The Last Word on Giffen Goods?

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February, 1996

Abstract

Giffen goods have long been a minor embarrassment to courses in microeconomic theory. The standard approach has been to dismiss Giffen goods as a theoretical curiosity without empirical content. This note points out that the underlying theory is itself seriously flawed.

KEYWORDS: Giffen goods, demand theory, general equilibrium theory.

1 Introduction

Giffen goods have long been a minor embarrassment to courses in microeconomic theory. The standard approach has been to dismiss them as a theoretical curiosity without empirical content. In teaching this material, one might, for example, cite the observation in Stigler (1947) that Giffen goods must be extremely rare to have evaded discovery for so long. This note points out that the underlying theory is itself seriously flawed.

A Giffen good, the reader will recall, is one for which the fixed income version of the Law of Demand fails: the quantity of the good demanded increases in the good's own price, holding nominal income fixed. Since our actual interest is presumably in endogenous income demand, the use of fixed income demand requires some defense. The standard rationale for using fixed income demand is that it is relatively tractable

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and that it provides a good approximation to endogenous income demand when income effects are small.[1] However, Giffen goods are, by their very nature, goods for which income effects are *large*. The study of Giffen goods is thus concerned with whether the fixed income version of the Law of Demand holds in situations where there is strong reason to believe that fixed income analysis is invalid. To underscore this point, this note will show that the standard empirical test for detecting whether a good is Giffen is based on comparative statics predictions which are almost precisely reversed once endogenous income (i.e. general equilibrium (GE)) effects are taken into account.

For concreteness, think of potatoes and Ireland. Model the potato famine of the 1840s as a negative shock to the supply of the potatoes (or, in a more sophisticated model, as a negative shock to the technology for producing potatoes). If potatoes had been Giffen then standard fixed income analysis predicts that the equilibrium price of potatoes would have *fallen*. Dwyer Jr. and Lindsay (1984) provides an example of this sort of analysis. Conversely, if potatoes had not been Giffen then the equilibrium price of potatoes would have risen. Thus, we could, in principle, test whether potatoes were Giffen for the 1840s Irish economy by checking whether potato prices fell during the years of drought and blight.[2]

The above comparative statics argument ignores two GE complications. The first, obvious, complication is that the supply shock lowered nominal income, which shifted demand. As a consequence, the price of potatoes could have fallen even if potatoes had been normal, provided the income effect from the supply shock was sufficiently large. Nevertheless, a fall in the price of potatoes would at least have been *consistent* with potatoes having been Giffen. Conversely, an increase in the price of potatoes would continue to indicate, unambiguously, that potatoes were *not* Giffen.

The second, more subtle, complication concerns the slope of demand. A basic GE fact is that the fixed and endogenous income versions of the Law of Demand are in large measure independent of each other. The Appendix contains a brief review. A striking example of this independence occurs in the case of a representative

[1] The approximation argument goes back to Marshall; see Vives (1987) for a modern treatment.

[2] According to Dwyer Jr. and Lindsay (1984), potato prices are not available for this period. However, my concern is not with implementing an empirical test of whether potatoes were Giffen but with the logic of the test itself.

consumer. At equilibrium, the fixed income Law of Demand can fail (for the usual reasons) but the endogenous income Law of Demand *must* hold. Thus, if there had been a representative consumer and if potatoes had been a Giffen good then fixed income demand would not have gotten even the *sign* of the slope of endogenous income demand correct. With a representative consumer, the price of potatoes could have dropped only if potatoes had been a *normal* good and income effects had been large, as described above. An empirical test for Giffeness based on fixed income analysis would therefore have identified potatoes as Giffen only if they had *not* been Giffen.

The remainder of this note is devoted to establishing a general version of the GE comparative statics result just sketched. Informally, the result states that if the endogenous income Law of Demand holds at equilibrium and if the equilibrium price of that good falls in response to a negative production shock for that good then the good must be *normal*, hence not Giffen. Conversely, if the good is inferior, even Giffen, its price must *rise*.[3] As mentioned, the Law of Demand assumption is automatically satisfied in the case of a representative consumer. More generally, the assumption is restrictive but, as discussed below, one can argue that it is a minimal prerequisite for comparative statics analysis.[4] The GE result may help explain why Giffen goods have eluded detection in market data. However, to the extent that our ultimate interest is in comparative statics predictions, the main import of the result is that it casts doubt on whether we should even be interested in whether a good is Giffen, as opposed to merely inferior, in the first place.

