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WORKING PAPER
NUMBER - ECIPA - peters -95-1

On The Equivalence of Walrasian and Non-Walrasian Equilibria in Contract Markets: Complete Contracts

by

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February, 1995
This version - February 19, 1995

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ISSN 0829-4909

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Abstract

This paper explores two models of an economy in which contracts are exchanged. In the first version contracts are exchanged on a competitive market in which traders expectations concerning conditions that prevail within specific markets adjust until markets 'clear'. In the second model contract designers compete directly against one another by offering alternate contracts. It is shown that the two models are equivalent in the sense that an allocation is supported by some equilibrium in one model if and only if there is an economy in which the allocation is supported by equilibrium in the other.

1 Introduction

In a recent series of papers, Gale (1992),(1994) has proposed a Walrasian theory for contract markets. In this theory each imaginable contract is traded on a separate market. Since contracts normally specify trading prices explicitly, the usual Walrasian notion in which prices adjust to clear markets must be adapted. In Gale's model traders form expectations about the conditions that will prevail in different markets, and the requirement that traders expectations be rational in all active markets replaces the standard market clearing condition.

Gale was primarily interested in whether or not 'Walrasian' contract markets would be efficient in the presence of adverse selection[1]. This paper is concerned with a special, but interesting sub-case of Gale's model in which contracts are *complete* in the sense that the value of the contract to any trader who accepts it is determined entirely by the terms of the contract. The primary contribution of the paper is a decentralization theorem that shows that in an economy consisting of an infinitely large number of traders, the allocations generated by Gale's Walrasian equilibrium coincide exactly with the equilibrium allocations of a non-Walrasian model which makes more conventional assumptions about traders' beliefs in off equilibrium situations.

The Walrasian model is conceptually simpler than any of the models that involve direct competition[2]. Indeed this is the primary motivation suggested by Gale for the study of Walrasian models. Yet even without adverse selection, the Walrasian theory requires additional, and very strong informational assumptions to generate results. The reason is that the theory only restricts traders' expectations in what Gale calls *active* markets. These are simply markets in which some transactions occur in equilibrium. This assumption is in the spirit of subjective equilibrium (Kalai 1993) since traders will never observe anything inconsistent with their beliefs if the condition is satisfied. Yet when expectations elsewhere are unrestricted many equilibria can be supported. If traders jointly believe that they will not be able to find trading partners in some market, then they will rationally avoid this market and their belief will be fulfilled. This makes the set of markets arbitrary.

This problem can be avoided by imposing two restrictions on traders' beliefs. First, traders must have common beliefs about inactive markets. Second, they must believe that there are many buyers and sellers in all markets. If buyers, for example, believe that they have zero probability of finding a partner in a specific market, this should be

[1]Adverse selection here means that when a contract is exchanged, the value of the contract to the trader on one side of the exchange depends on the private characteristics of the trader on the other side of the exchange.

[2]Though perhaps a better way of looking at it is that the Walrasian arguments are more familiar.

because they believe that too many buyers go to the market, not because no sellers do. Though it is simple enough to present these assumptions, it is not obvious why they should be true.

In the non-Walrasian model of direct competition presented in this paper, inactive markets correspond to deviations that sellers consider, but reject on the equilibrium path. For sellers to be able to reject these deviations, they must be able to correctly anticipate the effect that such deviations will have on their payoff. The decentralization theorem proved in this paper, in addition to showing the equivalence of the equilibrium allocations in the two models, shows that these correct conjectures in the game with direct competition coincide with one of the sets of beliefs about conditions in inactive markets that support the Walrasian equilibrium.

The Walrasian model of competition in contract markets provides a potentially useful tool for the analysis of applied problems. The essential economic argument behind it is that traders will perceive a market driven relationship between the value of the contract that they are searching for, and the probability that they will be able to find someone to exchange such a contract with them. Models with these properties are sometimes referred to a *queuing models*, though they are closely related to the standard search models as described in (Diamond 1986), (Mortensen 1986), and (Pissarides 1990).

The most striking difference between queuing models and the standard search models from which they are derived is their prediction that there should be some relationship between the wage rate that is offered in a market, and the level of congestion that workers will experience when they search within that market. The most direct evidence of this relationship is given in (Holzer, Katz, and Kreuger 1991) who find that there are more applicants for jobs at the minimum wage than there are applicant for jobs that pay wages slightly above or below the minimum wage.

In the first section of the paper it is shown how Gale's competitive arguments can be applied to this problem. The idea resembles the argument in Holzer et al. (1991) in that all jobs generate rents for both firms and workers. Jobs differ endogenously according to the speed with which they can be found or filled. An increase in the minimum wage raises the rents associated with minimum wage jobs. Some workers who were previously searching for higher paying jobs modify their search strategies to try to get the higher rents that are associated with the minimum wage jobs. The additional congestion that this creates means that the probability that a worker will find a job at the minimum wage declines, while the probability of finding a job at a slightly higher or lower wage rate increases. This process continues until workers earn the same expected utility by searching for a job at the minimum wage as they do by searching for jobs at wages slightly above and below the minimum wage.

The second section of the paper provides a more formal description of the Walrasian version of the model. The third section provides a non-Walrasian model in which firms compete directly against one another by publicly offering alternative contracts. Finally we prove equivalence of the equilibrium allocations in the Walrasian and non-Walrasian versions of the model.

1.1 Related Literature

Moen (1994) is the first paper that I can find that tries to apply these queuing arguments to the generation of equilibrium wage distributions. In his model, the set of possible markets is essentially equivalent to the set of possible wages that could be paid to workers. The market generates a relationship between the wage on offer and the probability with workers can find jobs. His primary result is that the additional equilibrium conditions that are generated when workers and firms have the opportunity to select among various wage probability combinations ensures an efficient outcome.

To my knowledge the first paper to apply Walrasian arguments explicitly to queuing problems of this kind was (Carlton 1978) who was concerned with the relationship between firms' prices and their capacity. Some firms could charge higher prices than others in his model, if these firms were more likely to be able to supply customers in cases when demand was unexpectedly high. This crucial ability to supply is endogenous and depends on how customers allocate themselves across firms. Carlton imposed the restriction that customers anticipate the same level of expected utility with every supplier and verified that this expectation was correct in equilibrium.

It is precisely this assumption that is at the heart of Walrasian models for queuing problems. The literature on non-competitive, or 'monopolistically competitive' models of queuing problems has been somewhat more active. The price capacity relationship has been the subject of a number of papers, including (Peters 1985) ,(Wilson 1989), and (Denekere and Peck 1992). Problems in which sellers compete directly with one another by offering more complicated kinds of contract, are discussed in (McAfee 1993), (Peters 1992), (Peters and Severinov 1993), (Peters 1994) and (Peck 1994). The results of this paper indicate that there is a very close link between these two strands of literature.

This paper is organized in the following way. The first section illustrates the queuing approach with an application to the problem of minimum wages in the labor market. The second section gives a more abstract description of the queuing problem in a fairly general economic environment. The third section describes the Walrasian equilibrium for the contract market, shows the conditions under which an equilibrium can be shown to exist. The fourth section describes the non-Walrasian model and proves the equivalence of it's

equilibrium allocations to that of the Walrasian model. Finally we discuss some possible issues associated with extending the model to the more general case with incomplete contracts.

2 Minimum Wages

Empirical work on minimum wages yields ambiguous results Baker and Benjamin (1994). Estimates of the impact of an increase in the minimum wage on overall employment can be positive or negative. Furthermore detected effects of changes in the minimum wage are normally statistically insignificant. This is somewhat surprising given the straightforward predictions of the theory. Higher minimum wage will move firms up their labor demand curves, lowering employment. Displaced workers may move to alternative markets in which higher wages are offered, but only part of this displaced workforce will be employed. The movement of workers to these alternative markets will result in wages being depressed slightly in these alternative markets. Consequently overall employment must fall when the minimum wage rises in either the partial or general equilibrium version of the standard competitive model.

Simple search theory tells a slightly different story. The increase in the minimum wage will mean that workers who sample firms who are affected by the wage increase will be more likely to accept employment with these firms. Hence employment should rise (at least in the short run) at firms who are affected by the increase in the minimum wage. This increase in the acceptance rate for jobs paying the minimum wage may result in a decline in the arrival rate of workers for other job vacancies. Since this effect is merely a displacement of workers from one sort of job to another, the overall level of employment cannot increase. It is possible that firms whose arrival rate declines as a result of the imposition of the minimum wage, will raise their wage offers to compensate. However, such firms are unlikely to want to maintain as high an acceptance rate for job offers at the higher wages that they are forced to pay. As a result, overall employment is again likely to fall. Furthermore, though the level of employment is supposed to rise at firms offering the minimum wage, the increase in the arrival rate that was detected by Holzer et al. (1991) is not supposed to occur.

As will be seen, simple queuing models give a rather good account of most of these phenomena. Since the queuing model gives workers information about wages available in different markets, the increase in the minimum wage must result in an immediate increase in the arrival rate at firms who offer the minimum. The increased arrival rate results in vacancies being filled more quickly, so that employment increases at the minimum wage. The increase in the arrival rate occurs because workers consciously choose to forgo other

alternatives. Hence the arrival rate at jobs that pay more than the minimum wage will decline until workers are indifferent between the two options. This may increase or decrease overall employment depending on the matching technology. We begin with a brief account of this process before describing the more general model in some detail.

We begin with the assumption that the set of markets consists of the set of feasible wage offers. There is a specific location at which each potential contract is traded - this location is referred to as the contract's *market*. Workers and firms with vacancies can attempt to trade on any of these feasible markets. The minimum wage will impose a restriction on the set of feasible markets.

Let the probability that a worker will receive a job offer in a specific *market* be given by a declining function $q(k)$, where k is the ratio of the number of searching workers to the number of employers with vacancies in the market. The variable k measures the level of congestion in a market. Along with the wage offered in the market, it is the key determinate of a workers willingness to search within that market. The larger k is, the less likely the worker will find a suitable match in any given period of time. Put another way, the larger k is, the longer it will take the worker to find a suitable match in a specific market. The worker may be quite happy to spend a long time searching within the market if the wage in the market is high.

To keep things simple, attention will be concentrated on a specific period in which firms and workers must choose one and only one of the feasible markets in which to try to organize an exchange. Workers and firms who fail to trade in this interval of time get a fixed level of utility that might be equal to the discounted surplus associated with trying to trade in the next period. A firm who wishes to offer a wage w_0 simply goes to the market indexed w_0 and searches there. Similarly workers choose a particular market. Both firms and workers recognize that there is no guarantee that they will find a trading partner in the market that they have chosen within the given time interval. Indeed the possibility of rationing is the primary determinate of their choice of markets.

