A Rudimentary Random-Matching Model with Divisible Money and Prices *

Edward J. Green
Research Department
Federal Reserve Bank of Minneapolis
250 Marquette Avenue
Minneapolis, MN 55401-2171

Ruilin Zhou
Department of Economics
University of Pennsylvania and
Federal Reserve Bank of Minneapolis
Philadelphia, PA 19104-6297

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Abstract

We consider a version of Kiyotaki and Wright's monetary search model in which agents can hold arbitrary amounts of divisible money. A continuum of stationary equilibria, indexed by the aggregate real-money stock, exist with all trading occurring at a single price. There is always a maximum level of the real money stock consistent with existence of such an equilibrium. In the limit as trading becomes faster relative to discounting, any real money stock becomes feasible in such an equilibrium. In contrast to the original Kiyotaki-Wright model, higher equilibrium real money stocks unambiguously correspond to higher welfare in this costless-production environment.

J.E.L Classification: D51, E40

I Introduction

Monetary theory deals with a feature of the economy from which Walrasian general equilibrium theory deliberately abstracts: the inability of all traders to meet concurrently to transact mutually beneficial trades. Models of economies with random pairwise meetings, but without double coincidence of wants, are suitable to study this problem. Kiyotaki and Wright (1989) provide a prototypical formal model of this sort.

The advantage of taking this approach is the ability to proceed with tractable, fully consistent models that deepen our understanding of fundamental issues. At present, though, these highly schematic models are still long way from being straightforwardly applicable to many policy questions. In this paper, we formulate and analyze a random-matching model that is less schematic in one important respect than its predecessors. Specifically, fiat money is modeled as being perfectly divisible and not subject to inventory constraints.[1]

Our model is closely similar to the Kiyotaki-Wright (1989) model in other respects, and it also broadly resembles more recent models by Shi (1995) and Trejos and Wright (1995). In all of those models, money is assumed to be indivisible and a trader is assumed not to be able to hold more than a single unit of it. This pair of assumptions has two restrictive consequences. It trivializes the “law of one price,” and it leads to difficulty in welfare analysis.

One of the goals of studying models of decentralized trade is to understand how approximately uniform terms of trade become established throughout an entire economy where agents do not deal with one another directly. If there is no auctioneer with whom all of the agents in the economy can and must deal simultaneously, and if moreover the decentralization of exchange amplifies the heterogeneity among agents by generating dispersion in money holdings, then is it consistent to assume that all transactions occur at identical terms? If so, does the process of exchange move agents systematically toward such uniformity? These are questions that can be addressed using a search model with divisible money, but that cannot even be posed in the context of models that incorporate a one-unit inventory constraint on the holding of an indivisible money object.[2] The only trading pairs

[1]Diamond and Yellin (1990) study a search model in which both money and goods are divisible, but they assume an exogenous cash-in-advance constraint for one class of agents rather than deriving the monetary nature of trade as an equilibrium phenomenon. Shi (*Econometrica*, forthcoming) studies a search model in which money is assumed to be divisible from the viewpoint of the household (which consists of infinitely many individual traders) but in which each transaction involves an indivisible unit of money. Molico (1996) computationally studies a model in which both goods and money are divisible.

[2]Camera and Corbae (1996), Hendry (1993) and Wallace (1996) relax the constraint, specifying an arbitrary finite upper bound, which is sufficient to produce heterogeneity among trading pairs. Such heterogeneity would also be consistent with the inventory constraint if, for example, utility or production opportunities were not time separable. However, time separability is another assumption that Kiyotaki

that can make mutually beneficial trades in those models (even if fiat money is valued) are those in which the buyer has exactly one unit of money, and this assumption plus the assumed symmetry of traders' preferences and technologies imply uniformity of the terms of trade. That is, the "law of one price" holds trivially. In contrast, because agents with different money holdings have different willingness to pay for consumption in the equilibrium of our model, the existence of a single-price equilibrium is nontrivial here.

The imposition of a one-unit inventory constraint limits the usefulness of currently available models for policy analysis. The most obvious instance is that an increase in the nominal money stock can be Pareto worsening in these models because the agents who hold the newly-minted money would be unable to gain from engaging in production. The possibility that an increase in nominal money can decrease welfare deserves serious consideration, but the crude mechanism by which occurs is incredible. Aiyagari, Wallace and Wright (1995) discuss further difficulties in this same vein, in the context of a model that builds on Trejos and Wright's (1995) model. Our present model avoids the problematic assumptions.

The spirit of our present investigation is to relax a particularly stringent assumption of the Kiyotaki-Wright (1989) model, and to examine the consequences. Three consequences are particularly striking. First, there can be equilibria in which the "law of one price" holds exactly, rather than only approximately as would be anticipated. Second, there exists a continuum of such steady-state equilibria that correspond to distinct real allocations. Third, although money is neutral (in the sense that the equilibrium conditions can be defined with respect to real money balances alone) and the underlying technology of pairwise meetings is exogenous, the equilibrium price level and velocity of money are indeterminate. The last two results, especially, pose challenges for the applicability of random-matching models to policy analysis. Deeper knowledge about the source of these results, and about whether or not they are robust, will clarify how serious these challenges are.