[3]The proposition statement requires that the price change belong to a particular set of possible price changes; a sufficient condition is that the prices of the other goods not change "too much" in response to the shock. This requirement is trivially satisfied if there are only two goods, hence only one relative price, but more generally it is restrictive. Given the complexity of GE comparative statics, some such restriction is to be expected.

[4]Dougan (1982) argues that Giffen goods are irrelevant because an equilibrium in which the fixed income Law of Demand failed would be dynamically unstable, hence would not be observed. I make a similar appeal to stability here. However, Giffen goods are compatible with GE dynamic stability, so my appeal to stability, unlike Dougan's, does not rule Giffen goods out. My point, rather, is that when Giffen goods are present, analysis based on fixed income demand is potentially invalid.

2 A Result on GE Comparative Statics.

A general reference for the technical material in this section is Mas-Colell (1985). Some of the material is also contained in Mas-Colell, Whinston, and Green (1995).

Consider a production economy with $1 \leq I < \infty$ consumers, $1 \leq K < \infty$ inputs, and $1 \leq L < \infty$ outputs. For notational convenience, I will assume that consumers are endowed only with inputs and have preferences only over outputs. This is less restrictive than it may appear. For example, an exchange economy in this setup is simply a production economy with $K = L$ and, for each k, a linear technology that transforms one unit of k into one unit of the corresponding ℓ. Let $\omega_i \in \mathbb{R}^K_{++}$ denote the endowment of consumer i, where $\mathbb{R}^K_{++}$ is the set of strictly positive vectors in $\mathbb{R}^K$. Let $p \in \mathbb{R}^L_{++}$ be the vector of output prices. As usual in GE systems, only relative prices matter. Normalize prices by fixing the price of the Lth output good at 1: $p^L = 1$. We will not need to consider input prices explicitly.

The Proposition below will be phrased in differential form. To this end, assume that preferences satisfy the standard assumptions of differentiable general equilibrium theory. In particular, assume that the demand of consumer i is given by a $\mathcal{C}^1$ (i.e. continuously differentiable) function $\phi_i : \mathbb{R}^{L+1}_{++} \to \mathbb{R}^L_{++}$, a function of p and the income of consumer i, m_i. Aggregate (i.e. market) demand is then given by $\phi : \mathbb{R}^{L+I} \to \mathbb{R}^L_{++}$, $\phi(p, m_1, \ldots, m_I) = \sum_i \phi_i(p, m_i)$.

In order to describe aggregate demand for a good as inferior or normal we must be able to write aggregate demand as a function of aggregate income $m = \sum_i m_i$, rather than as a function of the vector of individual incomes. A sufficient and essentially necessary condition for this is that ownership of endowments and firm shares be collinear across consumers, in which case there exist $\alpha_i > 0$, $\sum_{i=1}^I \alpha_i = 1$, such that $m_i = \alpha_i m$ for any vector of input and output prices. Aggregate demand is then $\phi(p, \alpha_1 m, \ldots, \alpha_I m)$. Abusing notation, I will write this simply as $\phi(p, m)$. In Remark 4 below, I discuss how the proposition must be modified if collinearity is dropped.

The assumption that firm ownership is collinear across consumers implies that it is sufficient to consider aggregate production; in particular, we do not need to track profits firm by firm in order to calculate demand. The aggregate production technology will be represented in dual form, via an aggregate revenue function ρ : $\mathbb{R}^{L+K}_{++} \to \mathbb{R}^L_{+}$, a function of p and the aggregate endowment $\omega = \sum_i \omega_i$. $\rho(p, \omega)$ is the solution to the program: maximize $p \cdot y$ such that output $y \in \mathbb{R}^L_+$ can be produced from inputs ω. Henceforth, fix an aggregate endowment $\bar{\omega}$ and suppress $\bar{\omega}$ as an argument of ρ.

