1-\alpha)-\delta)v} \frac{\psi(t+v)}{\psi(t)}\} ds \middle| \mathcal{F}_t \right] \psi(t) \\ &= \mathbb{E} \left[\int_0^{+\infty} e^{-(\delta+\alpha\beta)s} \beta \inf_{0 \leq v \leq s} \{Le^{-(\beta(1-\alpha)-\delta)v} \psi(v)\} ds \right] \psi(t) \end{aligned}$$

where the last equation holds true because X is a Lévy process. Thus, I^L does indeed solve Equation (17) for $M = M(L) > 0$ as defined in (22). Note that $M < +\infty$ because the infimum in its definition is always less than or equal to 1 and because $\delta + \alpha\beta > 0$ by Assumption 4.5. \square

By the same arguments as in the proof of Theorem 3.12, we now can show that the consumption plans C^L satisfy the first-order conditions (ii) and (iii) with $T = +\infty$. Moreover, it is easy to see that these conditions are sufficient for optimality also in the infinite horizon case:

Theorem 4.7 *A consumption plan C^* solves (2) for $T = +\infty$ if the following conditions hold true for some Lagrange multiplier $M \geq 0$:*

- (i) $\Psi(C^*) = w$,
- (ii) $\phi(C^*) \leq M\psi$ \mathbb{P} -a.s.,
- (iii) C^* is flat off $\{\phi(C^*) = M\psi\}$ \mathbb{P} -a.s., i.e.,

$$\mathbb{E} \int_0^{+\infty} 1_{\{\phi(C^*)(t) \neq M\psi(t)\}} dC^*(t) = 0$$

where $\phi(C)$ is defined for $C \in \mathcal{X}_+$ by (5) with $T = +\infty$.

PROOF : Without loss of generality we may assume $\mathbb{E}U(C^*) < +\infty$. Then a careful repetition of the argument for the sufficiency part of Theorem 3.3 with $T \triangleq +\infty$ shows that indeed $\mathbb{E}U(C^*) \geq \mathbb{E}U(C)$ for any other consumption plan $C \in \mathcal{A}(w)$. \square

We finally obtain

Corollary 4.8 *In the homogeneous setting described in Section 4.2, the consumption plan C^L is optimal given initial capital $w = \Psi(C^L)$, provided this value is finite.*

4.4 Prices and Utilities

The preceding section shows that — in the homogeneous setting of Section 4.2 — the consumption plans C^L ($L > 0$) are optimal in their respective class *provided their price is finite*. Hence, we still have to check for which parameter values of the problem this condition is satisfied. Furthermore, we should calculate the exact prices for varying Lagrange parameter $M > 0$ in order to find the plan whose price coincides with a given initial capital $w > 0$.

Concerning the condition of finite prices, we will prove the following characterization:

Theorem 4.9 *The following assertions are equivalent:*

- (i) *Finite prices: $\Psi(C^L) < +\infty$ for some (all) $L > 0$.*
- (ii) *Finite utilities: $\mathbb{E}U(C^L) < +\infty$ for some (all) $L > 0$.*
- (iii) *The investor's rate of time preference δ satisfies*

$$(23) \quad \delta > \hat{\delta} \triangleq \alpha r + (1 - \alpha)\pi \left(\frac{\alpha\theta}{1 - \alpha} \right) + \alpha\pi(-\theta).$$

If $\delta < \hat{\delta}$, then, for any initial wealth $w > 0$, there is a budget-feasible C with infinite expected utility, i.e., the optimization problem (2) is ill-posed.

The proof of this theorem will be prepared by the following lemmata 4.10, 4.11, and 4.12. Readers more interested in the computation of policy prices may skip this part at first reading and proceed from Remark 4.13 below.

Lemma 4.10 (i) *In terms of the increasing process A^L , we may express the price of plan C^L as*

$$(24) \quad \Psi^L \triangleq \Psi(C^L) = \frac{1}{\beta} \left(\mathbb{E}^* A^L(\tau^*) - \eta \right) \quad (L > 0),$$

where τ^ is an independent exponential random time with parameter $r + \beta > 0$.*

(ii) *We have $\Psi^L < +\infty$ for all $L > 0$ iff*

$$(25) \quad \mathbb{E}^* S(\tau^*) < \infty,$$

where τ^ is as in (i). In particular, the price of every policy C^L is finite if just one of these prices is finite.*

(iii) *$L \mapsto \Psi^L$ is a nonnegative, nondecreasing, and convex function. If prices are finite, $\Psi^L \uparrow +\infty$ as $L \downarrow 0$ and $\Psi^L \downarrow 0$ as $L \uparrow +\infty$. In particular, for every initial capital $w > 0$ there is a consumption policy C^L with price $\Psi^L = w$ in this case.*

PROOF : From $dC^L(t) = \frac{1}{\beta} e^{-\beta t} dA^L(t)$ and partial integration of the price functional, we deduce for all $L > 0$

$$(26) \quad \Psi^L = \frac{1}{\beta} \mathbb{E}^* \lim_{T \uparrow +\infty} \left(A^L(T) e^{-(r+\beta)T} - \eta + \int_0^T A^L(t) (r + \beta) e^{-(r+\beta)t} dt \right).$$