We assume that a firm that offers a job in the market associated with the wage w_0 is offering a contract that binds the firm to pay w_0 to any worker that it hires. Workers are all identical risk neutral expected wage maximizers. A worker who selects the market w_0 gets expected payoff $q(k)\,w_0 + (1 - q(k))\,\bar{u}$. Here $\bar{u}$ is the payoff received by an unmatched worker. This could be, for example, the discounted payoff that the worker will get by continuing to try to match in the next period. We could also allow $\bar{u}$ to include current wages if the worker is searching on the job.

Firms come in two types. There is a profitable firm who generates a surplus y_H when it fills it vacancy, and a less profitable firm that generates a surplus $y_L < y_H$. If either one of these firms decides to offer it's vacancy on the market w_0, it will fill the

vacancy with probability $q(k)k$ where k is the level of congestion that it expects[3]. The firm's expected payoff is then $q(k)k(v_i - w_0) + (1 - q(k)k)\bar{v}$ for $i = H, L$. Again $\bar{v}$ is the surplus of a firm who fails to trade in the current period.

Workers and firms believe that they can fill vacancies or find jobs in any of the continuum of available markets. The attractiveness of different markets depends on the level of congestion expected. It is natural to try to describe the relationship $\hat{k}(w)$ between the wage rate on offer in a market, and the ratio of the number of searching workers to vacancies within that market. The story becomes *Walrasian* if we add two critical assumptions. First we assume that all traders agree about what $\hat{k}(w)$ is in each possible market. This is the same as assuming that all traders agree on the price at which they can buy or sell any given commodity. Secondly, analogous to assuming that all goods are priced, we assume that $\hat{k}(w)$ is well defined for every feasible wage. This rules out coordination failures of the kind that occur when workers and firms do not try to trade at some wage simply because they rationally believe that no one else will try to transact at that wage.

The equilibrium for this problem is depicted in Figure 1. The convex locus WW is an indifference curve for workers, given by all the solutions to the equation

$$q(k)w + (1 - q(k))\bar{u} = \text{constant}$$

Workers' utility is increasing moving up and to the left in the diagram. The curves F_H and F_L are iso-profit curves for the two different kinds of firms in this economy. Firms like low wages and lots of workers competing for their vacancies, so firms' payoffs are increasing moving down and to the right.

The low profit firms settle in the market w_L while the high profit firms move to the market w_H. To support this we choose the worker's indifference curve WW to be the relationship $\hat{k}(w)$ that traders think prevails between the wage that they choose, and the level of congestion. A point like β in the diagram where the worker and firm indifference curves are not tangent, can not be an equilibrium. The reason is that workers' and firms' indifference curve must be tangent to the same 'budget line' given by the relationship $\hat{k}(w)$ that they think relates wages and congestion. This is not possible unless the indifference curves are tangent. This is essentially the result in Moen (1994).

To close the model, we still need to describe exactly which indifference curve we choose as the 'budget line'. This argument is quite important in the analysis of minimum wages. Let m be the measure of the set of workers who are searching for jobs. Let n_H and n_L

[3]The proportion of all the workers in the market who fill a vacancy is $q(k)$. Since there are k times as many workers in a market as there are available vacancies, the proportion of vacancies that are filled is $kq(k)$.

denote the corresponding measure of high and low profit sellers respectively. For each indifference curve WW that we choose to represent this budget line, the mass n_H of high profit firms will choose the point on WW where it cuts the locus AA of points at which worker and high profit firm indifference curves are tangent. The mass n_L of low profit sellers will pick the point where WW cuts BB, the corresponding locus where worker indifference curves are tangent to the indifference curves for low profit sellers. If we allocate workers to these points in such a way that the rational expectation condition holds, then WW must be chosen in such a way that $n_H k_H + n_L k_L = m$. This condition is essentially a *market clearing condition* in the market for contracts.

The total number of jobs that are filled during the period is $n_H k_H q(k_H) + n_L k_L q(k_L)$ with $n_L k_L q(k_L)$ of these jobs being at the lower wage rate.

The impact of a minimum wage set equal to $w_{\min}$ is analyzed in Figure 2. The new equilibrium in the figure is indicated by the heavier lines. The lighter lines depict the equilibrium outcome before imposition of the minimum wage for comparison. Workers and firms can no longer expect to fill vacancies at wages below $w_{\min} > w_L$. The low profit firms are then forced up to an inefficient outcome like $(w_{\min}, k_{\min})$ in the figure. The high profit firms end up paying higher wages equal to w'_H. If there had been a third type of firm paying wages below $w_{\min}$, this firm also would be forced up to $w_{\min}$. In this sense the lower bound on the set of feasible wages allows different types of firms to cluster in a single market. This clustering explains the phenomena often observed in minimum wage studies that there is a spike in the distribution of wages at the minimum wage Baker et al. (1994).

The key question is what happens to employment. It is not hard to see that the minimum wage must make workers better off in this model. If the indifference curve for workers along which the new equilibrium lies is below the equilibrium indifference curve for the case in which there are no minimum wages, then the equilibrium worker vacancy ratio in both markets would have to rise. This is incompatible with the market clearing condition described above.

The interesting implication of this rise in workers' utility is that the worker vacancy ratio ends up being higher at the minimum wage than it was in the old low wage market, while the corresponding ratio in the high wage market falls. As a consequence, more jobs will be filled at the minimum wage, while fewer will be filled at high wages. The minimum wage has a strong distributional effect. The effect on overall employment is ambiguous and depends on the curvature of the function q. If the marginal impact of changes is k is declining, then it seems plausible that the redistribution would increase the total number of vacancies that are filled, or raise employment.

The intuition for the result is also apparent. The imposition of the minimum wage

raises the rents that are associated with the low wage job (workers are indifferent between the low and high wage markets in equilibrium). This increased rent attracts buyers away from the higher wage jobs and induces them to concentrate their search efforts finding jobs at the minimum wage. Vacancies are filled more quickly at the minimum wage, and more slowly at higher wages.

As is apparent from this diagram, the Walrasian equilibrium for the contract market is efficient in the usual sense when there are no minimum wages, and it is inefficient after the minimum wage is imposed.

There are a number of important observations to be made at this point. The efficiency and uniqueness of equilibrium in the examples given in Figure 1 and 2 depend critically on the assumption that buyers and sellers have the same expectations about the relationship between the wage and the worker vacancy rate. It is easy to see how to support alternative outcomes if they do not. Simply draw different 'budget lines' for each type of trader to support any desired outcome. It is not at all clear what mechanism is used to ensure the coincidence of these beliefs. However rationalizing this assumption is one of the central problems addressed in this paper.

Secondly, there are many 'budget lines' that will support the equilibrium outcomes. On all of them traders have irrational beliefs about the worker vacancy ratio that prevails off the equilibrium path. Indeed, as the relationship is drawn, both workers and firms believe that they will be able to find potential trading partners in other markets, despite that fact that they should realize that no one has any incentive to offer contracts in these alternative markets given the common belief. In the usual competitive story, this is not much of an issue because traders are really quite badly informed about tastes of traders on the other side of the market. This is often argued to be one of the strengths of competitive analysis in that traders need only know prices in all markets to make the 'right' decision. They do not need to know much about the tastes of their competitors[4].

In contractual situations this assumption is not always so plausible. Firms may know quite a bit about workers marginal disutility for effort, for example. Such considerations make this irrationality of beliefs less convincing.

Before turning to the results on these issues, a more complete definition of the Walrasian equilibrium is in order.

[4]This is one of the problems with applying refinement criteria to restrict beliefs in problems with asymmetric information. The criteria can be in conflict with the orderly markets assumption.

3 The Economy

The economy consists of an ocean of buyers and sellers. Sellers are characterized by their type $s \in S$ where S is an interval $[s_0, s_1]$ in $\mathbf{R}^+$, while buyers are characterized by their type $b \in B \subset \mathbf{R}^+$ where again, B is assumed to be an interval $[b_0, b_1]$. Let the distribution functions G and F characterize the distributions of seller and buyer types respectively. It will be assumed that each of these functions can be written as the sum of a jump function with a *finite* support, and an absolutely continuous function[5]. The measures of S and B are assumed to be N and M respectively. We reserve the notation $\bar{k} = M/N$ to denote the ratio of the measure of buyers to the measure of sellers in the entire economy.

The notation $G(s)$ is the measure of the set of sellers whose types are less than or equal to s for each $s \in S$. If Σ is a subset of S, we will occasionally use the notation $G(\Sigma) = \int K_\Sigma(s) dG(s)$ to denote[6] the measure of the set of sellers whose types belong to the set Σ.

The names buyer and seller are arbitrary in this context. The important assumption is that a mutually beneficial exchange requires exactly one member from each group in the sense that a pair of buyers cannot profitably trade with one another. A *contract* between a buyer and seller is a vector $x \in \mathbf{R}^l$ for $l > 0$. We will assume that the set of feasible contracts, Ξ, is closed and bounded in $\mathbf{R}^l$, and that it contains points x^0 and x^1 such that $x^0 \leq x \forall x \in \Xi$, while $x^1 \geq x \forall x \in \Xi$.

Buyers and sellers maximize expected utility. Payoffs for a buyer or seller respectively who finds a trader on the other side of the market willing to exchange the contract x are given by $u(x, b)$ and $v(x, s)$ where b and s are the buyer's and seller's types. The functions u and v are assumed to be jointly continuous, and strictly quasi-convex. The payoffs for buyers or sellers who do not succeed in finding a trading partner are given by $\bar{u}$ for buyers and $\bar{v}$ for sellers.

In the Walrasian theory that is presented below, the decision problem that traders face is similar to the problem they face in a standard Diamond-Mortensen-Pissarides search problem. They will search in an attempt to find a trading partner. The success of this search process will depend on the number buyers and sellers who are searching. The difference is that, as in (Hosios 1990), buyers and sellers can choose between a continuum of search opportunities represented by Ξ.

To see how the standard search technology can be generalized, consider any (Lebesque)

[5] Any distribution function can always be decomposed into a jump function with a countable support and a *continuous* function. So the key assumptions here are finiteness of the support and absolute continuity.

[6] Here $K_{\bar{S}}(x)$ is the indicator function for the set S.

measurable subset $\bar{X}$ of the set of feasible contracts Ξ. Each such subset will be identified here as a *search opportunity* for buyers. Let m and n denote the measures of the sets of buyers and sellers respectively who choose to take advantage of this opportunity and search for partners who are willing to exchange contracts within the set $\bar{X}$. As in the usual matching problem, it will be assumed that the probability that buyers who are searching for a contract in the set $\bar{X}$ will find a seller willing to exchange it with them, will depend on the ratio of the measures of the sets of buyers and sellers who are also searching for trading opportunities within $\bar{X}$.