The proposition will be stated in terms of a shock to the production technology governing output of consumption good 1. The shock is parametrized by a number $\xi > 0$, with zero shock represented by $\xi = 1$. When $\xi = \xi^*$, output $(y^1, \ldots, y^L)$ is feasible if and only if output $(y^1/\xi^*, \ldots, y^L)$ is feasible when $\xi = 1$. A decrease in ξ thus represents a deterioration in the economy's ability to produce good 1. In the case of an exchange economy, a decrease in ξ corresponds to a decrease in the endowment of good 1. I will let $\rho(p, \xi)$ denote aggregate revenue at prices p and production shock ξ. Note that, for given (p^*, ξ^*), aggregate income is $m^* = \rho(p^*, \xi^*)$.

I will assume ρ is C^2. In the case of an exchange economy, $\rho(p, \xi) = p \cdot \bar{\omega} + p^1(\xi - 1)\bar{\omega}^1$, which is clearly C^2. More generally, ρ will be C^2, at least in a neighborhood of equilibrium, in differentiable strictly decreasing returns to scale economies and in "well behaved" linear economies (for example, in the linear economies of classical comparative statics results such as the Rybczynski and Stolper-Samuelson theorems). It follows almost immediately from its definition that ρ is convex in price. Therefore, for given (p^*, ξ^*), $D_{pp}\rho(p^*, \xi^*)$, the second derivative of ρ with respect to price, is positive semidefinite.

The argument will exploit some basic facts about the derivatives of ρ. Let $y^* = y(p^*, \xi^*)$ be the solution to the revenue maximization problem at (p^*, ξ^*). Thus, $\rho(p^*, \xi^*) = p^* \cdot y^*$. Let $e^1 = (1, 0, \ldots, 0)'$ denote the first unit vector, where $'$

denotes transpose. One can show that[5]

$$D_p\, \rho(p^*, \xi^*) = y^{*'}, \tag{1}$$

$$D_\xi\, \rho(p^*, \xi^*) = \frac{1}{\xi^*} p^{1*} y^{1*}, \tag{2}$$

$$e^{1'} D_\xi D_p\, \rho(p^*, \xi^*) \geq y^{1*}. \tag{3}$$

In view of (1), we can write the aggregate excess demand for consumption goods as

$$f(p, \xi) = \phi(p, \rho(p, \xi)) - D_p\, \rho(p, \xi)'.$$

In equilibrium, $f(p, \xi) = 0$. Fix a reference equilibrium price vector $\bar{p}$ of the $\xi = 1$ economy.

The Proposition is a statement about comparative statics in a competitive GE economy. For comparative statics to make sense, we must have some reason to believe that the economy will actually be in equilibrium, that is, that the equilibrium is dynamically stable. A sufficient and essentially also necessary condition for an equilibrium to be locally asymptotically stable under the Walrasian price tâtonnement is that the endogenous income version of the Law of Demand holds at equilibrium. Thus, an argument can be made that a minimal prerequisite for GE comparative statics analysis is that the endogenous income version of the Law of Demand holds at equilibrium.

Informally, the endogenous income Law of Demand states that changes in excess demand lie in the opposite direction from (form an obtuse angle with) changes in price, for any price change v that preserves the normalization. Formally:

[5]Expressions (1), (2) and (3) are variants of standard "envelope theorem" results. For (1) and (2), note that, for any (p, ξ),

$$\rho(p, \xi) - \left[p \cdot y^* + \left(\frac{\xi}{\xi^*} - 1 \right) p^1 y^{1*} \right] \geq 0,$$

since $(\xi/\xi^* y^{1*}, \ldots, y^{L*})'$ will always be feasible at (p, ξ) even if it is not revenue maximizing. Note also that the left-hand side attains its minimum, namely 0, at (p^*, ξ^*). Expressions (1) and (2) then follow from the first order conditions for minimization. For (3), let $y = y(p^*, \xi)$, with $\xi > \xi^*$. Then $p^* \cdot y \geq p^* \cdot (y^{1*}\xi/\xi^*, y^{2*}, \ldots, y^{L*})$, since, at (p^*, ξ), y is revenue maximal while $(y^{1*}\xi/\xi^*, y^{2*}, \ldots, y^{L*})$ is feasible. Similarly, $p^* \cdot y^* \geq p^* \cdot (y^1 \xi^*/\xi, y^2, \ldots, y^L)$. Manipulating these inequalities yields $y^1 \geq \xi/\xi^* y^{1*}$. The claimed inequality in (3) follows.