Hence,

$$(27) \quad \mathbb{E}^* A^L(\tau^*) = \mathbb{E}^* \int_0^{+\infty} A^L(t)(r + \beta)e^{-(r+\beta)t} dt < +\infty$$

is necessary for $\Psi^L < +\infty$. It is also sufficient since it implies

$$(28) \quad \lim_{T \uparrow +\infty} A^L(T)e^{-(r+\beta)T} = 0 \quad \mathbb{P}^*\text{-a.s.}$$

Indeed, otherwise we have $\limsup_{T \uparrow +\infty} A^L(T)e^{-(r+\beta)T} > 0$ with positive \mathbb{P}^* -probability. Thus, on a set with positive \mathbb{P}^* -measure, there is a random $\epsilon > 0$ such that

$$A^L(\sigma_n)e^{-(r+\beta)\sigma_n} \geq \epsilon$$

along a sequence of random times σ_n tending to $+\infty$ as $n \uparrow +\infty$. Without loss of generality we may assume that $\sigma_{n+1} - \sigma_n \geq 1$ for all n . Since A^L is nondecreasing we have $A^L(t)e^{-(r+\beta)t} \geq \epsilon e^{-(r+\beta)t} > 0$ whenever $t \in [\sigma_n, \sigma_{n+1})$ for some n . This implies $\int_0^{+\infty} A^L(t)(r + \beta)e^{-(r+\beta)t} dt = +\infty$ with positive \mathbb{P}^* -probability. Hence, (27) implies (28). Furthermore the preceding considerations yield that (i) is implied by (26).

For (ii) it remains to note that $\mathbb{E}^* A^L(\tau^*) < +\infty$ is equivalent to $\mathbb{E}^* S(\tau^*) < +\infty$. This follows from $A^L(\tau^*) \geq L^{-\frac{1}{1-\alpha}} S(\tau^*)$ and

$$\mathbb{E}^* A^L(\tau^*) = L^{-\frac{1}{1-\alpha}} \mathbb{E}^* S(\tau^*) + \mathbb{E}^* \left(\eta - L^{-\frac{1}{1-\alpha}} S(\tau^*) \right)^+ \leq L^{-\frac{1}{1-\alpha}} \mathbb{E}^* S(\tau^*) + \eta.$$

From (i) we deduce that Ψ^L is nonnegative, nondecreasing, and convex, since so is A^L . If prices are finite, $A^{L_0}(\tau^*)$ is \mathbb{P}^* -integrable. Thus, $\Psi^L \downarrow 0$ for $L \uparrow +\infty$ by dominated convergence. For $L \downarrow 0$, we have $\Psi^L \geq \Delta C^L(0) \uparrow +\infty$. This yields (iii). \square

The following is an analog of Lemma 4.10 for utilities instead of prices:

Lemma 4.11 (i) *In terms of the increasing process A^L , we may express the expected utility of plan C^L as*

$$(29) \quad \mathbb{E}U(C^L) = \frac{1}{\alpha(\delta + \alpha\beta)} \mathbb{E} \left(A^L(\tau) \right)^\alpha \quad (L > 0),$$

where τ is an independent exponential random time with parameter $\delta + \alpha\beta$.

(ii) *We have $\mathbb{E}U(C^L) < +\infty$ for all $L > 0$ iff*

$$(30) \quad \mathbb{E} (S(\tau))^\alpha < \infty,$$

where τ is as in (i). In particular, the expected utility of every policy C^M is finite if just one of these utilities is finite.

PROOF : Note first that, because of Assumption 4.5, we have $\delta + \alpha\beta > 0$, and, therefore, τ is well-defined. Now, (i) follows from $Y(C^L)(t) = e^{-\beta t} A^L(t)$ and the definition of the utility functional $U(\cdot)$. For (ii) we note that $(A^L(\tau))^\alpha \geq L^{-\frac{\alpha}{1-\alpha}} (S(\tau))^\alpha$ and

$$\mathbb{E} \left(A^L(\tau) \right)^\alpha \leq \eta^\alpha + L^{-\frac{\alpha}{1-\alpha}} \mathbb{E} (S(\tau))^\alpha.$$

\square

Lemma 4.10 and Lemma 4.11 are valid for any semimartingale state-price density which induces a constant interest rate. Yet, for the proof of Theorem 4.9, we will need the special Lévy-structure of ψ which allows us to prove the following technical

Lemma 4.12 (i) Let τ be an exponential random time independent of X . We have

$$(31) \quad \mathbb{E} \exp \left(\xi \sup_{0 \leq s \leq \tau} X(s) \right) \mathbb{E} \exp \left(\xi \inf_{0 \leq s \leq \tau} X(s) \right) = \mathbb{E} \exp (\xi X(\tau))$$

for all $\xi \in \mathbb{R}$.

(ii) If, in addition, X has no positive jumps and is neither a deterministic drift nor the negative of a subordinator, then $\sup_{0 \leq s \leq \tau} X(s)$ is exponentially distributed. The parameter ζ of its distribution is uniquely determined by $\pi(\zeta) = \xi$, where ξ is the parameter of the exponential distribution of τ .

