Testing for a unique equilibrium in applied general equilibrium models

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Abstract

This paper introduces a new and computationally inexpensive method to test for uniqueness of equilibrium in exchange economies.

1 Introduction

Exchange economies with multiple equilibria are easy to construct and can be very simple. Nevertheless, reports of multiplicity in applied work are rare, with the notable exception of the Denny, Hannan, and O'Rourke (1995) general equilibrium model of the Irish economy. The question is still open whether most Applied General Equilibrium (AGE) models truly possess only one equilibrium or whether multiplicity usually simply goes undetected.

The importance of the potential existence of multiple equilibria in AGE models arises from their implications for statistical inference as well as for comparative statics: price confidence intervals may be disconnected just

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as policy consequences can be discontinuous. Ignoring the possibility of multiplicity may lead to misinterpretation of computational results.

Kehoe and Whalley (1985) perform a test for uniqueness on the Fullerton, King, Shoven, and Whalley (1981) general equilibrium model of the U.S. economy by being able to reduce the system of excess demand functions sufficiently so as to allow their plotting on a grid over a one-dimensional price space. While, in theory, this technique may also be a valid approach for models with more goods, each additional relative price adds one dimension to the domain to be searched. Hence the number of numerical evaluations necessary for the grid search increases exponentially with each additional relative price, making the trade-off between accuracy and computational cost increasingly difficult.

This may be, in part, why Kehoe and Whalley suggest that without an *all-solutions algorithm,* "it is next to impossible to establish uniqueness of equilibrium for large dimensional models" (p. 247). Indeed, an effective all-solutions algorithm would constitute a test allowing modelers to ensure uniqueness ex-post without having to impose restrictive ad hoc and ex-ante assumptions, such as the weak axiom of revealed preference property. Although this assumption would be a sufficient condition for uniqueness, it is a very strong assumption.[1]

Unfortunately, all-solutions algorithms have not yet reached a stage where they are easily implementable by applied economists. One noteworthy possibility, suggested by Kehoe (1991), is to adapt Garcia and Zangwill's (1981) all-solutions algorithm for polynomial systems, which is implemented in *Polsys*, a subroutine of the *Hompack* package developed by Morgan, Sommese, and Watson (1989). The idea is to first approximate the system of excess demands by a polynomial system and then to solve the polynomial system using *Polsys*. Unfortunately, two problems arise: (1) there is no criterion to help decide what choice of polynomial degree will be large enough for a satisfactory approximation of the functions, and (2) since the approximation requires the evaluation of the excess demand functions over a grid on price space, this approach exhibits the same sort of difficulties as the grid search method used in Kehoe and Whalley and quickly becomes prohibitively costly.

In this paper I confine my attention to exchange economies and propose a test for uniqueness that is not a grid search and that does not require the use of an all-solutions algorithm. The test consists of searching for critical

[1] See Allingham (1987) for a short review of the conditions for uniqueness.

economies in a subset of the model's parameter space. The argument is as follows: multiplicity of equilibria for an economy implies the existence of a critical economy along the segment connecting the economy with multiple equilibria to an economy corresponding to a Walrasian allocation. Hence, if the search for a critical economy along that path is unsuccessful, uniqueness is confirmed.

I introduce the general model of an exchange economy and its notation in section 2 and describe the test procedure in section 3. In section 4, I present an application of the procedure to a simple exchange economy.

2 Model and Notation

The framework of smooth economies will be employed. There are L commodities (specified by subscripts $\ell = 1, \ldots, L$) and N consumers (specified by superscripts ($n = 1, \ldots, N$). To model preferences, I shall adopt Mas-Colell's (1985) notation: Let $\mathcal{P}^{\infty}_{b,sc}$ be the space of C^{∞}, strictly convex, and monotone preference relations on $\mathbb{R}^L_{++}$, written $\succsim$, with $\{z|z \succsim x\}$ closed in $\mathbb{R}^L_{++}$ for all x. Endow $\mathcal{P}^{\infty}_{b,sc}$ with the topology of uniform convergence on compacta.

