Sensitivity Analysis for Applied General Equilibrium Models in the Presence of Multiple Equilibria

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Abstract

Pagan and Shannon's (1985) widely used approach employs local linearizations of a system of non-linear equations to obtain asymptotic distributions for the endogenous parameters (such as prices) from distributions over the exogenous parameters (such as estimates of taste, technology, or policy variables, for example). However, this approach ignores both the possibility of multiple equilibria as well as the problem (related to that of multiplicity) that critical points might be contained in the confidence interval of an exogenous parameter.

Critical equilibria occur for parameter values that generate a singular excess demand Jacobian at the equilibrium prices. At such points, the equilibrium correspondence might not be lower hemi-continuous and the selection of equilibria made by a computation algorithm (or by a tâtonnement process) can jump. From a statistical viewpoint, the presence of critical economies means that statistical error in the parameter estimates can have a large and discontinuous impact on error in the endogenous variables, such as prices.

We generalize Pagan and Shannon's approach to account for critical economies and multiple equilibria by assuming that the choice of equilibrium is described by a continuous random selection. We develop an asymptotic theory regarding equilibrium prices, which establishes that their probability density function is multimodal and that it converges to a weighted sum of normal density functions.

An important insight is that if multiple equilibria exist but multiplicity is ignored, the computed solution will be an inconsistent estimator of the true equilibrium *even if* the computation algorithm tracks the same equilibrium as the economy's tâtonnement.

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1 Introduction

When performing sensitivity analyses, users of Applied General Equilibrium (AGE) models often use Pagan and Shannon's (1985) method, which employs local linearizations of a system of non-linear equations. This method allows derivation of asymptotic distributions for the endogenous parameters (such as prices) from distributions over the exogenous parameters (such as estimates of taste, technology, or policy variables, for example). However, the Pagan-Shannon approach ignores the possibility that the model might allow multiple equilibrium solutions; it also ignores the possibility that critical points might be contained in the confidence interval of an exogenous parameter.

For arbitrary models, existence of multiple equilibria and existence of critical economies almost always go hand in hand. Dakhlia (1997), for instance, proposes to test models for multiplicity by searching for critical economies.

Critical equilibria occur for parameter values that generate a singular excess demand Jacobian matrix at the equilibrium prices. At such points, jumps in equilibrium prices may occur. Presence of critical equilibria can cause disconnections in the equilibrium confidence intervals, which, if unnoticed, can lead to misinterpretations of computational results.

The importance of critical equilibria in AGE models arises from their potentially dramatic implications for statistical inference as well as for comparative statics: price confidence intervals may be disconnected just as policy consequences can be discontinuous. An interesting phenomenon related to the presence of multiple equilibria is known as the "transfer paradox". An endowment transfer from one country to another can, under certain parameter configurations, actually make the donor country better off and the recipient worse off. Trannoy (1986) shows that multiplicity of equilibria for an initial allocation is a necessary and sufficient condition for the existence of the transfer paradox.

Similarly, if there is a critical equilibrium in an interval of admissible tax rates, the outcome of a policy change could be quite surprising. Moreover, it could be largely irreversible, as is illustrated in Figure 1, where τ is an exogenous policy parameter, say an exogenous tax rate on a particular good; p denotes price; and the inverted-S shaped curve represents the equilibrium price manifold. Suppose that the original tax rate is τ_0 with equilibrium at point A. An increase in the tax rate from τ_0 to τ_1 will take the economy from point A to point D, with a jump at the singularity at B. Now, if the

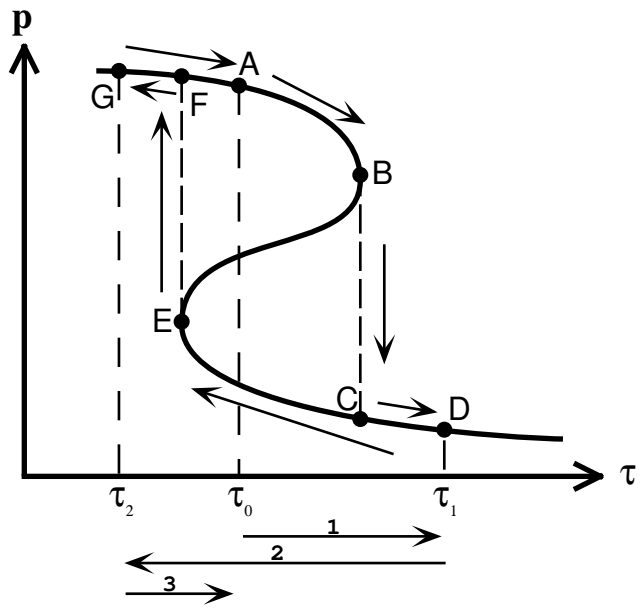

Figure 1: Irreversibility in policy

policymaker wished to reverse the policy and return to equilibrium A, she must first decrease the tax rate to a level τ_2 far below the original τ_0 so as to "pass through" singularity E and return to the upper sheaf of the manifold at equilibrium G.

Of course, this informal illustration makes implicit assumptions on the nature of the economy's tâtonnement dynamics. For our purposes we shall assume that the tâtonnement process is "tight" in the sense that a temporary disequilibrium price cannot "venture far" from the equilibrium manifold, hence that "jumps" will only occur at critical economies.

Although economic theorists are well aware of the possible existence of critical equilibria, applied economists seem to have, by and large, chosen to dismiss this possibility—along with the possibility of multiple equilibria to which it is related—as an unlikely (and unwelcome) complication of their research or even as an artifact of modeling. Clearly, however, there should be no a-priori reason to assume that a system as highly non-linear as those found in the average AGE model possesses but one solution. Nevertheless, reports of multiplicity in applied work are rare, with the notable exception perhaps of the Denny, Hannan, and O'Rourke (1995) general equilibrium

model of the Irish economy.

An important reason for this state of affairs may well be the lack of an implementable all-solutions algorithm, that is, an algorithm capable of *systematically* finding all solutions to a problem without recourse to a user-provided initial guess. Some notable advances have been made in this direction, such as the smooth homotopy approach for polynomial systems described in Garcia and Zangwill (1981) and contained in *Hompack*, a program developed by Morgan, Sommese, and Watson (1989), for example. In the absence of an implementable all-solutions algorithm, however, it might be safe practice to exclude the possibility of existence of multiple solutions by testing AGE models for uniqueness (see Dakhlia (1997)).

Although it will (almost) never happen that a general economic model is calibrated precisely at a critical economy (that is, an economy that possesses at least one critical equilibrium price), there can be a positive probability that such an economy is included in a parameter's confidence *interval*. This possibility has, so far, been largely ignored in the Applied General Equilibrium (AGE) literature. Pearson (1991) recognized this possibility, but did not attempt to address it.

From a statistical viewpoint, the presence of critical economies means that errors in the parameter estimates can have a very large and discontinuous impact on error in the endogenous variables. Thus, a Pagan-Shannon type linear approximation for an Unconditional Systematic Sensitivity Analysis (USSA),[1] for example, could be very misleading when performed in the proximity of a critical equilibrium. This paper proposes to give the asymptotic density function of prices for normally distributed parameter estimates as a (multimodal) sum of normal density functions weighted with a continuous random selection.