Definition. *The* (endogenous income) Law of Demand *holds at the equilibrium* $(\bar{p}, 1)$ *iff*

$$v' D_p f(\bar{p}, 1) v < 0$$

for any $v \in \mathbb{R}^{L-1} \times \{0\}, v \neq 0.$

In particular, taking v to be the ℓth unit vector, the endogenous income Law of Demand implies $\partial f^\ell / \partial p^\ell < 0$: own price effects are negative.

The endogenous income Law of Demand necessarily holds at equilibrium if there is a single consumer or a representative consumer. This is a standard GE fact, which, to make the discussion reasonably self-contained, I review briefly in the Appendix. More generally, one can show that the endogenous income Law of Demand holding at $(\bar{p}, 1)$ is essentially equivalent to excess demand obeying the Weak Axiom of Revealed Preference (WARP) in a neighborhood of $\bar{p}$. This is a weaker requirement than having excess demand obey WARP globally, which in turn is weaker (for $L > 2$) than having a representative consumer. Even if the endogenous income Law of Demand holds at $(\bar{p}, 1)$, the economy may have multiple equilibria.

The Proposition will be framed in terms of how a differential change in the endowment of good 1, which may be thought of as potatoes, alters the equilibrium price of good 1. To this end, we wish to express equilibrium prices as a C^1 function of ξ. Explicitly, we will assume that there is an open neighborhood $\mathcal{O}$ of $\xi = 1$ and a differentiable map $P : \mathcal{O} \to \mathbb{R}^L_{++}$ such that $P(1) = \bar{p}$ and for every $\xi \in \mathcal{O}$, $P(\xi)$ is an equilibrium price. By the Implicit Function Theorem, a sufficient condition for such $\mathcal{O}$ and P to exist is that $\bar{p}$ be a *regular* equilibrium of the $\xi = 1$ economy: that is, $D_p f(\bar{p}, 1)$ has maximal rank (namely $L - 1$). A sufficient condition for $\bar{p}$ to be regular, in turn, is that the Law of Demand hold at $(\bar{p}, 1)$.

Starting from $\xi = 1$, an increase in ξ (i.e. an improvement in the economy's ability to produce good 1) changes equilibrium prices by the vector $v = DP(1)$. The Proposition will apply to any price change v that belongs to a set W, an open half space of $\mathbb{R}^{L-1} \times \{0\}$, given by

$$W = \left\{ v \in \mathbb{R}^{K-1} \times \{0\} : D_p f^1(\bar{p}, 1) v < 0 \right\}.$$

To interpret W, suppose that W contains $e^1 = (1, 0, \ldots, 0)'$. If $DP(1) = v = \lambda e^1$, $\lambda > 0$, then an increase in the economy's ability to produce good 1 raises the price of good 1, leaving all other prices unchanged. Conversely, the equilibrium price of good 1 will *fall* following a *decrease* in the economy's ability to produce good 1, the case discussed in the Introduction. Since W is open, if it contains e^1 then it also contains an open neighborhood of e^1, consisting of price changes of the form, "the price of good 1 increases while other prices remain largely unchanged." W may, of course, also contain additional price changes which do not fit this interpretation.

It remains to verify whether, in fact, $e^1 \in W$. W will contain e^1 if the endogenous income Law of Demand holds at $(\bar{p}, 1)$, since, by the endogenous income Law of Demand, $e^{1'} D_p f(\bar{p}, 1) e^1 < 0$, hence $D_p f^1(\bar{p}, 1) e^1 < 0$. If the endogenous income Law of Demand fails, the Proposition still holds whenever $DP(1) \in W$, but the Proposition may no longer have the intended interpretation.