Prices will be given by $p \in \mathbb{R}^L_{++}$. Economies are parameterized by endowment $\omega \in \Omega = \mathbb{R}^{LN}_{++}$. Consumer n's demand for good ℓ is given by a function $\varphi^n_\ell : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}$, while aggregate excess demand is given by a function $f : \mathbb{R}^L_{++} \times \Omega \to \mathbb{R}^L$, written $f(p, \omega)$, where $f_\ell(p, \omega) = \sum_n \varphi^n_\ell(p, p \cdot \omega) - \sum_n \omega^n_\ell$. An *equilibrium price* for an economy ω is $p \in \mathbb{R}^L_{++}$ such that $f(p, \omega) = 0$. Let $\mathcal{C} \subset \Omega$ be the set of efficient allocations, also known as the contract curve.

Next, a standard trick will account for Walras's law. Assume that for all $\omega \in \Omega$ and all $p \in \mathbb{R}^L_{++}$, $p \cdot f(p, \omega) = 0$. Fix $p^L = 1$ and let the set of admissible prices be $\hat{P} \equiv \mathbb{R}^{L-1}_{++}$. In terms of notation, let $\hat{p} \in \hat{P}$ where $\hat{p} = (p_1, \ldots, p_{L-1})$. Finally, define $\hat{f}(\hat{p}, \omega) \equiv [f_1([\hat{p}, 1], \omega), \ldots, f_{L-1}([\hat{p}, 1], \omega)]$. It is clear that $\hat{f}(\hat{p}, \omega) = 0$ if and only if $[\hat{p}, 1]$ is an equilibrium price for ω. I shall abuse notation slightly and call $\hat{p}$ an equilibrium price in this case. Standard notation for derivatives will be used: $D_{\hat{p}}\hat{f}(\hat{p}, \omega)$ and $D_\omega \hat{f}(\hat{p}, \omega)$ are the Jacobians of $\hat{f}$ with respect to $\hat{p}$ and ω, respectively.

An economy ω is called *regular* if for every equilibrium price $\hat{p} \in \hat{P}$, $D_{\hat{p}}\hat{f}(\hat{p}, \omega)$ is nonsingular. Otherwise ω is called *critical.* If for some ω an equilibrium price $\hat{p}$ has the property that $D_{\hat{p}}\hat{f}(\hat{p}, \omega)$ is singular, then the corresponding equilibrium is called *critical.*

Let the equilibrium correspondence be the set-valued map $W : \Omega \to \hat{P}$ where $W(\omega) = \{\hat{p} \in \hat{P} \mid \hat{f}(\hat{p}, \omega) = 0\}$ and let $E = \{(\hat{p}, \omega) \in \hat{P} \times \Omega \mid \hat{p} \in W(\omega)\}$ be the equilibrium manifold. Finally, let $\Sigma = \{(\hat{p}, \omega) \in E \mid \det(D_{\hat{p}} \hat{f}(\hat{p}, \omega)) = 0\}$ be the set of critical points on the equilibrium manifold.

3 The Test

First I need to establish the intuitive result that multiplicity of equilibria for some economy implies the existence of a critical economy in the space of endowments. Hence, if the search for a critical equilibrium does not converge, then it may be safe to assume that the benchmark economy for which the test is performed has but one equilibrium solution.

Proposition 1 *Let $\omega \in \Omega$ be an economy with multiple equilibrium prices and let $x \in \mathcal{C}$ be any point on the contract curve. Then there exists a $t \in [0, 1]$, such that the economy $\omega_t \equiv (1 - t)\omega + tx$ is critical.*

The proof is based on the fact that any economy located on the contract curve has a unique price equilibrium. Hence, as we follow a path connecting an economy with multiple equilibria to a point on the contract curve, we must pass through a critical economy, that is, a "switch point" where the number of equilibria collapses to unity.

