

Individual and Aggregate Real Balances in a Random Matching Model*

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Abstract

This paper investigates the existence and properties of stationary single-price equilibrium in a monetary random matching model where agents can hold an arbitrary amount of divisible money, and where production is costly. For some parameter values of the model, there exists a continuum of single-price equilibria indexed by the aggregate real-money balance. At such an equilibrium, an agent accumulates money only up to a certain point where his marginal value of holding money drops below the cost of production. Different upper bounds on money holdings imply different distributions of money holdings. The coexistence of multiple equilibria with distinct upper bounds but identical aggregate real-money balances suggests the importance of money distribution in determining trade velocity, production and welfare.

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1 Introduction

Despite the success of research based on Kiyotaki and Wright (1989) in endogenizing monetary trade as a means of overcoming market frictions, there are apparent limitations with respect to questions such as how price is formed. Two key simplifying assumptions are the main restrictions: indivisibility of traded objects (including both money and goods) and unit-inventory constraint on the holdings of objects. Together, these assumptions imply that the only transaction pattern is one-for-one swapping of goods and/or money. Shi (1995) and Trejos and Wright (1995) relax the indivisibility constraint on goods by endowing each agent with the capacity to produce any amount of goods instantaneously with cost, while keeping both constraints on money. The equilibrium price is the outcome of bargaining between an agent holding one unit of money and an agent with production capacity. I will study a model that is dual to these models in the sense that both indivisibility and unit-inventory constraints are kept on goods, but money will be divisible and agents can carry any amount of money. This approach is adopted for a better understanding of the crucial aspects of the random-matching model, and for its potential in addressing issues such as the uniformity of terms of trade and indeterminacy and/or multiplicity of monetary equilibrium. Addressing these issues is a necessary step toward applying these models to a broad range of policy questions.

Green and Zhou (1995) study a version of Kiyotaki and Wright's monetary search model in which agents can hold an arbitrary amount of divisible money.¹ The main result of the paper is the existence of a continuum of stationary single-price equilibria in a costless-production environment. These equilibria are indexed by the aggregate real-money balance, and they are also welfare-ranked. The main problem of the paper is that money is not necessary to facilitate trades given that production is costless. In fact, a gift-giving non-monetary equilibrium always exists and Pareto-dominates any monetary equilibrium. The question is then whether the assumption of costless production is essential to the existence of a continuum of monetary equilibria.

To answer this question, I study the stationary single-price equilibrium in a costly-production environment.² The main results of the paper are as follows. First, at any single-price equilibrium, agents' money holdings are uniformly bounded. That is, there exists an endogenous upper bound on money holdings, at which the cost of production exceeds agents' marginal value of holding money, therefore "rich" people stop selling and accumulating money. Second, for some parameter

¹Some parallel work has been done by Camera and Corbae (1996), Molico (1996) and Wallace (1996). They endow agents ability to produce perfectly divisible good instantaneously, and adopt a full-information bargaining mechanism in pairwise trading. Camera and Corbae (1996) and Wallace (1996) allow agents to hold multiple units of money up to a finite upper bound. Molico (1996) computationally studies a model without upper bound on money holdings.

²I am indebted to Dean Corbae for correcting my initial erroneous conjecture that a stationary distribution of single-price equilibrium would not exist in a costly-production environment.

values of the model, a single-price equilibrium exists. Generically, if one such equilibrium exists where the price is p , for the same set of parameters, the equilibrium also exists in the neighborhood of p . In other words, the existence of a continuum of single-price equilibria indexed by price or aggregate real-money balance survives the introduction of costly production. Third, there is another dimension of multiplicity of equilibrium in this model. For a given real-money balance (i.e., nominal money balance and price), single-price equilibria with distinct upper bounds on money holdings may coexist. They differ in the distribution of money holdings, velocity of trading, and therefore welfare. This result suggests that the distribution of money may be an important contributing factor to the non-neutrality of money. Fourth, the equilibrium set with endogenous upper bound on money holdings in general is different from those with exogenous upper bound.

2 The Model

There is a continuum of infinitely-lived agents with mass of measure 1. The population is equally divided into k types, $k \geq 3$. Time is continuous. There are $k + 1$ goods. Of these goods, k (indexed by 1 through k) are indivisible, immediately perishable goods that are produced by the agents. The remaining good is a divisible, perfectly durable, fiat-money object, with a constant total nominal stock M . A type i agent can produce one unit of good $i + 1 \pmod{k}$ instantaneously and at a utility cost $c > 0$ any time. He consumes only good i , from which he derives an instantaneous utility $u > c > 0$. Each agent maximizes the discounted expected utility of his consumption stream, with discount rate γ . Note that two key assumptions of Kiyotaki and Wright on fiat money are relaxed here: money is divisible and agents can hold any amount of money at any time, even while they are producers. But both indivisibility and inventory constraints on consumption goods are maintained.

Agents meet pairwise randomly according to a Poisson process with parameter μ . The specialization in preference and production ensures a complete lack of double coincidence of wants between any pair of agents. Consumption goods cannot be used as commodity money because they are perishable. Thus trade must involve using fiat money as a medium of exchange. An agent is characterized by his type and the amount of money he holds, which varies across traders. Agents' types are known, but their money holdings and trading histories are private information. Pairwise trading is conducted through a seller-posting-price mechanism. When a type i agent who has fiat money (buyer) meets a type $i - 1$ agent who can produce his desired good (seller), the seller posts an offer at which he is willing to sell that the buyer must either choose to accept or reject. Trade occurs if and only if the offer is accepted, and in that case the buyer pays exactly the seller's offer price.

I will be focusing on stationary symmetric equilibrium where all agents of the same type with

the identical money holdings act alike, all of the k types are symmetric, and agents' trading strategies are time-invariant.

Let H denote the measure of stationary distribution of money holdings in the economy, with domain \mathbb{R}_+ . A generic type i agent's trading strategy consists of a pair of real-valued functions defined on \mathbb{R}_+ . An offer strategy specifies the offer that an agent will make as a seller when his current money holding is η and he meets a type $i + 1$ agent, denoted by $\omega(\eta)$. A reservation-price strategy specifies the maximum willingness to pay as a buyer when his current money holding is η and he meets a type $i - 1$ agent, denoted by $\rho(\eta)$. A buyer with money-holding η accepts an offer if it does not exceed his reservation-price $\rho(\eta)$. A buyer cannot accept an offer that exceeds his money holding, which is the feasibility constraint,

$$\rho(\eta) \leq \eta. \quad (1)$$

A symmetric strategy profile and a stationary distribution of money holdings imply stationary distributions of offers, called Ω ,

$$\Omega(x) = H\{\eta \mid \omega(\eta) \leq x\}, \quad (2)$$

and of reservation prices, called R ,

$$R(x) = H\{\eta \mid \rho(\eta) < x\}. \quad (3)$$

Note that, for convenience, R is defined to be continuous from the left, rather than from the right as is conventional.

The value function $\mathcal{V}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ specifies the expected discounted utility that an agent will receive as a function of his current money holding, if he adopts an optimal trading strategy. Consider an agent of type i with money holding η . The value of his money holding $\mathcal{V}(\eta)$ depends on its usage in potential trade meetings. With probability $1/k$, the agent meets a producer of his consumption good, from whom he might be able to make a purchase, pay an offer price x which decreases his money holding to $\eta - x$, and enjoy a utility gain of u . With equal probability $1/k$, the agent meets a consumer of his production good to whom he might be able to make a sale, receive a payment o of his offer price which increases his money holding to $\eta + o$, but suffer a disutility cost of production c . Formally, given the discount rate γ and the random-matching frequency μ , the value function can be written as follows (for detailed derivation, see Green and Zhou (1995)),

$$\begin{aligned} \mathcal{V}(\eta) = & \frac{\mu}{k\gamma + 2\mu} \left[\max_{r \in [0, \eta]} \left[\int_0^r (u + \mathcal{V}(\eta - x)) d\Omega(x) + (1 - \Omega(r))\mathcal{V}(\eta) \right] \right. \\ & \left. + \max_{o \in \mathbb{R}_+} \left[R(o)\mathcal{V}(\eta) + (1 - R(o))(\mathcal{V}(\eta + o) - c) \right] \right]. \quad (4) \end{aligned}$$

Equation (4) is the Bellman equation for \mathcal{V} . Standard arguments establish that there exists a unique solution in the space of bounded measurable functions, and that this solution specifies the optimal expected discounted value of each possible level of money holding.

A potential trade meeting between a buyer and a seller entails a sequential bargaining game (the seller posts an offer, and then the buyer accepts or rejects it). Extending the reduced-form representation of the buyers' strategy — the reservation price — back to the sequential setting, a buyer accepts an offer if and only if it does not exceed his full valuation of a unit of consumption good. That is, his reservation price is determined by his willingness to pay,

$$\forall o \in \mathbb{R}_+ [\rho(\eta) \geq o \iff u + \mathcal{V}(\eta - o) \geq \mathcal{V}(\eta)]. \quad (5)$$

Perfectness of the equilibrium requires that this condition be satisfied at any offer. This condition is particularly important because the equilibrium studied in this paper does not have full support.

The state of the environment is summarized by the distribution of money holdings. Implicitly, the money holding of each agent is a continuous-time, pure-jump Markov process on the state space \mathbb{R}_+ . The transition probabilities are the probabilities of transactions occurring, induced by the optimal strategies (ω, ρ) . The environment is stationary if the measure H is a stationary initial distribution of this process.

The equilibrium concept adopted here is stationary perfect Bayesian equilibrium. I will refer to this simply as stationary equilibrium.