II The Environment

The set of agents is a nonatomic mass of measure 1. There are $k \geq 3$ types of agent. Each type $i \in \{1, \ldots k\}$ has mass $1/k$ in the population. Time is continuous, and agents are infinite lived. There are $k+1$ goods. Of these goods, k (which we index by 1 through k) are indivisible, immediately perishable goods that are produced by the agents. The remaining good is a divisible, perfectly durable, fiat-money object. The total nominal stock of this fiat money is a constant M.

and Wright and subsequent researchers have adopted for analytical tractability.

An agent of type i can produce one unit of good $i + 1 \pmod{k}$ instantaneously and costlessly at any time. He consumes only good i, from which he derives instantaneous utility $u > 0$. Each agent maximizes the discounted expected utility of his consumption stream, with discount rate γ.

Meetings between agents are pairwise. Each agent meets other agents randomly according to a Poisson process with parameter μ. The distribution of partners' characteristics from which an agent' meetings are drawn matches the demographic distribution of characteristics in the entire population of the economy. A meeting partner has two characteristics, his type and the amount of money that he holds. An agent's type is observable, but not his money holding.

In this economy there is no double coincidence of wants (in the sense of trades that give strictly positive utility to both traders) between any pair of agents. Consumption goods cannot be used as commodity money because they are perishable. Thus trade must involve using fiat money as a medium of exchange. We assume that transactions occur according to a seller-posting-price protocol. When a type-i agent who possesses fiat money meets a type-$(i-1)$ trader who can produce his desired good, the seller (the type-$(i-1)$ agent) posts an offer that the buyer (the type-i agent) must either accept or reject. Trade occurs if and only if the offer is accepted, and in that case the buyer pays exactly the seller's offer price. This specific assumption about the trading protocol is crucial to the results which follow.

III Definition of Stationary Equilibrium

We are going to consider stationary equilibrium in the trading environment just described. Moreover we restrict attention to equilibria in which all agents with identical characteristics act alike, and in which all of the k types are symmetric. (Hereafter, all of our discussion will be in terms of a generic type i.) A stationary equilibrium can be characterized in terms of six theoretical constructs: agents' offer strategy and reservation-price strategy, the stationary measure on traders' money holdings, the stationary distributions of offers and reservation prices, and the value function for money holdings.

The domain of possible money holdings is $\mathbb{R}_+$, the set of nonnegative real numbers. A type-i agent's trading strategy is a pair of real-valued functions on $\mathbb{R}_+$, $\omega(\eta)$ that specifies the offer that he will make as a seller when his current money holding is η and he meets a type-$(i+1)$ agent, and $\rho(\eta)$ that specifies the maximum willingness to pay as a buyer when his current money holding is η and he meets a type-$(i-1)$ agent. Note that ρ is a reduced-form description of the actual decisions made by a buyer. The buyer with a particular money holding accepts certain offers and rejects others. It is obvious that *optimal* decisions will involve a threshold offer level, below which offers are accepted and above which they are rejected. $\rho(\eta)$ specifies this threshold for a buyer with money holding η.

A buyer must always be able to pay his reservation price, so we impose the feasibility constraint that

$$\rho(\eta) \leq \eta. \tag{1}$$

The stationary distribution of money holdings is given by a measure H defined on $\mathbb{R}_+$.

A strategy (or, more precisely, a symmetric strategy profile) and a stationary distribution of money holdings imply stationary distributions of offers and reservation prices. Note that a buyer's willingness to pay depends on his current money holding, so a trader's reservation price as a function of his money holding is the solution of an optimization problem. Thus we will often refer to a trader's optimal reservation price.

Define the stationary distribution of offers by

$$\Omega(x) = H\{\eta \,|\, \omega(\eta) \leq x\} \tag{2}$$

and the stationary distribution of reservation prices by

$$\mathrm{R}(x) = H\{\eta \,|\, \rho(\eta) < x\}. \tag{3}$$

(Note that, for convenience, we are defining R to be continuous from the left, rather than from the right as would be conventional.)

The value function $\mathcal{V}\colon \mathbb{R}_+ \to \mathbb{R}_+$ of money holdings specifies the expected discounted utility that an agent will receive, given his current money holding, if he adopts an optimal trading strategy.

The value function is studied in terms of its Bellman equation. Intuitively the Bellman equation states that $\mathcal{V}(\eta)$ is the discounted expected value of $W + \mathcal{V}(\eta')$, where η' is the agent's money holding immediately after his next meeting with a potential trading partner, and $W = 0$ if that transaction will not result in a purchase but $W = u$ if it will result in a purchase (and hence will be accompanied by consumption). By a potential trading partner of an agent of type i, we mean either an agent of type $i-1$ from whom the agent might make a purchase, or else an agent of type $i+1$ to whom he might make a sale. Since the mass of those two types together in the population is $2/k$, the Poisson parameter for the frequency of meetings with them is $2\mu/k$. Therefore the value function with appropriate discount rate is given by

$$\mathcal{V}(\eta) = \int_0^\infty e^{-\gamma t}\mathsf{E}[W + \mathcal{V}(\eta')|\eta]\frac{2\mu}{k}e^{-(2\mu/k)t}\, dt \tag{4}$$

where the first exponential expression inside the integrand is the discount rate, and the second is the exponential waiting time implied by the Poisson process. Note that the expectation does not have to be conditioned on t because, since we are assuming stationarity, the expectation does not depend on the time at which it is taken. Evaluation of the integral yields

$$\mathcal{V}(\eta) = \frac{2\mu}{k\gamma + 2\mu}\mathsf{E}[W + \mathcal{V}(\eta')|\eta]. \tag{5}$$