Lemma 1 *Let ω be a regular economy. Then there exists an ε-neighborhood of ω, $\mathcal{N}_\varepsilon(\omega)$, such that $\forall \tilde{\omega} \in \mathcal{N}_\varepsilon(\omega)$, $\tilde{\omega}$ is regular and $\#W(\tilde{\omega}) = \#W(\omega)$, where $\#$ denotes the cardinality of a set.*

Proof: See Debreu (1970, p.390) or, for example, Theorem 4.2.4, p.93, in Balasko (1988). ■

Proof of Proposition 1: An efficient allocation $x \in \mathcal{C}$ has a unique equilibrium price p. Indeed, suppose there is another equilibrium price $p' \neq p$: by preference maximization, for any consumer n, $\varphi^n(p', p' \cdot x^n) \succsim^n x^n$ (where $\succsim^n$ represents the n-th consumer's preference relation); on the other hand, by the First Welfare Theorem, $\varphi(p', p' \cdot x)$ cannot Pareto dominate $\varphi(p, p \cdot x) = x$; Hence $\forall n, \varphi^n(p', p' \cdot x^n) \sim^n \omega^n$, which violates our assumption of strictly convex preferences.

So, connect ω and x by a linear path $t \mapsto \omega_t \equiv (1 - t)\omega + tx$ and let $t^* \equiv \sup\{t \in [0, 1] \mid \#W(\omega_t) > 1\}$. Then $\forall \varepsilon > 0$, $\exists t \in [0, 1]$ such that $\omega_t \in \mathcal{N}_\varepsilon(\omega_{t^*})$ and $\#W(\omega_t) = 1$, i.e., $\#W(\omega_t) \neq \#W(\omega_{t^*})$. Then, by Lemma 1, t^* cannot be a regular economy.[2] ■

Having thus established a motivation, I can now address the issue of *detection* of critical economies along the path $t \mapsto (1-t)\omega + tx$. A straightforward way to solve for critical equilibria is to apply the usual fixed point computational algorithm to the augmented problem

$$\xi(\hat{p}, t) \equiv \begin{pmatrix} \hat{f}(\hat{p}, t) \\ \det(D_{\hat{p}}\hat{f}(\hat{p}, t)) \end{pmatrix} = 0 \qquad (1)$$

a system of L equations and L unknowns.

The successful application of the proposed procedure to a simple exchange economy with multiple equilibria is reported in section 4.

I finish section 3 by pointing to problem areas and opportunities for future research:

(1) The proposed method is not impervious to the generation of "false positives." Should the computer indeed find a critical equilibrium along the line segment, then one must be careful *not* to conclude that the tested economy ω necessarily has multiple equilibria: for example, if there exists a second "switch point," then the subset of economies $\{\omega_t\}$ along the line segment that do admit multiple equilibria may not include the tested economy ω. In this case, the modeler must content himself with the result that the model *can* admit multiple solutions, at least for *some* redistribution of endowments. A second potential source of "false positives" is a degenerate critical equilibrium that is just an inflection and does not generate multiplicity anywhere.

(2) Clearly, we are after a computational test for *existence*. Ideally, the search for a critical economy would make use of a globally convergent algorithm, the idea being that if an algorithm that is normally guaranteed to find a solution does in fact *not* converge, it must be the case that a solution does not exist. Therefore, if we can conclude non-existence of a critical economy along the path, we would also be able to conclude that the tested economy has a unique price equilibrium.

[2] The proof outline was suggested by Tim Kehoe in a private communication.

Smale (1976) proposes using a globally convergent algorithm based on an ordinary differential equation (also known as the "Global Newton" equation) to find economic equilibria. Smale's global convergence result relies on the satisfaction of a boundary condition and it is well known that this boundary condition is satisfied for systems of excess demand functions. However, our problem is different: while the boundary condition continues to hold in price space, it is not necessarily satisfied for the $t \in [0,1]$ variable.

One possibility would be to transform the original augmented system (1) in a way that will restrict the solution path to the domain $\hat{P} \times (0,1)$. In the example given in section 4, an effective "trick" to obtain this restriction is to divide all equations of system (1) by $t(1-t)$. Thus, let

$$\hat{\xi}(\hat{p}, t) \equiv \frac{1}{t(1-t)} \xi(\hat{p}, t). \tag{2}$$

Clearly, for $t \in (0,1)$, the zeros of $\hat{\xi}(\hat{p}, t)$ coincide with those of $\xi(\hat{p}, t)$. However, unlike the vector field generated by the Global Newton equation for the original system, the vector field for the transformed system, where for some $\lambda \in I\!R$,

$$v(\hat{p}, t) = -\lambda [D\hat{\xi}(\hat{p}, t)]^{-1} \hat{\xi}(\hat{p}, t), \tag{3}$$

is now properly behaved and pointing inwards at the boundary.