(iii) Under the risk-neutral measure \mathbb{P}^* induced by ψ , X is again a Lévy-process with finite exponential moments and its \mathbb{P}^* -Laplace exponent is given by

$$(32) \quad \pi^*(\xi) = \pi(\xi - \theta) - \pi(-\theta).$$

PROOF :

(i) For $t \geq 0$, let $\tilde{X}(t) \triangleq \sup_{0 \leq s \leq t} X(s)$. By Theorem VI.5(i) in Bertoin (1996), the random variables $\tilde{X}(\tau)$ and $\tilde{X}(\tau) - X(\tau)$ are independent. Hence,

$$(33) \quad \begin{aligned} \mathbb{E} \exp(\xi X(\tau)) &= \mathbb{E} \left[\exp(\xi \tilde{X}(\tau)) \exp(-\xi(\tilde{X}(\tau) - X(\tau))) \right] \\ &= \mathbb{E} \exp(\xi \tilde{X}(\tau)) \mathbb{E} \exp(-\xi(\tilde{X}(\tau) - X(\tau))). \end{aligned}$$

Using the Duality Lemma II.2 in Bertoin (1996) and the independence of X and τ , we see that

$$\tilde{X}(\tau) - X(\tau) = \sup_{0 \leq s \leq \tau} \{X((\tau - s)-) - X(\tau)\}$$

has the same law as

$$\sup_{0 \leq s \leq \tau} \{-X(s)\} = - \inf_{0 \leq s \leq \tau} X(s).$$

In connection with Equation (33), this yields (i).

(ii) This is Corollary VII.1.2 in Bertoin (1996).

(iii) By definition of ψ , the density process Z for \mathbb{P} and \mathbb{P}^* is given by

$$Z(t) \triangleq \frac{d\mathbb{P}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp(-\theta X(t) - \pi(-\theta)t) \quad (t \geq 0).$$

Hence, for $s, t \geq 0$, we may calculate the conditional \mathbb{P}^* -Laplace transform of the increment $X(t+s) - X(t)$ given \mathcal{F}_t as follows:

$$\begin{aligned} &\mathbb{E}^* \left[\exp(\xi(X(t+s) - X(t))) \mid \mathcal{F}_t \right] \\ &= \frac{1}{Z(t)} \mathbb{E} \left[\exp(\xi(X(t+s) - X(t))) Z(t+s) \mid \mathcal{F}_t \right] \\ &= \frac{1}{Z(t)} \mathbb{E} \left[\exp((\xi - \theta)(X(t+s) - X(t))) \mid \mathcal{F}_t \right] \exp(-\theta X(t) - \pi(-\theta)(t+s)) \\ &= \exp(s(\pi(\xi - \theta) - \pi(-\theta))). \end{aligned}$$

Since the last quantity is deterministic and does not depend on t , the above calculation shows that, also under \mathbb{P}^* , X has independent and stationary increments. Furthermore, we can easily read off Equality (32) for the \mathbb{P}^* -Laplace exponent $\pi^*(\cdot)$.

□

Now, we are in the position to give the

Proof of Theorem 4.9:

(i) \Leftrightarrow (iii) By Lemma 4.10 (ii), we know that (i) is equivalent to

$$\begin{aligned} \infty > \mathbb{E}^* S(\tau^*) &= \mathbb{E}^* \sup_{0 \leq s \leq \tau^*} \exp \left(\frac{\theta}{1-\alpha} X(s) + \left(\frac{\pi(-\theta) + r + \beta(1-\alpha) - \delta}{1-\alpha} \right) s \right) \\ &= \mathbb{E}^* \exp \left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq \tau^*} \{X(s) + \mu s\} \right) \end{aligned}$$

where τ^* is an independent exponential random time with parameter $r + \beta$, and

$$\mu \triangleq \frac{1}{\theta} (\pi(-\theta) + r + \beta(1-\alpha) - \delta).$$

By Lemma 4.12 (i), the above calculation yields that (i) holds true iff

$$\mathbb{E}^* \exp \left(\frac{\theta}{1-\alpha} \{X(\tau^*) + \mu \tau^*\} \right) < +\infty.$$

Since τ^* is independent of X and exponentially distributed with parameter $r + \beta$, we may use Fubini's Theorem to obtain equivalence of (i) and

$$r + \beta > \pi^* \left(\frac{\theta}{1-\alpha} \right) + \frac{\theta \mu}{1-\alpha}.$$

Using the transformation rule (32), it is easy to see that this condition is indeed equivalent to (iii).

(ii) \Leftrightarrow (iii) Making use of Lemma 4.11, we may follow a similar line of arguments as in the proof of (ii) \Leftrightarrow (iii). First, (ii) is equivalent to

$$\mathbb{E} \exp \left(\frac{\alpha \theta}{1-\alpha} \{X(\tau) + \mu \tau\} \right) < +\infty$$

where τ is an independent exponential random time with parameter $\delta + \alpha\beta > 0$, and where μ is defined as above. Using Fubini's Theorem allows us to conclude the equivalence of (ii) and

$$\delta + \alpha\beta > \pi \left(\frac{\alpha \theta}{1-\alpha} \right) + \frac{\alpha \theta \mu}{1-\alpha},$$

which, by an easy calculation, can be shown to be equivalent to (iii), too.