Let $q : \mathbf{R} \to [0, 1]$ denote this relationship. It is customary to assume that q satisfies

1. q is continuous and decreasing with $q(0) = 1$; $q(\infty) = 0$;

2. $q(k)k$ is increasing in k with limits 1 and 0 respectively as k goes to infinity and 0.

The number $q(k)$ denotes the probability that a buyer locates a trading partner within any measurable set of markets, when the ratio of the measure of the set of buyers assigned to those markets to the measure of the set of sellers assigned to those markets is k. Since m is the measure of buyers assigned to markets in $\bar{X}$, the total number of trades within $\bar{X}$ will then be $q(k)m$. This means that the probability that a seller searching in $\bar{X}$ trades is equal to $q(k)\frac{m}{n}$ or $q(k)k$, which explains the restrictions on this function.

Conceptually it is also useful to interpret this process in a temporal sense. Then $1/q(k)$ is the mean time that a buyer must search for a trader partner in a set of markets in which k is the ratio of the measure of searching buyers to the measure of searching sellers. There is no particular difficulty associated with making the model presented here dynamic. One simply assumes that value functions are independent of any current actions and adds a step to solve for these value functions. The equilibria presented below might be considered as steady states in a dynamic model. Since this paper is concerned more with the relationship between models than it is with the properties of dynamic paths, it seems worthwhile to simplify the notation by restricting attention to the static version.

Traders can now rank the different trading opportunities. A buyer's expected payoff in market x (i.e., the payoff to searching in the location where x is traded) is given by

$$U(x, b, k) = q(k)\, u(x, b) + [1 - q(k)]\, \bar{u}$$

while a seller's expected payoff in the same location is

$$V(x, s, k) = q(k)\, kv(x, s) + [1 - q(k)\, k]\, \bar{v}$$

Though the description of Walrasian equilibrium given in this paper applies to a broad class of matching technologies, in this paper interest will be focussed on two particular technologies. The *simple rationing rule* is the one that is implicit in market clearing models. In a market clearing model the traders on the short side of the market always trade, while every trader on the long side of the market is equally likely to trade. This means that $q(k) = 1$ if $k < 1$ while $q(k) = 1/k$ otherwise. This technology will be referred to henceforth as the *classical matching technology.*

An alternative technology is one that is derived from the decentralized model that is to be described below. This technology is given by

$$q(k) = \frac{1 - e^{-k}}{k}$$

This technology will be referred to as the *exponential matching technology.*

3.1 Allocations

Let Ψ^s and Ψ^b be distribution functions whose domains are $S \times \Xi$ and $B \times \Xi$ respectively. These distributions are called *allocations* if their corresponding marginal distributions on S and B coincide with G and F respectively, and if the marginal distributions σ^s and σ^b on the space of contracts Ξ are both integrable functions that can be written as the sum of a jump function with a finite support, and a function that is absolutely continuous with respect to Lebesque measure. The jump function deals with contracts that are allocated a non zero measure of traders, like the null contract. The function also makes it possible to deal with situations in which there are a finite number of types of buyers and sellers, an assumption that is used in examples, and throughout most of the literature.

The measure of the sets of buyers and sellers who are assigned to particular contracts will play an important role in what follows. Let X_j^{1i} denote the finite set of points at which the distribution σ^i has a 'jump'[7]. Let $\sigma_j^i(x)$ denote the *jumpsize* at x (note that $\sigma_j^i(x) = 0$ if $x \notin X_j^{1i}$). The distributions σ^i for $i = s, b$ can be written as

$$\sigma^i(x) = \sum_{y \in X_j^{1i}; y \leq x} \sigma_j^i(y) + \int_{x^0}^{x} \tilde{\mu}^i(y)\, dy$$

for some integrable functions $\tilde{\mu}^i$. Let X_j^1 denote the (finite) union of the supports of σ_j^s and σ_j^b. The *generalized derivative of* σ^i at x is defined to be

$$\mu^i(x) \equiv \begin{cases} \sigma_j^i(x) & if\ x \in X_j^1 \\ \tilde{\mu}^i(x) & otherwise \end{cases}$$

[7]Formally, let Σ^n be a declining sequence of measurable sets in Ξ whose limit superior is equal to x. Then the jumpsize of the distribution σ^i at x is defined to be the supremum over all such sequences of $\lim_{n\to\infty} \int K_{\Sigma^n} d\sigma^i$.

If there is a jump in the distribution σ^i at x, then we will interpret the jumpsize as the measure of the set of sellers assigned to x. If there is no jump in the distribution, then it is more reasonable to use the density, represented by the Radon Nikodym derivative, as the assignment.

As a slight abuse of notation we will refer to the marginal distributions σ^s and σ^b as allocations and take it for granted that these are derived from a joint distribution whose marginal on the space of buyers and sellers is F and G respectively.

4 Walrasian theory

The terms *market* and *contract* are basically synonymous in the Walrasian theory that is studied in this paper. Since contracts will typically include a specification of the trading price, markets are not 'cleared' in the usual sense of the word, though market clearing in the usual form may be one of the implications of equilibrium. Rather, equilibrium adjustments occur as buyers and seller move into or out of specific contract markets. Specifically, adjustments occur in the ratio k of the measure of buyers to sellers who are searching for trading partners within a given contract market.

In each particular contract market x, traders conjecture a value for k, say $k(x)$. This conjecture determines how attractive the contract market is to traders since it determines their probability of finding a partner there. As k increases, there are more buyers per seller, making it harder for buyers to find trading partners within the market.

Given their beliefs about the way that this level of congestion k varies across the various contract markets, traders allocate themselves optimally to the different markets. These allocations correspond to densities of say, $\mu^s(x)$ sellers and $\mu^b(x)$ buyers to market x.

In equilibrium traders' conjectures about k should be correct. This leads to the condition $\frac{\mu^b(x)}{\mu^s(x)} = k(x)$, or, in a more convenient form, to the *generalized market clearing condition* $\mu^b(x) = k(x)\mu^s(x)$. This condition reduces to the usual demand equal to supply condition in market clearing models when the simple rationing rule is in effect.

The power of the contract market approach comes from two key assumptions. First, traders must all share the same beliefs about the amount of congestion in each market. Secondly, traders must be able to formulate a well defined expectation for every possible market. These are basic competitive assumptions. The first is analogous to the assumption that all traders agree on the price at which specific commodities can be exchanged. The second assumption is analogous to the complete markets assumption in standard competitive theory.

Notice that the assumption that there is a well defined k in every market means that

the conjecture by buyers that there is no point going to a market because there will not be any sellers there is ruled out. Buyers believe that there are sellers in every market. If the chance of completing a trade in a market is small, it is because there are too many buyers in the market, not too few sellers. Gale (1994) calls this property *orderliness* and attributes it to Hahn (1978).

Now given the relationship $k(x)$, define the sets

$$S \supseteq \phi^s(x,k) = \{s' \in S : \exists x' \leq x; V(x', s', k(x')) \geq V(y, s', k(y)) \forall y \in X\}$$

and

$$B \supseteq \phi^b(x,k) = \{b' \in B : \exists x' \leq x; U(x', b', k(x')) \geq U(y, b', k(y)) \forall y \in X\}$$

Here $\phi^s(x,k)$ is just the set of seller types who can find a contract below x that is at least as good as any other available contract given the relationship that they believe prevails between x and k[8].

An allocation $\sigma = \left\{\sigma^s, \sigma^b\right\}$ and a trading rule $k(x)$ constitute a *Walrasian equilibrium for the contract market* if

- (σ^s is an aggregate supply function) $\sigma^s(\Xi) = N$ and for almost all $x \in \Xi$,
$$\sigma^s(x) \leq \int K_{\phi^s(x,k)} dG[s];$$
- (σ^b is an aggregate demand function) $\sigma^b(\Xi) = M$ and for almost all $x \in \Xi$,
$$\sigma^b(x) \leq \int K_{\phi^b(x,k)} dF[b];$$
- (generalized market clearing condition) for almost all x in the union of the supports of the distributions σ^s and σ^b,
$$\mu^b(x) = \mu^s(x) k(x).$$

The first condition requires, along with the obvious adding up requirement, that the measure of the set of sellers allocated to contracts below x, is no larger than the measure of the set of sellers whose optimal choice of contracts is below x given beliefs k. The second condition is the same thing for buyers. These are both analogous to the requirement that the aggregate demand and supply functions be the sum of individual demand and supply functions in the usual Walrasian story. The third condition is the generalized market clearing condition described above. It requires that buyers' and sellers' beliefs about the trading probability in each market are correct in each market which is assigned some buyers or sellers[9].

[8]For the moment, we will ignore the problem of whether or not these sets are measurable. The assumptions below will guarantee that they are intervals which will resolve this problem.

[9](Gale 1994) refers to this condition as *fullfilled expectations in active markets.*

4.1 An Existence Theorem

Unfortunately a competitive equilibrium for the contract market need not always exist. Before showing how problems can arise, we present what is probably the strongest kind of existence theorem that is available for this environment. Essentially without additional restrictions, we can only ensure existence when all traders on one of the two sides of the market have identical tastes[10]. Note once again, that the strong assumption is that all the traders on one side of the market have identical tastes. The various other regularity assumptions like continuity of F and G can be relaxed.

Assumption 1. Buyers all have identical tastes, i.e., $\exists b^* : F(b) = 0 \forall b < b^*; F(b) = 1$ *otherwise*.

2. The function $V(x, s, k) = kq(k)v(x, s) + (1 - kq(k))\bar{v}$ is strictly quasi-convex in (x, k) and satisfies the single crossing property: for any $(x', k') \neq (x, k); k' \geq k$, $s' > s$ then $V(x', s', k') \geq V(x, s', k) \Rightarrow V(x', s, k') > V(x, s, k)$.

3. The function $U(x, b^*, k) = q(k)u(x, b^*) + (1 - q(k))\bar{u}$ is strictly quasi convex in (x, k).

If the space of contracts were single dimensional, like the set of possible trading prices, then this assumption makes the indifference curves for buyers and sellers in (p, k) space convex downward and upward respectively. Furthermore, an increase in s would make the indifference curve flatter. Note that these assumptions involve a joint restriction on preferences and the matching technology. These restrictions are satisfied with the functional forms provided with the pricing example given above.