4 An Application

This section illustrates the test procedure and shows pictures that help understand the intuition. For clarity's sake, the global convergence issue is only addressed at the end.

4.1 A Simple Model

A simple example of an exchange economy with multiple equilibria is easily constructed in the 2×2 case by setting parameters such that the law of demand is violated at an equilibrium price. We shall use the same parameter values as in Kehoe (1991). The exchange economy consists of two consumers and two goods. Consumer i $(i = 1, 2)$ has a CES-type utility function given by

$$u^i(x_1, x_2) = \left(\sum_{j=1}^{2} a_j^i (x_j^i)^{b^i} \right)^{\frac{1}{b^i}},$$

where $a_j^i \geq 0$ and $b^i < 1$. Given an endowment vector (ω_1^i, ω_2^i), the resulting demand functions are

$$x_j^i = \varphi_j^i(p_1, p_2) = \frac{\gamma_j^i \sum_{k=1}^2 p_k \omega_k^i}{p_j^{\eta^i} \sum_{k=1}^2 \gamma_k^i p_k^{1-\eta^i}}, \quad i = 1, 2,\ j = 1, 2,$$

where $\gamma_j^i = (a_j^i)^{\eta^i}$ and $\eta^i = \frac{1}{1-b^i}$.

As in Kehoe (1991), parametric values are: $a_1^1 = a_2^2 = 1024$, $a_1^2 = a_2^1 = 1$, $b^1 = b^2 = -4$, $\omega_1^1 = \omega_2^2 = 12$, $\omega_1^2 = \omega_2^1 = 1$. Hence $\eta^1 = \eta^2 = 1/5$, $\gamma_1^1 = \gamma_2^2 = 4$, $\gamma_1^2 = \gamma_2^1 = 1$. For normalization purposes, restrict prices to the unit simplex $S = [0, 1]$ by setting $p_2 = 1 - p_1$ and write $p_1 = \hat{p}$.

Hence, aggregate excess demand function for good 1 is

$$\hat{f}(\hat{p}) = \frac{4(11\hat{p} + 1)}{\hat{p}^{1/5}(4\hat{p}^{4/5} + (1 - \hat{p})^{4/5})} + \frac{12 - 11\hat{p}}{\hat{p}^{1/5}(\hat{p}^{4/5} + 4(1 - \hat{p})^{4/5})} - 13, \quad (4)$$

which has multiple solutions, as can be seen in Figures 1 and 2.

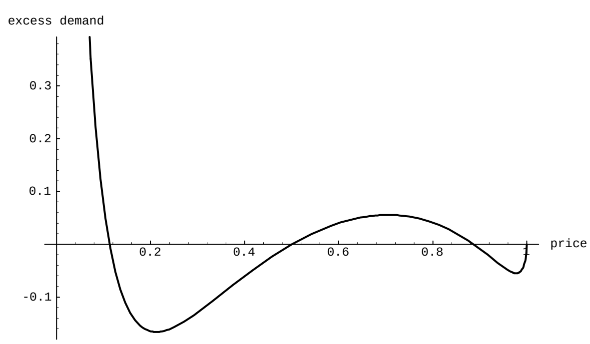

Figure 1: Excess demand function for the first good

4.2 Applying the Test Procedure

We are looking for possible critical equilibria in the Edgeworth Box, along the line segment connecting a point $x \in \mathcal{C}$ on the contract curve to the economy with the parametric values given above.

Therefore construct the following parametrization: once the applied general equilibrium problem has been run and solved a first time, an equilibrium price vector $p^* = p(\omega)$ and an efficient allocation $x = \varphi(p^*, \omega)$. are known.