DEFINITION. A *stationary equilibrium* consists of a time-invariant profile $\langle H, R, \Omega, \omega, \rho, \mathcal{V} \rangle$ that satisfies

- (I) H is stationary under trading strategies ω and ρ , and the reservation-price distribution R and the offer distribution Ω are derived from H , ω and ρ according to (2) and (3).
- (II) Given the distributions for money-holdings H , reservation-price R and offer Ω , the reservation-price strategy ρ satisfies feasibility condition (1) and perfectness condition (5), and trading strategies (ω, ρ) and value function \mathcal{V} solve Bellman equation (4). That is,

$$\begin{aligned} \mathcal{V}(\eta) = & \frac{\mu}{k\gamma + 2\mu} \left[\left[\int_0^{\rho(\eta)} (u + \mathcal{V}(\eta - x)) d\Omega(x) + (1 - \Omega(\rho(\eta)))\mathcal{V}(\eta) \right] \right. \\ & \left. + \left[R(\omega(\eta))\mathcal{V}(\eta) + (1 - R(\omega(\eta)))(\mathcal{V}(\eta + \omega(\eta)) - c) \right] \right]. \quad (6) \end{aligned}$$

3 The Characterization of Single-Price Equilibrium

In this section, I will study the existence of single-price equilibria. Consider a potential single-price equilibrium at which all trades occur at a single price p . I will first describe the set of trading

strategies for a single-price equilibrium. Assume that such strategies are adopted, I will characterize the stationary money-holding distribution, and solve the corresponding value function. Given the money-holding distribution and value function, I will then verify that the presumed trading strategies are indeed optimal. I will show that for any strictly positive production cost c , every single-price equilibrium is associated with an endogenous upper bound on agents' money holdings, above which no one will produce at the prevailing price p . I will give a set of sufficient conditions under which the single-price equilibrium exists, and discuss the choice of equilibrium among an infinite set with equilibrium refinement.

Consider the formulation of equilibrium above in the special case that all trades occur at price p . By definition, for any pair of trade meeting at equilibrium, either the buyer is not willing to pay the seller's offer price, or they trade at exactly price p . Formally, let \mathcal{H} denote the product measure on $\mathbb{R} \times \mathbb{R}$, with H in each coordinate. Then

DEFINITION. A stationary single-price equilibrium at which all trades occur at price p is a stationary equilibrium that satisfies an additional condition. For meetings between a buyer and a seller with money holdings $(\eta_b, \eta_s) \in \mathbb{R} \times \mathbb{R}$, the following two events occur with probability 1 (measurable with respect to \mathcal{H}): either $\rho(\eta_b) < \omega(\eta_s)$ thus no trade occurs, or $\omega(\eta_s) = p \leq \rho(\eta_b)$ so trade occurs at price p .

If all trades occur at price p , the effective money holdings for an agent are those that are integer multiples of p . I will consider money-holding distributions only of this kind. That is, the support of the distribution H is a subset of the discrete set $p^{\mathbb{N}} \equiv \{0, p, 2p, \dots\}$. For simplicity, define

$$h(n) \equiv H(\{np\}). \tag{7}$$

That is, $h(n)$ is the measure of the set of agents who hold precisely np units of fiat money. Now, instead of working with the measure H on \mathbb{R}_+ , we can work with the equivalent distribution h on \mathbb{N} . I will say that an agent is in state n if his money holding is np .

We are looking for an equilibrium where money has value and where p units of money can be exchanged for one unit of consumption goods. An agent can only obtain p units of money by incurring production cost c and selling his good. Presumably, the agent will be unwilling to incur the cost in exchange for the same amount of money if he has a lot of money in his pocket. That is, he will charge a higher price, at which he will not make any sale. This intuition suggests a potential upper bound on an agent's money holding. Let Np (N units of p) denote the maximum quantity of money that an agent will hold in equilibrium, which can be finite or infinite. If N is infinity, as in the case of Green and Zhou (1995) with zero production cost, an agent may hold any amount of money. Denote the effective support of money-holding distribution H , $\{0, p, 2p, \dots, Np\}$, by $\Lambda(p, N)$. I am going to show later that the above intuition can be justified:

with positive production cost, there is an endogenously determined finite N for every single-price equilibrium.

Given the upper bound np on an agent's money holdings, the set of optimal trading strategies for single-price equilibrium that are consistent with the above definition are as follows. Sellers who hold less than np units of money choose to post price p to sell their goods, and sellers with money balance np demand a higher price than p . But with probability 1, any price higher than p is never accepted by any buyers. In other words, if there is any seller posting higher than p price (in particular, sellers with money balance np), there is no buyer with money holding in the support $\Lambda(p, N)$ who is willing to pay that price. Therefore if an agent's money balance reaches np , he will wait until he succeeds in spending p on his consumption good before he will produce again. As an outcome, no agents' money balance exceeds the upper bound np . Furthermore, any buyer holding at least p units of money will be willing to buy at price p , since there is no lower price and not accepting p would only delay consumption which is costly. To summarize, the characteristics of the conjectured equilibrium strategies are as follows.

(i) All buyers with money holdings at least p accept offer p :

$$\forall n = 1, 2, \dots, \rho(np) \geq p;$$

(ii) All sellers with money holdings less than np offer price p :

$$\forall n = 0, 1, 2, \dots, N - 1 \quad \omega(np) = p;$$

(iii) Agents with money holdings greater or equal to np offer above p :

$$\forall n \geq N, \omega(np) > p;$$

(iv) Offers made by agents with money holdings np or more are not acceptable for any buyer:

$$\forall n \geq N, \omega(np) > \max_{k \leq N} \rho(kp).$$

Note that agents with money holdings not in the support of H , i.e., more than np , may be willing to accept a higher price than p . But since this group of agents has measure 0, such probability 0 event should not affect trading strategies (i)—(iv). However, the price that sellers in state N offer will affect the values of these states off the support. There are potentially many equilibria that are outcome-equivalent — all trades occur at price p with probability 1 — that have sellers in state N posting different prices. The choice among these equilibria will be made later with a specific equilibrium refinement.

For now, suppose that sellers in state N offer to sell at price jp . By (iv), no agents with money holding in the support will be willing to pay jp . But agents off the support, with money holdings more than some kp ($k > N$) might be willing to pay jp . That is, in addition to (i)—(iv), the following characteristics of the strategies are relevant,

(v) Sellers in state N sell at price jp : $\omega(np) = jp$;

(vi) There exists a least money balance kp ($k \geq N$), buyers with money holdings above which

are willing to pay $\mathcal{J}p$: $\rho(\kappa p) < \mathcal{J}p$, and $\forall n > \kappa$, $\rho(np) \geq \mathcal{J}p$.

Obviously, κ is determined by \mathcal{J} . The offer strategy implies the two types of positive-probability trading opportunities for agents holding more than $\mathcal{N}p$: buy consumption goods at price p , and buy consumption goods at price $\mathcal{J}p$ which they will engage only if their money holding is more than κp . From now on, I will refer to a single-price equilibrium at which all trades occur at price p , the support of money-holding distribution is $\Lambda(p, \mathcal{N})$, and all agents adopt trading strategies (i)—(vi), a single-price- \mathcal{N} - \mathcal{J} - p equilibrium.

3.1 Stationary Money-Holding Distribution

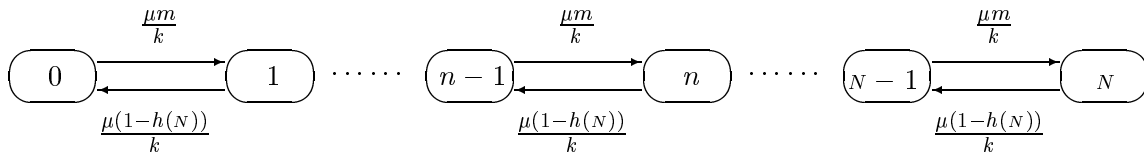
Assume that agents adopt the above set of trading strategies. Let the proportion of agents who hold positive money holdings to be m ,

$$m \equiv \sum_{n=1}^{\mathcal{N}} h(n). \quad (8)$$

Note that

$$h(0) = 1 - m. \quad (9)$$

Since all transactions are made at price p , an agent moves into state n (the state of having money holding np) only by either making a sale from state $n - 1$ or making a purchase from state $n + 1$. He moves out of state n by either making a purchase or a sale. Clearly a type i agent with a money balance less than $\mathcal{N}p$ will make a sale whenever he meets an agent of type $i + 1$ who has money, the Poisson frequency of such trading is $\mu m/k$. A type i agent with money will make a purchase whenever he meets an agent of type $i - 1$ who is willing to sell at price p (everybody except agents in state \mathcal{N}), and the Poisson frequency of such trading is $\mu(1 - h(\mathcal{N}))/k$. The trading flows and frequencies are represented in the following graph. Stationarity requires that the time rate of inflow to any state n equals the time rate of outflow from state n .



Or equivalently, as indicated by the graph, the time rate of inflow into all the states above an arbitrary n (state $n, n + 1$ up to \mathcal{N}) should be equal to the time rate of outflow from these states. That is, for any $n = 1, 2, \dots, \mathcal{N}$,

$$\frac{\mu m}{k} h(n - 1) = \frac{\mu(1 - h(\mathcal{N}))}{k} h(n). \quad (10)$$

Combining (8), (9) and (10), we can express the size of population in state n as a function of the proportion of agents having money m and the upper bound N , for any $n = 0, 1, \dots, N$,

$$h(n) = \left(\frac{m}{1 - h(N)} \right)^n (1 - m). \quad (11)$$

Note that equation (11) is consistent with the costless production case of Green and Zhou (1995) where $N = \infty$, the money-holding distribution is geometric and $h(N) = 0$. The fact that (11) holds for $n = N$ implies

$$h(N)(1 - h(N))^N = (1 - m)m^N. \quad (12)$$

Given the functional form of money-holding distribution (11), the nominal money stock M is the sum of money held by all agents. That is,

$$M = \sum_{n=1}^N np h(n) = p(1 - m) \sum_{n=1}^N n \left(\frac{m}{1 - h(N)} \right)^n. \quad (13)$$

Rearranging equation (13), we can solve $h(N)$ as a function of real-money balance $\frac{M}{p}$,

$$h(N) = \frac{m - (1 - m)\frac{M}{p}}{m(N + 1) - \frac{M}{p}}. \quad (14)$$

For a given N , real-money balance $\frac{M}{p}$ and the proportion of people holding money m have to jointly satisfy (12)–(14), as well as satisfy $0 \leq m \leq 1$ and $0 \leq h(N) \leq 1$. We can show that the solution exists, which together with (11) defines the distribution completely. Equations (11)–(14) are the equilibrium stationarity condition. More specifically, the distribution has the following features.