Since agents of types $i-1$ and $i+1$ are equally numerous in the population, the probability that the first potential trading partner will be a seller is $1/2$. If the type-i trader's reservation price is r, then the conditional expectation of $W + \mathcal{V}(\eta')$ in that event is

$$\int_0^r [u + \mathcal{V}(\eta - x)]\, d\Omega(x) + (1 - \Omega(r))\mathcal{V}(\eta). \tag{6}$$

In the complementary event that the first potential trading partner will be a buyer, if the type i trader makes offer o, then the conditional expectation of $W + \mathcal{V}(\eta')$ is

$$\mathrm{R}(o)\mathcal{V}(\eta) + (1 - \mathrm{R}(o))\mathcal{V}(\eta + o). \tag{7}$$

Substitution of the equally weighted average of the optimized values of (6) and (7) for the expectation in (5) yields

$$\begin{aligned}\mathcal{V}(\eta) \quad = \quad & \frac{\mu}{k\gamma + 2\mu}\Bigg[\max_{r\in[0,\eta]}\Big[\int_0^r [u + \mathcal{V}(\eta - x)]\, d\Omega(x) + (1 - \Omega(r))\mathcal{V}(\eta)\Big] \\ & + \max_{o\in\mathbb{R}_+}\Big[\mathrm{R}(o)\mathcal{V}(\eta) + (1 - \mathrm{R}(o))\mathcal{V}(\eta + o)\Big]\Bigg].\end{aligned} \tag{8}$$

Equation (8) is the Bellman equation for $\mathcal{V}$. Standard arguments establish that it has a unique solution in the space of bounded measurable functions, and that this solution does indeed specify the optimal expected discounted value of each possible level of money holding.

At this point we need to consider an implication of the way in which we have collapsed buyers' decisions into the reduced-form representation ρ. Literally, in the seller-posting-price protocol, for any possible offer o, the buyer must issue either an acceptance or a rejection. The criterion for making this decision optimally is to accept o if and only if the offer is below the buyer's full valuation of the good, that is,

$$\forall o \quad [\rho(\eta) \geq o \iff u + V(\eta - o) \geq V(\eta)]. \tag{9}$$

Perfectness of equilibrium requires that this condition be satisfied even for offers that are never made in equilibrium.

The state of the environment is summarized by the distribution of money holdings. Implicitly, the money holding of each agent is a continuous-time, pure-jump Markov process on the state space $\mathbb{R}_+$. The transition probabilities are the probabilities of transactions occurring, induced by the optimal strategies (ω, ρ). The environment is stationary if the measure H is a stationary initial distribution of this process.

The equilibrium concept that we will adopt is stationary perfect Bayesian Nash equilibrium. We will refer to this simply as stationary equilibrium.

DEFINITION. A *stationary equilibrium* consists of a time-invariant profile $\langle H, R, \Omega, \omega, \rho, \mathcal{V}\rangle$ that satisfies

(I) H is stationary under trading strategies ω and ρ, and the reservation-price distribution R and the offer distribution Ω are derived from H, ω and ρ according to (2) and (3).

(II) Given the distributions for money-holdings H, reservation-price R and offer Ω, the reservation-price strategy ρ satisfies feasibility condition (1) and perfectness condition (9), and trading strategies (ω, ρ) and value function $\mathcal{V}$ solve Bellman equation (8). That is,

$$\begin{aligned}\mathcal{V}(\eta) \quad = \quad & \frac{\mu}{k\gamma + 2\mu}\Bigg[\Bigg[\int_0^{\rho(\eta)}(u + \mathcal{V}(\eta - x))\; d\Omega(x) + (1 - \Omega(\rho(\eta)))\mathcal{V}(\eta)\Bigg] \\ & + \Big[\mathrm{R}(\omega(\eta))\mathcal{V}(\eta) + (1 - \mathrm{R}(\omega(\eta)))(\mathcal{V}(\eta + \omega(\eta)))\Big]\Bigg]. \qquad (10)\end{aligned}$$

In the equilibria that we are going to study in this paper, the support of H will be the discrete set $\{0, p, 2p, 3p, \ldots\}$. Giving an exact statement of the stationarity condition is much easier in this special case than in general, so we will state the formal condition of stationarity in the context of this class of equilibria.

IV Single-Price Equilibrium

In this section we are going to characterize a sufficient condition for a single-price equilibrium to exist. We are going to begin by supposing that all trades occur at a single price p, and that all agents' money holdings are integer multiples of p. Also we assume that agents always offer to sell at p, and that every agent who holds money is willing to purchase at p. We characterize the stationary measure on traders' money holdings under these assumptions. Then we find the corresponding solution for the value function and use it to calculate the optimal reservation-price function. Finally we find a sufficient condition such that the optimal offer function is constant at price p. Thus, under this condition, the offer function, reservation-price function, and stationary measure that we have found are an equilibrium.