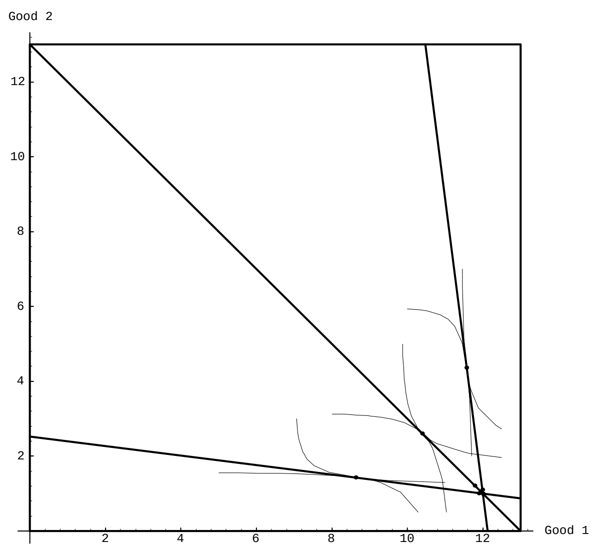

Figure 2: Edgeworth Box with three equilibria

Suppose, for instance, that the computer finds the first of the three solutions, that is, $(p_1, p_2) = (0.1129, 0.8871)$. Straightforward calculations obtain the following Walrasian allocations:

	Good 1	Good 2
Cons. 1	8.6313	1.4288
Cons. 2	4.3687	11.5712

Next, write $\hat{f}_i : S \times [0,1] \to I\!R$ as the parametrized aggregate excess demand function for the i-th good ($i = 1, \ldots, \ell - 1$), where

$$\begin{aligned} \hat{f}_i(\hat{p}, t) &= \sum_{j=1}^{n} \varphi_i^j \left(\hat{p}, tx^j + (1-t)\omega^j\right) - t\sum_{j=1}^{n} x_i^j - (1-t)\sum_{j=1}^{n} \omega_i^j \qquad (5) \\ &= \sum_{j=1}^{n} \varphi_i^j \left(\hat{p}, tx^j + (1-t)\omega^j\right) - \sum_{j=1}^{n} \omega_i^j. \end{aligned}$$

Clearly, $\hat{f}(\hat{p}, 0) = 0$ represents the original general equilibrium problem, while $\hat{f}(\hat{p}, 1) = 0$ is the system of equations at the no-trade economy.

In our particular example, the aggregate excess demand function for good 1, parameterized as in equation (5), becomes

$$\begin{aligned}\hat{f}(\hat{p},t) &= \frac{4(\hat{p}(12-3.3687t)+(1-\hat{p})(1+0.4288t))}{\hat{p}^{1/5}(4\hat{p}^{4/5}+(1-\hat{p})^{4/5})}+\\ &\quad \frac{\hat{p}(1+3.3687t)+(1-\hat{p})(12-0.4288t)}{\hat{p}^{1/5}(\hat{p}^{4/5}+4(1-\hat{p})^{4/5})}-13.\end{aligned}$$

By solving augmented system (1), a critical economy is found at $t = 0.03148$, which corresponds to the point in the Edgeworth Box where the first consumer's endowment parameters are $\omega_1^1 = 11.89$ and $\omega_2^1 = 1.014$. The excess demand function of this critical economy is shown in Figure 3.

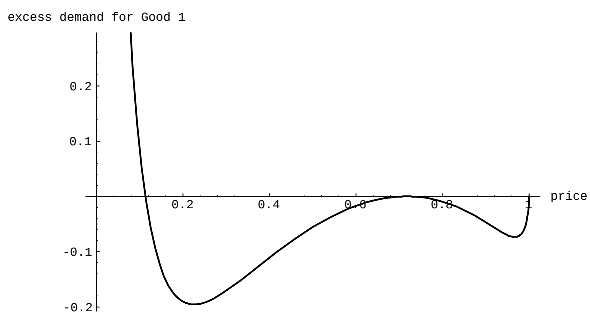

Figure 3: Excess demand for a critical economy

Figure 4 shows the collection of all excess demand functions with $t \in [0,1]$ and, as suggested by the implicit function theorem, the equilibrium manifold is easily recognized where the set of excess demand functions intersects the zero plane.