LEMMA 1 *Conditional on $0 \leq m \leq 1$ and $0 \leq h(N) \leq 1$, the following relationship holds among N , $\frac{M}{p}$, m and the distribution h .*

- (a) $N > \frac{M}{p}$.
- (b) Given N , m is uniquely determined by $\frac{M}{p} < N$ and vice versa.
- (c) The distribution h satisfies the following conditions: if $\frac{M}{p} < \frac{N}{2}$, then $m < \frac{N}{N+1}$, $1 - h(N) > m$ and the distribution $\{h(n)\}_{n=0}^N$ is a decreasing sequence of n ; if $\frac{M}{p} = \frac{N}{2}$, then $m = \frac{N}{N+1}$, and h is a uniform distribution, i.e., $\forall n = 0, 1, \dots, N$, $h(n) = \frac{1}{N+1}$; if $\frac{M}{p} > \frac{N}{2}$, then $m > \frac{N}{N+1}$, $1 - h(N) < m$ and the distribution $\{h(n)\}_{n=0}^N$ is an increasing sequence of n .

Lemma 1 asserts that for a given N , there is a one-to-one mapping between m and $\frac{M}{p}$, therefore either one uniquely determines the stationary distribution. Note that in case (c), the money-holding distribution is censored-geometric: the proportion of people holding a higher quantity of money declines geometrically before it hits the upper bound. The proof of lemma 1 is given in appendix A.

3.2 Equilibrium Value Function

Now we solve equation (6) for the equilibrium value function. The presumed optimal trading strategies (i)—(vi) postulate that sellers in states in the support except those in state N will sell at price p and no lower than price p ,

$$\Omega(p) = 1 - h(N); \quad \forall x < p \quad \Omega(x) = 0; \quad (15)$$

that agents in state N will sell at price np ,

$$\forall p < x < np, \quad \Omega(x) = \Omega(p); \quad \Omega(np) - \Omega(p) = h(N); \quad \forall x > np \quad \Omega(x) = 1; \quad (16)$$

and that all buyers with money will accept offer p , but not any higher offer made by agents with money holding np ,

$$1 - R(p) = m, \quad R(np) = 1. \quad (17)$$

For convenience, define $V(n) = \mathcal{V}(np)$. Given (15)—(17), the presumed trading strategies (i)—(iv) imply that the value function (6) can be written as the following set of equations:

$$V(0) = \frac{\mu}{k\gamma + 2\mu} \left(V(0) + (1 - m)V(0) + m(V(1) - c) \right), \quad (18)$$

for all $n = 1, 2, \dots, N - 1$,

$$V(n) = \frac{\mu}{k\gamma + 2\mu} \left((1 - h(N))(u + V(n - 1)) + h(N)V(n) + (1 - m)V(n) + m(V(n + 1) - c) \right), \quad (19)$$

and

$$V(N) = \frac{\mu}{k\gamma + 2\mu} \left((1 - h(N))(u + V(N - 1)) + h(N)V(N) + V(N) \right). \quad (20)$$

The value function for states off the support, $n > np$, can be defined similarly given (i)—(vi). For all $n = N + 1, \dots, K$,

$$V(n) = \frac{\mu}{k\gamma + 2\mu} \left((1 - h(N))(u + V(n - 1)) + h(N)V(n) + V(n) \right), \quad (21)$$

for $n \geq K + 1$,

$$V(n) = \frac{\mu}{k\gamma + 2\mu} \left((1 - h(N))(u + V(n - 1)) + h(N)(u + V(n - j)) + V(n) \right). \quad (22)$$

Note that the parameters μ , γ and k function as a single parameter in the form of $k\gamma/\mu$. For simplicity, define

$$\phi = \frac{k\gamma}{\mu}. \quad (23)$$

The system of equations (19) can be rewritten in the matrix form,

$$\begin{pmatrix} V(n+1) \\ V(n) \\ B \end{pmatrix} = \mathcal{D} \begin{pmatrix} V(n) \\ V(n-1) \\ B \end{pmatrix} \quad (24)$$

where $B = (1 - h(N))u - mc$, and

$$\mathcal{D} = \begin{pmatrix} \left[\frac{\phi}{m} + \frac{1-h(N)}{m} + 1 \right] & -\frac{1-h(N)}{m} & -\frac{1}{m} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

Equation (24) is a second-order linear difference equation. Its family of solutions is given in terms of eigenvectors of the matrix \mathcal{D} , which have three real eigenvalues, $\lambda_0 (= 1)$, $\lambda_1 \in (0, 1)$ and $\lambda_2 > 1$. The precise solution is determined by the two endpoint equations (18) and (20). Appendix B solves the difference equation in detail. To summarize the result, the value function V takes the following form: for all $n = 0, 1, \dots, N$,

$$V(n) = \theta_0/\phi + \theta_1\lambda_1^n + \theta_2\lambda_2^n \quad (26)$$

where θ_0 , θ_1 and θ_2 are functions of N , m , $h(N)$, and the parameters of the model u , c and ϕ . The explicit functional forms for λ_1 , λ_2 , θ_0 , θ_1 and θ_2 are given in appendix B.

By (21) and (22), the value function for $n > N$ can be solved recursively through the following, for $n = N + 1, \dots, K$,

$$V(n) = \frac{1 - h(N)}{\phi + 1 - h(N)}(u + V(n - 1)), \quad (27)$$

for $n \geq K + 1$,

$$V(n) = \frac{1}{\phi + 1} \left((1 - h(N))(u + V(n - 1)) + h(N)(u + V(n - J)) \right). \quad (28)$$

When we analyze the explicit solution of the value function (26), (27) and (28), it is easy to show the following.

LEMMA 2 *The value function V is increasing in n .*

Although we characterize a class of equilibria in which each trader's money holding is always an integer multiple of p , the value function is defined for non-integer multiples as well. Given the presumed optimal trading strategy, the value function is evidently a step function. Specifically, if $[x]$ denotes the integer part of x (that is, $x = [x] + \epsilon$ for some $\epsilon \in [0, 1)$) then

$$\mathcal{V}(\eta) = V([\eta/p]). \quad (29)$$

This completes the derivation of the value function.

3.3 Equilibrium Strategy

Given the characterization of the distribution of money holdings and value function, I will first show the existence of the finite upper bound N on money holdings in a single-price equilibrium.

THEOREM 1 *For any given ϕ and $c > 0$, there is no single-price equilibrium at which all agents' money holdings are unbounded.*

Proof. Suppose to the contrary, there is a single-price equilibrium where all transactions occur at price p and all agents' money holdings are unbounded, i.e., $N = \infty$. Since the only way for an agent to acquire money is by selling his product, the supposition implies that the agent must be willing to sell at price p regardless of his money holding. That is, for any $n \in \mathbb{N}$, $V(n+1) - V(n) \geq c$. As $N \rightarrow \infty$, $h(N) \rightarrow 0$ by (14). For a given ϕ , $\lambda_1 \in (0, 1)$ and $\lambda_2 > 1$. By (26), for any $n = 0, 1, \dots, N$, $V(n+1) - V(n) = \theta_1 \lambda_1^n (\lambda_1 - 1) + \theta_2 \lambda_2^n (\lambda_2 - 1)$, therefore it holds for $n = \lfloor N/2 \rfloor$. It is easy to show that as $N \rightarrow \infty$, θ_1 is finite but $\lambda_1^{\lfloor N/2 \rfloor} \rightarrow 0$ since $\lambda_1 < 1$, and $\theta_2 \lambda_2^{\lfloor N/2 \rfloor} \rightarrow 0$, thus, $V(\lfloor N/2 \rfloor + 1) - V(\lfloor N/2 \rfloor) = \theta_1 \lambda_1^{\lfloor N/2 \rfloor} (\lambda_1 - 1) + \theta_2 \lambda_2^{\lfloor N/2 \rfloor} (\lambda_2 - 1) \rightarrow 0$. But this contradicts the implication of the supposition that $V(n+1) - V(n)$ is at least $c > 0$ for any n . Therefore, there exists some finite N such that for $n \geq N$, an agent with money holding np will not be willing to sell at p . Whatever his posting price will be, trade will not take place at equilibrium because such trade conflicts with the definition of *single-price* equilibrium. That is, an agent will not be able to sell while holding np units of money. He can succeed only to purchase his consumption good at p and therefore reduces his money balance. Such a trading pattern implies that no agent's money holding will exceed N . Q.E.D.

Now we can study the set of single-price equilibrium strategies in the context of a finite money-holding upper bound N . I will characterize the set of sufficient conditions under which the presumed optimal trading strategies (i)—(vi) of the single-price- N - J - p equilibrium are optimal relative to the value function derived above for an agent, given the money-holding distribution H , the reservation-price distribution R and the offer distribution Ω . The choice of J will be discussed in the next subsection.

Suppose that a buyer of type i with money holding η meets a seller of type $i-1$ who produces the buyer's consumption good. Without the knowledge of each other's money holding, the seller makes an offer o to sell his product, and the buyer chooses a reservation price r . If $r \geq o$, the buyer accepts the offer and pays o units of money in exchange for one unit of his consumption good. Otherwise, the offer is rejected and the buyer and the seller part ways without trade. The buyer should choose an optimal reservation price, given the sellers' offer distribution Ω , that solves the maximization problem with respect to r of the Bellman equation (4), subject to the perfectness constraint (5). The solution is,

$$\rho(\eta) = \max\{r \in [0, \eta] \mid u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\}. \quad (30)$$

This reservation price reflects the agent's full value for a unit of his consumption good. The seller also chooses an optimal offer strategy, given the buyers' reservation-price distribution R , that

solves the maximization problem with respect to o of the Bellman equation (4),

$$\begin{aligned}\omega(\eta) &= \arg \max_{o \in \mathbb{R}_+} \left[R(o)\mathcal{V}(\eta) + (1 - R(o))(\mathcal{V}(\eta + o) - c) \right] \\ &= \arg \max_{o \in \mathbb{R}_+} \left[(1 - R(o))(\mathcal{V}(\eta + o) - c - \mathcal{V}(\eta)) \right].\end{aligned}\tag{31}$$

Since the value function is a step function defined by (29), we can eliminate some obvious weakly dominated trading strategies with (30) and (31).

LEMMA 3 *Given the support of money-holding distribution $\Lambda(p, N)$, the equilibrium offer and reservation-price strategies involve only integer multiples of p . More precisely, for every positive integer multiple np of p , the reservation price $\rho(np)$ is an integer multiple of p ; and given all agents' reservation prices are integer multiples of p , the optimal offer $\omega(\eta)$ is an integer multiple of p for any η .*

See Green and Zhou (1995), lemma 2 and lemma 4 for proof. We can now focus on the set of offer and reservation-price strategies that are integer multiples of p .