We require one last piece of notation. Let aggregate expenditure on goods other than good 1 be denoted

$$z(p, m) = \sum_{\ell=2}^{L} p^\ell \phi^\ell(p, m).$$

For fixed prices $p^2, \ldots, p^L$, z is a Hicksian composite commodity. In particular, one may refer to z as an inferior or normal good depending on the sign of $D_m z$, the marginal propensity to consume z out of aggregate income.

Proposition. *Consider a differentiable production economy with collinear endowments and profit shares. As described above, the production technology governing output of the first good is parametrized by a number ξ. Let $\bar{p}$ be an equilibrium price vector when $\xi = 1$. Suppose that there exist $\mathcal{O} \subset \mathbb{R}$ and $P : \mathcal{O} \to \mathbb{R}^L_{++}$, as described above. If $DP(1) \in W$ then*

1. *aggregate consumption of good 1 is normal:*

$$D_m \phi^1(\bar{p}, \bar{m}) > 0; \tag{4}$$

2. *aggregate consumption of composite commodity z is inferior:*

$$D_m z(\bar{p}, \bar{m}) < 0. \tag{5}$$

Proof. For any $\xi \in \mathcal{O}$

$$f(P(\xi), \xi) = 0.$$

Differentiating with respect to ξ and evaluating at $\xi = 1$ yields

$$D_p f(\bar{p}, 1)\ DP(1) + D_m \phi(\bar{p}, \bar{m})\ D_\xi\, \rho(\bar{p}, 1) - D_\xi D_p\, \rho(\bar{p}, 1) = 0.$$

In view of (2) and the fact that $(\bar{p}, 1)$ is an equilibrium, this can be rewritten as

$$D_p f(\bar{p}, 1)\ DP(1) + D_m \phi(\bar{p}, \bar{m})\ \phi^1(\bar{p}, \bar{m}) \bar{p}^1 - D_\xi D_p\, \rho(\bar{p}, 1) = 0.$$

Premultiplying by $e^{1'}$ and using (3) and the fact that $(\bar{p}, 1)$ is an equilibrium, we can derive

$$\bar{p}^1 D_m \phi^1(\bar{p}, \bar{m}) \geq 1 - \frac{D_p f^1(\bar{p}, 1)\ DP(1)}{\phi^1(\bar{p}, \bar{m})}. \tag{6}$$

Since $DP(1) \in W$, the right-hand side of (6) is greater than 1, which proves (4).

To prove (5), differentiate Walras's Law, $p^1 \phi^1(p, m) + z(p, m) = m$, with respect to m and evaluate at $(p, m) = (\bar{p}, \bar{m})$. This yields

$$\bar{p}^1 D_m \phi^1(\bar{p}, \bar{m}) + D_m z(\bar{p}, \bar{m}) = 1.$$

In view of (6), (5) then follows. ■

To interpret the Proposition, note that the equilibrium supply of the first good is $e^{1'} D_p \rho(P(\xi), \xi)$, a function of ξ. The derivative of this, evaluated at $\xi = 1$, is equal to

$$e^{1'} D_{pp}\ \rho(\bar{p}, 1)\ DP(1) + e^{1'} D_\xi D_p\ \rho(\bar{p}, 1).$$

In view of (3), this derivative will be positive so long as $DP(1) \in V = \{v \in \mathbb{R}^{L-1} \times \{0\} : e^{1'} D_{pp}\ \rho(\bar{p}, 1) v \geq 0\}$. Thus, so long as $DP(1) \in V$, the equilibrium supply of good 1, hence the equilibrium consumption of good 1, will increase in response to an increase in the economy's ability to produce good 1.

Let $W^* = W \cap V$. Since $D_{pp}\ \rho(\bar{p}, 1)$ is positive semidefinite, $e^1 \in V$. Thus $e^1 \in W$ implies $e^1 \in W^*$. W^* will in fact contain an open neighborhood of e^1 provided V contains an open neighborhood of e^1. This is the standard case; it will obtain when production is linear (so that $D_{pp}\ \rho(\bar{p}, 1) = 0$) or exhibits strictly decreasing returns to scale (so that $D_{pp}\ \rho(\bar{p}, 1)$ is positive definite on $\mathbb{R}^{L-1} \times \{0\}$).