To prove that problem (2) is ill-posed in case $\delta < \hat{\delta}$, let $\bar{\delta} > \hat{\delta}$ and consider the consumption plan \bar{C}^L obtained from reflecting the level of satisfaction at the lower boundary process

$$\bar{I}^L(t) \triangleq \left(L e^{\bar{\delta} t} \psi(t) \right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0).$$

The corresponding increasing process \bar{A}^L is given by

$$\bar{A}^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp \left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq t} \{X(s) + \bar{\mu} s\} \right) \quad (t \geq 0)$$

where

$$\bar{\mu} \triangleq \frac{1}{\theta} \left(\pi(-\theta) + r + \beta(1 - \alpha) - \bar{\delta} \right).$$

From (i) \Leftrightarrow (iii) we know that the price of every policy \bar{C}^L is finite because $\bar{\delta} > \hat{\delta}$. Hence, for any initial wealth $w > 0$, we can find $L = L(w)$ such that $\bar{C}^{L(w)}$ is budget-feasible.

By the same arguments as in the proof of Lemma 4.11, one can now show that the expected utility of the plan $\bar{C}^{L(w)}$ is finite iff

$$\mathbb{E} \exp \left(\frac{\alpha\theta}{1 - \alpha} \sup_{0 \leq s \leq \tau} \{X(s) + \bar{\mu}s\} \right) < \infty$$

where τ is, as usual, an independent exponential random time with parameter $\delta + \alpha\beta$. From equality (31), we deduce that the above relation holds true iff

$$\mathbb{E} \exp \left(\frac{\alpha\theta}{1 - \alpha} \{X(\tau) + \bar{\mu}\tau\} \right) < \infty.$$

Since τ is independent of X and exponentially distributed, this is equivalent to

$$(34) \quad \delta + \alpha\beta > \pi \left(\frac{\alpha\theta}{1 - \alpha} \right) + \frac{\alpha\theta\bar{\mu}}{1 - \alpha}.$$

Now, note that, for $\bar{\delta} \downarrow \hat{\delta}$, the right side of this inequality increases to

$$(35) \quad \pi \left(\frac{\alpha\theta}{1 - \alpha} \right) + \frac{\alpha\theta\hat{\mu}}{1 - \alpha} = \hat{\delta} + \alpha\beta > \delta + \alpha\beta,$$

where

$$\hat{\mu} \triangleq \lim_{\bar{\delta} \downarrow \hat{\delta}} \bar{\mu} = \frac{1}{\theta} \left(\pi(-\theta) + r + \beta(1 - \alpha) - \hat{\delta} \right).$$

Equality (35) follows by use of the definition of $\hat{\delta}$. Hence, there are $\bar{\delta} > \hat{\delta}$ for which inequality (34) is violated and for which, therefore, the associated plans \bar{C}^L have infinite expected utility, even though their price is finite. \square

Remark 4.13 *Note that there is a slight 'gap' in Theorem 4.9, since it leaves open whether or not the optimization problem is well-posed in case $\delta = \hat{\delta}$. However, as shown by Lemma 4.15 below, this case can be treated under some additional assumption.*

In order to obtain closed-form solutions for (2), it still remains to calculate all prices $\Psi(C^L)$ ($L > 0$) and to identify the parameter $L(w)$ for which $\Psi(C^{L(w)}) = w$. To this end, let us introduce the Lévy process

$$Z(t) \triangleq X(t) + \mu t \quad (t \geq 0),$$

where

$$(36) \quad \mu \triangleq \frac{1}{\theta} (\pi(-\theta) + r + \beta(1 - \alpha) - \delta).$$

This allows us to rewrite A^L in the form

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp \left(\frac{\theta}{1 - \alpha} \check{Z}(t) \right) \quad (t \geq 0)$$

where

$$\tilde{Z}(t) \triangleq \sup_{0 \leq s \leq t} Z(s) = \sup_{0 \leq s \leq t} \{X(s) + \mu s\} \quad (t \geq 0).$$

Now, we can compute the prices $\Psi(C^L)$ and utilities $\mathbb{E}U(C^L)$ ($L > 0$) explicitly in two cases:

Assumption 4.14 (i) Z has ever decreasing paths.

(ii) Z does not have ever decreasing paths and all its jumps are nonpositive ($\Delta Z \leq 0$).

Both Assumption 4.14 (i) and Assumption 4.14 (ii) ensure that $\tilde{Z}(\tau)$ and $\tilde{Z}(\tau^*)$ are exponentially distributed under \mathbb{P} and \mathbb{P}^* respectively, where τ and τ^* are independent exponential random times with parameters $\delta + \alpha\beta$ and $r + \beta$. In fact, if Assumption 4.14 (i) holds true, we evidently have $\tilde{Z}(t) \equiv 0$ which corresponds to the parameter values $\zeta = \zeta^* = 0$ for the respective exponential distribution. Under Assumption 4.14 (ii), we may apply Lemma 4.12 (ii) with Z instead of X to identify the exponential parameters as the unique positive solutions to

$$(37) \quad \pi(\zeta) + \mu\zeta = \delta + \alpha\beta \quad \text{and} \quad \pi^*(\zeta^*) + \mu\zeta^* = r + \beta$$

respectively.