Note that the choice of x from the set of Walrasian allocations of ω, the economy to be tested, was a perilous one. Had the algorithm initially come up with $p = 0.5$, the (unstable) equilibrium with index -1, we would have been led to construct a non-regular parametrization as can be seen in Figure 5. The problem with such a parametrization is that the Jacobian $D\xi(\hat{p},t)$ is singular at the solution. While this is not a problem in theory, in practice path tracking becomes increasingly difficult as it approaches such a singularity. Figure 5 is akin to Figure 4 in that it shows the collection of all excess demand functions with $t \in [0,1]$ and the set of price equilibria is easily recognized where the set of excess demand functions intersects the

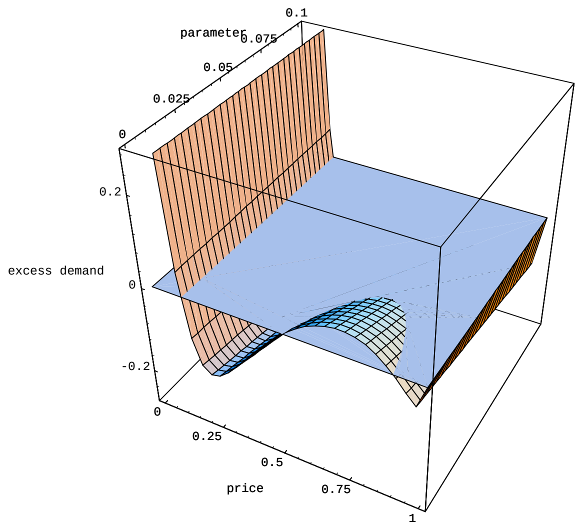

Figure 4: Parameterized excess demand

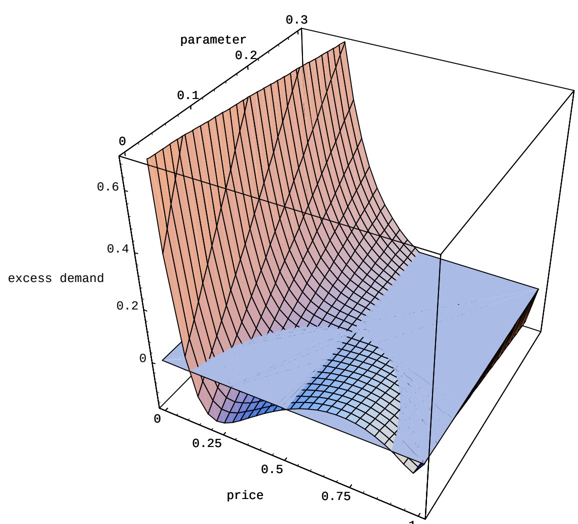

Figure 5: Non-regular parameterized excess demand

zero plane. Clearly, the set of price equilibria does not constitute a manifold since the intersection point constitutes a non-regular equilibrium. However, a small perturbation, shown in Figure 6, restores the manifold structure of

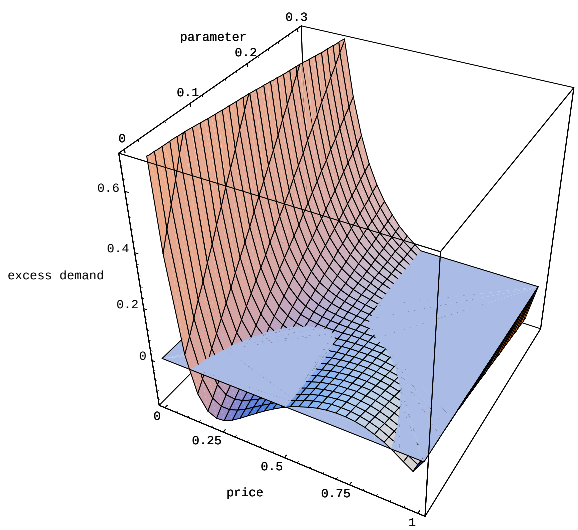

Figure 6: Regular parameterized excess demand

the set of price equilibria. Therefore, as a general rule, x should be chosen *at random* from the set of efficient allocations $\mathcal{C}$.