The following section specifies model and notation. In section 3 we state minor extensions of some results known informally in the literature but not previously placed in the context of the smooth economies framework. These results give asymptotic distributions for equilibrium prices, quantities, and other relevant endogenous variables for economies with a unique equilibrium. Section 4 presents the main result on economies with multiple equilibria. In Section 5, finally, we apply our theory to a simple example and compare our results with those of a Pagan-Shannon approximation.

[1] See Pagan and Shannon (1985) and also Wigle (1991) for a review.

2 The Model

The framework of smooth economies (see Mas-Colell, 1985) will be employed in this paper, but we shall abstract from the primitives of the framework and use only derived concepts. There are L commodities in all economies, where L is finite and integer. Prices will be given by $p \in \mathbb{R}^L_{++}$. Economies are parameterized by $\alpha \in \mathcal{K} = \mathbb{R}^K$, a finite set of parameters, some of which might be known and some of which might be estimated. These consist of policy variables, which will generally have degenerate distributions, and unknown parameters, estimates of which are made before computation is attempted. For example, α could index policy variables such as tax rates or estimated parameters such as endowments, preferences, production functions, or elasticities.

Aggregate excess demand is given by a function $f : \mathbb{R}^L_{++} \times \mathcal{K} \to \mathbb{R}^L$, and we write $f(p, \alpha)$. It is assumed throughout that f is C^1. We do not attempt here to justify this framework with assumptions on primitives, since this is well-known (see, for example, Mas-Colell (1985, p. 85)). We only note that the assumption includes smooth exchange economies as well as many smooth production economies (see Mas-Colell (1985, p. 249)) that are used in the context of computation of equilibrium. An *equilibrium price* for an economy α is $p \in \mathbb{R}^L_{++}$ such that $f(p, \alpha) = 0$.

Next we employ a standard trick to account for Walras' law. We assume that for all $\alpha \in \mathcal{K}$ and all $p \in \mathbb{R}^L_{++}$, $p \cdot f(p, \alpha) = 0$. Let superscripts represent vector components. Fix $p^L = 1$, and let the set of admissible prices be $\hat{P} \equiv \mathbb{R}^{L-1}_{++}$. In terms of notation, let $\hat{p} \in \hat{P}$. Finally, define $\hat{f}(\hat{p}, \alpha) \equiv [f^1([\hat{p}, 1], \alpha), \ldots, f^{n-1}([\hat{p}, 1], \alpha)]$. It is clear that $\hat{f}(\hat{p}, \alpha) = 0$ if and only if $[\hat{p}, 1]$ is an equilibrium price for α. We shall abuse notation slightly and call $\hat{p}$ an equilibrium price in this case. Standard notation for derivatives will be used: for example $D_1\hat{f}(\hat{p}, \alpha)$ and $D_2\hat{f}(\hat{p}, \alpha)$ are the Jacobians of $\hat{f}$ with respect to the first term (here $\hat{p}$) and the second term (here α), respectively.

The equilibrium correspondence is the set-valued map $W : \mathcal{K} \to \hat{P}$ where $W(\alpha) = \{\hat{p} \in \hat{P} \mid \hat{f}(\hat{p}, \alpha) = 0\}$. An economy α is called *regular* if for every equilibrium price $\hat{p} \in \hat{P}$, $D_1\hat{f}(\hat{p}, \alpha)$ is nonsingular. Otherwise α is called *critical*. If for some α an equilibrium price $\hat{p}$ has the property that $D_1\hat{f}(\hat{p}, \alpha)$ is singular, then the corresponding equilibrium is called *critical*. That α is generically regular is well-known in this context; since the genericity of regular economies has been studied in great detail elsewhere, it is unnecessary to belabor this point. Let $E = \{(\hat{p}, \alpha) \in \hat{P} \times \mathcal{K} \mid \hat{f}(\hat{p}, \alpha) = 0\}$ be the equilibrium manifold and, finally, let $\Sigma = \{(\hat{p}, \alpha) \in E \mid \det(D_1\hat{f}(\hat{p}, \alpha)) = 0\}$ be

the set of critical equilibria. ($\det(M)$ means the determinant of M.)

The stochastic aspects of the problem are specified next. The first obstacle encountered in formulating the results is the obvious fact that, in practice, the estimates of the parameters of the economy generally come from different sources and samples, and thus also involve different sample sizes. In terms of notation, we shall use an index $t = 1, 2, \ldots, T$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let ω be a generic element of Ω. Let the random variables $X_1^k(\omega), \ldots, X_T^k(\omega)$ be measurable maps from $(\Omega, \mathcal{F}, P)$ to $\mathcal{K}$ that form a random sample from a distribution with mean α^k (the true value of parameter k) and variance $(\sigma^k)^2$. For brevity of notation, we shall drop ω whenever appropriate. Let $\bar{X}_T^k = 1/T \sum_{t=1}^T X_t^k$ be the sample mean, which is an estimate of parameter α^k.

Denote by M' the transpose of a matrix M. Define $\hat{S}_T \equiv [X_v - \bar{X}_T] \cdot [X_w - \bar{X}_T]'$. Moreover, define the $K \times K$ matrix $\Gamma_T \equiv \hat{S}_T^{-1/2}$, as is standard.

3 Economies with a Unique Equilibrium

3.1 The Pagan-Shannon Approach

The type of result detailed in this section is well-known and is inspired by Pagan and Shannon (1985), but it has not been related to the smooth economies literature nor stated precisely in terms of regularity. In addition, an extension is proposed here to samples from different sources of different sizes.

Figure 2 displays an equilibrium correspondence for a one-dimensional parameter space and roughly illustrates Pagan and Shannon's approach, which is an application of the δ-method to the general equilibrium framework. The (asymptotically normal) probability density function (p.d.f.) Φ of an estimator of the exogenous parameter, $\bar{X}_T$, is "reflected" against the tangent to the price manifold evaluated at mean α, thereby inducing an (asymptotically normal) p.d.f. for the endogenous price variable p.
We now proceed more formally:

Definition 1 *Define* $u_T \equiv \Gamma_T \cdot [\bar{X}_T - \alpha]$, $\Sigma_T \equiv \Gamma_T^{-1}\sqrt{T}$, *and* $v_T \equiv \sqrt{T}[\hat{p}(\bar{X}_T) - \hat{p}(\alpha)]$. *Let* V_T *be the distribution function of* v_t.