A *Stationary Measure on Traders' Money Holdings*

Consider the formulation of equilibrium just given, in the special case that all trades occur at a single price p, and that the support of the population measure H of money holdings is on the discrete set of points $p^{\mathbb{N}} = \{0, p, 2p, 3p, \ldots\}$.[3] In this case, define

$$h(n) = H(\{np\}). \tag{11}$$

That is, $h(n)$ is the measure of the set of agents who hold precisely np units of fiat money. Now, instead of working with the measure H on $\mathbb{R}_+$, we can work with the equivalent measure h on $\mathbb{N}$. We will say that a trader is in state n when his money holding is np. The proportion of agents who hold positive money holdings is defined to be

$$m \equiv \sum_{n=1}^{\infty} h(n). \tag{12}$$

Note that

$$h(0) = 1 - m. \tag{13}$$

In a single-price equilibrium, an agent only moves into state n (the state of having money holding np) by either making a sale from state $n-1$ or making a purchase from state $n+1$. He moves out of state n by either making a purchase or a sale. Clearly an agent of type i will make a sale whenever he meets an agent of type $i+1$ whose money holding is positive, and he will make a purchase whenever he meets an agent of type $i-1$ if his own money holding is positive. Stationarity requires that the sum of time rates of inflow to state n from all other states must equal the time rate of outflow from state n. The time rate of a population flow is the instantaneous transition probability for an individual multiplied by the population of the state from which the transition occurs. The time derivative of $h(n)$, for all $n > 0$ is thus

$$\dot{h}(n) = \mu h(n+1) + \mu m h(n-1) - \mu(m+1)h(n). \tag{14}$$

The time derivative of $h(0)$ is

$$\dot{h}(0) = \mu h(1) - \mu m h(0). \tag{15}$$

[3]This is evidently the simplest stationary equilibrium. We do not know whether there are other single-price equilibria in which the support of H is not $p^{\mathbb{N}}$.

Setting these two derivatives equal to zero and arguing recursively, it is seen that the only candidates for stationary measures are of the form

$$\forall n \in \mathbb{N} \quad h(n) = m^n(1-m) \tag{16}$$

for some $m \in (0,1)$. Given this geometric functional form specified by (16), the quantities p, m, and M are related by the equation

$$M = p\sum_{n=1}^{\infty} nh(n) = \frac{m}{1-m}p. \tag{17}$$

This characterization of stationarity by equations (16) and (17) is the remaining equilibrium condition that was postponed from the end of the preceding section.

B *Equilibrium Value Function*

Now we solve equation (10) for the equilibrium value function. To begin, recall that the presumed optimal strategy in single-price equilibrium is that agents are always willing to sell at p, and that every agent who holds money is willing to purchase at p. Formally this assumption means that

$$\Omega(p) = 1 \quad \text{and} \quad \forall x < p \;\; \Omega(x) = 0 \tag{18}$$

and

$$\mathrm{R}(p) = 1 - m. \tag{19}$$

It will be convenient to define $V(n) = \mathcal{V}(np)$. Then, using (18) and (19), (10) simplifies for $n = 0$ to

$$V(0) = \frac{\mu}{k\gamma + 2\mu}[V(0) + (1-m)V(0) + mV(1)], \tag{20}$$

which yields

$$V(0) = \frac{\mu m}{k\gamma + \mu m}V(1). \tag{21}$$

For all positive n, (10) simplifies to

$$V(n) = \frac{\mu}{k\gamma + 2\mu}[[u + V(n-1)] + [(1-m)V(n) + mV(n+1)]]. \tag{22}$$

The system of equations (21) and (22) defines the value function implicitly in terms of five parameters: μ, k, γ, m, and u. Note that $k\gamma/\mu$ actually functions as a single parameter. To simplify further computations, define

$$\phi = \frac{k\gamma}{\mu}. \tag{23}$$

Equation (22) can be rewritten in matrix form as

$$\begin{pmatrix} V(n+1) \\ V(n) \\ u \end{pmatrix} = \begin{pmatrix} \left[\frac{\phi}{m} + \frac{1}{m} + 1\right] & \frac{-1}{m} & \frac{-1}{m} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V(n) \\ V(n-1) \\ u \end{pmatrix} \tag{24}$$

Equation (24) is an inhomogeneous second-order linear difference equation. Its family of solutions is given in terms of eigenvectors of the matrix

$$\mathcal{D} = \begin{pmatrix} \left[\frac{\phi}{m} + \frac{1}{m} + 1\right] & \frac{-1}{m} & \frac{-1}{m} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

in (24), and the correct solution is determined by means of two endpoint conditions.[4] One of these endpoint conditions is equation (21). The other condition is that V is bounded. It is bounded below because it is nonnegative and above because, even if a trader's rate of consumption were not constrained by his need to pay for the goods that he acquires in trade, he would still have only discrete consumption opportunities that would occur at times separated by μ on average, and the utility of which would thus be discounted.