Theorem 1 *Under Assumptions 1, 3 and 2, there exists a Walrasian equilibrium for the contractual market.*

The proof of this and the other theorems in the paper is given in the appendix. In outline the proof is straightforward. It exploits the fact that all buyers have the same preferences. Pick u_0 and let $k(x) = \{x : U(x, b^*, k) = u^0\}$ if such a solution exists and $k(x) = 0$ otherwise. Now let sellers allocate themselves on the surface $\{x, k(x)\}$ as they desire. If buyers believe that the relationship between contracts and matching probabilities is given by $k(x)$ then they will not care which contract they choose. Thus it makes sense to allocate buyers across the various markets until the ratio of the density of buyers allocated to market x to the density of sellers who choose market x is just equal

[10] This is sometimes a convincing assumption. In the labour market, for example, free entry of firms should give them access to the same technology, so that it would not be reasonable to assume that the preferences of all firms were identical.

to $k(x)$. Of course this might not be feasible, so we manipulate u_0 until the integral of all the buyers allocated across all the markets is equal to the measure of the set of buyers. This integral turns out to be a continuous function of u_0 under our assumptions which is sufficient to make it possible to prove existence.

When workers can also have different types, existence becomes problematic. Since all traders in the market are supposed to perceive the same 'budget line' in equilibrium, all of the workers who search within the same market must have the same type. If there are two different types of workers in the same active market, both of their indifference curves must be tangent to this common budget line. The budget line in turn must be tangent to the indifference curve for some seller. The single crossing property then implies that one of the workers cannot be choosing a search market optimally.

The tangency of workers and firms indifference curves, along with the fact that the ratio of the measure of workers of a given type to the measure of firms of a given type is given exogenously, completely determines the array of markets that can be open in equilibrium. However the array of open markets must satisfy the additional restriction of incentive compatibility, i.e., each trader must prefer the market to which he is assigned to any alternative market. Unless there is some way to manipulate the worker firm ratio in different markets, it may be impossible to satisfy all of these conditions simultaneously. In the cases where the conditions cannot be satisfied, equilibrium fails to exist.

It is possible to get around the difficulties created in this situation by adding a *free entry* condition. For example suppose that workers will enter this contractual market as long as their expected payoff from participating is better than it is at their best alternative. Take the envelope of buyer indifference curves that generate this alternative payoff for each type of buyer. This envelope generates a well defined feasible set. Sellers can locate themselves along it as they wish and the resulting outcome will be an equilibrium. We will not pursue this formally here as the details are straightforward[11].

5 A Decentralized Model of Contractual Markets

As may be apparent from the preceding discussion, there are two restrictions on beliefs that do most of the work in contract market equilibrium. The first restriction is that all traders agree about the value for k that is expected to prevail in inactive markets. Perhaps more important, traders don't really believe in inactive markets at all. They persist in expecting to trade with positive probability in markets which provide no incentive for traders to visit. Apart from the fact that these restrictions are analogous to standard competitive restrictions, and give the model nice properties, it is not obvious

[11] In fact this free entry condition is critical in Gale's work,

why they should be true. The purpose of this section is to show the degree to which these assumptions make sense.

A class of games is described in which sellers have varying preferences while buyers are all identical[12]. The games in the class differ according to the number of buyers and sellers who play. In each of the games in the class, the distribution of preferences is approximately equal to the distribution of preferences in the continuum for which we have described a Walrasian Equilibrium.

In each of these games, sellers simultaneously and publicly offer buyers contracts in the first stage of the game. In the second stage, buyers simultaneously choose one and only one of the contracts on offer. Since contractual arrangements must be bilateral, there is a chance that many buyers will choose the same contract offer. If this happens, the seller selects one of the buyers at random as a trading partner. Buyers realize that the ex ante value of any contract offer depends on the number of other buyers who select it.

In a game like this, there is no such thing as active or inactive markets. However, in any equilibrium, sellers can consider deviating from the offer they are supposed to make. If they do, they will be able to calculate the expected number of buyers who will respond to such an offer by using their information about equilibrium strategies. Of course, buyers should also understand exactly how many other buyers will respond to such a deviation, again from their knowledge of equilibrium strategies.

Within this class of games, it is possible to consider the version of the game with a infinite number of buyers and sellers by taking limits in an appropriate way of all of the payoff functions in the finite game. Notice that there is a one to one correspondence between the set of feasible offers that the sellers can make in the games under consideration, and the set of markets from which buyers and sellers choose in the Walrasian equilibrium. We show that in the equilibrium of the infinite version of the game, every contract that is offered gives buyers and sellers the same expected payoff that they get by choosing the corresponding contract market in an appropriate Walrasian equilibrium.

Furthermore, for each possible deviation from the equilibrium contract array that might be offered by sellers, the payoff that sellers expect to get when they make this deviation, and the payoff that buyers expect to get when they choose the deviator are both equal to *one* of the expected payoffs that can be used to support the Walrasian equilibrium for the contract market[13].

[12]This restriction is designed to get around the difficulties associated with proving existence. These were discussed in the last section.

[13]Recall that they may be many beliefs about payoffs in inactive markets that can be used to support a particular Walrasian equilibrium for the contract market.

For the purposes of this section, we will assume that the matching technology is the classical one in which the long side of the market is rationed at random, i.e.,

$$q(k) = \begin{cases} 1 \; if \; k \leq 1 \\ \frac{1}{k} \; otherwise \end{cases}$$

Finally before describing the offer game form, it is necessary to describe sellers' preferences in each of the games in the class. As the distribution of preferences in each of the games under consideration is supposed to *approximate* the distribution of preferences in the Walrasian version of the contract market, the following procedure is employed. First index the games by the number J of participating sellers. The number of buyers will be assumed to be $\bar{k}J$ where $\bar{k}$ is the ratio of the measure of the set of buyers to the measure of the set of sellers in the entire economy.

Second, let $s^j \equiv \{s : G(s) = nj/J\}$ for $j = 1, \ldots, J$. For each of the J sellers in the game, it is assumed that j's preferences are given by $\tilde{V}_j(x, k) \equiv V(x, k, s^j)$. This assignment of preferences approximates the preferences in the distribution G in the sense that the distribution function G_J of preferences that applies in the finite version of the game consisting of J sellers and kJ buyers, converges point-wise, and hence weakly to the distribution G. Notice that G does not have to be an absolutely continuous distribution function in this formulation. In the interesting case in which all sellers are identical, the distribution G is degenerate at some point. The finite approximation in that case simply involves a finite game with J identical sellers.

The game form that governs the interaction between buyers and sellers may now be described. The sellers begin the procedure by simultaneously and publicly offering contracts to buyers. Contract offers must come from the space Ξ which describes feasible markets in the Walrasian version of the model. The buyers, after seeing the contracts offered by each of the sellers, simultaneously choose one and only one of the sellers as a potential trading partner. If many buyers choose the same seller, then the seller selects one of the buyers at random and executes the contract x that he offered with that buyer. The other buyers whom were not selected by the seller are out of luck, and are assumed to end the procedure without trading.

5.1 Equilibrium in the Decentralized Model

Let $\tilde{x} = \{x', x_{-j}\}$ denote the vector of contract offers made by each of the sellers, where x' denotes the offer by the seller indexed j, and x_{-j} denotes the vector of offers by each of the other sellers. Attention in this section will be (somewhat arbitrarily) restricted to equilibria in which all buyers behave symmetrically. So let $\pi_J^j(\tilde{x})$ denote the (common) probability with which *each* of the buyers selects seller j. Of course, since this choice is

made after the buyers observe the vector of seller offers, the choice should be conditioned on x. Let $\pi_J(\cdot)$ denote the vector of such probability functions, assumed to satisfy $\sum_{i=1}^{J} \pi_J^i(\tilde{x}) = 1$ for all $\tilde{x} \in \Xi$.

If many buyers select a particular seller j, then only one of them is chosen to trade with seller j.[14] The probability with which a buyer trades with seller j is readily shown to be given by

$$\sum_{l=0}^{kJ-1} \binom{kJ-1}{l} \left(\pi_J^j(\tilde{x})\right)^l \left(1-\pi_J^j(\tilde{x})\right)^{kJ-1-l} q(l+1)$$

$$= \frac{1}{kJ\pi_J^j(\tilde{x})}\left[1-\left(1-\pi_J^j(\tilde{x})\right)^{kJ}\right] \equiv \tilde{q}_J^j(\tilde{x})$$

since $q(l+1) = 1/(l+1)$ from the properties of the classical matching technology.

The probability with which the seller trades is just

$$\left[1-\left(1-\pi_J^j(\tilde{x})\right)^{kJ}\right] \equiv \tilde{r}_J^j(\tilde{x})$$

The matching technology is said to be *incentive consistent* if for all $\tilde{x}$, $\pi_J^j(\tilde{x}) > 0$ implies that

$$\sum_{j=1}^{J} \pi_J^j \left\{\tilde{q}_J^j(\tilde{x})u(x_j) + (1-\tilde{q}_J^j(\tilde{x}))\bar{u}\right\} \geq \tilde{q}_J^i(\tilde{x})u(x_i) + (1-\tilde{q}_J^i(\tilde{x}))\bar{u} \tag{1}$$

for all $i = 1, \ldots, J$. We will also say that a vector of choice probabilities $\left\{\pi_J^j\right\}$ is incentive consistent at x if 1 holds for $\tilde{x} = x$, and

$$\frac{1}{kJ\pi_J^j}\left[1-\left(1-\pi_J^j\right)^{kJ}\right] \equiv \tilde{q}_J^j(x)$$

for each $j = 1, \ldots, J$.

Notice that a matching technology is incentive consistent if the assignment probabilities that it specifies constitute a continuation equilibrium for the subgame that is played when buyers simultaneously choose among the sellers.

The pair $\{x^*, \pi_J^*\}$ is an *equilibrium* if π_J^* is incentive consistent if for each j, and each possible deviation x_j,

$$\tilde{r}_J^j(x_j^*, x_{-j}^*)v_j(x_j^*) + (1-\tilde{r}_J^j(x_j^*, x_{-j}^*))\bar{v} \geq$$

$$\tilde{r}_J^j(x_j, x_{-j}^*)v_j(x_j) + (1-\tilde{r}_J^j(x_j, x_{-j}^*))\bar{v}$$

This much is straightforward. The purpose of this section is to try to extend these definitions to the limit where there are an infinite number of buyers and sellers. The

[14] The assumption that each of the buyers is equally likely to be selected by seller j might be motivated by the assumption that the buyers arrive randomly at different times and the seller simply trades with the first buyer who arrives.

limit is of interest because the strategic considerations that normally make Walrasian reasoning inappropriate, disappear when there are an infinite number of buyers and sellers. In principle, it is easy enough to see how to do this. Simply take limits of each of the functions presented so far and require that the corresponding equilibrium conditions hold in the limit. The problem with this approach is that it is not clear how the array of offers should change as J increases. We turn next to this problem.