A Geometric Intuition

The price equilibrium manifold E over the Edgeworth Box is shown in Figure 7. By repeating the above search for critical economies for varying values of a second endowment parameter, one can–with a sequence of points–actually *track* an entire boundary of critical equilibria, Σ, over the Edgeworth Box. Σ, which is embedded in the price equilibrium manifold E, is shown in Figure 8 (blown up to larger scale to focus on the lower right hand corner of the Edgeworth Box). The natural projection of the singular boundary Σ onto the Edgeworth Box generates the cusp in the lower, right corner of Figure 9. All economies contained in the small diamond-shaped area are economies with multiple equilibria. (The other curve in Figure 9

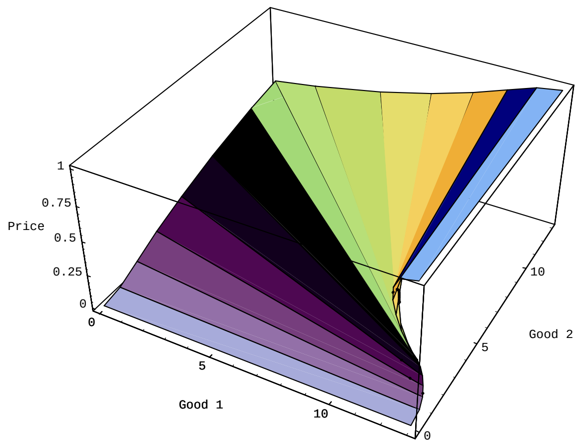

Figure 7: Equilibrium manifold E

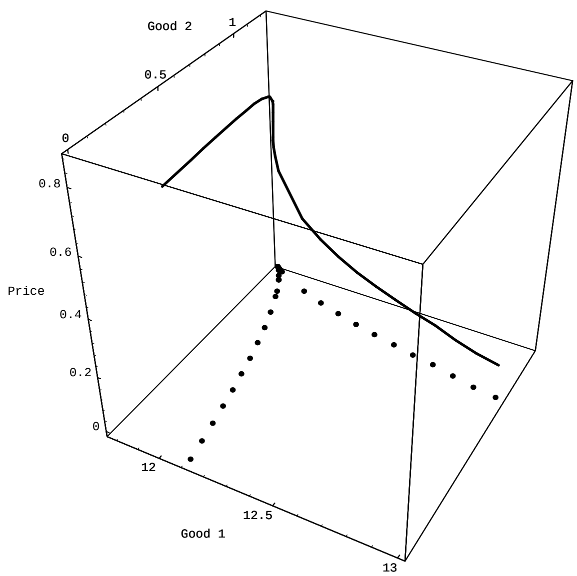

Figure 8: Singular boundary Σ and its projection into Ω

is the set of efficient allocations $\mathcal{C}$, or contract curve.) Clearly any line connecting an economy with multiple equilibria to an economy on the contract curve must cross the singularity boundary.

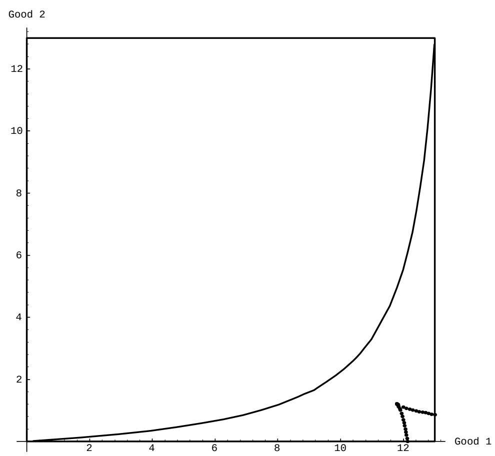

Figure 9: Edgeworth Box with contract curve and critical economies

4.3 Global Convergence

The following plots of vector fields illustrate the difference between the convergence property of the original system $\xi(\hat{p}, t) = 0$ and that of the transformed system

$$\hat{\xi}(\hat{p}, t) \equiv \frac{1}{t(1-t)}\xi(\hat{p}, t) = 0.$$

Figure 10 shows the vector field for the original system of equations. The solution is on the lower left corner roughly at $(t, p) = (0.05, 0.3)$ with surrounding vectors pointing in its direction. However, for many starting values (for example $(1, 1)$), the solution path will cross the $t = 0$ boundary and converge to a solution outside the space of interest. In contrast, the vector field for the transformed system is shown in Figure 11. Here all vectors at the boundary are pointing inwards, hence all paths must lead to the solution at $(t, p) = (0.05, 0.3)$.