The matrix $\mathcal{D}$ has three distinct eigenvectors, all of which have real eigenvalues. The solution of equation (24) is therefore determined by a linear combination of the eigenvectors for which (because V is bounded) the eigenvalue has absolute value at most 1. Two of the eigenvectors satisfies this criterion. They can be expressed as

$$\mathcal{D} \begin{pmatrix} 1/\phi \\ 1/\phi \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\phi \\ 1/\phi \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix} . \tag{26}$$

where

$$\lambda = \frac{1}{2} \left(\frac{\phi}{m} + \frac{1}{m} + 1 - \sqrt{\left(\frac{\phi}{m} + \frac{1}{m} + 1 \right)^2 - \frac{4}{m}} \right) \in (0,1). \tag{27}$$

Now, by Lefschetz (1977), there are two coefficients θ_1 and θ_2 such that

$$\begin{pmatrix} V(1) \\ V(0) \\ u \end{pmatrix} = \theta_1 \begin{pmatrix} 1/\phi \\ 1/\phi \\ 1 \end{pmatrix} + \theta_2 \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix} . \tag{28}$$

It follows from (21) and (28) that

$$\theta_1 = u\,; \qquad \theta_2 = V(0) - \frac{u}{\phi}\,; \qquad V(0) = \frac{(1-\lambda)m}{\phi + (1-\lambda)m} \frac{u}{\phi}. \tag{29}$$

[4]See Lefschetz (1977, Chapter III) for a discussion of the continuous-time theory, which is completely analogous.

Moreover, by induction, for all $n \geq 0$

$$\begin{pmatrix} V(n+1) \\ V(n) \\ u \end{pmatrix} = \mathcal{D}^n \begin{pmatrix} V(1) \\ V(0) \\ u \end{pmatrix}$$

$$= u \begin{pmatrix} 1/\phi \\ 1/\phi \\ 1 \end{pmatrix} + \lambda^n \left(V(0) - u/\phi\right) \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}. \quad (30)$$

In particular, the second row states that for all n

$$V(n) = \lambda^n V(0) + (1 - \lambda^n)u/\phi. \quad (31)$$

Equations (29) and (31) imply the following lemma.

LEMMA 1 V is increasing and satisfies the concavity condition that, for all $j > 0$, $V(n+j) - V(n)$ is a decreasing function of n.

Although we are characterizing a class of equilibria in which each trader's money holding is always an integer multiple of p, the value function is defined for non-integer multiples as well. Given the presumed optimal trading strategy, the value function is evidently a step function. Specifically, if $[x]$ denotes the integer part of x (that is, $x = [x] + \epsilon$ for some $\epsilon \in [0, 1)$) then

$$\mathcal{V}(\eta) = V([\eta/p]). \quad (32)$$

This completes the derivation of the value function.

C *Equilibrium Strategy*

Suppose that an agent a of type i with money holding η meets a trading partner of type $(i-1)$ with money holding η'. Each observes the other's type but neither observes the other's money holding. Independently of one another, a chooses a reservation price r and the partner posts an offer o. If $r \geq o$, then the partner supplies a unit of good i to a in exchange for amount o of money, and otherwise no transaction takes place. Agent a should choose an optimal reservation price that solves the maximization problem with respect to r that occurs on the right hand side of the Bellman equation (8). The solution to this maximization problem may not be unique. The solution that satisfies the perfectness condition (9) can be written as

$$\rho(\eta) = \max\{r \in [0, \eta] | u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\}. \quad (33)$$

This reservation price is a's full value for a unit of good $(i+1)$. (Analogously, in Vickrey's (1960) analysis of a second-price auction, to bid this full value is the buyer's weakly dominant action.) An alternative assumption, that a buyer only accepts an offer when he gains strictly from it, would not change our general conclusions. The following lemma provides further information about the reservation-price function ρ that will be used below.

LEMMA 2 The reservation-price function ρ specified by equation (33) satisfies the conditions that $\rho(p) = p$ and that, for every positive integer multiple np of p, $\rho(np)$ is an integer multiple of p. It also satisfies the condition that $[\rho(\eta)/p]$ is a nondecreasing function of η.

Proof. Equations (29) and (33) imply that $\rho(p) = p$. By (32) and (33), $\mathcal{V}(\eta) = V([\eta/p])$ and $\eta - \rho(\eta) = jp$ for some $j \in \mathbb{N}$. To prove that $[\rho/p]$ is nondecreasing, suppose that $\eta < \eta'$. By (33), $\mathcal{V}(\eta) - \mathcal{V}(\eta - \rho(\eta)) \leq u$. By (32), then, $V([\eta/p]) - V((\eta - \rho(\eta))/p) = V([\eta/p]) - V([\eta/p] - [\rho(\eta)/p]) \leq u$. By the concavity of V established in lemma 1, $V([\eta'/p]) - V([\eta'/p] - [\rho(\eta)/p]) \leq u$. That is, this concavity condition implies that $\mathcal{V}(\eta') - \mathcal{V}(\eta' - p[\rho(\eta)/p]) \leq u$. Thus by (33) and the increasingness assertion in lemma 1, $\rho(\eta') \geq p[\rho(\eta)/p]$. Therefore $[\rho(\eta)/p] \leq [\rho(\eta')/p]$. Q.E.D.