Together, the Proposition and the above observation imply that, whenever $DP(1) \in W^*$, if the equilibrium price and equilibrium consumption of good 1 both move in the same direction as a result of a change in the economy's ability to produce good 1 then aggregate consumption of good 1 must be normal. This confirms the claim made in the Introduction.

Remark 1. The role played in the Proposition by the Law of Demand is implicit: it guarantees that the equilibrium is regular, which in turn assures the existence of $\mathcal{O}$ and P, and it assures that W (and W^*) has the intended interpretation. *Some* form of behavioral restriction along these lines is required. It is not hard to construct exchange economy examples in which the Law of Demand fails and a drop in endowment lowers the equilibrium price of an inferior good. Since the Law of demand necessarily holds at equilibrium if there is a representative consumer, any such example must have at least two consumers. □

Remark 2. The assumption that endowments are collinear is, of course, unrealistic. However, broadly similar results obtain even if collinearity is dropped. To avoid undue notational complication, I focus on the exchange case. Suppose that we parameterize shifts away from the initial vector of endowments, $(\bar{\omega}_1, \ldots, \bar{\omega}_I)'$, by

$$\omega_i(\lambda) = (\lambda\bar{\omega}_i^1, \bar{\omega}_i^2, \ldots, \bar{\omega}_i^L)',$$

the same λ for all i. Writing the equilibrium price function as $P(\lambda)$, an argument similar to the one above establishes that if $DP(1) \in W$, then the sum of individual marginal propensities to consume good 1, weighted by expenditure shares for good 1, is positive:

$$\sum_{i=1}^{I} D_{m_i}\phi_i^1(\bar{p}, \bar{m}_i)\frac{p^1\omega_i^1}{p^1\omega^1} > 0.$$

Similarly, the sum of individual marginal propensities to spend money on other goods, again weighted by expenditure shares for good 1, is negative:

$$\sum_{i=1}^{I} D_{m_i}\left(\sum_{\ell=2}^{L} p^\ell\phi_i^\ell(\bar{p}, \bar{m}_i)\right)\frac{p^1\omega_i^1}{p^1\omega^1} < 0.$$

This generalizes the result for collinear exchange economies: if endowments are collinear, then $p^1\omega_i^1/p^1\omega^1 = \alpha_i$. □

Remark 3. Proposition 17.G.2 in Mas-Colell, Whinston, and Green (1995) states that, in response to a parameter shock, excess demand (holding prices fixed) moves in the same direction as equilibrium prices provided the equilibrium price change belongs to a set analogous to W. (This interpretation of Proposition17.G.2 can be inferred from equation 17.G.1.) The Proposition of this note belongs to the same general family of comparative statics results but addresses a different question. $\square$

APPENDIX: The Law of Demand at Equilibrium

The material here is provided for completeness. As above, I will assume that the distribution of endowments and profit shares is collinear across consumers, so that aggregate demand can be written as a function of aggregate income. Similar, but notationally more involved, analysis applies when endowments are not collinear; see, for example, Section 5.7 of of Mas-Colell (1985) or Section 17.F of Mas-Colell, Whinston, and Green (1995).

Endogenous Income Demand.

Expanding $D_p f$:

$$D_p f(\bar{p}, 1) = D_p \phi(\bar{p}, \bar{m}) + D_m \phi(\bar{p}, \bar{m}) D_p \rho(\bar{p}, 1)' - D_{pp} \rho(\bar{p}, 1). \tag{7}$$