Thus, proceeding from formulae (24) and (29), we now can compute

$$(38) \quad \Psi(C^L) = \frac{1}{\beta} \cdot \begin{cases} \left(L^{-\frac{1}{1-\alpha}} - \eta\right)^+ & \text{if } \zeta^* = 0 \\ \frac{(1-\alpha)\zeta^*}{(1-\alpha)\zeta^* - \theta} L^{-\frac{1}{1-\alpha}} - \eta & \text{if } \eta \leq L^{-\frac{1}{1-\alpha}}, \zeta^* > 0 \\ \frac{\theta}{(1-\alpha)\zeta^* - \theta} \eta^{-\frac{(1-\alpha)\zeta^* - \theta}{\theta}} L^{-\frac{\zeta^*}{\theta}} & \text{else} \end{cases}$$

and

$$\mathbb{E}U(C^L) = \frac{1}{\alpha(\delta + \alpha\beta)} \cdot \begin{cases} \eta^\alpha \vee L^{-\frac{\alpha}{1-\alpha}} & \text{if } \zeta = 0 \\ \frac{(1-\alpha)\zeta}{(1-\alpha)\zeta - \alpha\theta} L^{-\frac{\alpha}{1-\alpha}} & \text{if } \eta \leq L^{-\frac{1}{1-\alpha}}, \zeta > 0 \\ \eta^\alpha + \frac{\alpha\theta}{(1-\alpha)\zeta - \alpha\theta} \eta^{-\frac{(1-\alpha)\zeta - \alpha\theta}{\theta}} L^{-\frac{\zeta}{\theta}} & \text{else.} \end{cases}$$

Hence, an agent with initial wealth $w > 0$ optimally follows the consumption plan $C^{L(w)}$ with

$$L(w) \triangleq \begin{cases} (\beta w + \eta)^{-(1-\alpha)} & \text{if } \zeta^* = 0 \\ \left(\frac{(1-\alpha)\zeta^* - \theta}{(1-\alpha)\zeta^*} (\beta w + \eta)\right)^{-(1-\alpha)} & \text{if } w \geq \hat{w}, \zeta^* > 0 \\ \left(\frac{(1-\alpha)\zeta^* - \theta}{\theta} \eta^{\frac{(1-\alpha)\zeta^* - \theta}{\theta}} \beta w\right)^{-\frac{\theta}{\zeta^*}} & \text{else} \end{cases}$$

where $\hat{w} \triangleq \frac{1}{\beta} \frac{\theta}{(1-\alpha)\zeta^* - \theta} \eta$.

Furthermore, using Lemma 4.12 (iii), one can show that $\zeta^* = \zeta + \theta$ by a straightforward calculation. This allows us to represent the agent's maximal utility (the value $v(w)$ of the

program (2)) by

$$v(w) = \frac{1}{\alpha(\delta + \alpha\beta)} \cdot \begin{cases} (\beta w + \eta)^\alpha & \text{if } \zeta^* = 0 \\ \zeta \left(\frac{1 - \alpha}{(1 - \alpha)\zeta - \alpha\theta} \right)^{1-\alpha} \left(\frac{\beta w + \eta}{\zeta + \theta} \right)^\alpha & \text{if } w \geq \hat{w}, \zeta^* > 0 \\ \eta^\alpha + \alpha\eta^{-\frac{(1-\alpha)\zeta - \alpha\theta}{\zeta + \theta}} \left(\frac{\theta\beta w}{(1 - \alpha)\zeta - \alpha\theta} \right)^{\frac{\zeta}{\zeta + \theta}} & \text{else.} \end{cases}$$

The above formulae give us the desired explicit solution to the investor's utility maximization problem (2) in the homogeneous setting of Section 4.2.

As pointed out by Remark 4.13, Theorem 4.9 does not characterize completely the parameter values for which problem (2) is well-posed in the present context. However, under Assumption 4.14, this problem can be solved:

Lemma 4.15 *Under Assumption 4.14, the parameter restriction $\delta > \hat{\delta}$ of Lemma 4.9 (iii) is also necessary for problem (2) to be well-posed. More precisely, suppose that Assumption 4.14 and that the parameters of the problem are such that*

$$(39) \quad \delta \leq \hat{\delta} = \alpha r + (1 - \alpha)\pi \left(\frac{\alpha\theta}{1 - \alpha} \right) + \alpha\pi(-\theta).$$

Then we have

$$\sup_{C \in \mathcal{A}(w)} \mathbf{E}U(C) = +\infty$$

for any initial capital $w > 0$.

PROOF : As in the proof of Theorem 4.9, choose some $\bar{\delta} > \hat{\delta}$ and consider, for every $L > 0$, the lower boundary process \bar{I}^L obtained from I^L by replacing δ with $\bar{\delta}$. Again, the corresponding consumption plans will be denoted by \bar{C}^L , and we will write \bar{S} for the analog of the supremum process S . For simplicity, we assume that $\eta = 0$.

We have

$$\Psi(\bar{C}^L) = \frac{L^{-\frac{1}{1-\alpha}}}{\beta} \mathbf{E}^* \bar{S}(\tau^*)$$

and

$$\mathbf{E}U(\bar{C}^L) \geq \mathbf{E} \int_0^\infty e^{-\bar{\delta}t} \frac{1}{\alpha} \left(e^{-\beta t} L^{-\frac{1}{1-\alpha}} \bar{S}(t) \right)^\alpha dt = \frac{L^{-\frac{1}{1-\alpha}}}{\alpha(\bar{\delta} + \alpha\beta)} \mathbf{E} \bar{S}(\tau)^\alpha$$

where τ^* and τ are independent exponential random times with parameters $r + \beta > 0$ and $\bar{\delta} + \alpha\beta > 0$ respectively.