Theorem 1 *Let* H_T *be the distribution function of the* $(L-1)$*-variate normal random variable* h_T *with mean zero and dispersion matrix*

$$\Delta_T \equiv Z_T \Sigma_T (Z_T \Sigma_T)',$$

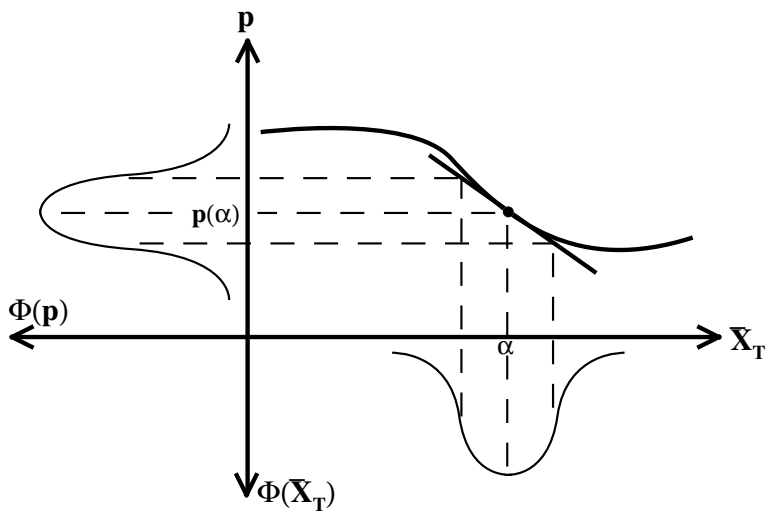

Figure 2: δ-method for regular economies

where

$$Z_T \equiv [D_1 \hat{f}(\hat{p}(\bar{X}_T), \bar{X}_T)]^{-1} \cdot D_2 \hat{f}(\hat{p}(\bar{X}_T), \bar{X}_T).$$

Then for all regular economies α with a unique equilibrium, $\lim_{T\to\infty}(v_T - h_T) = 0$*, where the limit is taken to be convergence in distribution. Alternatively, for all regular economies α,* $\lim_{T\to\infty} \sup_{\hat{p}} |V_T - H_T| = 0$.

Remark: The loose but conventional interpretation of this result is that the distribution of $\hat{p}(\bar{X}_T)$ is *approximately* $\mathcal{N}(\hat{p}(\alpha), \Delta_T \cdot T^{-1})$, from which confidence intervals can be calculated.

Theorem 1 will follow directly from Theorem 2, our generalization of the Pagan-Shannon method to economies with multiple equilibria.

3.2 An Extension to Functions of Equilibrium Prices

Confidence levels around prices are not of primary interest. Of more interest are confidence intervals around values of functions of (equilibrium) prices and the parameters of the economy, such as demands, supplies, consumer surpluses, marginal tax burdens, and comparative statics. Generally, we represent any of these by a C^1 function $g : \hat{P} \times \mathcal{K} \to I\!R^q$, and we write

$g(\hat{p}, \alpha)$. In fact, since we are only interested in values of g at equilibrium, we can write $g(\hat{p}(\alpha), \alpha)$.

Definition 2 *Define* $y_T \equiv \sqrt{T} \cdot [g(\hat{p}(\bar{X}_T), \bar{X}_T) - g(\hat{p}(\alpha), \alpha)]$. *Let* Y_T *be the distribution function of* y_T.

Corollary 1 *Let* E_T *be the distribution function of the* $(L-1)$*-variate normal random variable* e_T *with mean zero and dispersion matrix*

$$\beta_T \equiv G_T \Sigma_T (G_T \Sigma_T)',$$

where

$$G_T \equiv D_1 g(\hat{p}(\bar{X}_T), \bar{X}_T) \cdot Z_T + D_2 g(\hat{p}(\bar{X}_T), \bar{X}_T).$$

Given the assumptions of Theorem 1, for all regular economies α *with a unique equilibrium,* $\lim_{T\to\infty} \sup_{\hat{p}} |Y_T - E_T| = 0$. *Alternatively, for all regular economies* α *with a unique equilibrium,* $\lim_{T\to\infty}(y_T - e_T) = 0$, *where the limit is taken to be convergence in distribution.*

Again, the proof is omitted since Corollary 1 will be a trivial consequence of Corollary 2 in section 4.

Remark: Of course, higher order expansions can be used to obtain more accurate estimates of the confidence intervals as in Pagan and Shannon (1985). In general, the closer $\hat{p}(\alpha)$ is to a critical equilibrium, the closer $D_1\hat{f}$ is to singularity, and the larger are the confidence intervals.

What we have said so far is neither very surprising nor very novel; the main innovation is the connection to the smooth economies literature.

4 Economies with Multiple Equilibria

4.1 Intuition

Before we proceed with the formal discussion, it might be helpful to provide an intuitive understanding of our approach. Figure 3 is akin to Figure 2, except that the equilibrium correspondence now includes singularities, that is, two critical economies α_0 and α_1. Suppose we wish to find an (approximate) p.d.f. over price for an economy $\alpha \in [\alpha_0, \alpha_1]$, which in this example admits three distinct equilibria. To apply the δ-method as shown in the previous section, we would need to know which equilibrium is realized. Unfortunately, we are faced with a new kind of uncertainty in that we do not know which one of the three equilibria will be selected.

For example, if we assume that the tâtonnement process is such that unstable equilibria[2] are never selected, then in the absence of any other additional information justifying a bias in equilibrium selection, it may seem natural to assign a constant prior probability of 1/2 to the event that the economy selects one over the other stable price equilibrium. We would then be tempted to apply the δ-method to each equilibrium and simply weigh the p.d.f.'s so obtained with the assigned probabilities of selection. The idea that Figure 3 should convey is that, in the presence of singularities, the resulting p.d.f. over estimators of price can be **multimodal**.

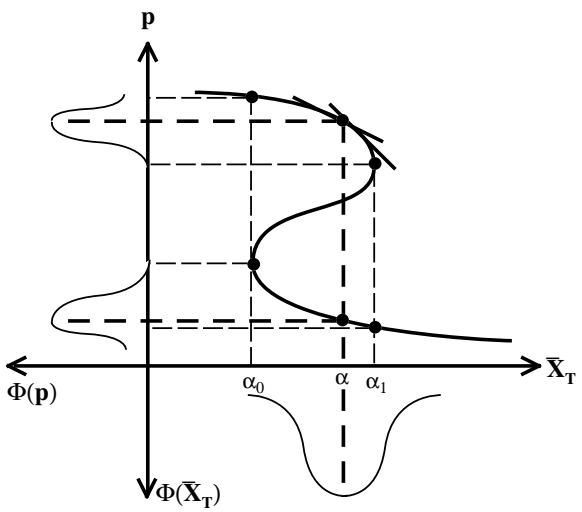

Figure 3: δ-method for multiple equilibria with constant probability of selection

More generally, however, the selection probabilities may not equal 1/2 and may not even be constants; they may, instead, be *functions* of the exogenous parameter α. Hence Figure 3 only reveals *one* consequence of the role of the probabilistic equilibrium selection, namely the existence of multiple modes for the p.d.f. over price space. *Another* consequence of a non-constant selection probability, now shown in Figure 4, is the **bias** the selection rule induces on each price mode. Let α_{up} be the expected value of the exogenous random variable $\bar{X}_T$ *conditional* on inducing an equilibrium on, say, the upper "branch" of the price equilibrium manifold and, as before, let α be the (unconditional) mean of $\bar{X}_T$. Since in general we should expect

[2]In Figure 2, the unstable equilibria are the median ones.

that $\alpha_{\text{up}} \neq \alpha$), we should not apply the δ-method at $(\hat{p}_{\text{up}}(\alpha), \alpha)$, where $\hat{p}_{\text{up}}$ is the local functional form of the equilibrium correspondence at the upper "branch", but rather at $(\hat{p}_{\text{up}}(\alpha_{\text{up}}), \alpha_{\text{up}})$. Hence the expected value of price *conditional on selecting the upper "branch"* would be $\hat{p}_{\text{up}}(\alpha_{\text{up}})$ and not $\hat{p}_{\text{up}}(\alpha)$.