6 Analysis of the Equilibrium of the Decentralized Model

The convexity assumptions that we have made ensure that the relation

$$x(\tilde{u}, s) \equiv \arg\max_{x} \{v(x, s) : u(x) = \tilde{u}\}$$

is a continuous function. This means that once we decide what utility payoff a seller wishes to offer to buyers to attract them, we can use the function $x(\tilde{u}, s)$ to work back to the contract. It is then possible to think of the seller offering a level of utility instead of a contract.

To begin we show a simple result that relates competition in contracts to competition in levels of utility. Let $u_0 = \min_{x \in X} u(x)$, while $u_1 = \max_{x \in X} u(x)$.

Lemma 2 *Suppose that buyer and seller preferences satisfy assumptions 2 and 3 on the space $X \times \mathbf{R}$. Then these preferences satisfy the same assumptions on the space $[u_0, u_1] \times \mathbf{R}$.*

6.1 The Infinite Game

The goal of this section is to show that one can think about the infinite game by imagining a seller making a utility offer against a fixed distribution of utility offers H made by the other sellers. This distribution H will generate an expected payoff for buyers that is subsequently independent of the utility or contract offer that is made by any single deviating seller. To motivate this interpretation a finite approximation of the limit game is described in which the distribution of utility offers made by all but one of the sellers, is approximately equal to the limit distribution H that is of interest. The one additional seller is a deviator who makes an arbitrary utility offer. The payoff to the deviator from making this arbitrary offer in the limit game is then given by the limit of his payoffs from this arbitrary offer as the sequence of finite approximations converges to H.

To approximate the distribution H, we begin by finding each of the J tiles of the distribution H. We then locate one seller at each of these J tiles. That is, we have one

seller offer a contract that is better in utility terms than the fraction $1/J$ of all offers in the distribution H, one seller offers a contract that is better than $2/J$ of the offers in H, and so on. Then for any deviation u', we remove the seller from the largest tile that does not exceed u' and replace him with the deviator offering u'[15].

The payoff that this deviator gets can then be calculated for each of the finite approximations to H that are indexed by J. This sequence of payoffs to the deviator has a limit, that defines the payoff for the deviating seller in the limit game.

Since this procedure yields a payoff for each possible utility offer when it is made against a distribution H of offers by the other sellers, it is possible to calculate a distribution of best replies to H. The fixed point of this mapping from distributions into distributions will constitute an equilibrium for the limit game.

To be specific, for any H and any deviation u', let $\tilde{u}_j(H,J) = \{u : H(u) = jN/J\}$ (recall that N is the measure of the set of sellers) for $j = 1, \ldots, J$. Let i be the largest index for which $\tilde{u}_j(H,J) \leq u'$. Let $\tilde{u}(H,J)$ denote the array of offers

$$\tilde{u}(H,J) \equiv \{\tilde{u}_1(H,J), \ldots \tilde{u}_{i-1}(H,J), u', \ldots, \tilde{u}_J(H,J)\}$$

The empirical distribution function that this implies is given by

$$H_J(u,u') = \frac{\#\{\tilde{u} \in \{\tilde{u}_1(H,J), \ldots \tilde{u}_{i-1}(H,J), u', \ldots, \tilde{u}_J(H,J)\}; \tilde{u} \leq u\}}{J} N$$

Notice that this distribution is a simple function that converges point-wise, and hence weakly to the distribution H as $J \to \infty$ no matter what u' is. Our first result is the following

Theorem 3 *For any distribution H of utilities offered by sellers, if the matching technology is incentive consistent, then there is a utility level $u^*(H)$ for buyers such that for any contract offer x', any j and any s, the limiting payoff to seller j is given by*

$$\begin{aligned} &\lim_{J\to\infty} \left\{ \left(1 - \left(1 - \pi_J^j(x', \tilde{x}_{-j}(H,J))\right)^{kJ}\right) v(x',s) + \right. \\ &\left. \left[1 - \left(1 - \left(1 - \pi_J^j(x', \tilde{x}_{-j}(H,J))\right)^{kJ}\right)\right] \bar{v} \right\} \\ &= \left\{ \begin{array}{c} \bar{v} \ if \ u(x') < u^*(H) \\ \left(1 - e^{-k'}\right) v(x',s) + e^{-k'} \bar{v} \ otherwise \end{array} \right\} \end{aligned}$$

where k' is the unique solution to the equation

$$\frac{1 - e^{-k}}{k} u(x') + \left[1 - \frac{1 - e^{-k}}{k}\right] \bar{u} = u^*(H) \tag{2}$$

[15] This awkward procedure ensures that the sequence of approximating distributions to H approaches it from below in an appropriate sense.

Note the resemblance to the result in the section on the Walrasian theory of markets. As the 'deviating' seller considers all of the possible contract offers that he might propose at the beginning of the game, he is in the same position as the seller who tries to select a market in the Walrasian version of the model. As he considers all these possibilities he calculates a trade-off between the contract that he offers and the parameter k that determines the probability with which he will trade. This relationship behaves much like the anticipated trade-off in the pure Walrasian version of the model, especially in regard to the fact that the implied 'budget line' corresponds to an indifference curve for one of the buyers.

Note further that the theorem applies to any contract offer that the seller considers making. Since we have not yet calculated the equilibrium distribution H, there is no sense in which this payoff applies only in active markets.

With the help of the key relation $H \to u^*(H)$ described by this theorem, it is now possible to give a complete description of equilibrium in the limit game. To keep down the notation, which is already dense enough, we will not give the most general definition. Define the relations

$$\hat{u}(s,u) : s \to \underset{u'}{\operatorname{argmax}} \left\{ \left[1 - e^{-k}\right] v(x(u',s),s) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}[u' - \bar{u}] \geq u \right\}$$

and

$$\hat{x}(s,u) : s \to \underset{x}{\operatorname{argmax}} \left\{ \left[1 - e^{-k}\right] v(x,s) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}[u(x) - \bar{u}] \geq u \right\}$$

It is readily shown that these are continuous functions. Furthermore, from the single crossing property of seller preferences, the function $\hat{x}$ has a continuous inverse.

Since $\hat{u}$ and $\hat{x}$ are continuous, they are measurable, so for each sub-interval $[u_0, u']$, the set $\{s : \hat{u}(s, u^*(H)) \in [u_0, u']\}$ is measurable. Then we have the following definition

Definition 1 *The distribution H of utilities is an* equilibrium for the limit game *if for all u', $H(u') = G(\{s : \hat{u}(s, u^*(H)) \in [u_0, u']\})$.*

The definition says that a distribution H is an equilibrium if the measure of the set of sellers who offer contracts yielding utility less than or equal to u' for buyers is equal to the measure of the set of sellers whose best reply to the distribution H is to offer a contract yielding buyers a level of utility no larger than u'.

The allocation that is supported by the equilibrium in the infinite game is just the induced allocation of buyers and sellers to contracts. Again, using the measurability of the function $\hat{x}(s, u^*(H^*))$, define for each $x \in \Xi$

$$\sigma^s(x) = G(\{s : \hat{x}(s, u^*(H^*)) \leq x\})$$

The continuity and invertability of $\hat{x}$, and the fact that G is a distribution function, guarantee that σ^s is continuous from above, and therefore is a distribution function. Define the transformation T from the subsets of Ξ to the subsets of $[u_0, u_1]$ as

$$\Sigma \to \hat{u}\left[\hat{x}^{-1}\left(\Sigma, u^*\left(H^*\right)\right), u^*\left(H^*\right)\right]$$

. The transformation T is measurable as the composition of two measurable transformations. Now define

$$\sigma^b(x) = \int_{x_0}^{x} k^* \cdot T d\sigma^s(y)$$

where $k^*(u)$ is equal to the solution to the equation

$$\bar{u} + \frac{1 - e^{-k}}{k}[u - \bar{u}] = u^*(H^*) \tag{3}$$

if a solution exists and $k^*(u) = 0$ otherwise. The distribution functions σ^s and σ^b are referred to as the *allocations supported by the equilibrium in the infinite game.*

Theorem 4 *Under the assumptions 1,2, and 3 used to prove Theorem 1, an allocation can be supported as an equilibrium allocation for the limit game in an economy with the classical matching technology if and only if it can be supported as a Walrasian allocation for an economy with the exponential matching technology.*

Note that there is no need to prove existence of a decentralized equilibrium. Simply apply Theorem 1 to the Walrasian economy with the exponential matching technology to verify the existence of a Walrasian equilibrium. Then use Theorem 4 to verify that the corresponding equilibrium allocation is also an equilibrium allocation for the decentralized problem.

The rationalization for the use of the Walrasian equilibrium concept in an economy with the exponential matching technology, then arises from the fact that the payoff that all traders expect to get when they go to inactive markets is equal to the payoff that they would actually get in the non-Walrasian equilibrium for the problem.

One restrictive feature of the result described above should be mentioned. The Walrasian equilibrium for the contract market can be defined for arbitrary matching technologies (though the existence of equilibrium may not be so easy unless restrictions are put on these technologies). The correspondence between Walrasian and non-Walrasian models only works with the classical and exponential matching technologies. Though it would seem plausible to expect this result to generalize, I do not have any formal result on this.

There is another generalization of the theorem that seems feasible. In each of the finite approximations to the limit economy, one might compute exact subgame perfect

equilibrium for the two stage process by which contracts are determined. If these exact equilibria exist for each approximation, then by the usual upper hemicontinuity of the Nash correspondence, the limit of these equilibria should also be an equilibrium for the limit game that is discussed in this paper. This would imply that at least one of the equilibria for the non-Walrasian version of the model discussed here would be close to some exact equilibrium for the large finite approximation. In turn, the theorem applied above then implies that at least one of the Walrasian equilibrium will be close to an exact equilibrium for a large finite approximation. It is not too hard to give conditions under which the Walrasian equilibrium is unique. In this sense the Walrasian equilibrium is an approximation to the non-Walrasian equilibrium for a large but finite economy.

The real difficulty with this argument is that I have not yet been able to show that exact equilibrium exist for the large finite approximations to the infinite economy. The game that is played in the non-Walrasian version of the model involves two stages. The continuation equilibrium for the second stage involves an incentive consistency condition which will be met by matching process that need not be continuous in the contract offers that are made in the first stage of the game. Continuity of the first stage payoffs is required to verify existence for all the well known fixed point arguments.

Finally, the theorem presented here has no immediate extension to problems involving adverse selection. If contract offers convey information about the types of the sellers offering them, there will be multiple equilibria in both the Walrasian and non-Walrasian versions of the model. Whether there will be a correspondence between these two sets of equilibria is an open question.