In order to meet the budget-constraint, we have to choose L such that $\Psi(\bar{C}^L) = w$. Note that this is indeed possible because of $\bar{\delta} > \hat{\delta}$. By the above calculations, this gives us

$$v(w) \triangleq \sup_{C \in \mathcal{A}(w)} \mathbf{E}U(C) \geq \frac{(\beta w)^\alpha}{\alpha(\bar{\delta} + \alpha\beta)} \frac{\mathbf{E} \bar{S}(\tau)^\alpha}{(\mathbf{E}^* \bar{S}(\tau^*))^\alpha}.$$

Hence, to show that $v(w) \equiv +\infty$, it suffices to prove

$$(40) \quad \frac{\mathbf{E} \bar{S}(\tau)^\alpha}{(\mathbf{E}^* \bar{S}(\tau^*))^\alpha} \rightarrow +\infty \quad \text{as } \bar{\delta} \downarrow \hat{\delta}.$$

Using Lemma 4.12 (ii), it is easy to see that

$$\mathbb{E}\bar{S}(\tau)^\alpha = \frac{\bar{\zeta}}{\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}} \quad \text{and} \quad \mathbb{E}^*\bar{S}(\tau^*) = \frac{\bar{\zeta}^*}{\frac{\theta}{1-\alpha} - \bar{\zeta}^*}$$

where $\bar{\zeta}$ and $\bar{\zeta}^*$ are determined by

$$(41) \quad \pi(\bar{\zeta}) + \bar{\mu}\bar{\zeta} = \hat{\delta} + \alpha\beta \quad \text{and} \quad \pi^*(\bar{\zeta}^*) + \bar{\mu}\bar{\zeta}^* = r + \beta$$

with $\bar{\mu} \triangleq \frac{1}{\theta}(\pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta})$. A straightforward calculation based on Lemma 4.12 (iii) shows that $\bar{\zeta}^* = \bar{\zeta} + \theta$ by a straightforward calculation.

This allows us to conclude that

$$(42) \quad \frac{\mathbb{E}\bar{S}(\tau)^\alpha}{(\mathbb{E}^*\bar{S}(\tau^*))^\alpha} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha} \frac{(\frac{\theta}{1-\alpha} - \bar{\zeta}^*)^\alpha}{\frac{\alpha\theta}{1-\alpha} - \bar{\zeta}} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha} \left(\frac{\alpha\theta}{1-\alpha} - \bar{\zeta} \right)^{-(1-\alpha)}.$$

Using the definition of $\hat{\delta}$, one can show that $\hat{\zeta} \triangleq \frac{\alpha\theta}{1-\alpha}$ is the unique solution to

$$(43) \quad \pi(\hat{\zeta}) + \hat{\mu}\hat{\zeta} = \hat{\delta} + \alpha\beta$$

with $\hat{\mu} \triangleq \frac{1}{\theta}(\pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta})$. Since, by definition, $\bar{\zeta}$ depends continuously on $\bar{\delta}$ and because Equation (43) is the limit of Equation (41) for $\bar{\delta} \downarrow \hat{\delta}$, this shows that $\bar{\zeta} \rightarrow \hat{\zeta} = \frac{\alpha\theta}{1-\alpha}$ as $\bar{\delta} \downarrow \hat{\delta}$. Now, the claimed convergence (40) can be read off Equation (42). \square

Remark 4.16 *Theorem 4.9 and Lemma 4.15 show that our method provides the complete solution to the utility maximization problem (2).*

4.5 Examples: Brownian Motion and Poisson Process

This section illustrates the preceding results by two case studies, X being either a Brownian motion or a Poisson process. For the special case of Brownian motion, our results allow to recover (and extend) the results by Hindy and Huang (1993). In particular, we will recover the singularity of optimal consumption plans with respect to Lebesgue measure in this setting. By contrast, in the Poissonian case, optimal consumption may occur in gulps and at rates.

4.5.1 Geometric Brownian Motion

For $X = (W(t), t \geq 0)$, a Brownian motion, the parameter condition (23) ensuring well-posedness of our problem (2) takes the form

$$(44) \quad \delta > \hat{\delta} = \alpha r + \frac{1}{2} \frac{\alpha\theta^2}{1-\alpha}.$$

Note that this is exactly the regularity assumption also needed in the context of the classical Merton portfolio problem; compare, e.g., Karatzas and Shreve (1998), Remark 3.9.23, Merton (1990), Section 4.6, or Korn (1997), Corollary 3.3.7.

As already indicated, the Brownian case is also studied by Hindy and Huang (1993) and, indeed, their solution may be recovered from ours. Recall that their result is obtained by use of the Bellmann methodology under the additional parameter restriction

$$(45) \quad \delta < r + \beta(1 - \alpha);$$

confer their Equation (41). Our approach, however, shows that this assumption can be dispensed with. Only the natural well-posedness condition (44) has to be required; compare Theorem 4.9 and Lemma 4.15.