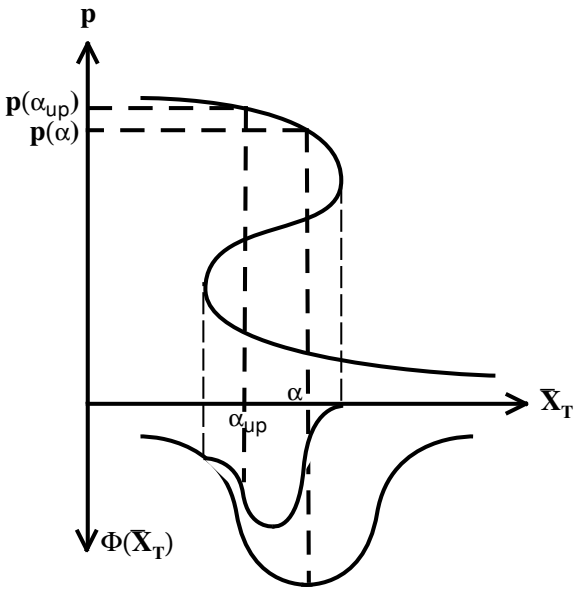

Figure 4: Bias on upper price induced by continuous random selection

In this paper, we shall refrain from justifying specific probabilities for the equilibrium selection process; however, we shall restrict our attention to the class of models where the probabilities of selection vary *continuously* with α. Continuous random selections, which are well-known in the mathematical economics literature, were first applied by Allen (1985a) to prove existence of a competitive equilibrium in Roberts' (1980) Nash-Cournot setting. Moreover, Mas-Colell and Nachbar (1991) have shown that, generically, a continuous random selection exists for arbitrary parameterizations of exchange economies. Further discussion requires a formal definition:

Definition 3 *Let $\mathcal{M}(\hat{P})$, the space of probability measures on $\hat{P}$, be endowed with the topology of weak convergence. A* **continuous random selection** *from the equilibrium correspondence W is a continuous function $r : \mathcal{K} \to \mathcal{M}(\hat{P})$ such that $\forall \alpha \in \mathcal{K}, r(\alpha)(W(\alpha)) = 1$. Let $r_i(\alpha)$ denote the*

probability of selecting economy α's i-th equilibrium.

Note that the continuity assumption for the random selection implies that *conditional* on being at a critical economy where the number of equilibria changes, the probability of selecting a critical *equilibrium* must be zero; otherwise, there would be a jump in the selection probability of a regular equilibrium of that critical economy, violating the continuity assumption. Moreover, the probability of being on a particular "branch" of the equilibrium manifold smoothly converges to zero as we approach a critical equilibrium at the end of that "branch". Hence the inherent difficulty of finding the induced asymptotic distribution on price estimates in the presence of critical equilibria is twofold: as we move along the price equilibrium manifold and towards a critical equilibrium, not only does the slope at the manifold change dramatically, but so does the probability of realizing that equilibrium.

Ideally, we should fully account for these changes, but the requirements on information (such as knowing the exact functional form of the equilibrium manifold, for instance) would be prohibitive. However, just as in the previous section, the focus on *estimators* of our parameters, rather than on the parameters *per se*, allows the use of asymptotics and permits the derivation of approximate results. As in the previous section, the asymptotics associated with the weak law of large numbers (WLLN) justifies a linear approximation of the manifold (which constitutes the δ-method) around each equilibrium solution and thus takes care of the slope: the closer the equilibrium is to a singularity, the steeper the slope. The remaining difficulty is to correctly take into account the skewing effects of the continuous random selection, that is, to correctly compute the bias for each price mode.

4.2 The Formal Approach

We shall now proceed formally to extend the results of section 3 and generalize the δ-method to economies with multiple solutions: let $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ be two probability spaces and let ω be a generic element of Ω and ω' be a generic element of Ω'. The random variable $X_t(\omega)$ is defined as in the previous section, while $A_t(X_t(\omega), \omega')$ is the random variable of selection taking values $i = 1, 2, \ldots, m$; in other words, if $A_t(X_t(\omega), \omega') = i$ then the i-th price equilibrium is selected. We shall assume that m, the maximal number of equilibria, is finite. The specification of two distinct probability spaces is necessary to model some degree of independence between realiza-

tions of A_t and realizations of X_t. Finally, let $I()$ be an indicator function taking the value 1 if its argument is true and the value 0 otherwise.

Definition 4 *Let $J_i = \{X_t \mid A_t(X_t(\omega), \omega') = i\}$ be the set of samples X_t for which the i-th equilibrium is selected. Let $T_i(T) \equiv \#(J_i)$ $(\leq T)$ be the corresponding sample size, where $\#$ denotes cardinality. If $X_t \in J_i$, we shall write $X_t = X_{i,t}$ to indicate that it is associated with the i-th equilibrium. Let α_i be the true mean and σ_i^2 be the true variance of the distribution from which $X_{i,t}$ is drawn. Finally, define the sample mean of $X_{i,t}$ as*

$$\bar{X}_{i,T} = \frac{\sum_{t=1}^{T} X_t(\omega) I\left[A_t(X_t(\omega), \omega') = i\right]}{\sum_{t=1}^{T} I\left[A_t(X_t(\omega), \omega') = i\right]}.$$

Definition 5 *Define Q_i as the* a priori *probability of selecting an event pair $(\omega, \omega') \in \Omega \times \Omega'$ such that $A(X_t(\omega), \omega') = i$.*

Clearly, if ϕ is the probability density function for the random variable X_t, then

$$Q_i = \int_{-\infty}^{+\infty} r_i(X_t)\phi(X_t)dX_t.$$

Moreover,

$$\lim_{T \to \infty} T_i(T)/T = Q_i.$$

Definition 6 *For a regular economy α, let $\hat{p}_i : \mathcal{K} \to \hat{P}$ denote the functional form of the equilibrium manifold in a neighborhood of the i-th equilibrium $(\hat{p}_i, \alpha)$. For each i, define the random variable*

$$\hat{\Pi}_{i,T} = \hat{p}_i(\bar{X}_{i,T}).$$

Definition 7 *Define $u_{i,T} \equiv \Gamma_{i,T} \cdot (\bar{X}_{i,T} - \alpha_i)$, $\Sigma_{i,T} \equiv \Gamma_{i,T}^{-1}\sqrt{T_i}$, and $v_{i,T} \equiv \sqrt{T_i}(\hat{\Pi}_{i,T} - \hat{p}_i(\alpha_i))$. Let $V_{i,T}$ be the distribution function of $v_{i,T}$.*

Theorem 2 *Let $H_{i,T}$ be the distribution function of the $(L-1)$-variate normal random variable $h_{i,T}$ with mean zero and dispersion matrix*

$$\Delta_{i,T} \equiv Z_{i,T}\Sigma_{i,T}(Z_{i,T}\Sigma_{i,T})',$$

where

$$Z_{i,T} \equiv [D_1\hat{f}(\hat{p}_i(\bar{X}_{i,T}), \bar{X}_{i,T})]^{-1} \cdot D_2\hat{f}(\hat{p}_i(\bar{X}_{i,T}), \bar{X}_{i,T}).$$

Then, if α_i is a regular economy, we have

$$\lim_{T\to\infty}(v_{i,T} - h_{i,T}) = 0,$$

where the limit is taken to be convergence in distribution. Alternatively, if α_i is a regular economy, then $\lim_{T\to\infty}\sup_{\hat{p}_i}|V_{i,T} - H_{i,T}| = 0$.