Lemma 2 has the following, intuitively obvious, consequence.

LEMMA 3 For an agent whose money holding is at least p, it is optimal to accept an offer of p if all sellers' offers are almost surely at price p.

Lemma 3 establishes one of the two presumptions about the equilibrium strategy, stated at the beginning of section B, that we have used to derive the value function. The other presumption is that all offers are made at price p. The following lemma establishes a sufficient condition for this latter presumption to characterize agents' optimizing behavior in an equilibrium.

LEMMA 4 If all agents' reservation prices are integer multiple of p, then the optimal offer $\omega(\eta)$ is an integer multiple of p for every η. If the proportion of agents with positive money holdings in a stationary measure of form (16) is $m \leq 1/2$, and if all agents with positive money holdings have reservation price at least p, then it is optimal for an agent always to offer to sell at p.

Proof. The first assertion is obvious. To prove the second assertion, define the set of money holdings η at which an offer at price o would be accepted by $A(o) = \rho^{-1}([o, \infty)) \subseteq \mathbb{R}$, and define $a(o) = \min\{n | np \in A(o)\}$. Note that $a(o) \geq [o/p]$ because an agent's reservation price cannot exceed his money holding. By lemma 1 and (33), $\{n | np \in A(o)\} = \{a(o), a(o)+$

$1, \ldots\}$. Thus, by (11) and (16), $H(A(o)) = m^{a(o)} \leq m^{[o/p]}$. If the seller's money holding is η, then his expected value of offering o is

$$\begin{aligned} W(\eta, o) &= H(A(o))\mathcal{V}(\eta + o) + (1 - H(A(o)))\mathcal{V}(\eta) \\ &\leq m^{[o/p]}\mathcal{V}(\eta + o) + (1 - m^{[o/p]})\mathcal{V}(\eta). \end{aligned} \tag{34}$$

By the first assertion of this lemma, there must be an optimal offer of the form $o = jp$, so we can restrict attention to this case and also assume that $\eta = np$, and simplify the upper bound on the expected value of offering o to $W(\eta, o) \leq m^j V(n+j) + (1-m^j)V(n)$. By the concavity of V established in lemma 1, $V(n+j) < V(n) + j(V(n+1) - V(n))$. Therefore $W(\eta, o) \leq V(n) + jm^j(V(n+1) - V(n))$. If $m \leq 1/2$, then $jm^j \leq m$ for all $j > 1$, so $W(\eta, o) \leq V(n) + m(V(n+1) - V(n)) = H(A(p))\mathcal{V}(\eta + p) + (1 - H(A(p)))\mathcal{V}(\eta) = W(\eta, p)$. Q.E.D.

D *A Continuum of Equilibria*

We began section IV by making two assumptions about the form of a possible stationary equilibrium. We assumed that all offers are made at a single price p and that reservation prices of all agents with positive money holdings are at least p. Under these assumptions, we have shown in section A that every geometrically-distributed measure on money holdings that are nonnegative integer multiples of p is stationary. In section B, we have characterized the equilibrium value function. Finally, in section C, we have shown that the characterizations of stationarity and optimality imply that our assumptions regarding reservation prices and offers are implied by agents' optimizing decisions if $m \leq 1/2$. Equation (17) establishes that $m \leq 1/2$ if $p \geq M$. Thus we have proved the following theorem.

THEOREM 1 In every trading environment described in section II, for every price $p \geq M$, there is a stationary equilibrium in which all transactions occur at price p and all traders' money holdings are integer multiples of p. The proportion of agents with positive money holdings is an increasing function of M/p, so there is a continuum of distinct stationary-equilibrium allocations.

Theorem 1 states that if the price level p is not below M (that is, essentially, if the per capita real money stock is not greater than 1), then a stationary single-price equilibrium exists. This is a sufficient condition, not a necessary condition for existence. The question remains, then, whether there is any maximum real money stock that is consistent with equilibrium. In fact, for any ϕ, such a maximum real money stock does exist. We prove this via two lemmas.

LEMMA 5 Suppose that the following condition holds.

$$m^{a(2)} > \frac{1}{1+\lambda}. \tag{35}$$

Then there exists an n such that $\omega(np) \geq 2p$.

Proof. Using notation developed in the proof of lemma 4, a seller with money holding np will offer at least $2p$ if $W(np, 2p) > W(np, p)$. Substituting (34) into this inequality yields $m^{a(2)}[V(n+2) - V(n)] > m^{a(2)+1}[V(n+2) - V(n)] > m[V(n+1) - V(n)]$. Thus the inequality between the second and third terms is a sufficient condition for an offer of at least $2p$. Applying (31) yields the equivalent condition (35). Q.E.D.

LEMMA 6 For any given ϕ, there exists some $J \in \mathbb{N}$ such that, for all m, $a(2) \leq J$.