Let us now focus on the economic interpretation of the results in the Brownian case. Recall that the agent consumes whenever the process

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq t} \{X(s) + \mu s\}\right)$$

increases, where now, since X is Brownian motion, μ is given by

$$\mu = \frac{1}{\theta} \left(\frac{1}{2} \theta^2 + r + \beta(1 - \alpha) - \delta \right).$$

From this we immediately infer the following fundamental difference between the classic time-additive models and the Hindy-Huang-Kreps (HHK) approach: While, in the time-additive case, agents typically consume all the time, in the HHK-framework, it is typical that optimal consumption occurs periodically (and can even be singular with respect to Lebesgue measure as in the Brownian case considered here). This has already been pointed out by Hindy and Huang (1993). In their case, i.e., when (45) holds true, the agent never refrains from consumption totally. In fact, our analysis shows that this is the case iff $\mu \geq 0$, i.e., iff

$$\delta \leq r + \beta(1 - \alpha) + \frac{1}{2} \theta^2.$$

It is interesting to see what happens if this inequality does not hold true. In this case, the overall supremum of the Brownian motion with drift ($W(t) + \mu t$, $t \geq 0$) is finite almost surely. Therefore, the investor's cumulated total consumption is finite with probability one as well, even though the time horizon is infinite. Moreover, there is an almost surely finite last time of consumption. However, since this is not a stopping time the agent will not consume all his wealth at that time because he does not know for sure that there will not be another opportunity for consumption! To illustrate this point further, we calculate next the portfolio used by an agent in a standard Samuelson-type model of the asset market.

Portfolios Consider now a complete market with one risky asset P which evolves according to

$$P(0) > 0, \quad dP(t) = P(t) (\sigma dW(t) + (r + \theta\sigma) dt) \quad (t \geq 0)$$

for some $\sigma > 0$. The agent uses the asset P and the bond to finance his consumption plan C^L . Under \mathbb{P}^* ,

$$W^*(t) \triangleq W(t) + \theta t \quad (t \geq 0)$$

becomes a Brownian motion and the discounted asset price $\bar{P} = (e^{-rt}P(t), t \geq 0)$ is a \mathbb{P}^* -martingale with

$$d\bar{P} = \sigma \bar{P} dW^*$$

as usual. Denote by

$$V^L(t) \triangleq \mathbb{E}^* \left[\int_t^{+\infty} e^{-r(s-t)} dC^L(s) \middle| \mathcal{F}_t \right]$$

the present value of the remaining consumption at time $t \geq 0$. The portfolio strategy π^L in the asset we are looking for has to satisfy

$$dV^L(t) = \pi^L(t) d\bar{S}(t) - e^{-rt} dC^L(t) \quad (t \geq 0).$$

Theorem 4.17 *The agent puts a constant fraction of his wealth in the risky asset:*

$$\frac{\pi^L(t)\bar{S}(t)}{V^L(t)} \equiv \frac{\zeta^*}{\sigma},$$

where ζ^* is as in (37).

PROOF: We are interested in the representation of the martingale part of V^L as a stochastic integral with respect to W^* . We will therefore compute V^L explicitly.

We have $V^L(0-) = \Psi^L$, which has been computed in (38). For $t > 0$ we proceed along the same lines (compare the proof of Theorem 4.9 and the calculation of (38)) to obtain:

$$V^L(t) = \frac{e^{-\beta t}}{\beta} \left(\mathbb{E}^* \left[\int_t^{+\infty} (r + \beta) e^{-(r+\beta)(s-t)} A^L(s) ds \middle| \mathcal{F}_t \right] - A^L(t) \right).$$

The above expectation can be rewritten as

$$\mathbb{E}^* \left[A^L(t) \vee L^{-\frac{1}{1-\alpha}} \exp \left(\frac{\theta}{1-\alpha} \left(W(t)^* + \mu^* t + \sup_{0 \leq s \leq \tau^*} \{W^*(t+s) - W^*(t) + \mu^* s\} \right) \right) \middle| \mathcal{F}_t \right]$$

where τ^* is an independent exponential random variable with parameter $r + \beta$ and

$$\mu^* \triangleq \frac{r + \beta(1-\alpha) - \delta}{\theta} - \frac{1}{2}\theta.$$

The Markov property of Brownian motion and Lemma 4.12 (ii) allow us to conclude that this is equal to

$$A^L(t) + \frac{L^{-\zeta^*/\theta}}{\nu} e^{\zeta^* \mu^* t} A^L(t)^{-\nu} e^{\zeta^* W^*(t)}$$

where ζ^* is determined by (37) and $\nu \triangleq \frac{(1-\alpha)\zeta^* - \theta}{\theta}$, a strictly positive constant because of condition (44). The present value of the consumption policy C^L is therefore

$$(46) \quad V^L(t) = \frac{L^{-\zeta^*/\theta}}{\beta\nu} e^{(\zeta^* \mu^* - \beta)t} A^L(t)^{-\nu} e^{\zeta^* W^*(t)}.$$

Hence,

$$dV^L(t) = V^L(t)\zeta^* dW^*(t) + \text{terms of bounded variation}$$

and we conclude that at each time $t \geq 0$ the investor must hold

$$\pi^L(t) \triangleq \frac{\zeta^* V^L(t)}{\sigma \bar{S}(t)}$$

shares of the risky asset in his portfolio in order to finance the consumption policy C^L .