If a single-price equilibrium exists, where agents' money holdings are bounded by np , the sellers' offer strategy must change distinctly at money holding np . In particular, a seller offers to sell at price p in state $N - 1$, and he is not willing to offer at p in state N . Given the reservation-price distribution (17), these offer strategies imply

$$V(N) - c \geq V(N - 1),\tag{32}$$

$$V(N + 1) - c < V(N).\tag{33}$$

The conditions can be expressed as functions of the parameters of the model by applying the solutions for the value function (26), (27) and (28). This pair of inequalities defines precisely the upper bound N on money holdings as the incremental value of p units money drops below the cost of production c . The necessity of these two conditions in defining the upper bound N implies that they are necessary conditions for the existence of the single-price equilibrium with that upper bound. Assuming these two conditions hold, let's examine the rest of equilibrium trading strategies.

First, condition (32) implies the following characteristics of the value function.

LEMMA 4 *At equilibrium, the value function V is concave in n up to K . That is, for all $1 \leq n \leq K$, $V(n) - V(n - 1)$ is decreasing in n .*

Lemma 4 is proved in appendix C. The value function may not be concave above K . Remember that K is the state above which buyers accept offer jp made by sellers in state N , $K \geq N$. Given lemma 4, it is easy to show the existence of K .

LEMMA 5 *For any finite $J > 1$, at equilibrium there exists a finite K such that buyers in state K reject offer jp and buyers in state $K + 1$ accept offer jp .*

Proof. Suppose K is infinite, that is, no money holding is high enough for agents to accept jp . By lemma 4, the value function is concave in n throughout. By (27), it is easy to show that for a finite J , $V(n) - V(n - J) \rightarrow 0$, as $n \rightarrow \infty$. Therefore there exists an n^* such that $V(n^*) - V(n^* - J) > u$ but $V(n^* + 1) - V(n^* + 1 - J) \leq u$. By (30), then, $\rho((n^* + 1)p) \geq jp$, contradicts to the supposition. The threshold n^* is the K we are looking for. That is,

$$K = \max\{n \mid V(n) - V(n - J) > u\} \quad (34)$$

which is finite and unique for a given J . Q.E.D.

Now we are ready to verify the presumed equilibrium reservation-price strategy (i).

LEMMA 6 *If condition (32) holds, it is optimal for agents with money holdings at least p to accept offer p .*

Proof. By lemma 4, condition (32) guarantees that the value function is concave up to K . Consider buyers in state n . If $n = 1$, subtracting (18) from (19) for $n = 1$ and rearranging the equation, we have

$$(1 - h(N))(u + V(0) - V(1)) = \phi(V(1) - V(0)) + m((V(1) - V(0)) - (V(2) - V(1))). \quad (35)$$

The right hand side of (35) is positive because V is increasing by lemma 2 and concave in n .³ Therefore $u + V(0) > V(1)$. By the reservation-price function (30), $\rho(p) = p$.

If $2 \leq n \leq K$, again since the value function V is concave in n , $V(n) - V(n - 1) < V(1) - V(0) < u$. Therefore, $u + V(n - 1) > V(n)$.

If $n \geq K + 1$, from (28),

$$u + V(n - 1) - V(n) = \phi V(n) + h(N)(V(n) - V(n - J)). \quad (36)$$

The right hand side of (36) is positive since V is increasing. So $u + V(n - 1) > V(n)$.

Combining all three cases, by (30), we find for all $n \geq 1$, $\rho(np) \geq p$. That is, it is optimal that buyers with money holding at least p set their reservation prices no lower than p . Q.E.D.

Next, I examine the validity of sellers' offer strategy profile (ii)—(iv) as equilibrium strategy profile.

³In the special case where $N = K = 1$, (35) is replaced by $(1 - h(N))(u + V(0) - V(1)) = \phi V(1) > 0$. No concavity condition is needed here.

LEMMA 7 *Given that conditions (32) and (33) hold, if*

$$V(N) - V(N - 2) > u, \quad (37)$$

then it is optimal for buyers with money holdings in the support pay no more than p , for sellers with money holdings less than np to offer p , and for sellers with money holdings at least np but less than kp to offer higher than p .

Proof. By lemma 4, condition (32) guarantees the concavity of the value function up to κ . If $V(N) - V(N - 2) > u$ holds, by (30) agents with money holdings np do not pay $2p$ for their consumption goods. By the concavity of the value function, $\forall n = 2, 3, \dots, N$, $V(n) - V(n - 2) > u$. Also by the increasingness of V , for any $l = 2, 3, \dots, N$ and $n = l, \dots, N$, $V(n) - V(n - l) > u$. Thus no agents will be willing to pay higher than p price. By lemma 6, all agents with money holdings at least p accept p for trade, therefore, for all $n = 1, 2, \dots, N$, $\rho(np) = p$, or $R(p) = 1 - m$ and for any $l > 1$, $1 - R(lp) = H\{\eta \mid \rho(\eta) \geq lp\} = 0$.

Consider a seller with money holding np , denote his expected value of offering lp by $W(np, lp)$. For $l = 0, 1, \dots, N$,

$$\begin{aligned} W(np, lp) &= R(lp)V(n) + (1 - R(lp))(V(n + l) - c) \\ &= V(n) + (1 - R(lp))(V(n + l) - c - V(n)). \end{aligned} \quad (38)$$

Since $V(N) - V(N - 1) \geq c$, by concavity $V(n) - V(n - 1) > c$ for all $n = 1, 2, \dots, N$. Together with the result regarding the reservation distribution R above, for any $n < N$ and $l > 1$, $W(np, p) = V(n) + m(V(n + 1) - c - V(n)) > W(np, lp) = V(n) > W(np, 0)$. Thus for all $n < N$, $\omega(np) = p$. That is, all agents holding less than np units of money should offer exactly p .

Given $V(N + 1) - V(N) < c$, the concavity of V up to κ implies that for all $N \leq n < \kappa$, $V(n + 1) - V(n) < c$. Applying these inequalities to (38), we have for all $N \leq n < \kappa$ and any $l > 1$, $W(np, 0) < W(np, p) = V(n) + m(V(n + 1) - c - V(n)) < W(np, lp) = V(n)$. Therefore, the optimal offer should be higher than p , for $N \leq n < \kappa$, $\omega(np) > p$. Q.E.D.

Condition (37) ($V(N) - V(N - 2) > u$) is imposed as a sufficient condition for (iv). It eliminates all higher value (than p) of reservation price, and therefore ensures $R(x) = 1$ for any $x > p$ which is stronger than (iv) requires. It is conceivable that there is a single-price equilibrium that does not satisfy (37). For example, there may be agents with reservation price $2p$, but the optimal offer is $3p$ or higher for agents holding np . In this case, (iv) is satisfied, but (37) is not. On the other hand, if we impose $V(N) - V(N - J) > u$, i.e., no agents are willing to pay jp , it may not be enough to guarantee that there is no trade conducted at a price other than p with positive probability. In case there are agents willing to pay some lp which is above p but below jp , it may be the best response for agents holding np units of money to offer lp .

Since the imposition of (37) implies that for any $l > 1$, $1 - R(lp) = 0$, agents with money holding Np are in fact indifferent among posting any offer jp for $J > 1$. To simplify the analysis temporarily, let's assume that a seller can post 'infinity' as a price, and so make it impossible for a buyer with any finite money holding to purchase his consumption good, i.e., κ is infinity as well. This is the simplest single-price equilibrium possible. The value function in the states above N is given by (27), and is concave in n throughout. In this case, lemma 3—lemma 7 completely characterize the set of equilibrium strategies, with conditions (32), (33), (37) as the set of sufficient conditions for single-price equilibrium of this kind to exist. In fact, when the instantaneous utility of consumption u and production cost c are such that $u \leq 2c$ ($u > c$ is assumed in the beginning), (32) and (33) become the necessary and sufficient conditions for the existence of the single-price- N - ∞ - p equilibrium ((32) implies condition (37)). To summarize,

THEOREM 2 *For a given set of the parameters M , u , c , ϕ in the above environment,*

- (a) *For any $J > 1$, if a single-price- N - J - p equilibrium exists, conditions (32) and (33) hold.*
- (b) *A single-price- N - ∞ - p equilibrium exists if conditions (32), (33) and (37) are satisfied.*
- (c) *When $u \leq 2c$, a single-price- N - ∞ - p equilibrium exists if and only if (32) and (33) hold.*

Readers who are not interested in equilibrium refinement can skip the next subsection.

3.4 Equilibrium Refinement

There are many equilibria at which all offer prices are finite. We can shrink the equilibrium set a bit by imposing ex-post individual rationality for sellers. An offer o is ex-post individually rational if it does not make the seller worse off in case a trade does occur at the offer price, or

$$\mathcal{V}(\eta + o) - c \geq \mathcal{V}(\eta). \quad (39)$$

But the set of single-price- N - J - p equilibria (indexed by J), with the offer jp in state N satisfying (39), i.e., $V(N + J) - c \geq V(N)$, is still infinite. One can stop here, pick a J at random, and study that particular single-price- N - J - p equilibrium. Here, I will attempt to use an equilibrium refinement concept, Myerson's *properness* (1991) in essence, heuristically to select a particular equilibrium, or more specifically, select an offer strategy among the set that agents with money holding Np are indifferent.

Under condition (37), the optimal reservation price is p for all agents with money holdings in the support. Suppose instead of assuming that agents accept only offer p with probability 1, we assume that they might make mistakes, and that the chance of a small mistake (small loss of utility) is bigger than the chance of a big mistake (big loss of utility). More precisely, denote the net gain of a buyer with money holding η who purchases at an offer o ($o \leq \eta$) to be $\tau(\eta, o)$, $\tau(\eta, o) = u + \mathcal{V}(\eta - o) - \mathcal{V}(\eta)$. The net gain can be positive or negative. Suppose that buyers

adopt the following “tremble” strategy, for some $\varepsilon \in (0, 1)$, for any money holding η and any offer $o \leq \eta$,

$$\begin{array}{llll} \text{if } o \leq p & \text{accept } o & \text{wp } 1 - \varepsilon^{1+\tau(\eta,o)} & \text{if } o > p & \text{accept } o & \text{wp } \varepsilon^{1-\tau(\eta,o)} \\ & \text{reject } o & \text{wp } \varepsilon^{1+\tau(\eta,o)} & & \text{reject } o & \text{wp } 1 - \varepsilon^{1-\tau(\eta,o)}. \end{array}$$

That is, when $o \leq p$, the net gain for the buyer is positive, and he should accept it. He rejects it only mistakenly with a tremble probability $\varepsilon^{1+\tau(\eta,o)}$, which is smaller as the offer o becomes lower and the net gain becomes higher.⁴ When $o > p$, the net gain of the deal for the buyer is negative and he should reject it. He accepts it only mistakenly with a tremble probability $\varepsilon^{1-\tau(\eta,o)}$, which is smaller as the offer o becomes higher and the net loss becomes higher.