Remark: Speaking very loosely, the common interpretation of this result is that the conditional approximate density function of each $\hat{\Pi}_{i,T}$ is normal with mean $\hat{p}_i(\alpha_i)$ and dispersion matrix $\Delta_{i,T}\cdot T_i^{-1}$. It follows, moreover, that the *non-conditional* (hence multimodal) approximate density function over price must be a weighted sum of the conditional p.d.f.'s, where the weights are Q_i, which can be approximated by T_i/T.

Proof: In order to find the asymptotic distribution of $v_{i,T}$, we apply the "δ-method" on the equilibrium manifold near $(\hat{p}_i(\alpha_i),\alpha_i)$. By definition, $\hat{f}(\hat{p}_i(\alpha_i),\alpha_i) = 0$. Since α_i is assumed to be regular, $D_1\hat{f}(\hat{p}_i(\alpha_i),\alpha_i)$ is nonsingular. Using the Implicit Function Theorem, $\hat{p}_i(x)$ is *locally* C^1 near α_i and the tangent to the equilibrium manifold at α_i is

$$\frac{d\hat{p}_i}{dx}(\alpha_i) = -[D_1\hat{f}(\hat{p}_i,\alpha_i)]^{-1}\cdot D_2\hat{f}(\hat{p}_i,\alpha_i).$$

By Breiman's (1968, Theorem 11.10) multivariate central limit theorem, the random variable $u_{i,T}$ converges in distribution to a random vector u_i as T tends to infinity, where u_i is distributed $\mathcal{N}(0,I)$. Notice that $u_{i,T} = \sqrt{T}(T^{-1/2}\cdot\Gamma_{i,T}\bar{X}_{i,T} - T^{-1/2}\cdot\Gamma_{i,T}\alpha_i)$, so if $w_{i,T} \equiv \sqrt{T}(\bar{X}_{i,T} - \alpha_i)$, then $w_{i,T} = \Sigma_{i,T}\cdot u_{i,T}$. Let $H_{i,T}$ be the distribution function for a random variable $h_{i,T}$ distributed $\mathcal{N}(0,\Delta_{i,T})$.

Applying Rao (1973, pp. 388–389 (iv)), $\lim_{T\to\infty}\sup_{\hat{p}_i}|V_{i,T} - H_{i,T}| = 0$ and $\lim_{T\to\infty}(v_{i,T} - h_{i,T}) = 0$, where the limit in the second formulation is convergence in distribution. ■

The conclusion is *not* that prices are distributed approximately normal around some mean $\bar{p}$. Even though this may be true for average prices, it is not very informative. Note that unlike $\hat{\Pi}_{i,T}$, an estimator of average price such as

$$\bar{\Pi}_T = (1/T)\sum_{t=1}^{T}\sum_i \hat{p}_i(X_t)I\left[A_t(X_t(\omega),\omega') = i\right]$$

is not a consistent estimator of *any* price equilibrium.

Again, the most significant insight is that multiplicity of equilibria induces a bias on the modes of the distribution; in other words, a computed equilibrium is a biased estimate of the true price, *even if the computation algorithm has "picked" the correct "branch" of the equilibrium manifold.* This will become clear with the example in section 5.

4.3 An Extension to Functions of Equilibrium Prices

As in section 3.2, we now turn to confidence intervals around values of functions of (equilibrium) prices and the parameters of the economy. Again, consider a C^1 function $g : \hat{P} \times \mathcal{K} \rightarrow I\!R^q$, and since we are only interested in values of g at equilibrium, we shall focus on values of $g(\hat{p}_i(\alpha_i), \alpha_i)$, $i = 1, 2, \ldots$ Clearly, if there exist multiple price equilibria, then the p.d.f. over values of g will be multimodal as well.

Definition 8 *Define* $y_{i,T} \equiv \sqrt{T_i} \cdot [g(\hat{p}_i(\bar{X}_{i,T}), \bar{X}_{i,T}) - g(\hat{p}_i(\alpha_i), \alpha_i)]$. *Let* $Y_{i,T}$ *be the distribution function of* $y_{i,T}$.

Corollary 2 *Let* $E_{i,T}$ *be the distribution function of the* $(L-1)$*-variate normal random variable* $e_{i,T}$ *with mean zero and dispersion matrix*

$$\beta_{i,T} \equiv G_{i,T}\Sigma_{i,T}(G_{i,T}\Sigma_{i,T})',$$

where

$$G_{i,T} \equiv D_1 g(\hat{p}_i(\bar{X}_{i,T}), \bar{X}_{i,T}) \cdot Z_{i,T} + D_2 g(\hat{p}_i(\bar{X}_{i,T}), \bar{X}_{i,T}).$$

Given the assumptions of Theorem 2, if α_i *is a regular economy, then* $\lim_{T\rightarrow\infty} \sup_{\hat{p}_i} |Y_{i,T} - E_{i,T}| = 0$. *Alternatively, if* α_i *is a regular economy, then* $\lim_{T\rightarrow\infty}(y_{i,T} - e_{i,T}) = 0$, *where the limit is taken to be convergence in distribution.*

Proof: Up to the use of the Chain Rule to derive the Jacobian $G_{i,T}$, the proof is identical to the proof of Theorem 2. ■

Remark: The rough interpretation of this result is that the approximate density function of values of g, *conditional on the economy selecting the i-th equilibrium,* is a normal p.d.f. with mean $g(\hat{p}_i(\alpha_i))$ and dispersion matrix $\beta_{i,T} \cdot T_i^{-1}$. Moreover, it follows that the *non-conditional* (hence multimodal) approximate density function over values of g is a weighted sum of the conditional p.d.f.'s, where the weights are Q_i. Those weights can be approximated by T_i/T.