□

Remark 4.18 *If $\sigma = \zeta^*$, the agent invests all his wealth in the risky asset. This case can be viewed as a single-agent equilibrium of the stock market for this type of investors.*

Consider again the case when there is an almost surely finite, yet imperceptible last time of consumption. This occurs, as we pointed out above, iff

$$\delta > r + \beta(1 - \alpha) + \frac{1}{2}\theta^2.$$

In this case, the investor's level of satisfaction eventually decreases at rate β forever, inducing an ever increasing appetite. His wealth, however, decreases at a higher rate, namely $|\zeta^*\mu^*| + \beta$, as can be read off (46). Thus, the investor's relative level of satisfaction — the fraction of his level of satisfaction and his wealth V — remains large. This in turn drives him to wait for better times to come. He keeps being engaged in the risky asset although he knows that with probability one he will be 'unlucky' from some point in time on. This illustrates that, as already noted by Hindy and Huang (1993), an investor whose preferences exhibit local substitution is less risk averse than his time-additive counterpart, because he obtains utility from past consumption which makes him less dependent on current consumption. He can afford to invest in the risky asset and to refrain from consumption for a while in order to speculate on a higher future level of satisfaction.

4.5.2 Geometric Poisson Processes

Let us now study Poisson price processes, i.e., we let $X = (\pm N(t), t \geq 0)$. A jump of the process N corresponds to an unpredictable 'price shock' or, in the terminology of Hindy and Huang (1993), an 'information surprise'. We distinguish the two cases where the shocks are 'good' (price decrease) or 'bad' (price increase).

Upward Price Shocks First we consider the case of 'bad' upward price shocks, i.e., $X = (-N(t), t \geq 0)$, a Poisson process with downward jumps and intensity $\lambda > 0$ under the objective probability \mathbb{P} .

For this choice of X , the optimization problem (2) is well-posed iff

$$\delta > \hat{\delta} = \alpha r + \lambda \left((1 - \alpha)e^{-\frac{\alpha\theta}{1-\alpha}} + \alpha e^\theta - 1 \right).$$

As in the Brownian case,

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp \left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq t} \{X(s) + \mu s\} \right),$$

but now

$$\mu \triangleq \frac{1}{\theta} \left(\lambda(e^\theta - 1) + r + \beta(1 - \alpha) - \delta \right).$$

In contrast to the Brownian case, it now may happen that $Z = (X(t) + \mu t, t \geq 0)$ has ever decreasing paths. Indeed, this is the case iff $\mu \leq 0$, i.e., iff

$$\delta \geq \lambda(e^\theta - 1) + r + \beta(1 - \alpha).$$

Hence, a very impatient agent (characterized by a high rate of time preference δ), optimally consumes his whole wealth by one single gulp at time $t = 0$. If the agent is not that impatient, then, apart from a possible initial gulp, he only consumes at rates

$$dC^L(t) = \frac{1}{\beta} e^{-\beta t} dA^L(t) = \frac{r + \beta(1 - \alpha) - \delta + \lambda\theta}{\beta(1 - \alpha)} e^{-\beta t} A^L(t) 1_{\{\dot{A}^L(t) \neq 0\}} dt \quad (t > 0)$$

until an upward price shock makes him refrain from consumption. After a while, when his wealth and appetite have become large enough again, he restarts consumption until the next shock, etc.

Downward Price Shocks In the second Poisson example, there are 'nice' downward price shocks, i.e., $X = (N(t), t \geq 0)$ with N as before.

As usual,

$$A^L(t) = \eta \vee L^{-\frac{1}{1-\alpha}} \exp\left(\frac{\theta}{1-\alpha} \sup_{0 \leq s \leq t} \{X(s) + \mu s\}\right)$$

where, in this case,

$$\mu \triangleq \frac{1}{\theta} (\lambda(e^{-\theta} - 1) + r + \beta(1 - \alpha) - \delta).$$

Note that now X has positive jumps and, therefore, both Assumption 4.14 (i) and Assumption 4.14 (ii) do not hold true. Hence, the closed-form expressions for the prices of optimal consumption plans and their utility as derived at the end of Section 4.4 are no longer valid here.

However, we still have that the utility maximization problem (2) is well-posed if

$$(47) \quad \delta > \hat{\delta} = \lambda(e^{-\theta} - 1) + r + \beta(1 - \alpha).$$

Remark 4.19 *We conjecture, but cannot yet prove that condition (47) is also necessary for well-posedness of problem (2) in the case considered here. Note, however, that we know by Theorem 4.9 that the problem is ill-posed if $\delta < \hat{\delta}$. Thus the only open case is $\delta = \hat{\delta}$.*

Economically, it is interesting to note that, depending on the parameter values, two types of (optimal) consumption behavior can emerge in the presence of downward price shocks considered here:

- If we have $\mu \geq 0$, then, once the investor has started consumption, he consumes continually at rates

$$\dot{C}^L(t) = \frac{\lambda(e^{-\theta} - 1) + r + \beta(1 - \alpha)}{\beta(1 - \alpha)} e^{-\beta t} A^L(t)$$

and takes a gulp

$$\Delta C^L(t) = \frac{e^{\frac{\theta}{1-\alpha}}}{\beta} e^{-\beta t} A^L(t-) \Delta N(t)$$

whenever a nice price shock occurs. This is due to the fact that prices decline very fast and the relative wealth of the consumer increases.

- If the world is not such a comfortable one, that is if $\mu < 0$, then the agent consumes only in gulps, namely every time a nice price shock causes A^L to reach a new maximum.

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