Given this tremble strategy of the buyers and given that the value function is a step function defined by (29), first of all, all non-integer multiples of p are weakly dominated by the integer multiples of p as offers. Then, among the set of offers that are integer multiples of p , one can choose ε small enough such that for sellers holding Np , if selling at lp and $(l+1)p$ both are ex-post individually rational, posting the lower offer lp yields a higher expected return, therefore dominates posting the higher offer $(l+1)p$. That is,

$$\omega(Np) = p \min\{l \mid V(N+l) - c \geq V(N)\}. \quad (40)$$

Taking ε to 0, the tremble strategy converges to the original buyers optimal strategy, represented by its reduced-reform reservation-price strategy (30), and $\omega(Np)$ remains the offer strategy for agents in state N . That is, (40) in addition to (30) and (31) are the set of equilibrium strategies that survive this equilibrium refinement process. To distinguish single-price- N - J - p equilibrium with $J = \omega(Np)$ as in refinement condition (40), I will call it single-price- N - J - p *proper* equilibrium.

To illustrate the workings of the single-price- N - J - p proper equilibrium, let's consider the simple case of $J = 2$. Given (33), $V(N+1) - c < V(N)$, (40) requires

$$V(N+2) - c \geq V(N). \quad (41)$$

Given agents in state N posting offer $2p$, condition (37), which guarantees that no agents are willing to pay $2p$, becomes a necessary condition for no trades occur at a price other than p . Now we are ready to verify the rest of the presumed optimal strategies (iii) and (vi) for agents off the support with money holdings more than κp for the case of $J = 2$. We have already shown that it is optimal for agents to accept an offer of p in states higher than κ by lemma 6, and to accept an offer of $2p$ in state $\kappa + 1$ by lemma 5.

⁴The 1 is added to ensure the probability of rejection when $o \leq p$ and the probability of acceptance when $o > p$ are not 1 in case the net gain is zero.

LEMMA 8 *In addition to condition (32), (33), (37) and (41), if*

$$V(N+2) - c < V(N+1), \quad (42)$$

then it is optimal for agents holding at least kp units of money to offer a price above p , and for agents holding more than kp units of money to accept an offer of $2p$.

The proof is in appendix D. Condition (42) is imposed to ensure that agents with money holding more than kp do not offer to sell at p when $K = N$ or $K = N + 1$. In case of $K \geq N + 2$, (42) is always satisfied. Note that depending on the value of K , $V(N+1)$ and $V(N+2)$ may be different. If $K = N$, both $V(N+1)$ and $V(N+2)$ are computed through (28). If $K = N + 1$, $V(N+1)$ is given by (27) and $V(N+2)$ by (28). If $K \geq N + 2$, both $V(N+1)$ and $V(N+2)$ are given by (27).

For the cases of $J > 2$, the analogous of conditions (41) and (42) can be constructed accordingly. Combining the previous results with lemma 8, we have

THEOREM 3 *For a given set of the parameters M, u, c, ϕ in the above trading environment, a single-price- N - 2 - p proper equilibrium exists if conditions (32), (33), (37), (41) and (42) are satisfied.*

We can compare the sets of sufficient conditions for the existence of single-price- N - J - p proper equilibrium and that of the single-price- N - ∞ - p equilibrium discussed in the end of section 3.3.

COROLLARY 1 *For any $J > 1$, the existence region of the single-price- N - J - p proper equilibrium is a subset of the existence region of the corresponding single-price- N - ∞ - p equilibrium.*

Proof. By theorem 2, the sufficient conditions for the existence of single-price- N - ∞ - p equilibrium are (32), (33), and (37), which are a subset of the equilibrium conditions for single-price- N - J - p proper equilibrium. Among this subset of conditions, (32) and (37) involve value function only in the support, therefore are common to both equilibria. Condition (33), $V(N+1) - c < V(N)$, differs in two equilibria only if $V(N+1)$ is computed differently. This can happen only if $K = N$ in the proper equilibrium, in which case $V(N+1)$ is computed through (28), denoted by $V^J(N+1)$, but it is computed through (27) in the N - ∞ - p equilibrium for $K = \infty$, denoted by $V^\infty(N+1)$. By (34), $K = N$ requires $u + V(N+1 - J) \geq V^J(N+1)$, which also implies $u + V(N+1 - J) \geq V^\infty(N+1)$. Under these conditions, it is easy to show that $V^J(N+1) \geq V^\infty(N+1)$. That is, (33) is a tighter constraint for the N - J - p proper equilibrium. If $K > N$, (33) is the same for both equilibria. Therefore, in general, if a single-price- N - J - p proper equilibrium exists, a single-price- N - ∞ - p equilibrium exists as well. Q.E.D.

Note that if a single-price equilibrium (with or without proper refinement) exists, generically, all conditions hold with strict inequality. Then for the same N , these conditions will hold for

prices in a neighborhood of p as well. That is, single-price equilibria with the same money-holding upper bound N , but different prices in the neighborhood of p , coexist. Thus, if one single-price equilibrium exists, a continuum of similar single-price equilibria with different prices exist in general.

4 Existence of Continuum of Single-Price Equilibria

We have discussed the set of sufficient conditions for the existence of a single-price equilibrium. In this section, I will explore some features of the single-price equilibrium with a special case of $N = 1$ and some numerical examples for higher N . These features include the existence of a continuum of single-price equilibria, the implication of endogenous upper bound on money holdings at equilibrium, and the coexistence of single-price equilibria with different money-holding distributions.

4.1 Single-Price-1- ∞ - p Equilibrium

The reason to study a single-price equilibrium with money-holding upper bound $N = 1$ is that it corresponds, in outcome, to the monetary equilibrium with exogenous unit-inventory constraint that has been widely studied. The choice of $J = \infty$, thus equilibrium without proper refinement, is made because this equilibrium has the largest existence region (by corollary 1) among all single-price- N - J - p equilibria. If this equilibrium does not exist, there is no other single-price equilibrium with price p and money-holding bounded at $1 \cdot p$.

Consider an environment defined by an arbitrary set of parameters M , ϕ , u and c . At a single-price-1- ∞ - p equilibrium, all trades occur at price p , and all traders hold either no money (in which case they can not buy) or p units of money (in which case they choose not to sell). The stationary money-holding distribution is simple:

$$h(1) = m = \frac{M}{p}, \quad \text{and} \quad h(0) = 1 - \frac{M}{p}. \quad (43)$$

The value function can be solved from (26) and (27).

Since at equilibrium, all agents hold at most p units of money, no agent with money holding in the support can afford to pay more than p units of money. Therefore, condition (37) which ensures that all agents pay no more than p is not needed. By theorem 2, then, conditions (32) and (33) fully characterize the set of single-price-1- ∞ - p equilibria. With the solution of the value function, conditions (32) ($V(1) - V(0) \geq c$) and (33) ($V(2) - V(1) < c$) are reduced to

$$\Psi_2 < \frac{M}{p} \leq \Psi_1 \quad (44)$$

where

$$\Psi_1 = 1 - \frac{\phi}{\left(\frac{u}{c} - 1\right)}, \quad \Psi_2 = \frac{1}{2\left(\frac{u}{c} - 1\right)} \left(2\left(\frac{u}{c} - 1\right) - \phi - \sqrt{\left(4\left(\frac{u}{c} - 1\right) + 1\right)\phi^2 + 4\phi\left(\frac{u}{c} - 1\right)} \right)$$

and $\Psi_1 > \Psi_2$ always holds. One can show that $\Psi_1 \leq 1$ and $\Psi_2 \leq 1$, but they may not always be positive. If $\phi < \frac{u}{c} - 1$, $\Psi_1 > 0$. Let $\Psi_0 \equiv \max\{\Psi_2, 0\}$. By theorem 2,

COROLLARY 2 *If $\phi < \frac{u}{c} - 1$, a single-price-1- ∞ - p equilibrium exists if and only if the price p is such that the real-money balance $\frac{M}{p} \in (\Psi_0, \Psi_1]$.*

Note that under the condition $\phi < \frac{u}{c} - 1$, the interval $(\Psi_0, \Psi_1]$ is not empty. Given that both the money-holding upper bound N and price p are equilibrium choice variables, corollary 2 states the existence condition such that one single-price equilibrium exists.

This particular equilibrium demonstrates several interesting points. One is that the money-holding upper bound $1 \cdot p$ (i.e., $N = 1$) is endogenous here. This upper bound results from the fact that it is not optimal to sell a unit of goods as one holds p units of money, since the value of an additional p units of money $V(2) - V(1)$ is lower than the cost of producing the good c . Had an exogenous upper bound on an agent's ability to hold money been imposed, namely, an agent *can* hold at most one unit of goods *or* p units of money, a monetary equilibrium exists if $\frac{M}{p} \leq \Psi_1$. Condition $\Psi_2 < \frac{M}{p}$ when $\Psi_2 > 0$, which corresponds to (33), is not needed since nobody is able to carry more money anyway. Therefore, as one eliminates the inventory holding constraint and endows each agent with the ability to hold any amount of money, the equilibrium region will shrink. If the proper equilibrium concept is adopted, the equilibrium existence region for $\frac{M}{p}$ is a subset of $(\Psi_0, \Psi_1]$ (or empty if the equilibrium does not exist). Thus, the conclusion that the equilibrium existence region might be too large with exogenous upper bound continues to hold.