5 An Illustration

A Simple Model

We contemplate a simple exchange economy with multiple equilibria.[3] An economy consists of two consumers and two goods. Consumer i ($i = 1, 2$) has a CES-type utility function given by

$$u_i(x_1, x_2) = \left(\sum_{j=1}^{2} a_j^i (x_j^i)^{b_i} \right)^{\frac{1}{b_i}},$$

where $a_j^i \geq 0$ and $b_i < 1$. Given an endowment vector (e_1^i, e_2^i), the resulting demand functions are

$$x_j^i(p_1, p_2) = \frac{\gamma_j^i \sum_{k=1}^{2} p_k e_k^i}{p_j^{\eta_i} \sum_{k=1}^{2} \gamma_k^i p_k^{1-\eta_i}}, \quad i = 1, 2, \; j = 1, 2. \tag{1}$$

Here $\gamma_j^i = (a_j^i)^{\eta_i}$ and $\eta_i = \frac{1}{1-b_i}$.

As in the original source, parametric values are: $a_1^1 = a_2^2 = 1024$, $a_1^2 = a_2^1 = 1$, $b_1 = b_2 = -4$, $e_1^1 = e_2^2 = 12$, $e_1^2 = e_2^1 = 1$. Hence $\eta_1 = \eta_2 = 1/5$, $\gamma_1^1 = \gamma_2^2 = 4$, $\gamma_1^2 = \gamma_2^1 = 1$. For normalization purposes, restrict prices to the unit simplex $S = [0, 1]$ by setting $p_2 = 1 - p_1$. To simplify notation, p_1 is written p.

Now let e_2^2 be the free parameter, henceforth called e. Hence, aggregate excess demand function for good 1 is

$$\zeta(p; e) = \frac{4(11p + 1)}{p^{1/5}(4p^{4/5} + (1-p)^{4/5})} + \frac{p + (1-p)e}{p^{1/5}(p^{4/5} + 4(1-p)^{4/5})} - 13.$$

By solving system

$$\xi(\hat{p}, e) \equiv \begin{pmatrix} \hat{f}(\hat{p}, e) \\ \det(D_1 \hat{f}(\hat{p}, e)) \end{pmatrix} = 0, \tag{2}$$

we find that critical equilibria occur for $e = 11.58$ and $e = 12.56$. The S-shaped equilibrium manifold is shown in Figure 5.

Next, we partition the equilibrium manifold into three "branches": the lower branch (A) with natural projection $(-\infty, 12.56]$, the middle branch

[3] This example is the same as the one used in Kehoe (1991).

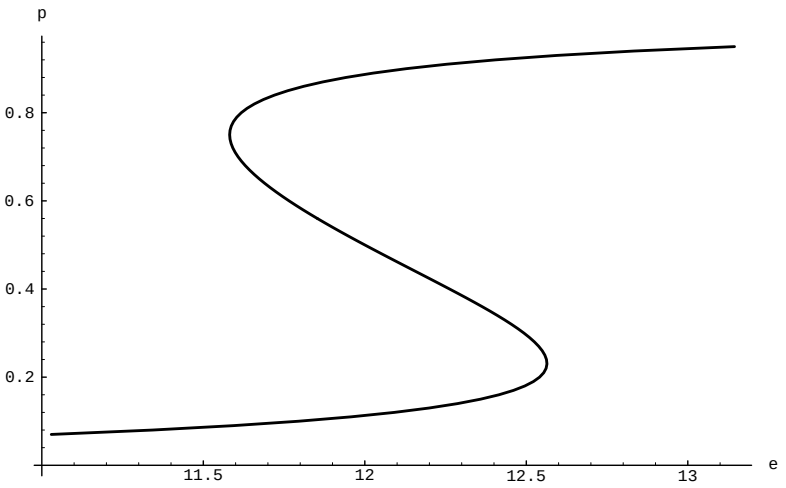

Figure 5: Equilibrium Price Manifold

(B) of unstable equilibria with projection $[11.58, 12.56]$, and the upper branch (C) with projection $[11.58, \infty)$. To simplify matters, we shall assume a zero probability of selection for any unstable equilibrium and will therefore only need to deal with branches A and C.

Assume that e is uniformly distributed on the interval $[11, 13]$, a distribution with mean 12 and variance $1/3$. Moreover, suppose that its estimate $\bar{X}_T$ is based on 30 observations.

A Continuous Random Selection

For simplicity's sake, let the probability of selecting branch A be a piecewise linear function taking the form

$$r_A(e) = \begin{cases} 1 & \text{for } e \leq 11.58 \\ 1 - \frac{e-11.58}{12.56-11.58} & \text{for } 11.58 < e < 12.56 \\ 0 & \text{otherwise} \end{cases} .$$

Of course, $r_C(e) = 1 - r_A(e)$.

We can now calculate $\bar{X}_{A,T}$ and $\bar{X}_{C,T}$, the sample means of the exogenous endowment parameter conditional on selecting branch A and branch C, respectively. For instance, the distribution of $X_{A,t}$ is uniform on the interval $[11, 11.58]$ and triangular on the interval $(11.58, 12.56]$. Computing its mean and variance is a straightforward, if somewhat tedious, exercise and we find

$\bar{X}_{A,T} = 11.55$ and $\text{var}(X_{A,t}) = \sigma_A^2 = 0.1345$. Similarly, $\bar{X}_{C,T} = 12.50$ and $\text{var}(X_{C,t}) = \sigma_C^2 = 0.1110$. Furthermore, K_A, the a priori probability of selecting a price equilibrium from branch A (i.e., when X_t is unknown), equals 0.535. By the same token, $K_C = 0.465$.

Next we compute both equilibrium prices A and C as well as the slopes of the branches at points $(\bar{X}_{A,T}, p_A(\bar{X}_{A,T}))$ and $(\bar{X}_{C,T}, p_C(\bar{X}_{C,T}))$, respectively. This is done with the help of the GAMS/MPSGE software package; slopes $\frac{dp_i}{dx}(\bar{X}_{i,T})$ are approximated by $(p_i(\bar{X}_{i,T} + \varepsilon) - p_i(\bar{X}_{i,T}))/\varepsilon$.[4] We find $p_A(\bar{X}_{A,T}) = 0.08841$ and $p_C(\bar{X}_{C,T}) = 0.92531$; moreover, $\frac{dp_A}{dx}(\bar{X}_{A,T}) \approx 0.043210$ and $\frac{dp_C}{dx}(\bar{X}_{C,T}) \approx 0.052714$.

From the Central Limit Theorem, we know that

$$\sqrt{T_i}(\bar{X}_{i,T} - \alpha_i) \xrightarrow{d} \mathcal{N}(0, \sigma_i^2), \quad i = A, C,$$

specifically,

$$\begin{aligned} \sqrt{T_A}(\bar{X}_{A,T} - 11.55) &\xrightarrow{d} \mathcal{N}(0, 0.1345), \text{and} \\ \sqrt{T_C}(\bar{X}_{C,T} - 12.50) &\xrightarrow{d} \mathcal{N}(0, 0.1110). \end{aligned}$$

With this, we can finally apply Theorem 2 to conclude that

$$\sqrt{T_i}(p_i(\bar{X}_{i,T}) - p_i(\alpha_i)) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{dp_i}{dx}(\bar{X}_{i,T})\right)^2 \sigma_i^2\right), \quad i = A, C,$$

specifically,

$$\begin{aligned} \sqrt{T_A}(p_A(\bar{X}_{A,T}) - 0.08841) &\xrightarrow{d} \mathcal{N}(0, 0.000251), \text{and} \\ \sqrt{T_C}(p_C(\bar{X}_{C,T}) - 0.92531) &\xrightarrow{d} \mathcal{N}(0, 0.000308). \end{aligned}$$

The rough interpretation of these results is that, conditional on branch A being selected, the price estimate $p_A(\bar{X}_{A,T})$ is distributed approximately normal with mean 0.08841 and variance $(2.51 \times 10^{-4})/T_A$. A proxi for T_A is $T \times K_A = 30 \times 0.535 = 16.05$, hence the variance estimate is 1.56×10^{-5}. Similarly, conditional on branch C being selected, $p_C(\bar{X}_{C,T})$ is distributed approximately normal with mean 0.92531 and variance $(3.08 \times 10^{-4})/(T \times K_C) = 2.21 \times 10^{-5}$.