7 Appendix-Proofs of Theorems

For the proof of Theorem , we will make use of the relations

$$\hat{u}(s,u) : s \to \underset{u'}{\operatorname{argmax}} \left\{ \left[1 - e^{-k}\right] v(x(u',s),s) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}[u' - \bar{u}] \geq u \right\}$$

and

$$\hat{x}(s,u) : s \to \underset{x}{\operatorname{argmax}} \left\{ \left[1 - e^{-k}\right] v(x,s) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}[u(x) - \bar{u}] \geq u \right\}$$

described earlier in the paper. These relations are continuous functions and $\hat{x}$ has a continuous inverse function.

7.1 Proof of Theorem 1

Proof. The relation

$$\tilde{k}\left(x,u\right)=\begin{cases} k:U\left(x,k,b^{*}\right)=u\ if\ u\left(x,b^{*}\right)>u \\ 0\ otherwise \end{cases}$$

is readily shown to be a continuous function. In addition, this selection has the property that for any sequence $u^n \to u^0$, the function $\tilde{k}(\cdot, u^n)$ converges point-wise to the function $\tilde{k}(\cdot, u^0)$.

Now recall that the set

$$\phi^s(x,\tilde{k}\left(\cdot,u\right)) = \\ \left\{s' \in S : \exists x' \leq x; V(x',s',\tilde{k}(x,u)) \geq V(y,s',\tilde{k}(x,u)) \forall y \in X\right\}$$

Since $\tilde{k}$ is continuous, this set is measurable for each x. So define the distribution function

$$\tilde{G}(x,u) = \int K_{\phi^s(x,\tilde{k}(\cdot,u))}(s)dG(s)$$

Let u^* be a solution to the equation

$$M = \int \tilde{k}(x,u)d\tilde{G}(x,u) \tag{4}$$

Lemma 5 *The expectations function $\tilde{k}(x,u^*)$ along with the allocations*

$$\sigma^{*s}(x) = \tilde{G}\left(x,u^*\right)$$

and

$$\sigma^{*b}(x) = \int_{x_0}^{x} \tilde{k}\left(x',u^*\right) d\tilde{G}\left(x',u^*\right)$$

constitute a Walrasian equilibrium for the contract market.

Proof. First, to verify that σ^{*s} is an aggregate supply function, all we really need to show is that it is an allocation, in the sense that it can be written as the sum of jump function with a finite support and an absolutely continuous function. The other properties of the aggregate supply function are fulfilled by σ^{*s} by construction.

Now writing

$$\sigma^{*s}\left(x\right) = \tilde{G}\left(x,u^*\right) = \int K_{\phi^s\left(x,\tilde{k}(\cdot,u^*)\right)}\left(s\right) dG\left(s\right)$$

$$= \sum_{s\in S_1} K_{\phi^s\left(x,\tilde{k}(\cdot,u^*)\right)} G_j\left(s\right) + \int_S K_{\phi^s\left(x,\tilde{k}(\cdot,u^*)\right)} g\left(s\right) ds$$

$$= \sum_{x\in X_1} G_j\left(\hat{x}^{-1}\left(x\right)\right) + \int_{\Xi} g\left(\hat{x}^{-1}\left(x\right)\right) h\left(x\right) dx \tag{5}$$

where $X_1 = \{x : x = \hat{x}(s, u^*) \ some \ s \in S_1\} \cap \left\{x : \tilde{k}(x, u^*) > 0\right\}$ and h is some measurable function. The first line follows from the fact that G can be decomposed, while the second line follows from a standard change of variable theorem Burrill (1978).

By the construction of $\tilde{k}$, buyers are indifferent among all the contracts for which $\tilde{k}(x, u^*) > 0$. Hence for each x, $\phi^b(x, \tilde{k}(\cdot, u))$ is either empty, or equal to b^* (the common type for all the buyers). Since

$$\int_{x_0}^{x} \tilde{k}(x', u^*) \, d\tilde{G}(x', u^*)$$

is a non-decreasing function of x, and $M = \int_{x_0}^{x_1} \tilde{k}(x, u^*) d\tilde{G}(x, u^*)$ by the choice of u^*, the integral is always less than or equal to the measure of the set $\phi^b(x, \tilde{k}(\cdot, u))$.

Finally, to verify that σ^{*b} is an allocation, we have that

$$\sigma^{*b}(x) = \int_{x_0}^{x} \tilde{k}(x', u^*) \, d\tilde{G}(x', u^*)$$

$$= \sum_{X_1} \tilde{k}(x', u^*) \, G_j\left(x^{-1}(x)\right) + \int_{x_0}^{x} \tilde{k}(x', u^*) \, g\left(\hat{x}^{-1}(x)\right) h(x) \, dx \tag{6}$$

Comparing this last expression with (5) shows that the rational expectations condition must hold for almost all x. ■

This argument implies that an equilibrium will exist whenever the equation (4) has a solution. So we will now show that the expression $\int \tilde{k}(x, u) d\tilde{G}(x, u)$ is continuous in u, provided that $u > \bar{u}$.

Now consider the set $\Phi^n(k, x) = \left\{y \in X : \tilde{k}(y, u^n) \leq k\right\}$ and the distribution functions $F^n(k) = \int K_{\Phi^n(k,x)}(s) d\tilde{G}(s, u^n)$ and $F^*(k) = \int K_{\Phi^*(k,x)}(s) d\tilde{G}(s, u^*)$. The value $F^n(k)$ is the measure of the set of sellers who are allocated to contracts by the distribution $\tilde{G}$, for which the ratio of the measure of buyers to sellers allocated to the same market is less than or equal to k. Then changing variables in the integration, $\int \tilde{k}(s, u^n) d\tilde{G}(s, u^n) = \int k dF^n(k)$. The key point is that the integral on the left hand side of this equation converges as $u^n \to u$ if F^n converges weakly to F^*.[16]

To show this latter point, we show that $F^n \to F$ at each continuity point of F. To see this, note that

$$F^n(k) = \int K_{\left\{x : \tilde{k}(x, u^n) \leq k\right\}}(s) \, d\tilde{G}(s, u^n)$$

$$= \int K_{\left\{s : \hat{k}(s, u^n) \leq k)\right\}}(s') \, dG(s')$$

$$= 1 - G(\tilde{k}^{-1}(k, u^n))$$

The final equation follows from the single crossing property of seller preferences. Since $\tilde{k}$ is continuous, F^0 is continuous at k if and only if G is continuous at $\tilde{k}^{-1}(k, u^0)$.

[16](Hildenbrand 1974) p. 52

Now let k^0 be any point at which $G(\tilde{k}^{-1}(k^0, u^0))$ is continuous. Let s be any type of seller for whom $\hat{k}(s, u) < k^0$. The function $G(\tilde{k}^{-1}(k^0, u^0))$ represents the measure of the set of such sellers. Now for any such s, $\exists n^*$ large enough that $\tilde{k}(s, u^n) < k^0$ for all $n > n^*$. This follows from the joint continuity of $\hat{k}$ in s and u. Since $\tilde{k}$ is a decreasing function, this statement can be strengthened to read, for and $s : \tilde{k}(s, u) < k^0$, there is an n^* large enough that $\tilde{k}(s', u^n) < k^0$ for all $n > n^*$ and $s' > s$.

Now for any $\epsilon > 0$, $\exists s^\epsilon : \tilde{k}(s^\epsilon, u^0) < k^0$ and $F^0(k^0) \geq 1 - G(s^\epsilon) \geq F(k^0) - \epsilon$. By the reasoning of the last paragraph, there exists an n^* large enough that $F^n(k^0) \geq 1 - G(s^\epsilon)$ for all $n > n^*$. Since this is true for all positive ϵ, we have $\lim_{n\to\infty} F^n(k^0) \geq F(k^0)$. The reverse inequality is proved in exactly the same way.

This implies that the distribution function $F^n(\cdot)$ converges weakly to $F^*(\cdot)$ as $u^n \to u^*$. But then

$$\int \tilde{k}(s, u^n) d\tilde{G}(s, u^n) = \int k dF^n(k) \to \int k dF^*(k) = \int \tilde{k}(s, u^*) d\tilde{G}(s, u^*)$$

Finally, to prove the existence of equilibrium, note that when $u = \bar{u}$, the constraint set in the maximization problem used to define F^n consists of all contracts satisfying $u(x, b^*) = \bar{u}$, along with all possible values of k. Since sellers payoffs are increasing in k, we must have $\hat{k}(s, \bar{u}) = \infty$ for all $s \in S$.

On the other hand let $u > \max_x u(x, b^*) - \epsilon$. Then for all feasible x, there must exist a k such that

$$q(k)u(x, b^*) + (1 - q(k))\bar{u} > \max_x u(x, b^*) - \epsilon$$

which gives

$$q(k) > 1 - \frac{\epsilon}{\max_x u(x, b^*) - \bar{u}}$$

By the continuity of q, for each $\delta > 0$, there is an $\epsilon > 0$, such that if $u > \max_x u(x, b^*) - \epsilon$, then all feasible values for k will be less than δ.

The existence of a solution to

$$\int \tilde{k}(s, u^n) d\tilde{G}(s, u^n) = m$$

then follows from the mean value theorem. ■

7.2 Proof of Lemma 2

Proof. The convexity of the buyer's preferences is immediate. For the quasi-convexity of the seller's preferences, let (u_1, k_1) and (u_2, k_2) denote a pair of outcomes in $[u_0, u_1] \times \mathbf{R}$ to which a seller of type s is indifferent. Then

$$q(\lambda k_1 + (1 - \lambda) k_2) v(x(\lambda u_1 + (1 - \lambda) u_2, s), s) + [1 - q(\lambda k_1 + (1 - \lambda) k_2)] \bar{v}$$

$$\geq q\left(\lambda k_1+(1-\lambda)k_2\right)v\left(\lambda x\left(u_1,s\right)+(1-\lambda)x\left(u_2,s\right),s\right)+\left[1-q\left(\lambda k_1+(1-\lambda)k_2\right)\right]\bar{v}$$

$$\geq \min_{i=1,2}\left[q\left(k_i\right)v\left(x\left(u_i,s\right),s\right)+\left[1-q\left(k_i\right)\right]\bar{v}\right]$$

$$=q\left(k_1\right)v\left(x\left(u_1,s\right),s\right)+\left[1-q\left(k_1\right)\right]\bar{v}$$

The first inequality follows by the properties of maximization, the second by the quasi-convexity of the seller's preferences on $X\times\mathbf{R}$.