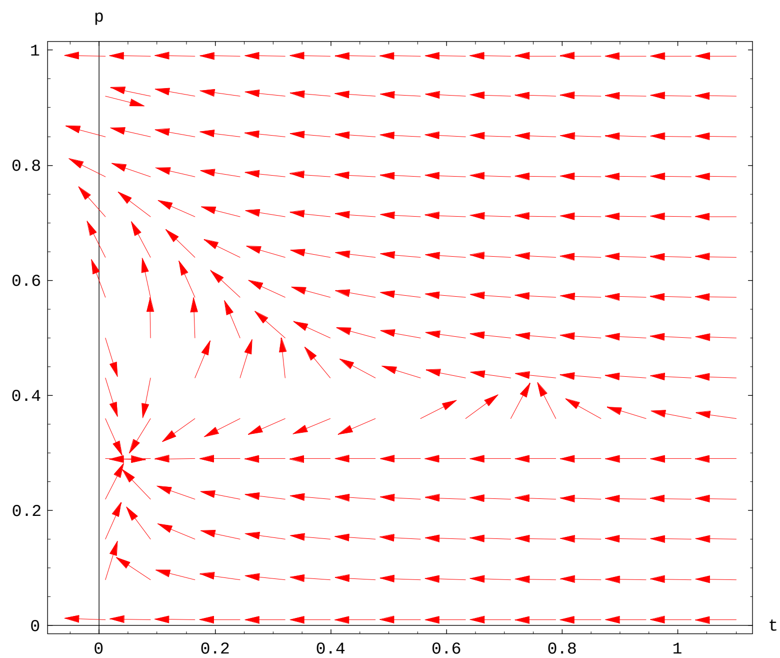

Figure 10: Vector field generated by $\xi(\hat{p}, t) = 0$

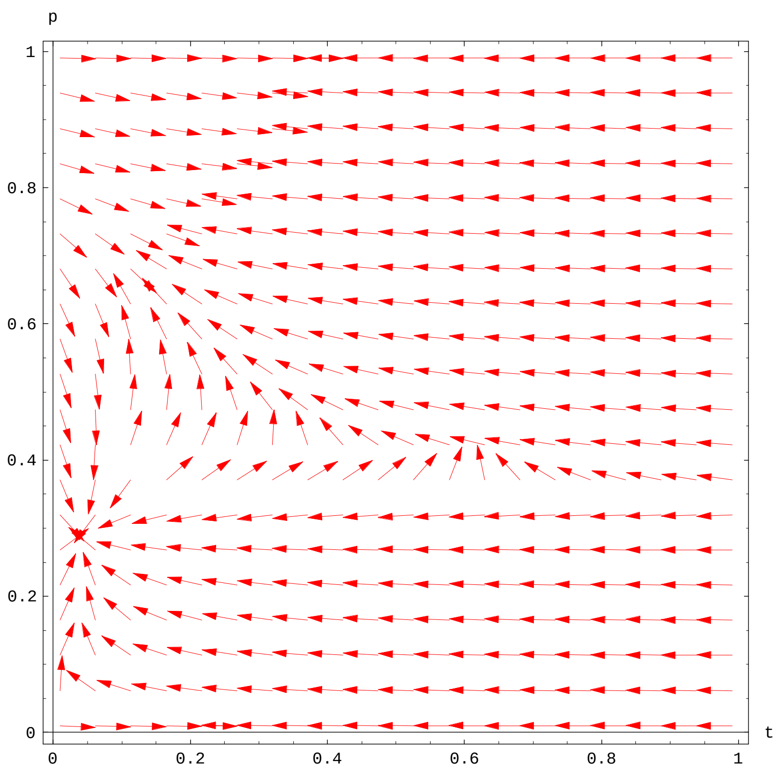

Figure 11: Vector field generated by $\hat{\xi}(\hat{p}, t) = 0$

5 Conclusion

I have introduced a new approach to testing for uniqueness of equilibrium in exchange economies. Future research in this area might address issues such as global convergence to a critical economy if it exists, generalization of the test procedure to economies with production, and applications to full-sized AGE models.

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