[4] To control equilibrium selection, we only had to specify a close enough initial guess, i.e., we set our initial guess for the price of the first good equal to 0.1 to find a low price equilibrium and to 0.9 to find a high price equilibrium.

The **unconditional** approximate p.d.f over price estimates is then

$$0.535\phi(0.08841, 1.56 \times 10^{-5}) + 0.465\phi(0.92531, 2.21 \times 10^{-5}),$$

where $\phi(a, b)$ now stands for the normal p.d.f. with mean a and variance b. The p.d.f. is shown in figure 6.[5]

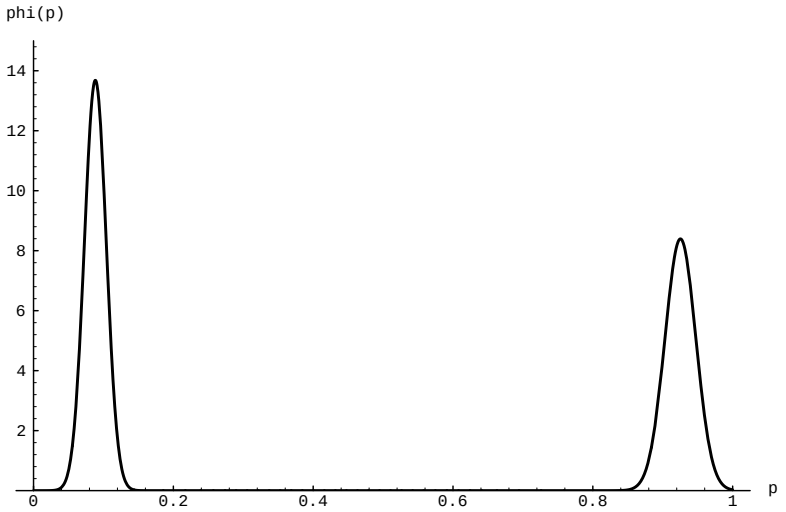

Figure 6: Unconditional approximate p.d.f. over price

We wish to point out here that the same result was obtained by a Monte-Carlo simulation; the short GAMS/MPSGE program is appended.

How does this result differ from a simple Pagan-Shannon approximation?

There is more than one way in which a sensitivity analysis can be performed incorrectly.

Case 1

The most innocent and also most likely occurrence of incorrect sensitivity analysis is that a modeler is simply not aware of the existence of multiple solutions. Suppose then, that the computational algorithm always converges to the low price equilibrium on branch A. We compute and find $p(12) =$

[5] To better see the effects of the various approaches, all variances used to plot the next four graphs (figures 6 through 9) have been multiplied by a factor of 10.

0.112923822 and $p(12.001) = 0.112996132$; hence the approximate slope of the manifold at point $(12, p(12))$ is $s = (0.112996132 - 0.112923822)/0.001 = 0.0723$. Since the endowment parameter is drawn from a distribution with variance $1/3$ and its estimator is based on 30 observations, we conclude that the price estimate $p(\bar{X}_T)$ is distributed approximately normal with mean 0.1129 and variance $s^2\sigma^2/T = 0.0723^2(1/3)/30 = 5.81 \times 10^{-5}$.

Hence compared to our earlier prediction regarding the low price equilibrium, this estimator is **biased** and differs from the unbiased one by 28%; moreover, its standard deviation is about twice that of the correct estimator. The reason for the discrepancy is the "overlap" in the exogenous variable X_t and a failure to distinguish the branches. The effect is shown in Figure 7, where the p.d.f. obtained with the correct approach (solid line) is compared to the p.d.f. obtained with the wrong approach (dashed line).

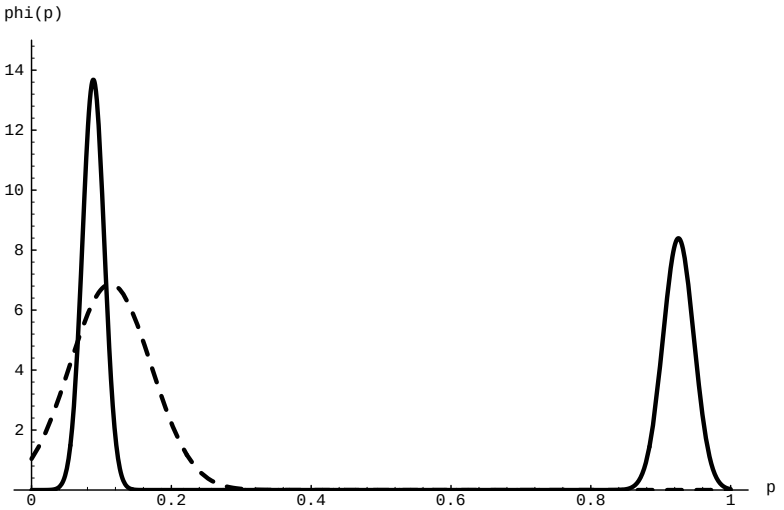

Figure 7: Case 1

Case 2

A not so subtle mistake would be to use a large change, ε, to approximate the slope s of the price manifold. This is particularly fatal if a critical economy is included in the interval $[12, 12 + \varepsilon]$, since there will be a *jump* in the equilibrium price. For $\varepsilon = 1$, we compute $s \approx (p(13) - p(12))/1 = .9457 - .1129 = .833$. We would erroneously conclude that the price estimate $p(\bar{X}_T)$ is distributed approximately normal with mean 0.1129 and variance $s^2\sigma^2/T = 0.833^2(1/3)/30 = 7.71 \times 10^{-3}$. This corresponds to a standard

deviation that is about twenty times that of the correct estimator. The effect is shown in Figure 8, where the solid line represents the correct and the dashed line represents the wrong result.