Proof. By equation (33), an agent with money holding np will have reservation price at least $2p$ if $u + V(n-2) \geq V(n)$. That is,

$$u \geq \frac{1}{\phi + (1-\lambda)m} u(\lambda^{n-2} - \lambda^n). \tag{36}$$

For this inequality to hold, it is sufficient that $u \geq u\lambda^{n-2}/\phi$, or

$$n \geq 2 + \frac{\ln(\phi)}{\ln(\lambda)}. \tag{37}$$

Noting that λ is a function of m and that the right hand of (37) reaches a finite maximum at $m = 1$, J can be taken to be the first natural number greater than that maximum. Since $a(2)$ is the smallest number satisfying (36) and J satisfies (36), $a(2) \leq J$. Q.E.D.

THEOREM 2 For any given ϕ, there is some $m^* < 1$ such that, for any $m > m^*$, a stationary single-price equilibrium does not exist.

Proof. Since $a(2) \leq J$ (where J is described in lemma 6), $m^{a(2)} \geq m^J$. Combining this inequality with (35),

$$m^J > \frac{1}{1+\lambda} \tag{38}$$

is a sufficient condition for there to exist an n such that $\omega(np) \geq 2p$. Both sides of (38) are continuous functions of m, and condition (38) is not satisfied at $m = 1/2$ but it is satisfied at $m = 1$. These two facts, combined with the fact that the set of values m at which (38) is not satisfied is closed, imply that the set has a maximum m^* which is strictly less than 1. Q.E.D.

V The Limiting Case

Theorem 2 shows that, for any given ϕ and for sufficiently high m (or equivalently, for sufficiently high M/p), a stationary single-price equilibrium cannot exist. However, in an economy where the parameter ϕ is small, reflecting high frequency of meetings or insignificance of discounting, the minimum price level for such an equilibrium to exist is actually arbitrarily low.

The intuition for this result is that, if a buyer is confident that he will almost immediately meet another seller whose offer is very close to the minimum offer in the market, then he should be unwilling presently to accept a high offer unless his money holding is huge. The key to the formal derivation is a closer examination of the optimal reservation price, characterized in (33) by

$$\rho(\eta) = \max\{r \in [0, \eta] | u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\}.$$

It is clear from this formula and the formula (32) for $\mathcal{V}$ that, if η is an integer multiple of p, then $\rho(\eta)$ must also be an integer multiple of p. Define, for the optimal reservation-price function ρ in an economy with parameters μ, k, and γ satisfying (23) and with proportion m of agents having positive money holdings,

$$\rho(np) = r(n, \phi)p \tag{39}$$

We want to study what happens in the limit as ϕ approaches zero. As in the preceding section, we assume that all offers are at price p and then verify that (asymptotically, in this section) such offers are indeed optimal given agents' optimal reservation-price functions.

LEMMA 7 If all offers are at price p, then for every natural number $n \geq 3$, there exists a $\phi_n \in \mathbb{R}$ such that

$$\forall \phi \leq \phi_n \quad \forall j \in \{1, \ldots, n-1\} \quad r(j, \phi) \leq 1. \tag{40}$$

Proof. By lemma 2, it is sufficient to show that $\forall \phi \leq \phi_n \;\; r(n-1, \phi) \leq 1$. Equation (33) implies that, if $r(n-1, \phi) > 1$ (that is, the optimal reservation price is at least 2), then $V(n-1) - V(n-3) \leq u$. Note that, by equations (29) and (31),

$$\begin{aligned} V(n-1) - V(n-3) &= [u/\phi - V(0)](\lambda^{n-3} - \lambda^{n-1}) \\ &= \frac{u(1-\lambda^2)\lambda^{n-3}}{\phi + (1-\lambda)m} \, . \end{aligned} \tag{41}$$

Substitution of the value of λ defined in (27) into (41), and application of l'Hôpital's rule, yield $\lim_{\phi\to 0}(V(n-1)-V(n-3)) = 2u$. Thus the lemma follows from lemma 2 and (33), since $\lim_{\phi\to 0}(V(n-1)-V(n-3)) > u$ implies that the trader's unique optimal reservation price is 1 for sufficiently small ϕ. Q.E.D.

Now we use lemma 7 to show that, for sufficiently small values of ϕ, the optimum offer for all sellers is p. Recall that $W(\eta, o)$ was defined in (34) to be the expected value, to a seller of type i holding quantity η of money, of making an offer of o to a buyer of type $(i+1)$.

THEOREM 3 For every $m \in (0,1)$ there exists a $\phi^*_m \in \mathbb{R}$ such that

$$\forall \phi \leq \phi^*_m \quad \forall \eta \in \mathbb{R}_+ \quad W(\eta, p) = \max_{o\in\mathbb{R}_+} W(\eta, o). \tag{42}$$