Using the usual Slutsky decomposition of $D_p \phi(\bar{p}, \bar{m})$, and using the fact that $(\bar{p}, 1)$ is assumed to be an equilibrium, we derive:

$$\begin{aligned} D_p f(\bar{p}, 1) = \sum_i S_i(\bar{p}, \alpha_i \bar{m}) - \sum_i D_{m_i} \phi(\bar{p}, \alpha_i \bar{m}) \; \phi_i(\bar{p}, \bar{m})' \\ + D_m \phi(\bar{p}, \bar{m}) \; \phi(\bar{p}, \bar{\omega})' - D_{pp} \rho(\bar{p}, 1), \end{aligned}$$

where $S_i(\bar{p}, \alpha_i \bar{m})$ is the substitution matrix of consumer i. As noted in Section 3, $D_{pp}\rho(\bar{p}, 1)$ is positive semidefinite. Moreover, under standard assumptions, the substitution terms $S_i(\bar{p}, \alpha_i \bar{m})$ are all negative definite on $\mathbb{R}^{L-1} \times \{0\}$. Thus, the endogenous income Law of Demand can fail only if the aggregate income term,

$$-\sum_i D_{m_i} \phi(\bar{p}, \alpha_i \bar{m}) \; \phi_i(\bar{p}, \bar{m})' + D_m \phi(\bar{p}, \bar{m}) \; \phi(\bar{p}, \bar{m}).$$

fails to be negative semidefinite on $\mathbb{R}^{L-1} \times \{0\}$.

If there is a representative consumer then the income terms exactly cancel and we are left with

$$D_p f(\bar{p}, 1) = S(\bar{p}, \bar{m}) - D_{pp} \rho(\bar{p}, 1).$$

Thus, if there is a representative consumer then the endogenous income Law of Demand necessarily holds at equilibrium.

More generally, with some manipulation we can derive:

$$
\begin{aligned}
D_p f(\bar{p},1) = &\sum_{i=1}^{I} S_i(\bar{p},\bar{m}) \\
&- \sum_{i=1}^{I} \left[D_{m_i}\phi_i(\bar{p},\bar{m}_i) - D_m\phi(\bar{p},\bar{m})\right]\left[\phi_i(\bar{p},\bar{m}_i) - \alpha_i\phi(\bar{p},\bar{m})\right]' \\
&\qquad\qquad\qquad\qquad\qquad\qquad - D_{pp}\rho(\bar{p},1).
\end{aligned}
$$

Loosely, the endogenous income Law of Demand can fail for good ℓ only if consumers with higher than average marginal propensities to consume ℓ tend to purchase less than their "share," as measured by $\alpha_i\phi^\ell$. Whether ℓ is inferior for some or even all consumers is irrelevant.

Fixed Income Demand.

Fixed income excess demand is given by

$$
g(p,m,\xi) = \phi(p,m) - D_p\,\rho(p,\xi)
$$

The fixed income Law of Demand requires that $D_p\,g(p,m,\xi)$ be positive definite on $\mathbb{R}^L$. Expanding $D_p\,g$ at the equilibrium $(\bar{p},\bar{m},1)$ yields

$$
D_p\,g(\bar{p},\bar{m},1) = D_p\phi(\bar{p},\bar{m}) - D_{pp}\,\rho(\bar{p},1).
$$

Using the Slutsky decomposition of $D_p\phi(\bar{p},\bar{m})$ we derive

$$
D_p g(\bar{p},\bar{m},1) = \sum_i S_i(\bar{p},\alpha_i\bar{m}) - \sum_i D_{m_i}\phi(\bar{p},\alpha_i\bar{m})\;\phi_i(\bar{p},\bar{m})' - D_{pp}\rho(\bar{p},1).
$$

Again, under standard assumptions, the $S_i(\bar{p},\alpha_i\bar{m})$ are negative definite on $\mathbb{R}^{L-1}\times\{0\}$ while $D_{pp}\rho(\bar{p},1)$ is positive semidefinite. Thus the fixed income Law of Demand will hold provided the income term,

$$
-\sum_i D_{m_i}\phi(\bar{p},\alpha_i\bar{m})\;\phi_i(\bar{p},\bar{m})',
$$

is, loosely speaking, sufficiently negative definite. In particular, a sufficient condition for the own-price fixed income Law of Demand to hold for good ℓ is that good ℓ be normal for every consumer. (However, normality is not a sufficient condition for the fixed income Law of Demand to hold for general price changes.)

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