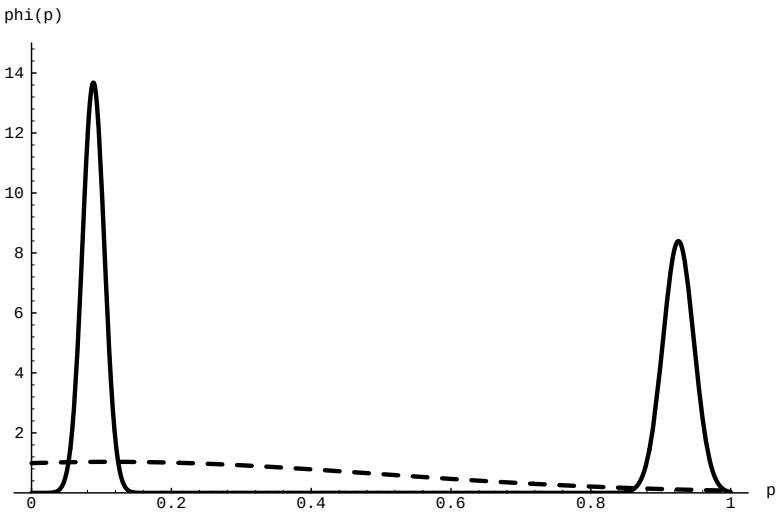

Figure 8: Case 2

Case 3

Finally, it is conceivable that the modeler is fully aware of the existence of multiple solutions, but that he has not incorporated the role of the equilibrium selection process into his sensitivity analysis. In this case he will apply the Pagan-Shannon method separately first near point $(12, p_A(12))$, then near point $(12, p_C(12))$. The prediction regarding the low price equilibrium will be the same as in Case 1 above and the incorrect prediction regarding the high price equilibrium will be that its estimate $p_C(\bar{X}_T)$ is distributed approximately normal with mean 0.8871 and variance $s^2\sigma^2/T = 0.1132^2(1/3)/30 = 1.42 \times 10^{-4}$.

Compared to the *correct* prediction regarding the high price equilibrium, this estimator is **biased** and differs from the unbiased one by 4%. Moreover, its standard deviation is about two and a half times that of the correct estimator. The effect is illustrated in Figure 9. Once again, the solid line shows the correct inference, while the dashed line is the one obtained under the implicit assumption of a uniform (hence discontinuous) random selection.

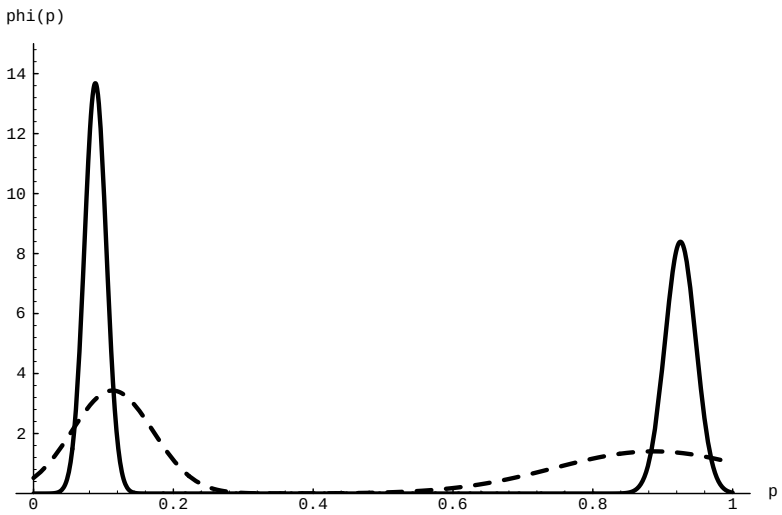

Figure 9: Case 3

6 Conclusion

Over a given range of parametrized economies with multiple equilibria, the probability of selecting a particular equilibrium may not be constant. Uncertainty over the exact location of an economy in such a range induces not only uncertainty over the realization of each of the multiple equilibrium prices, but it also induces a bias on the expected value of each price solution. In other words, an equilibrium price of the expected economy will generally not be equal to the expected value of that equilibrium price. This is but one of many complications that must be taken into account when analyzing models with multiple equilibria.

Appendix

```
FILE MC_OUT / mc.dat / ;

OPTION SEED = 30170671 ;

$TITLE  Monte Carlo simulation for an economy with multiple equilibria

*       This is a simple model of an exchange economy with
*       multiple (up to three) isolated equilibria, depending on
*       parametric values.  There are two commodities
```

```
*       and two consumers with CES preferences.
*
*       MILES may find any of the three equilibria depending
*       on the starting point.  For each run, the starting point
*       is determined probabilistically, according to the specified
*       continuous random selection.


SET     G       GOODS /G1*G2/
        C       CONSUMERS /C1*C2/

PARAMETER
        PROB   Selection probability for low price equilibrium ;

TABLE E(G,C) Factor endowments
                C1      C2
        G1      12       1
        G2       1      12

TABLE ALPHA(G,C)  Budget shares
                C1      C2
        G1      1024    1
        G2      1       1024

$ONTEXT

$MODEL: MC

$COMMODITIES:
        P(G)

$CONSUMERS:
        H(C)

$DEMAND:H(C)   s:  0.2
    E:P(G)      Q:E(G,C)
    D:P(G)      Q:1    P:ALPHA(G,C)

$OFFTEXT

$INCLUDE MC.HDR

SET   ITER     SET TO DRIVE ITERATIONS  /1*1000/;
```

```
LOOP
(ITER,

* sample endowment from uniform distribution
E("G2","C2") = UNIFORM(11,13) ;

* define continuous random selection (prob of selecting lower branch)
PROB = MIN(MAX(0,(12.56 - E("G2","C2"))/(.98)),1) ;

* control selection w/ help of initial guess (starting point)
IF(PROB LT UNIFORM(0,1),
P.L("G1") = 10 ;
ELSE P.L("G1") = .1 ; );

* price normalization
P.FX("G2") = 1 ;

$INCLUDE MC.GEN

SOLVE MC USING MCP;

PUT MC_OUT ;
IF(mc.modelstat eq 1,
PUT ITER.TL:5, E("G2","C2"):6:3, P.L("G1"):13:7 / ;
ELSE
PUT$(MC.modelstat eq 2) ITER.TL:5, E("G2","C2"):6:3, "    2" / ;
PUT$(MC.modelstat eq 3) ITER.TL:5, E("G2","C2"):6:3, "    3" / ;
PUT$(MC.modelstat eq 4) ITER.TL:5, E("G2","C2"):6:3, "    4" / ;
PUT$(MC.modelstat eq 5) ITER.TL:5, E("G2","C2"):6:3, "    5" / ;
PUT$(MC.modelstat eq 6) ITER.TL:5, E("G2","C2"):6:3, "    6" / ;
PUT$(MC.modelstat eq 7) ITER.TL:5, E("G2","C2"):6:3, "    7" / ;
PUT$(MC.modelstat eq 8) ITER.TL:5, E("G2","C2"):6:3, "    8" / ;
PUT$(MC.modelstat eq 9) ITER.TL:5, E("G2","C2"):6:3, "    9" / ;
PUT$(MC.modelstat eq 10) ITER.TL:5, E("G2","C2"):6:3, "   10" / ;
PUT$(MC.modelstat eq 11) ITER.TL:5, E("G2","C2"):6:3, "   11" / ;
PUT$(MC.modelstat eq 12) ITER.TL:5, E("G2","C2"):6:3, "   12" / ;
PUT$(MC.modelstat eq 13) ITER.TL:5, E("G2","C2"):6:3, "   13" / ;
);
);
```

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