The single crossing property requires that if $(u',k')\neq(u,k);k'\geq k,s'\geq s$ then $\tilde{V}\left(x\left(u',s'\right),k',s'\right)\geq\tilde{V}\left(x\left(u,s'\right),k,s'\right)\Rightarrow\tilde{V}\left(x\left(u',s\right),k',s\right)>\tilde{V}\left(x\left(u,s\right),k,s\right)$. From the single crossing property which applies on the space of contracts, we know that

$$\tilde{V}\left(x\left(u',s'\right),k',s'\right)\geq\tilde{V}\left(x\left(u,s'\right),k,s'\right)\Rightarrow\tilde{V}\left(x\left(u',s'\right),k',s\right)>\tilde{V}\left(x\left(u,s'\right),k,s\right)$$

Then because $x\left(u,s\right)$ is the optimal contract offer for type s, we have

$$\tilde{V}\left(x\left(u',s'\right),k',s'\right)\geq\tilde{V}\left(x\left(u,s'\right),k,s'\right)\Rightarrow\tilde{V}\left(x\left(u',s'\right),k',s'\right)\geq\tilde{V}\left(x\left(u,s\right),k,s'\right)$$

Then by the single crossing property on the space of contracts, we have

$$\tilde{V}\left(x\left(u',s\right),k',s\right)\geq\tilde{V}\left(x\left(u',s'\right),k',s\right)>\tilde{V}\left(x\left(u,s\right),k,s\right)$$

■

7.3 Proof of Theorem 3

To study the properties of incentive consistent matching technologies, it will be useful to approach the problem in a slightly different way. It is straightforward to show that $\frac{1}{kJ\pi}\left[1-(1-\pi)^{kJ}\right](u-\bar{u})$ is a continuous and monotonically declining function of π whenever $u>\bar{u}$. The limit of the function as π goes to zero is $u-\bar{u}$, while its limit when π goes to one is $(u_j-\bar{u})/kJ$. Define the inverse function

$$g_J(u,u_j)\equiv\begin{cases}0\ if\ u>u_j\\ 1\ if\ u<u_j/kJ\\ \pi:\frac{1}{kJ\pi}\left[1-(1-\pi)^{kJ}\right](u_j-\bar{u})=u-\bar{u}\ otherwise\end{cases}\tag{7}$$

The inverse function describes the assignment probability which will yield buyers any given level of expected utility. This inverse function is continuous and monotonically decreasing on the interval $[u_j/kJ,u_j]$. Now define the function

$$G_J(u,u_1,\ldots u_J)=\sum_{j=1}^{J}g_J(u,u_j)$$

This function[17] is continuous, as the sum of continuous functions. Furthermore, it is monotonically decreasing on the interval

$$\left[\min_j \frac{u_j}{kJ}, \max_j u_j\right]$$

Lemma 6 *For any array $\tilde{u} = \{\tilde{u}_1, \ldots, \tilde{u}_J\}$ of utility offers by the sellers, the vector of choice strategies $[\pi_1, \ldots, \pi_J]$ is incentive consistent for $\tilde{u}$ if and only if there is a utility level u^* such that*

$$G_J(u^*, \tilde{u}) = \sum_{j=1}^{J} g_J(u^*, \tilde{u}_j) = 1$$

and

$$g_J(u^*, \tilde{u}_j) = \pi_j$$

for each j.

Proof. First suppose that $\pi = [\pi_1, \ldots, \pi_J]$ is incentive consistent for $\tilde{u}$. Then by standard properties of the symmetric mixed strategy equilibrium, it must be that for each positive π_j,

$$\frac{1}{kJ\pi_J^j}\left[1 - \left(1 - \pi_J^j\right)^{kJ}\right]\tilde{u}_j + \left\{1 - \frac{1}{kJ\pi_J^j}\left[1 - \left(1 - \pi_J^j\right)^{kJ}\right]\right\}\bar{u} \geq$$

$$\frac{1}{kJ\pi_J^i}\left[1 - \left(1 - \pi_J^i\right)^{kJ}\right]\tilde{u}_i + \left\{1 - \frac{1}{kJ\pi_J^i}\left[1 - \left(1 - \pi_J^i\right)^{kJ}\right]\right\}\bar{u}$$

for each $i = 1, \ldots, J$. This implies that for each j for which $\pi_j > 0$, the left hand side of this equation has the same value $u^* < \tilde{u}_j$. Furthermore, for each j for which $\pi_j = 0$, we must have $\tilde{u}_j \leq u^*$. If $\pi_j = 1$ for some j, then $u^* = \tilde{u}_j/kJ$ and the result is immediate. If there are two or more j for which $\pi_j > 0$, then $u^* > \tilde{u}_j/kJ$ for all j. This implies that the functions $g_J(u^*, \tilde{u}_j)$ are all monotonic at u^*, implying that $g_J(u^*, \tilde{u}_j) = \pi_j$.

Alternatively, let u^* be any solution to $G(u, \tilde{u}) = 1$. Then by definition, there exists a vector $\pi = [\pi_1, \ldots, \pi_J]$ such that

$$G_J(u^*, \tilde{u}_1, \ldots \tilde{u}_J) = \sum_{j=1}^{J} g_J(u^*, \tilde{u}_j) = \sum_{j=1}^{J} \pi_j = 1$$

and

$$\sum_{j=1}^{J} \pi_j \left\{\frac{1}{kJ\pi_J^j}\left[1 - \left(1 - \pi_J^j\right)^{kJ}\right]\tilde{u}_j + \left\{1 - \frac{1}{kJ\pi_J^j}\left[1 - \left(1 - \pi_J^j\right)^{kJ}\right]\right\}\bar{u}\right\} \leq u^*$$

[17] This function is exactly analogous to the sum of the marginal cost functions that is used to allocate production in a multiplant firm.

$$\leq \max_j \frac{1}{kJ\pi_J^j}\left[1-\left(1-\pi_J^j\right)^{kJ}\right]\tilde{u}_j + \left\{1-\frac{1}{kJ\pi_J^j}\left[1-\left(1-\pi_J^j\right)^{kJ}\right]\right\}\bar{u}$$

so that the corresponding probabilities are incentive consistent for $\tilde{u}$. ■

Now consider again the vector $\{u', \tilde{u}_1(H,J), \ldots \tilde{u}_{J-1}(H,J)\}$ that is derived from the distribution H of offers along with the deviation u' by one of the sellers. Now for any J and u,

$$g_J(u,u') + \sum_{i=1}^{J-1} g_J(u, \tilde{u}_i(H,J))$$

$$= \frac{J}{J} g_J(u,u') + \frac{J\sum_{i=1}^{J-1} g_J(u,\tilde{u}_i(H,J))}{J} = \int b_J(\tilde{u},u)dH_J(\tilde{u},u')$$

where for each u in the interval $\left[\frac{u_{\max}}{kJ}, u_{\max}\right]$, $b_J(\tilde{u},u)$ is a map from $[\bar{u}, u_{\max}]$ defined as

$$b_J(\tilde{u},u) = \begin{cases} 0 \; if \; \tilde{u} < u \\ b : \frac{1-(1-b/J)^{kJ}}{kb}(\tilde{u}-\bar{u}) = (u-\bar{u}) \; otherwise \end{cases}$$

Notice that for each $i = 1, \ldots, J$, $b_J(\tilde{u}_i(H,J), u) = Jg_J(u, \tilde{u}_i(H,J))$.

The function b_J is relatively simple. In particular, it converges point-wise as J gets large to the inverse function of $\frac{1-e^{-bk}}{bk}$. Furthermore, $H_J(\cdot, u')$ is converging point-wise to the distribution function H as J gets large for any possible deviation u', as has been mentioned above.

Define

$$b(\tilde{u},u) = \begin{cases} 0 \; if \; \tilde{u} < u \\ b : \frac{1-e^{-bk}}{bk} = (\tilde{u}-\bar{u}) \; otherwise \end{cases}$$

Lemma 7 *For each $u > \bar{u}$, and each possible deviation u', $\lim_{J\to\infty} \int b_J(\tilde{u},u)\,dH_J(\tilde{u},u') = \int b(\tilde{u},u)\,dH(\tilde{u})$.*

Proof. If $u > \bar{u}$, there is some J^* large enough that $u > \frac{u_{\max}-\bar{u}}{kJ^*}$. We will restrict attention to the class of functions $\{b_J\}$ for $J > J^*$. Since this sequence of functions converges point-wise to $b(\tilde{u},u)$ on $[\bar{u}, u_{\max}]$, it converges in distribution in the sense of (Hildenbrand 1974)[18].

Second, note that the function

$$\frac{1-(1-b/J)^{kJ}}{bk}$$

is decreasing in J. To see this note that the derivative of this function with respect to J is given by

$$-\left(1-\frac{b}{J}\right)^{kJ}\left(k\ln(1-\frac{b}{J}) + bkJ^{-1}\left(1-\frac{b}{J}\right)^{-1}\right)b^{-1}k^{-1}$$

[18] A sequence of functions $\{f^J\}$ from and interval $[a,b]$ in $\mathbf{R}$ into $\mathbf{R}$ converges in distribution to $f : [a,b] \to \mathbf{R}$ if for every ϵ, the Lebesque measure of the set $\{\mu : \left|f^J(\mu) - f(\mu)\right| > \epsilon\}$ converges to zero as $J \to \infty$.

The sign of this expression is determined by the sign of the expression

$$\ln(1-x)+x\,(1-x)^{-1}$$

for $x \leq 1$ which is readily shown to be positive.

Now when $J = J^*$, $\sup_{\tilde{u}} b_{J^*}(\tilde{u}, u) = b_{J^*}(\tilde{u}_{\max}, u) < J^*$. This implies that for all $J > J^*$, $\int b_J(\tilde{u}, u)\, dH_J(\tilde{u}, u') < J^*$ which makes the family of functions $\{b_J\}$ uniformly integrable.

By the construction of the approximation, $H_J(u) \leq H(u)$. Then for all x, and all $J \geq J^*$,

$$H\{\tilde{u} : b(\tilde{u}) \geq x\} \geq H\{\tilde{u} : b_J(\tilde{u}) \geq x\} \geq H_J\{\tilde{u} : b_J(\tilde{u}) \geq x\}$$

The first inequality follows because of the fact that $b_J(\tilde{u})$ is a decreasing function, while the sequence $b_J(x)$ is a declining sequence for all x. The second inequality follows as above from the construction of the partition.

The convergence of the integral then follows from the Generalized Lebesque theorem given in Hildenbrand (1974)(p 52). ■

Lemma 8 *The sequence of solutions $\{u_J\}$to the equation*

$$\sum_{i=1}^{J} g_J(u, \tilde{u}_i(H, J)) = 1$$

has a limit u^* that is given by the solution to the equation

$$\int b(\tilde{u}, u)\, dH(\tilde{u}) = 1$$

.