Corollary 2 shows that for a set of parameters M , ϕ , u and c where the single-price-1- ∞ - p equilibrium exists, there is a continuum of such equilibria indexed by the real-money balance $\frac{M}{p}$, $\frac{M}{p} \in (\Psi_0, \Psi_1]$. This answers the question posted in the beginning of the paper. Namely, the existence of a continuum of single-price equilibria is robust to the introduction of costly production. But not all results in Green and Zhou (1995) are robust to such change. An important difference is in welfare implications of the model. For this equilibrium, the average welfare can be computed as

$$\text{Welfare} = h(0)V(0) + h(1)V(1) = \frac{1}{\phi} \frac{M}{p} \left(1 - \frac{M}{p}\right)(u - c). \quad (45)$$

Therefore, among the continuum of equilibria, the closer an equilibrium's real-money balance $\frac{M}{p}$ or the proportion of people holding money m to one half, the higher its average welfare. This is essentially the same result as many papers with indivisible goods and money and exogenous unit upper bound on inventory holdings have gotten (e.g., Kiyotaki and Wright (1991) and (1993)).

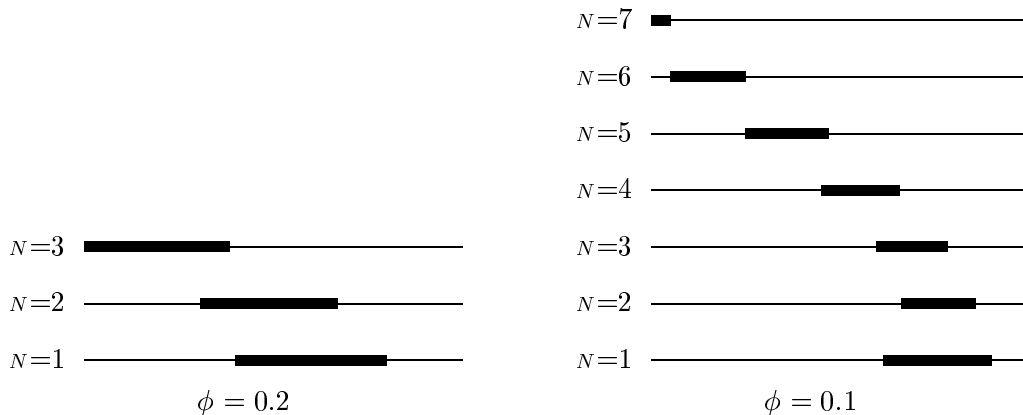
But it differs from the costless production case of Green and Zhou (1995), in which higher real-money balance gives higher welfare. Here the welfare loss from higher real-money balance stems from the slowing down of trade by the fact that agents with money holdings $1 \cdot p$ do not produce. The cost of production (obtaining money) induces agents to economize on the amount of money to hold, which simultaneously shrinks the productive population as well.

4.2 Coexistence of Single-Price- N -2- p Proper Equilibria

In this section, I will return to the proper equilibrium concept studied in section 3.4 and look at examples of single-price- N -2- p proper equilibria.

Similar results regarding the existence of a continuum of single-price equilibria hold here for $N \geq 2$. The explicit conditions such as those in corollary 2 on parameters of the model are more complicated for higher values of N , but the equilibrium is always well defined by theorem 3 where the conditions are implicit functions of the parameters of the model. Figure 1 shows the equilibrium existence region in terms of the proportion of people holding money m for some numerical examples. In all the examples, $u = 1$ and $c = 0.5$. The first panel represents single-price- N -2- p proper equilibrium with $N = 1, 2, 3$ for $\phi = 0.2$. The second panel shows the equilibrium with $N = 1, 2, 3, 4, 5, 6, 7$ for $\phi = 0.1$. Since the equilibrium conditions in theorem 3 are only sufficient, the equilibrium range may be larger than what is shown on the graph. Each thin line represents the value that m can take, namely the unit interval. The heavy bar on top of each thin line indicates the single-price equilibrium range of m for each case. For example, the bottom line of the first panel shows that the single-price-1-2- p proper equilibrium exists for $m \in (0.4, 0.8)$ (the heavy bar runs from 0.4 to 0.8). Since there is a one-to-one mapping between m and $\frac{M}{p}$ by lemma 1, the equilibrium existence region in m can be easily translated into that of $\frac{M}{p}$.

Figure 1. Equilibrium Existence Range for m
($u = 1, c = 0.5$)

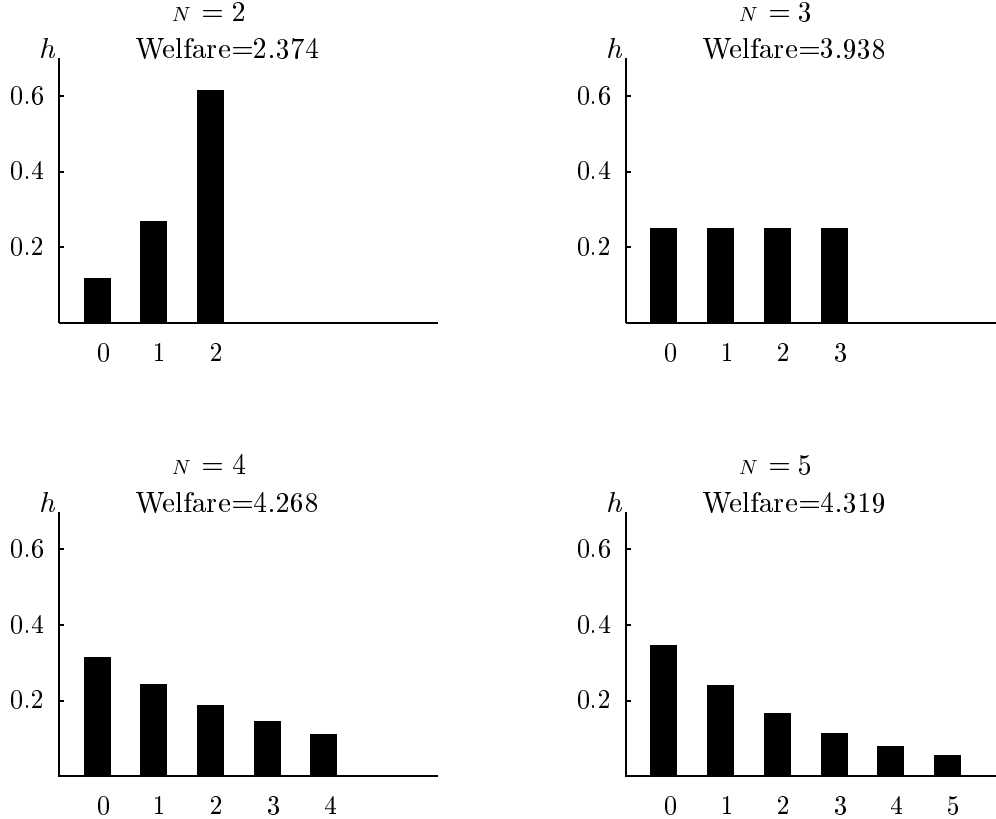


These examples confirm once again the previous result. In each case, the equilibrium existence region is an interval rather than a single point. That is, for the fixed environment represented by parameters M , ϕ , u and c , for a given N , there is a continuum of single-price- N - 2 - p proper equilibria with different prices p (uniquely determined by m).

Another observation from figure 1 is that equilibria with different money-holding upper bound N but the same m may coexist, although the price p and money-holding distribution are different. A more interesting comparison among coexisting equilibria with different money-holding upper bound N can be made for those with the same real-money balance $\frac{M}{p}$ (as well as other parameters u , c and ϕ). Figure 2 shows the distributions of money holdings h of four coexisting single-price equilibria for $N = 2, 3, 4, 5$ where $u = 1$, $c = 0.5$, $\phi = 1/14$ and $\frac{M}{p} = 1.5$. These examples confirm the result of lemma 1: the money-holding distribution is increasing in n for $N = 2$, uniform for $N = 3 = 2\frac{M}{p}$, and decreasing in n for $N = 4, 5$.

Figure 2. Distribution h of Coexisting Equilibria

($u = 1$, $c = 0.5$, $\phi = 1/14$, $\frac{M}{p} = 1.5$)



In these particular examples, the welfare is increasing in money-holding upper bound N , but this is not true in general. There are cases where welfare is higher with a lower value of N even though $\frac{M}{p}$ remains constant. By increasing the money-holding upper bound N , it has the welfare enhancing effect of reducing the size of un-productive population $h(N)$, simultaneously it has the welfare worsening effect of enlarging the group of people who do not have money $h(0)$. It is possible for either effect to dominate.

The coexistence of equilibrium allocations of consumption with different welfare levels in an environment with all identical exogenous variables, and also with identical real-money balances strongly suggests the non-neutrality of money. The channel for such non-neutrality is the distribution of money holdings. Different distributions generate different velocities of trading which dictate the speed of the production and consumption, therefore the level of welfare.

5 Concluding Remarks

I have studied a random matching model with divisible money and costly production. The model confirms the main findings of Green and Zhou (1995) regarding the existence of a continuum of single-price equilibria. I show that the phenomenon is robust to the introduction of costly production.

The mechanism through which the single-price equilibrium is reached in this costly production environment is somewhat different from that of the costless production model. In Green and Zhou (1995), agents prefer posting the prevailing price given the nondegenerate reservation-price distribution because a higher offer price reduces potential trading opportunities too much relative to the gain of getting more money. Here, the cost of production induces agents to ask for more when their money holdings reach a level at which the prevailing price is not enough to compensate for the cost of production. To maintain common price for all trades, the reservation price function has to be flat and equal to the posting price of agents with money holdings in the support. The nontrivial decision here for an agent is when to ask for more. Although in both environments, the self-fulfilling belief of the prevailing price plays the crucial role for the existence of single-price equilibria.

In addition to the continuum of single-price equilibria with different aggregate real-money balances, there exist multiple equilibria with identical aggregate real-money balances but different money-holding distributions. My conjecture for this multiplicity of equilibria is that it is a result of decentralized trade. Several distinct distributions can reinforce themselves through different trading patterns provided by the decentralized trading environment. The result shows that it is the distribution of money holdings, not the aggregate real-money balance, which provides the summary statistics of trading opportunities. The result suggests that the velocity of money is

endogenously determined rather than an exogenous feature of the economy. Furthermore, for a given aggregate real money balance, the equilibrium velocity may not be unique. It is through the velocity that an important aspect of the environment, the distribution of money, plays out its role on shaping economic activities.