Proof. Note first that $\lim_{x\to\infty} xm^x = 0$. Thus we can choose $n \in \mathbb{N}$ such that $\max_{n\leq x} xm^x < m$. Let ϕ^*_m be the value of ϕ_n satisfying (40). As in the proof of theorem 1, we can restrict attention to the case where η and o are both integer multiples of p. Specifically let $\eta = ip$ and $o = jp$. Then

$$W(\eta, o) = \begin{cases} V(i) & : \quad j = 0; \\ V(i) + m(V(i+1) - V(i)) & : \quad j = 1; \\ V(i) + m^{a(o)}(V(i+j) - V(i)) & : \quad j > 1; \end{cases} \tag{43}$$

where $a(o)$, which has been defined formally in the proof of lemma 4, corresponds to the least level of money holding at which an offer of o will be less than or equal to the optimal reservation price. If $1 < j \leq n$, $a(2p) \geq n$ by lemma 7. This implies that for all $o \geq 2p$, $a(o) \geq n$. Therefore, $m^{a(o)} \leq m^n$. Also since V is concave,

$$V(i) + m^{a(o)}(V(i+j) - V(i)) \leq V(i) + nm^n(V(i+1) - V(i)). \tag{44}$$

If $j > n$, the following bound can be derived as in the proof of lemma 4,

$$V(i) + m^{a(o)}(V(i+j) - V(i)) \leq V(i) + jm^j(V(i+1) - V(i)). \tag{45}$$

With these bounds, p is seen to be the optimal reservation price by choice of n. Q.E.D.

VI Welfare

This version of the Kiyotaki-Wright model with divisible money, and without inventory constraints on the holding of it, provides a more natural environment to study welfare questions. In contrast to the original version, here stationary equilibria with higher real money stocks always provide higher levels of welfare. Intuitively, the fewer agents there are without money, the fewer trading opportunities will be foregone, and therefore the higher welfare will be.

To show this formally, we consider the standard welfare measure of summing agents' utility levels. That is, our welfare measure is

$$U(m, \phi, u) = \sum_{n=0}^{\infty} h(n)V(n) = (1-m)\sum_{n=0}^{\infty} m^n V(n). \tag{46}$$

Substituting the values of $V(0)$ and $V(n)$ given in (29) and (31) into equation (46) yields

$$U(m, \phi, u) = \frac{mu}{\phi}. \tag{47}$$

Given this equation and the fact that ϕ and u are parameters of the model, welfare would be maximized by selecting the highest level of m consistent with existence of equilibrium. By equation (17), welfare is maximized by minimizing the price level given a fixed nominal money stock M (that is, by maximizing the real money stock in economy).

Our strong conclusion that a higher real money stock unambiguously corresponds to higher steady-state welfare arguably results from our having abstracted from production costs. Zhou (1996) models production costs, and shows that welfare need not be monotonic in the aggregate real-money stock. The reason is that traders do not produce when they are holding sufficiently high real money balances. An intuition for this result is that the cost of production must be borne immediately, while the benefit from acquiring additional money would be discounted to the time of its expenditure. Since an increase in the economy's real money stock can lead to an increase in the number of traders holding high real money balances, such an increase might cause aggregate production to decrease. In contrast to the technological incompatibility between money holding and production in models incorporating the Kiyotaki-Wright inventory constraint, such an equilibrium incentive effect of the real money stock on production would presumably disappear in the limit as ϕ approaches zero.

VII Discussion

We have studied a very preliminary, schematic version of a search-equilibrium model with divisible money. We show that there always exists a continuum of stationary monetary equilibria where all transactions occur at a single price. Agents in this model economy set their prices strategically rather than taking prices to be parametric as in the Walrasian model. The prevalence of a single price results from self-fulfilling beliefs of the agents. Besides the single-price equilibria of the model, we conjecture that there may be other stationary equilibria in which prices are dispersed. Nonstationary equilibria may also exist.

The existence of a single-price equilibrium might be viewed in either a positive or a negative light. On the positive side, the model provides support for conclusions drawn from the Kiyotaki-Wright analysis in which the parity of exchange between money and goods is exogenous. On the negative side, the existence of an exact single-price equilibrium seems to be a consequence of extreme features of the model such as the seller-posting-price protocol (rather than a trading protocol that splits the difference between seller's and buyer's reported valuations) and the indivisibility of the consumption good. While privacy of traders' information about their money holdings also play a role, we regard this as being a more appealing assumption than the others.

We conjecture that, even if our model were amended in ways that would eliminate exact single-price equilibrium from occurring, equilibrium price dispersion would vanish in the limit as ϕ is taken to zero. If a buyer is confident that he will almost immediately meet a seller whose offer is very close to the minimum offer in the market, then he should be unwilling to post a high reservation price and trade with any sellers who would force him to make a higher payment, unless his money holding is huge. This is the same insight on which our present analysis of frequent meetings is based.

As in all search-equilibrium models without production costs, there is a non-monetary equilibrium in this model. Each agent simply gives his good away for free to anyone he meets who wants it. It is noteworthy that this equilibrium has a greater amount of trade (and hence provides a higher level of welfare) than does any monetary equilibrium. However, this is a suspect equilibrium, because its existence is directly traceable to our schematic assumption that production of goods is completely costless. Introduction of any production cost, however small, removes it from the equilibrium set.

Monetary equilibrium in a random-matching environment with costly production is studied by Zhou (1996). The extension to costly production requires more careful attention to the equilibrium concept, including use of a perfectness-type refinement. Despite this refinement, the existence of a continuum of single-price equilibria persists. The concluding section of Zhou (1996) discusses this, as well as other issues regarding the economic interpretation of our model.

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