Proof. Suppose the sequence $\{u_J\}$ has a limit point $u_\infty > u^*$ for example. Let $\delta = |u_\infty - u^*|$. Since the integral $\sum_{i=1}^{J} g_J(u, \tilde{u}_i(H, J))$ converges point-wise in u by Lemma 7, and is continuous, it converges uniformly on the interval $[u^* - \delta, u^* + \delta]$. Hence for any ϵ there is a J^* large enough that $\left|\sum_{i=1}^{J} g_J(u, \tilde{u}_i(H, J)) - \int b(\tilde{u}, u)\, dH(\tilde{u})\right| < \epsilon$ for all $J > J^*$ and for all $u \in [u^* - \delta, u^* + \delta]$. Let $\epsilon = |\int b(\tilde{u}, u^*)\, dH(\tilde{u}) - 1| > 0$. Then this implies we can choose J^* large enough that $\sum_{i=1}^{J} g_J(u, \tilde{u}_i(H, J)) < 1$ for all $J > J^*$, a contradiction. ■

As the approximation to H becomes finer and finer as J goes to infinity, the payoff that buyers receive in the equilibrium continuation must converge to a payoff u^* that is independent of the utility level offered by the deviating seller. For any distribution H, let $u^*(H)$ denote this limit payoff. Let u' be the utility level that is offered by the

deviating seller. If the deviating seller is chosen with positive probability, then the payoff that buyers receive if they select the deviator is converging to

$$u^*(H) = \lim_{J\to\infty} \frac{1-(1-\pi'_J)^{kJ}}{kJ\pi'} (u' - \bar{u}) + \bar{u}$$

where π' is the probability with which all buyers select the deviator in the continuation equilibrium. On the other hand, the probability with which the deviating seller trades is given by the probability that at least one of the buyers selects him. This implies that the limiting value of the probability with which the deviating seller trades is given by

$$\lim_{J\to\infty} \left[1-(1-\pi')^{kJ}\right]$$

Lemma 9 *For any deviation $u' < u^*(H)$,*

$$\lim_{J\to\infty} \left[1-(1-\pi')^{kJ}\right] = \begin{cases} 0 \; if \; u' < u^*(H) \\ 1-e^{-k'} \; otherwise \end{cases}$$

where k' is the solution to the equation

$$\frac{1-e^{-k}}{k}(u' - \bar{u}) = u^*(H) - \bar{u}$$

Proof. First, for $u' < u^*(H)$ there must be some large J^* such that the equilibrium payoff in the continuation strictly exceeds u'. Since $\frac{1-(1-\pi'_J)^{kJ}}{kJ\pi'_J}$ is bounded above by 1, then the payoff to a buyer who visits the deviator is smaller than the equilibrium payoff in the continuation no matter what the value of π'_J is. Hence in this case, the Lemma is immediate.

So suppose instead that $u' \geq u^*$. Then we must have $\lim_{J\to\infty} \pi'_J = 0$, for otherwise the buyers' trading probability with the deviator goes to zero, the payoff with the deviator goes to $\bar{u}$, and incentive consistency is violated. Let $\tilde{J}_J$ and $\tilde{k}_J$ solve $\pi'_J = 1/\tilde{J}_J$ and

$$\frac{1-(1-\pi'_J)^{kJ}}{kJ\pi'_J} = \frac{1-\left(1-\frac{1}{\tilde{J}_J}\right)^{\tilde{k}_J\tilde{J}_J}}{\tilde{k}_J}$$

Since $\lim_{J\to\infty} \tilde{J}_J = \infty$ and

$$\lim_{\tilde{J}_J\to\infty} \frac{1-\left(1-\frac{1}{\tilde{J}_J}\right)^{\tilde{k}_J\tilde{J}_J}}{\tilde{k}_J}(u' - \bar{u}) = (u^* - \bar{u})$$

the result follows. ■

This argument completes the proof of the Theorem.

7.4 Proof of Theorem 4

Proof. Let σ^s and σ^b denote an allocation that can be supported as an equilibrium allocation for the limit game in an economy with the classical matching technology. We will now construct the corresponding Walrasian allocation for the economy with the exponential matching technology.

First note that the equilibrium for the limit game generates an equilibrium distribution H^* and equilibrium utility level $u^*(H^*)$. Let us begin by assigning to each contract x in Ξ the expectation $k^*(u(x))$ defined as the solution to the equation

$$\bar{u} + \frac{1-e^{-k}}{k}\left[u(x) - \bar{u}\right] = u^*(H^*)$$

Then evidently

$$\sigma^s(x) = G(\{s : \hat{x}(s, H^*) \leq x\})$$

$$= G\left\{\left\{s : \operatorname{argmax}\left[\left[1-e^{-k}\right] v(x,s) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}\left[u(x) - \bar{u}\right] \geq u^*(H^*)\right] \leq x\right\}\right\}$$

$$\equiv G\{s' \in S : \exists x' < x; V(x', s', k^*(u(x'))) \geq V(y, s', k^*(u(y)))\forall y \in X\}$$

Under the assumption 2, the set $\{s : \hat{x}(s, H^*) \leq x\}$ is union of intervals, so for some such union S^x, σ^s can be written as $\sigma^s(x) = G(S^x)$. Now let S^1 denote the set of points at which the distribution function G jumps. Then

$$\sigma^s(x) = \sum_{y \in S^x \cup S^1} G^j(s) + \int_{S^x} g(s)\, ds$$

This equation uses the assumption that G can be decomposed into a jump function and an absolutely continuous function. Now using a standard change of variable theorem for measurable transformations ((Burrill 1978)), this can be written as

$$= \sum_{x \in \hat{x}\{[s', s''] \cup S_1\}} G^j\left(\hat{x}^{-1}(x)\right) + \int_{\hat{x}(s')}^{\hat{x}(s'')} g\left(\hat{x}^{-1}(y)\right) e(y)\, dy \tag{8}$$

for some measurable function $e(y)$. This verifies the decomposition of σ^s. The distribution function σ^s is then an aggregate supply function for the Walrasian economy with the exponential matching technology.

It remains to show that the induced allocation σ^b is an aggregate demand function in the sense of the definition of Walrasian equilibrium, and that the generalized market clearing condition holds.

Now for any interval $[u_0, u']$, in equilibrium

$$H^*(u') = G(\{s : \hat{u}(s) \leq u'\}) = G(\{s : u(\hat{x}(s)) \leq u'\})$$

$$= \sigma^s \left(T^{-1} \left([u_0, u'] \right) \right).$$

Now $H^*(u) = 0$ if $u < u^*(H^*)$; $H^*(u^*(H^*)) = G(\underline{s})$ where $\underline{s} = \sup \{ s : \hat{u}(s) \leq u^*(H^*) \}$. Otherwise, since $\hat{u}$ is strictly increasing on the rest of it's domain, $H^*(u') = G(s')$ for some s'. It follows that $H^*(u)$ can be written as

$$H^*(u) = G(s') = G(\underline{s}) + \sum_{S^1 \cap (\underline{s}, s']} G^j(s) + \int_{\underline{s}}^{s'} g(y)\, dy$$

$$= G\left(\hat{u}^{-1}(u^*(H^*)) \right) + \sum_{u \in \hat{u}(\{S^1 \cap (\underline{s}, s']\})} G^j\left(\hat{u}^{-1}(u) \right) + \int_{u^*}^{\hat{u}(s')} g\left(\hat{u}^{-1}(u) \right) f(u)\, du \tag{9}$$

for some measurable function $f(u)$[19]. This verifies that H^* can be written as the sum of a jump function with a finite support and an absolutely continuous function.

It follows from a standard change of variables (Burrill 1978, Theorem 8.2) that

$$\sigma^b \left(T^{-1} \left([u_0, u'] \right) \right) = \int_{T^{-1}[u_0, u']} k^* \cdot T d\sigma^s(y) = \int_{u_0}^{u'} k^*(u)\, dH^*(u)$$

and using the decomposition of H^* given in (9), this equals

$$\int_{u_0}^{u'} k^*(u)\, d\left[H_j^*(u) + \int_{u^*}^{u} h(u')\, du' \right]$$

where $H_j^*(u) = G(\hat{u}^{-1}(u^*(H^*))) + \sum_{u \in \hat{u}(\{S^1 \cap (\underline{s}, s']\})} G^j(\hat{u}^{-1}(u))$ and $h(u) = g(\hat{u}^{-1}(u))$. This gives

$$\sigma^b \left(T^{-1} \left([u_o, u'] \right) \right) = \sum_{u \in \hat{u}(\{S^1 \cap (\underline{s}, s']\})} k^*(u)\, G^j\left(\hat{u}^{-1}(u) \right) + \int_{u_0}^{u'} k^*(u)\, g\left(\hat{u}^{-1}(u) \right) f(u)\, du$$

because of the fact that $k^*(u^*) = 0$. The right hand side of this equation equals

$$\sum_{S^1 \cap (\underline{s}, s']} k^*(\hat{u}(s))\, G^j(s) + \int_{\underline{s}}^{s'} k^*(\hat{u}(y))\, g(y)\, dy$$

Finally using (8), this is equal to

$$= \sum_{x \in \hat{x}\{[s', s''] \cup S_1\}} k^*(u(x))\, G^j\left(\hat{x}^{-1}(x) \right) + \int_{\hat{x}(s')}^{\hat{x}(s'')} k^*(u(y))\, g\left(\hat{x}^{-1}(y) \right) e(y)\, dy \tag{10}$$

Now σ^b can be written

$$\int_{u_0}^{u'} \bar{k} b(u, u^*)\, dH^*(u)$$

[19] For example, if $\hat{u}$ is differentiable, then $f(u) = \left. \frac{1}{\hat{u}'(s)} \right|_{\hat{u}(s)=u}$. See (Burrill 1978, Theorem 7-8b).

where $\bar{k}b(u,u^*) \equiv k^*(u)$. Since this integral is increasing in u', and since

$$\int_{u_0}^{u^1} \bar{k}b(u,u^*)\,dH^*(u) = \bar{k}N \equiv M$$

by Lemma 7, $\sigma^b(\Xi) = M$. Finally, since buyers are identical, and expectations are chosen so that buyers are indifferent among all contracts,

$$F\left\{b' \in B : \exists x' < x; U(x', k^*(u(x'))) \geq U(y, k^*(u(y)))\forall y \in X\right\}$$

is equal either to 0 or M. Hence the allocation of buyers is an aggregate demand function.

To verify the rational expectation condition simply take the generalized derivatives of (10) and (8).

The converse argument is straightforward. The Walrasian equilibrium with identical buyers generates an equilibrium level of utility which buyers get in all active markets. This is the utility level u^* for the equilibrium in the decentralized model. The distribution of payoffs implied by the Walrasian equilibrium is the equilibrium distribution H. The details are straightforward and are omitted. ■

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