The existence of a continuum of equilibria is a lot more difficult to explain. It might be a robust phenomenon, or it might be caused by the modelling assumptions, e.g., the indivisibility and the unit-inventory constraints of goods. An important point made by Wallace (1996) should be emphasized here. What I have shown here is the existence of a continuum of *stationary* equilibria, not equilibrium indeterminacy in the sense that all of these equilibria can be reached from the same initial distribution of money holdings. It is possible that the set of equilibrium paths from any given initial state is finite. Without the ability to characterize the full equilibrium path, moreover, without understanding the fundamental source of the problem, one cannot assess this class of models' applicability in substantial policy analysis.

Appendix

A. The Proof of Lemma 1

This lemma states the relationship among N , m and $\frac{M}{p}$, three variables determining the stationary distribution of money holdings h , that satisfies (12)–(14), $0 \leq m \leq 1$ and $0 \leq h(N) \leq 1$. First note that for any given N , $(m, h(N), \frac{M}{p}) = (0, 1, 0)$ and $(m, h(N), \frac{M}{p}) = (\frac{N}{N+1}, \frac{1}{N+1}, \frac{N}{2})$ are solutions to (12)–(14). While the former is a distribution for a non-monetary equilibrium, the latter corresponds to a uniform distribution of money holdings, i.e., $\forall n = 0, 1, \dots, N, h(n) = \frac{1}{N+1}$. We are looking for other nontrivial stationary distributions for potential monetary equilibrium. For a given N , I will show that m uniquely determines $\frac{M}{p}$ and vice versa.

Take a constant m as given. Rewrite $h(N)$ as $h(N; \frac{M}{p})$ to index the dependence of $h(N)$ on $\frac{M}{p}$,

$$h(N; \frac{M}{p}) = \frac{m - (1-m)\frac{M}{p}}{m(N+1) - \frac{M}{p}}, \quad 1 - h(N; \frac{M}{p}) = \frac{m(N - \frac{M}{p})}{m(N+1) - \frac{M}{p}}. \quad (\text{A.1})$$

Define

$$T(\frac{M}{p}) \equiv h(N; \frac{M}{p}) \left(1 - h(N; \frac{M}{p})\right)^N, \quad S(x) \equiv (1 - x)x^N. \quad (\text{A.2})$$

$S(m)$ is a positive constant given m . Instead of finding solution $\frac{M}{p}$ to (12), we will look for solutions for $T(\frac{M}{p}) = S(m)$. This can be done in the following two cases, depending on the values of N and m .

Case 1. $m(N+1) < N$.

Divide \mathbb{R}_+ into three segments: $I_1 = (0, m(N+1)]$, $I_2 = (m(N+1), N)$, $I_3 = [N, \infty)$.

If $\frac{M}{p} \in I_3$, it is easy to show that $T(\frac{M}{p})$ is increasing on I_3 , $T(N) = 0$, and $T(\frac{M}{p}) \rightarrow S(m)$ as $\frac{M}{p} \rightarrow \infty$. That is, $\forall \frac{M}{p} \in I_3$ $T(\frac{M}{p}) < S(m)$, therefore there is no solution on I_3 .

If $\frac{M}{p} \in I_2$, by (A.1), $1 - h(N) < 0$. Thus, there is no solution on I_2 .

If $\frac{M}{p} \in I_1$, we can show that $T(\frac{M}{p})$ is decreasing on I_1 , $T(0) = \frac{N^N}{(N+1)^{N+1}} > 0$, and $T(\frac{M}{p}) \rightarrow S(m)$ as $\frac{M}{p} \rightarrow m(N+1)$. Since $S(x)$ is an increasing function when $x < \frac{N}{N+1}$ and by assumption $m < \frac{N}{N+1}$, $S(m) < S(\frac{N}{N+1}) = T(0)$. Thus, there exists a $\frac{M}{p}^* \in (0, m(N+1))$ such that $T(\frac{M}{p}^*) = S(m)$.

Case 2. $m(N+1) > N$.

Similarly, divide \mathbb{R}_+ into three intervals: $I_1 = (0, N]$, $I_2 = (N, m(N+1))$, $I_3 = [m(N+1), \infty)$.

If $\frac{M}{p} \in I_3$, again we can show that $T(\frac{M}{p})$ is increasing on I_3 , $T(\frac{M}{p}) \rightarrow -\infty$ as $\frac{M}{p} \rightarrow m(N+1)$, and $T(\frac{M}{p}) \rightarrow S(m)$ as $\frac{M}{p} \rightarrow \infty$. That is, $\forall \frac{M}{p} \in I_3$ $T(\frac{M}{p}) < S(m)$, therefore there is no solution on I_3 .

If $\frac{M}{p} \in I_2$, by (A.1), $1 - h(N) < 0$. Thus, there is no solution on I_2 .

If $\frac{M}{p} \in I_1$, we can show that $T(\frac{M}{p})$ is decreasing on I_1 , $T(0) = \frac{N^N}{(N+1)^{N+1}} > 0$, and $T(N) = 0 < S(m)$. Since $S(x)$ is a decreasing function when $x > \frac{N}{N+1}$ and by assumption $m \geq \frac{N}{N+1}$, $S(m) < S(\frac{N}{N+1}) = T(0)$. Thus, there exists a $\frac{M^*}{p} \in (0, N)$ such that $T(\frac{M^*}{p}) = S(m)$.

The case of $m(N+1) = N$ corresponds to the special uniform distribution above. To summarize, for a given N and m , there exists a unique $\frac{M^*}{p} < N$, such that $T(\frac{M^*}{p}) = S(m)$.

Now let's consider the reverse case of finding m for a given $\frac{M}{p}$. Substituting $h(N)$ of (14) into (12), we have

$$\frac{\left(m(\frac{M}{p}+1) - \frac{M}{p}\right) m^N (N - \frac{M}{p})^N}{\left(m(N+1) - \frac{M}{p}\right)^{N+1}} = (1-m)m^N. \quad (\text{A.3})$$

Define

$$O(m) \equiv \frac{(1-m) \left(m(N+1) - \frac{M}{p}\right)^{N+1}}{m \left(\frac{M}{p}+1\right) - \frac{M}{p}}, \quad Q \equiv \left(N - \frac{M}{p}\right)^N. \quad (\text{A.4})$$

Then (A.3) can be written as

$$O(m) = Q. \quad (\text{A.5})$$

So instead of looking for solutions for (12), we will solve for m for (A.5). Given the above result ($(\frac{M}{p})^* < N$ for a given m), we only need to study the case of $\frac{M}{p} < N$.

Given $\frac{M}{p} < N$, we have $0 < \frac{M}{p} \frac{1}{N+1} < \frac{M}{p} \frac{1}{\frac{M}{p}+1} < \frac{N}{N+1}$. If $m \in (0, \frac{M}{p} \frac{1}{N+1})$, $1 - h(N) < 0$. If $m \in [\frac{M}{p} \frac{1}{N+1}, \frac{M}{p} \frac{1}{\frac{M}{p}+1})$, $h(N) < 0$. So the only possible solution of m is in $[\frac{M}{p} \frac{1}{\frac{M}{p}+1}, 1]$. On this interval, $O(m) \rightarrow +\infty$ as $m \rightarrow \frac{M}{p} \frac{1}{\frac{M}{p}+1}$, and $O(1) = 0 < Q$. Also

$$\frac{\partial O}{\partial m} = \frac{\left(m(N+1) - \frac{M}{p}\right)^N}{\left(m(\frac{M}{p}+1) - \frac{M}{p}\right)^2} (m(N+1) - N) \left(\frac{M}{p}(N+2) - m(N+1)\left(\frac{M}{p}+1\right)\right). \quad (\text{A.6})$$

The solutions for $\frac{\partial O}{\partial m} = 0$ are

$$m_1 = \frac{N}{N+1}, \quad m_2 = \frac{\frac{M}{p}(N+2)}{\left(\frac{M}{p}+1\right)(N+1)}, \quad (\text{A.7})$$

and $O(m_1) = \left(N - \frac{M}{p}\right)^N = Q$. Note that m_1 gives the special uniform distribution discussed above, which requires $\frac{M}{p} = \frac{N}{2}$. In this case $m_1 = m_2$ is saddle point of O which is the only solution of (A.5). Let's consider the cases where $\frac{M}{p} \neq \frac{N}{2}$.

If $\frac{M}{p} < \frac{N}{2}$, $m_1 > m_2$. Since O is a continuous function of m , there exists a $m^* \in (\frac{M}{p} \frac{1}{\frac{M}{p}+1}, m_2)$ such that $O(m^*) = \left(N - \frac{M}{p}\right)^N = Q$. That is, $m^* < m_2 < m_1$, and by (14), $1 - h(N) > m$. Then by (11), $\{h(n)\}_{n=0}^N$ is a decreasing sequence of n .

If $\frac{M}{p} > \frac{N}{2}$, $m_1 < m_2$. Again since O is a continuous function of m , there exists a $m^* \in (m_2, 1)$, such that $O(m^*) = \left(N - \frac{M}{p}\right)^N = Q$. That is, $m^* > m_2 > m_1$, and by (14), $1 - h(N) < m$. Then

by (11), $\{h(n)\}_{n=0}^N$ is an increasing sequence of n .

This concludes the second part of the proof: for a given $\frac{M}{p}$, there is a unique m^* that satisfies (A.5). Q.E.D.

B. The Derivation of the Value Function

The solution to the linear difference equation (24) can be represented as a linear combination of the eigenvectors of matrix \mathcal{D} . The matrix \mathcal{D} has three distinct eigenvectors, all of which have real eigenvalues. The eigenvalues λ_0 , λ_1 and λ_2 are

$$\begin{aligned} \lambda_0 &= 1 \\ \lambda_1 &= \frac{1}{2} \left(\frac{\phi}{m} + \frac{1-h(N)}{m} + 1 - \sqrt{\left(\frac{\phi}{m} + \frac{1-h(N)}{m} + 1 \right)^2 - \frac{4(1-h(N))}{m}} \right) \in (0, 1) \\ \lambda_2 &= \frac{1}{2} \left(\frac{\phi}{m} + \frac{1-h(N)}{m} + 1 + \sqrt{\left(\frac{\phi}{m} + \frac{1-h(N)}{m} + 1 \right)^2 - \frac{4(1-h(N))}{m}} \right) > 1. \end{aligned} \quad (B.1)$$

The corresponding eigenvectors ξ_0 , ξ_1 and ξ_2 can be expressed as

$$\xi_0 = \begin{pmatrix} 1/\phi \\ 1/\phi \\ 1 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} \lambda_1 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \lambda_2 \\ 1 \\ 0 \end{pmatrix} \quad (B.2)$$

that satisfy the following relations,

$$\mathcal{D}\xi_0 = \xi_0, \quad \mathcal{D}\xi_1 = \lambda_1\xi_1, \quad \mathcal{D}\xi_2 = \lambda_2\xi_2. \quad (B.3)$$

Define

$$w^{n+1} \equiv \begin{pmatrix} V(n+1) \\ V(n) \\ B \end{pmatrix}. \quad (B.4)$$

Let θ_0 , θ_1 and θ_2 be the parameters satisfies

$$w^1 = \theta_0\xi_0 + \theta_1\xi_1 + \theta_2\xi_2 \quad (B.5)$$

or

$$\left(\frac{\phi}{m} + 1\right)V(0) + c = \theta_0/\phi + \theta_1\lambda_1 + \theta_2\lambda_2 \quad (B.6)$$

$$V(0) = \theta_0/\phi + \theta_1 + \theta_2 \quad (B.7)$$

$$B = \theta_0 \quad (B.8)$$

where (B.6) is derived by substituting $V(1)$ solved from (18). By (B.3) and (B.5), induction on w^{n+1} for $n = 0, 1, \dots, N-1$ implies,

$$w^{n+1} = \theta_0\xi_0 + \theta_1\lambda_1^n\xi_1 + \theta_2\lambda_2^n\xi_2 = \theta_0 \begin{pmatrix} 1/\phi \\ 1/\phi \\ 1 \end{pmatrix} + \theta_1\lambda_1^n \begin{pmatrix} \lambda_1 \\ 1 \\ 0 \end{pmatrix} + \theta_2\lambda_2^n \begin{pmatrix} \lambda_2 \\ 1 \\ 0 \end{pmatrix}. \quad (B.9)$$

Take the second row of (B.9) we have for $n = 1, 2, \dots, N$,

$$V(n) = \theta_0/\phi + \theta_1\lambda_1^n + \theta_2\lambda_2^n \quad (B.10)$$

which is the general representation of value function (26). Rewrite equation (20),

$$(\phi + 1 - h(N))V(N) = (1 - h(N))(u + V(N - 1)) \quad (B.11)$$

and substitute (B.10) for $n = N - 1$ and $n = N$ into (B.11), we have

$$(\phi + 1 - h(N))\left(\theta_0/\phi + \theta_1\lambda_1^N + \theta_2\lambda_2^N\right) = (1 - h(N))\left(u + \theta_0/\phi + \theta_1\lambda_1^{N-1} + \theta_2\lambda_2^{N-1}\right). \quad (B.12)$$

Equations (B.6), (B.7), (B.8) and (B.12) are a four-equation linear system with four unknowns $\theta_0, \theta_1, \theta_2$ and $V(0)$. Solving this linear system, we have

$$\begin{aligned} \theta_0 &= (1 - h(N))u - mc \\ \theta_1 &= \frac{1}{G} \left((\lambda_2 - 1 - \frac{\phi}{m})mc - (\frac{1}{m}\theta_0 + c)F \right) \\ \theta_2 &= \frac{1}{G} \left((1 + \frac{\phi}{m} - \lambda_1)mc + (\frac{1}{m}\theta_0 + c)E \right) \\ V(0) &= \frac{1}{G} \left((\lambda_2 - \lambda_1)(\theta_0 F/\phi + mc) - (c + (\lambda_2 - 1)\theta_0/\phi)(F - E) \right) \end{aligned} \quad (B.13)$$

where

$$\begin{aligned} E &= \phi\lambda_1^N - \lambda_1^{N-1}(1 - h(N))(1 - \lambda_1) \\ F &= \phi\lambda_2^N + \lambda_2^{N-1}(1 - h(N))(\lambda_2 - 1) \\ G &= (1 - \lambda_1 + \frac{\phi}{m})F + (\lambda_2 - 1 - \frac{\phi}{m})E. \end{aligned}$$

Equation (B.10) and (B.13) fully characterize the value function in terms of the parameters of the model u, c and ϕ , the upper bound on money holdings N , and the distribution parameter m and $h(N)$.

C. The Proof of Lemma 4

The value function is concave in n for $N \leq n \leq K$ can be directly shown by equation (20) and (21), that is, for $k = 0, 1, 2, \dots, K$

$$V(N + k) - V(N + k - 1) > V(N + k + 1) - V(N + k). \quad (C.1)$$

I will focus on the case $n < N$. From (19) and (20),

$$(\phi + m + 1 - h(N))V(N - 1) = (1 - h(N))(u + V(N - 2)) + m(V(N) - c), \quad (C.2)$$

$$(\phi + 1 - h(N))V(N) = (1 - h(N))(u + V(N - 1)). \quad (C.3)$$

Subtracting (C.3) from (C.2) and rearranging the equation, we have

$$\begin{aligned} (1 - h(N))\left((V(N - 1) - V(N - 2)) - (V(N) - V(N - 1))\right) = \\ \phi(V(N) - V(N - 1)) + m(V(N) - c - V(N - 1)). \end{aligned} \quad (C.4)$$

The right hand side of (C.4) is positive since $V(N) > V(N-1)$ by lemma 2, and $V(N)-c \geq V(N-1)$ by (33). Therefore,

$$V(N-1) - V(N-2) > V(N) - V(N-1). \quad (C.5)$$

Similarly, subtracting equation (19) for n from that of $n-1$ and rearranging the difference, we have

$$(1-h(N))\left((V(n-1) - V(n-2) - (V(n) - V(n-1)))\right) = \phi(V(n) - V(n-1)) \\ + m\left((V(n) - V(n-1)) - (V(n+1) - v(n))\right). \quad (C.6)$$

Again the right hand side of (C.6) is positive for $n = N-1$ by lemma 2 and by inequality (C.5). So $V(N-2) - V(N-3) > V(N-1) - V(N-2)$. Induction on n backward using (C.6), we have for $n = 1, 2, \dots, N-1$,

$$V(n) - V(n-1) > V(n+1) - V(n). \quad (C.7)$$

Inequalities (C.1) and (C.7) give the statement of lemma 4. Q.E.D.

D. The Proof of Lemma 8

The proof of this lemma involves three cases depending on the value of κ . Compute the value of $V(N+1)$ by (27). If $V(N+1) - V(N-1) \leq u$, by (30), agents in state $N+1$ are willing to accept an offer of $2p$. This is the first case, $\kappa = N$. If $V(N+1) - V(N-1) > u$, compute next the value $V(N+2)$ by (27). If $V(N+2) - V(N) \leq u$, then agents in state $N+2$ are willing to accept an offer of $2p$. Then, $\kappa = N+1$. Otherwise, $\kappa \geq N+2$. I will consider each case separately.

Case 1. $\kappa = N$, $V(N+1) - V(N-1) \leq u$.

From (20) and (22), we have

$$(\phi+1)\left(V(N+2) - V(N)\right) = (1-h(N))\left(V(N+1) - V(N-1)\right) + h(N)u \leq u \quad (D.1)$$

because $V(N+1) - V(N-1) \leq u$. Therefore $V(N+2) - V(N) < u$. That is, agents in state $N+2$ are willing to accept an offer of $2p$. Similarly, for $n > N+2$, from (22) we can derive the following,

$$(\phi+1)\left(V(n) - V(n-2)\right) = (1-h(N))\left(V(n-1) - V(n-3)\right) + \\ h(N)\left(V(n-2) - V(n-4)\right). \quad (D.2)$$

Since $V(N+1) - V(N-1) \leq u$ and $V(N+2) - V(N) < u$, by (D.2), $V(N+3) - V(N+1) < u$. Applying (D.2) recursively, we have $V(n) - V(n-2) < u$ for any $n > N+2$. Combining all cases,

we find that it is optimal for agents with money holding more than $\kappa p (= np)$ accept an offer of $2p$.

In regard to sellers, by (33), $V(N+1) - c < V(N)$, and by (42), $V(N+2) - c < V(N+1)$. For $n > N+2$, from (22)

$$(\phi + 1)(V(n) - V(n-1)) = (1 - h(N))(V(n-1) - V(n-2)) + h(N)(V(n-2) - V(n-3)). \quad (D.3)$$

Again by applying (D.3) recursively, we have $V(n) - c < V(n-1)$ for all $n > N+1$. Then given (37), the expected value of making an offer, $W(np, \cdot)$ for all $n > \kappa$, satisfies $W(np, 0) < W(np, p) = V(n) + m(V(n+1) - c - V(n)) < W(np, lp) = V(n)$ for any $l > 1$. Therefore, it is optimal for agents with money holding at least κp offer higher than p .

Case 2. $\kappa = N+1$, $V(N+2) - V(N) \leq u$.

In this case, given $V(N+2) - V(N) \leq u$, we can show that $V(N+3) - V(N+1) < u$ by way of showing $V(N+2) - V(N) < u$ as in case 1. Then applying (D.1) recursively for $n = N+4, N+5, \dots$, we have $V(n) - V(n-2) < u$. That is, it is optimal for agents with money holding more than κp accept an offer of $2p$.

For offer strategy of the sellers with money holding at least κp , the same argument as in case 1 applies here.

Case 3. $\kappa \geq N+2$, $V(N+2) - V(N) > u$.

The proof for the optimality of buyers with money holding more than κp accepting an offer of $2p$ is similar as in the previous two cases.

For the sellers, by the definition of κ in (34), $V(\kappa+1) - V(\kappa-1) \leq u < V(\kappa) - V(\kappa-2)$. Rewrite this inequality,

$$V(\kappa+1) - V(\kappa) < V(\kappa-1) - V(\kappa-2). \quad (D.4)$$

By (33), $V(N+1) - V(N) < c$. Given $\kappa \geq N+2$, and the value function is concave up to κ , $V(\kappa-1) - V(\kappa-2) < c$. Then by (D.4), $V(\kappa+1) - V(\kappa) < c$. We also know $V(\kappa) - V(\kappa-1) < c$ from the concavity of V . Starting with the last two inequalities, applying (D.3) recursively, we can show $V(n) - c < V(n-1)$ for all $n \geq \kappa+1$. Following the rest of the proof as in case 1, it is optimal for agents with money holding at least κp offer higher than p .

Note that condition (42) is not needed in the last case. The concavity of the value function below κ is sufficient to show that no agent is willing to offer p . But (42) is needed for the other two cases. Q.E.D.

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